

On Mostow's Rigidity Theorem

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Abstract

This is a survey surrounding Mostow's rigidity theorem on hyperbolic manifolds. We will present Mostow's proof for the compact case. Then we will give some applications of the theorem.

1 Introduction

A Riemannian manifold is called hyperbolic if it is connected and has constant sectional curvature -1 . It is called complete if all the geodesics can be extended indefinitely.

This article is a survey surrounding the following theorem:

Theorem 1.1 (Mostow-Prasad). *Suppose M, N are complete hyperbolic n -manifolds of finite volume, $n \geq 3$. Suppose $f : M \rightarrow N$ is a homotopy equivalence. Then f is homotopic to an isometry from M to N .*

This theorem was proved by Mostow in 1968 for compact manifolds. In 1971, Prasad generalized Mostow's result to the finite volume cases.

By Cartan-Hadamard theorem, the universal covers of M and N are diffeomorphic to \mathbb{R}^n , thus $\pi_i(M) \cong \pi_i(N) \cong 0$ when $i \geq 2$, and hence M, N are Eilenberg-MacLane spaces. Therefore, the homotopy types of M and N are completely determined by their fundamental groups, and theorem 1.1 implies the following (seemingly stronger) result:

Corollary 1.2. *If M, N are complete hyperbolic n -manifolds of finite volume, $n \geq 3$, and suppose that $\pi_1(M) \cong \pi_1(N)$. Then M and N are isometric.*

Mostow's rigidity theorem is essential for the understanding of flexibility and rigidity of hyperbolic structures.

In two dimensional hyperbolic geometry the hyperbolic structures are flexible. That is to say, a hyperbolic structure on a fixed differentiable manifold can be perturbed to a different hyperbolic structure. In fact, when $g \geq 2$, the hyperbolic structures on a closed genus g surface form a moduli space of real dimension $(6g - 6)$. (We will discuss it in more detail in the appendix)

However, Mostow's rigidity theorem states that such flexibility does not exist in dimension at least three. In that case, the hyperbolic

structure on a fixed differentiable manifold is fixed: there is at most one complete hyperbolic structure of finite volume up to isometries homotopic to the identity map.

However, for three manifolds, there is another side of the coin. Although it is impossible to (non-trivially) perturb the hyperbolic structure on a fixed manifold, the hyperbolic structure can be perturbed if we are allowed to change the topology of the manifold. To make it precise, there is a topology defined on the set of all the complete hyperbolic 3 manifolds, called the geometric topology. It is proved by Thurston that under the geometric topology, many complete hyperbolic manifolds of finite volume are cluster points. The full theorem is stated as follows:

Theorem 1.3 (Thurston-Jorgensen). *Let \mathcal{A}_0 be the set of complete hyperbolic 3-manifolds whose volumes are finite. Let \mathcal{A}_{i+1} be the set of cluster points of \mathcal{A}_i under the geometric topology ($i = 0, 1, \dots$). Then $\mathcal{A}_i = \{M \in \mathcal{A} \mid M \text{ has } i \text{ cusps}\}$. In particular, \mathcal{A}_i are not empty.*

The part that $\mathcal{A}_i \subseteq \{M \in \mathcal{A} \mid M \text{ has } i \text{ cusps}\}$ in theorem 1.3 was proved by a construction described as follows: suppose M is a non-compact complete hyperbolic manifold of finite volume, there is a sequence $\{M_i\} \subseteq \mathcal{A}$ so that $M_i \rightarrow M$ in the geometric topology. M_i is obtained by first perturb the metric of M then take the metric completion under the new metric. M_i has one less cusps than M , and their topologies are obtained by Dehn fillings of M . This part of theorem 1.3 is often referred to as ‘‘Thurston’s non-rigidity theorem’’.

We will not present the proof of theorem 1.3. The reader may refer to [4], chapter E for a proof.

When the dimension is at least 4, even the flexibility given by Thurston’s non-rigidity theorem does not exist. In fact, H.-C. Wang has proved the following result in 1972:

Theorem 1.4 (H.-C. Wang). *For each positive number $V > 0$ and $n \geq 4$, there are only finitely many complete hyperbolic n -manifolds with volume at most V .*

Since the volume function is continuous under the geometric topology, Thurston’s non-rigidity theorem cannot hold in dimension at least 4. Theorem 1.4 suggests that hyperbolic n -manifolds with $n \geq 4$ are very rare. Again, we will not to prove this theorem here. For a proof, see [4], Theorem E.3.2.

The above theorems suggest that 3 dimensional hyperbolic geometry is special: on the one hand, by Mostow's rigidity theorem, the hyperbolic structure on a compact 3-manifold is uniquely determined by its topology; on the other hand, unlike the case of dimension ≥ 4 , there are plenty of examples of hyperbolic 3-manifolds. This observation vaguely indicates that 3-dimensional hyperbolic geometry might play an important role in the topology of 3-dimensional manifolds.

In 1982, Thurston proposed his Geometrization Conjecture in [9]. The conjecture states that every compact 3 dimensional topological manifold can be decomposed into pieces so that each has one of eight types of geometry structure. Among the eight types of geometries, seven are considered as the degenerate cases and manifolds with such geometries are very well understood. The last case, which is hyperbolic geometry, is the richest one and it is considered as the essential case. Therefore, according to the conjecture, hyperbolic 3-manifolds form the main ingredient of topological 3-manifolds, and a generic 3-manifold would be hyperbolic. For these manifolds, Mostow's rigidity theorem states that their hyperbolic structures are uniquely determined by its topology, so their hyperbolic invariants (such as volume, injective radius) are also topological invariants.

Mostow's rigidity theorem can also be applied to define invariants for knots and links. Let $L \subseteq S^3$ be a link. Then, $S^3 - L$ is a 3-manifold. In many cases, $S^3 - L$ can be endowed with a complete hyperbolic structure of finite volume. If this is true then L is called a hyperbolic link. By Mostow's rigidity theorem, the hyperbolic structure is unique if it exists. Therefore, every hyperbolic invariant can be used to define a link invariant. Such invariants are called hyperbolic invariants in the theory of knots and links. In particular, the hyperbolic volume (i.e. the volume of $S^3 - L$ under the corresponding hyperbolic structure) is a hyperbolic invariant for knots and links. For a discussion of other kinds of hyperbolic invariants, the reader may refer to [2].

Hyperbolic invariants of knots and links may not be impressive at first sight. By Mostow's rigidity theorem, these invariants are weaker than the knot and link groups, and they are not always well defined. However, Thurston has proved that non-hyperbolic knots are either torus knots or satellite knots, thus in some sense the majority of prime knots are hyperbolic ([1], page 119). Furthermore, hyperbolic invariants are easy to compute and compare: they are usually real or complex numbers which can be computed by computers. In [2], for example, 6 types of hyperbolic invariants for all knots with no more than 10

crossings and all links with no more than 9 crossings are numerically computed. Although hyperbolic invariants are weaker than the knot and link groups, they are sensitive enough in many cases. Even the hyperbolic volume is usually sensitive enough. As said on [1] (page 124), empirically the examples of different hyperbolic knots with the same hyperbolic volume are rare.

In this article we will give a proof of theorem 1.1 for the compact case, and discuss some of its applications. In sections 2, 3, and 4 we will present background knowledge in hyperbolic geometry, quasi-conformal maps and ergodic theory which are necessary for the proof. In section 5 we will present the proof, which was originally given by Mostow. Our exposition follows the outline given in the fifth chapter of [8]. In section 6, we will discuss some applications of theorem 1.1. A method of computing hyperbolic invariants of knots and links will be introduced in that section.

2 Basic hyperbolic geometry

In this section we recall some basic facts from hyperbolic geometry. They are necessary in the discussions of section 5 and section 6.2.

2.1 The ball model and the half space model

By Cartan-Hadamard theorem, the universal cover of every complete hyperbolic n -manifold is diffeomorphic to \mathbb{R}^n . By a well known fact of Riemannian geometry, there is only one simply-connected Riemannian manifold up to isometry which has constant sectional curvature -1 . Therefore, for any dimension n , there is a simply-connected Riemannian n -manifold which is the universal cover of every complete hyperbolic n -manifold.

This universal cover can be constructed explicitly in many different ways. We are going to need two constructions, which are called the “ball model” \mathbb{D}^n and the “half space model” \mathbb{H}^n .

The ball model \mathbb{D}^n is the open ball

$$\{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$$

endowed with a Riemann tensor:

$$ds^2 = 4 \frac{\sum_i dx_i^2}{(1 - \sum_i x_i^2)^2}$$

The upper half space model \mathbb{H}^n is the open half space

$$\{(x_1, \dots, x_n) \in \mathbb{R}^n | x_n > 0\}$$

endowed with a Riemann tensor:

$$ds^2 = \sum_i \frac{dx_i^2}{x_n^2}$$

It is well known in Riemannian geometry that both \mathbb{D}^n and \mathbb{H}^n are complete Riemannian manifolds with constant sectional curvature -1 . Since they are both simply-connected, \mathbb{D}^n and \mathbb{H}^n are isometric and they are the universal cover of every complete hyperbolic n -manifold. We can explicitly write down an isometry from \mathbb{D}^n to \mathbb{H}^n . Let $p = (0, \dots, 0, -1)$. Then the map

$$\begin{aligned} \mathbb{D}^n &\rightarrow \mathbb{H}^n \\ x &\mapsto p + \frac{2(x-p)}{\|x-p\|^2} \end{aligned}$$

is an isometry from \mathbb{D}^n to \mathbb{H}^n .

Sometimes it is useful to consider the boundaries of \mathbb{D}^n and \mathbb{H}^n . The boundary of \mathbb{D}^n is defined to be its boundary as a subset of \mathbb{R}^n , which is the unit sphere S^{n-1} . The boundary of \mathbb{H}^n is defined to be the $\{(x_1, \dots, x_n) | x_n = 0\} \cup \infty$. We denote them by $\partial\mathbb{D}^n$ and $\partial\mathbb{H}^n$. Both of them can be identified with a $(n-1)$ -sphere. Given a simply-connected complete hyperbolic manifold M , we can define its “sphere at infinity” or “boundary at infinity” by first identify M with \mathbb{D}^n or \mathbb{H}^n , then take the boundary of \mathbb{D}^n and \mathbb{H}^n . We will see in the next subsection that every isometry between \mathbb{D}^n and \mathbb{H}^n , and every self-isometry of \mathbb{D}^n and \mathbb{H}^n can be extended to conformal homeomorphisms on the boundaries. Therefore, the boundary at infinity of M is well defined and has a conformal structure.

The geodesics of \mathbb{D}^n and \mathbb{H}^n are lines or circles (of the ambient Euclidean space) which are orthonormal to the boundary. For a geodesic in \mathbb{D}^n or \mathbb{H}^n , we call its intersection with the boundary the “ends” of that geodesic. We will see in the next subsection that the ends of geodesics are equivariant under isometries of \mathbb{D}^n and \mathbb{H}^n , thus we can define the ends of a geodesic on any simply-connected complete hyperbolic space. Given two different points on the boundary, there is a unique geodesic connecting them.

2.2 Isometries of \mathbb{D}^n and \mathbb{H}^n

Suppose p is a point in \mathbb{R}^n , r is a positive number. We define the inversion centered at p with radius r to be the following map:

$$\begin{aligned} i_{p,r} : \mathbb{R}^n - \{p\} &\rightarrow \mathbb{R}^n - \{p\} \\ x &\mapsto p + \frac{r^2(x-p)}{\|x-p\|^2} \end{aligned}$$

$i_{p,r}$ is a conformal map, and it can be extended to a conformal self-homeomorphism of $\overline{\mathbb{R}^n}$.

When $n \geq 3$, we have the following theorem due to Liouville:

Theorem 2.1 (Liouville). *Every conformal diffeomorphism between two connected open subsets of \mathbb{R}^n is of the form*

$$x \mapsto \lambda Ai(x) + b \tag{1}$$

where $\lambda > 0$, $A \in O(n)$, i is an inversion or the identity.

Now consider the isometries of \mathbb{D}^n and \mathbb{H}^n . Both \mathbb{D}^n and \mathbb{H}^n can be conformally embedded to \mathbb{R}^n as its open sub-manifolds. When $n \geq 3$, by Liouville's theorem, isometries on \mathbb{D}^n and \mathbb{H}^n are given by formula (1). When $n = 2$, the same result can be proved by the Schwartz lemma. Such maps extend to conformal self-homeomorphisms of $\overline{\mathbb{R}^n}$, therefore any isometry on \mathbb{D}^n and \mathbb{H}^n gives a conformal homeomorphism of the boundaries.

It can be verified that every homeomorphism of \mathbb{D}^n and \mathbb{H}^n with form (1) is an isometry for the hyperbolic metric.

By Liouville's theorem again, every conformal self-homeomorphism of $\overline{\mathbb{R}^n}$ is given by formula (1), provided that n is at least 3. When $n = 2$, by complex analysis we have that any conformal self homeomorphism of $\overline{\mathbb{C}}$ is either a Möbius transformation or its conjugate, hence the same result still holds. Therefore every conformal self-homeomorphism of $\partial\mathbb{D}^n$ or $\partial\mathbb{H}^n$ can be extended to a conformal self-homeomorphism on $\overline{\mathbb{D}^n}$ or $\overline{\mathbb{H}^n}$, which is also an isometry for the hyperbolic metric.

It can be verified that the correspondence between isometries of $\mathbb{D}^n, \mathbb{H}^n$ and the conformal homeomorphisms of the boundaries is one-to-one.

In conclusion, when $n \geq 3$, there is a one-one correspondence between isometries of complete simply-connected hyperbolic n -manifolds and conformal homeomorphisms of their boundaries at infinity. For an isometry from \mathbb{D}^n to \mathbb{D}^n , the corresponding boundary map is its continuous extension to the boundary.

2.3 Horospheres and ideal simplices

This section discusses horospheres and ideal simplices. The materials in this section will not be used in the proof of Mostow's theorem in section 5, but they will be used in section 6.2.

Let M be a simply-connected complete hyperbolic n -manifold. Let p be a point on its boundary at infinity. For any $x \in M$, there is a unique geodesic $l_{x,p}$ passing through x and has p as one of its ends. Define $V_{x,p}$ as the orthogonal complement of $l_{x,p}$ in $T_x M$. Then, $\{V_{x,p}\}_{x \in M}$ is a distribution on M . We claim that such distribution is integrable, and its integrate manifolds has Euclidean metric when take the restricted metric from M .

The claim can be proven by taking $M = \mathbb{H}^n$ and $p = \infty$. Then, $V_{x,p}$ is always parallel to the boundary $(n - 1)$ plane of \mathbb{H}^n , and the claim is obvious.

The integrate manifolds of $\{V_{x,p}\}_{x \in M}$ are called horospheres centered at p . In the ball model, they are spheres inside \mathbb{D}^n which are tangent to p .

Horospheres can be used to study the properties of ideal simplices. Here we only consider the 3 dimensional case. Let M be a simply-connected complete hyperbolic 3-manifold. Let \overline{M} be the union of M and its boundary at infinity. Then for every two different points in \overline{M} there is a unique geodesic segment connecting them, thus the concept of convexity is well-defined on \overline{M} . By definition, an ideal simplex is the convex hull of 4 points on the infinite boundary of M .

Suppose T is an ideal simplex with vertices A, B, C, D . Let's denote $\angle AB$ to be the angle between the two surfaces of T containing the edge AB . Denote similarly for other edges. For a horosphere centered at A and close enough to A , its intersection with T is a Euclidean triangle, (this can be seen from figure 1, where M is taken to be \mathbb{H}^3 and p is taken to be ∞), let's denote it by S_A . The angles of S_A equals $\angle AB, \angle AC, \angle AD$. Therefore, we have

$$\angle AB + \angle AC + \angle AD = \pi \tag{2}$$

Similarly we can obtain three other equations from the vertices B, C and D . Combining these equations, we will get:

$$\angle AB = \angle CD \tag{3}$$

$$\angle AC = \angle BD \tag{4}$$

$$\angle AD = \angle BC \tag{5}$$

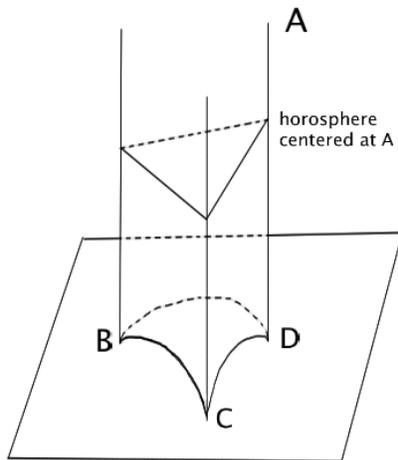


Figure 1: An ideal simplex in \mathbb{H}^3 , with a vertex equals ∞

Conversely, it can be easily proved that for any given (positive) angles satisfying equations (2), (3),(4),(5), there is a unique ideal simplex up to isometry realizing these angles. (For a proof, take the half space model again and take one vertex to be ∞ .)

We introduce the Lobachevsky function:

$$\Lambda(\theta) = - \int_0^\theta \log |2 \sin t| dt \quad (\theta \in \mathbb{R})$$

Then we have the following formula for the volume of an ideal simplex:

$$\text{Volume}(T) = \Lambda(\angle AB) + \Lambda(\angle AC) + \Lambda(\angle AD) \quad (6)$$

A proof of this formula is given in [4], Section C.2.

3 n dimensional quasi-conformal mappings

This section introduces some basic results about n -dimensional quasi-conformal maps. They are essential for Mostow's proof of the rigidity theorem. The proofs of the theorems in this section would lead us too far away, thus we will leave out the proofs and only give references.

The reader may refer to [10] for a self-contained treatment of what is presented here.

Definition 3.1. Let $f : \Omega_1 \rightarrow \Omega_2$ be a homeomorphism between two open subsets of \mathbb{R}^n . For a point $x \in \Omega$, define

$$\begin{aligned} L(x, f) &= \limsup_{y \rightarrow x} \frac{|f(y) - f(x)|}{|y - x|} \\ l(x, f) &= \liminf_{y \rightarrow x} \frac{|f(y) - f(x)|}{|y - x|} \\ H(x, f) &= \limsup_{r \rightarrow 0} \frac{\sup_{|y-x|=r} |f(y) - f(x)|}{\inf_{|y-x|=r} |f(y) - f(x)|} \end{aligned}$$

If f is differentiable at x , define $J(x, f)$ to be the determinant of the Jacobian of f at x .

Definition 3.2. Let Ω be an open subset of \mathbb{R}^n , and let f be a continuous function defined on Ω . Then f is called *ACL (Absolutely Continuous on Lines)* if f is absolutely continuous on almost every line segment in Ω parallel to the coordinate axes.

By standard results from real analysis, any ACL function f has partial derivatives almost everywhere, the partial derivatives are locally integrable, and they are also the distributional partial derivatives. The converse is also true. In fact, we have the following theorem:

Theorem 3.3 ([3], page 19). *If a continuous function f has locally integrable distributional derivatives, then it is ACL.*

The proof given in [3] is for the dimension 2 case, but it also works for the general case.

A map from an open set of \mathbb{R}^n to \mathbb{R}^m is called ACL all of its coordinate functions are ACL.

Since the distributional derivatives satisfy the coordinate transformation formulas, the ACL property is invariant under coordinate changes. Therefore, the ACL property can be defined for maps between manifolds.

Now we define quasi-conformal maps:

Definition 3.4 (Analytic definition of quasi-conformal maps). Let $f : \Omega_1 \rightarrow \Omega_2$ be a homeomorphism of two open subsets of \mathbb{R}^n . Then f

is called quasi-conformal if $\exists K > 0$, such that f is ACL, differentiable a.e., and for a.e. $x \in \Omega_1$, the following inequality holds:

$$\frac{L(x, f)^n}{K} \leq |J(x, f)| \leq K l(x, f)^n$$

We have the following theorem:

Theorem 3.5 ([10] theorem 34.1, theorem 34.6). *Let $f : \Omega_1 \rightarrow \Omega_2$ be a homeomorphism of two open subsets of \mathbb{R}^n . Then the following two statements are equivalent:*

- (1) $H(x, f)$ is bounded.
- (2) $\exists K > 0$, so that f is K -quasi-conformal

An important theorem by Gehring and Reshetnjak states that all 1-quasi-conformal maps are conformal. We will need the following special case of it in section 5:

Theorem 3.6. *Any 1-quasi-conformal mapping from S^n to S^n is conformal, thus it has the form of 1*

A short proof of this theorem can be found in Mostow's paper [7], page 101-102. The prove uses a geometric characterization of quasi-conformal maps, which we did not present here.

4 Basic ergodic theory

The goal of this section is to prove the following theorem:

Theorem 4.1. *The geodesic flow of a compact hyperbolic manifold is ergodic.*

First let's give a definition to ergodicity.

Let G be a locally compact second countable group, (X, μ) be a σ -finite measure space.

Definition 4.2. A measurable action of G on X is called *measure class preserving*, if for any measurable subset A of X , $\mu(gA) = 0$ if and only if $\mu(A) = 0$. G is called *measure preserving* if for any measurable subset A of X , $\mu(A) = \mu(gA)$.

Definition 4.3. An measure class preserving action of G on X is called *ergodic* if for every G -invariant measurable subset A of X , either $\mu(A) = 0$ or $\mu(X - A) = 0$.

Suppose G acts on X by a measure preserving action, then G unitarily acts on $L^2(X, \mu)$ as by

$$G \times L^2(X, \mu) \rightarrow L^2(X, \mu), (g, f) \mapsto f \circ g^{-1}$$

We have the following property:

Proposition 4.4. *Suppose $\mu(X)$ is finite. A measure preserving action of G on X is ergodic if and only if any element in $L^2(X, \mu)$ invariant under G is a.e. constant.*

Proof. If the action is not ergodic, then there is a measurable set $A \subset X$ such that $\mu(A) > 0, \mu(X - A) > 0$. Therefore χ_A is a non-constant G -invariant function in $L^2(X, \mu)$.

Conversely, if f is a G -invariant function in $L^2(X, \mu)$ which is not a.e. constant, then there exists a real number a such that the set $A = \{x \in X | f(x) > a\}$ satisfies $\mu(A) > 0, \mu(X - A) > 0$, and A is invariant under G . Thus the action is not ergodic. \square

In the following, we will take $G = \mathbb{R}$ under the additive group. We will need the following result in the proof of theorem 4.1:

Theorem 4.5 (von Neumann). *Suppose the action of \mathbb{R} on X preserves the measure. Let $F \subseteq L^2(X, \mu)$ be space of invariant elements. Since \mathbb{R} acts unitarily on $L^2(X, \mu)$, F is a closed subspace. Let P be the orthogonal projection of $L^2(X, \mu)$ onto F .*

Suppose the action of \mathbb{R} on $L^2(X, \mu)$ is continuous, thus the integration

$$\int_0^N t \cdot f dt$$

is well defined.

Then we have:

$$Pf = \lim_{N \rightarrow +\infty} \frac{1}{N} \int_0^N t \cdot f dt \tag{7}$$

The limit is taken with the L^2 norm.

Proof. We prove the theorem in three steps.

First, we show that

$$F = \text{span}\{t - t \cdot g | g \in L^2(X, \mu), t \in \mathbb{R}\}^\perp \tag{8}$$

In fact, for any $f \in F$, $g \in L^2(X, \mu)$, we have

$$\begin{aligned}\langle f, g - t \cdot g \rangle &= \langle f, g \rangle - \langle (-t) \cdot f, g \rangle \\ &= \langle f, g \rangle - \langle f, g \rangle \\ &= 0\end{aligned}$$

On the other hand, if $f \in \{t - t \cdot g | g \in L^2(X, \mu), t \in \mathbb{R}\}^\perp$, then for any $g \in L^2(X, \mu)$:

$$\begin{aligned}\langle f - t \cdot f, g \rangle &= \langle f, g \rangle - \langle f, (-t) \cdot g \rangle \\ &= \langle f, g - (-t) \cdot g \rangle \\ &= 0\end{aligned}$$

Thus $f \in F$ and (8) is proved.

Next, we prove (7) for elements in F and $\{t - t \cdot g | g \in L^2(X, \mu), t \in \mathbb{R}\}$.

If $f \in F$, then

$$Pf = f = \frac{1}{N} \int_0^N t \cdot f \, dt$$

for any N , hence (7) holds.

If $f = t - t \cdot g$, then by (8) $Pf = 0$. On the other hand, when $N > 2t$,

$$\begin{aligned}\left| \int_0^N t \cdot f \, dt \right| &= \left| \int_0^t t \cdot g \, dt - \int_{N-t}^N t \cdot g \, dt \right| \\ &\leq 2t \|g\|_{L^2(X)}\end{aligned}$$

Therefore

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \int_0^N t \cdot f \, dt = 0$$

and (7) holds.

Finally, we prove (7) for every $f \in L^2(X, \mu)$. By (8), the subspace spanned by F and $\{t - t \cdot g | g \in L^2(X, \mu), t \in \mathbb{R}\}$ is dense in $L^2(X, \mu)$. Hence for $\forall f \in L^2(X, \mu), \varepsilon > 0$, there $\exists f_0 \in L^2(X, \mu)$ such that (7)

holds for f_0 and $\|f - f_0\| < \varepsilon$. We have

$$\begin{aligned}
& \limsup_{N \rightarrow +\infty} \left\| Pf - \frac{1}{N} \int_0^N t \cdot f \, dt \right\| \\
& \leq \limsup_{N \rightarrow +\infty} \left\| P(f - f_0) - \frac{1}{N} \int_0^N t \cdot (f - f_0) \, dt \right\| \\
& \quad + \limsup_{N \rightarrow +\infty} \left\| Pf_0 - \frac{1}{N} \int_0^N t \cdot f_0 \, dt \right\| \\
& = \limsup_{N \rightarrow +\infty} \left\| P(f - f_0) - \frac{1}{N} \int_0^N t \cdot (f - f_0) \, dt \right\| \\
& \leq 2\|f - f_0\| \\
& \leq 2\varepsilon
\end{aligned}$$

Since ε can be arbitrarily small, this proves that (7) holds for any $f \in L^2(X, \mu)$. \square

Remark. If we apply theorem 4.5 to another action ρ of \mathbb{R} defined by

$$\rho : \mathbb{R} \times X \rightarrow X, (t, x) \mapsto (-t) \cdot x$$

We will have

$$Pf = \lim_{N \rightarrow +\infty} \frac{1}{N} \int_{-N}^0 t \cdot f \, dt \quad (9)$$

Now we discuss the ergodicity of geodesic flow.

Suppose M is a compact Riemannian manifold. Let T_1M be the unit tangent bundle of M , that is, the manifold consists of all the unit tangent vectors of X . For $\forall(x, v) \in T_1M$, where $x \in M$ and $v \in T_xM$, there exists a unique geodesic $\gamma_{x,v} : \mathbb{R} \rightarrow M$, such that $\gamma_{x,v}(0) = x$ and $\dot{\gamma}_{x,v}(0) = v$. Define an action of \mathbb{R} on T_1M as follows:

$$\begin{aligned}
\mathbb{R} \times T_1M & \rightarrow T_1M \\
(t, (x, v)) & \mapsto (\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t))
\end{aligned}$$

Such an action is called the geodesic flow on T_1M .

A Riemannian metric can be defined on T_1M as follows. Suppose

$$\alpha(t) = (x(t), v(t)), \beta(t) = (y(t), w(t))$$

are two curves in T_1M , and $\alpha(0) = \beta(0)$. For a fixed t , let $\tilde{v}(t)$ be the parallel translation of $v(t)$ to the point $x(0)$ along the curve x , and

$\tilde{w}(t)$ be the parallel translation of $w(t)$ to the point $y(0)$ along the curve y . Define the inner product of $\dot{\alpha}(0)$ and $\dot{\beta}(0)$ as

$$\langle \dot{\alpha}(0), \dot{\beta}(0) \rangle = \left\langle \frac{d}{dt} \tilde{v}(t)|_{t=0}, \frac{d}{dt} \tilde{w}(t)|_{t=0} \right\rangle + \langle \dot{x}(0), \dot{y}(0) \rangle \quad (10)$$

The inner product the on right hand side of (10) are taken under the Riemannian tensor of M . It can be easily verified that the right hand side of (10) only depends on $\dot{\alpha}(0)$ and $\dot{\beta}(0)$, and $\langle \dot{\alpha}(0), \dot{\alpha}(0) \rangle$ is always positive when $\dot{\alpha}(0) \neq 0$. Thus (10) defines a Riemannian metric on T_1M . Such metric is called the Liouville metric. Its volume form defines a measure, which is called the Liouville measure. It is a well known result that the Liouville measure is invariant under the geodesic flow. (For a proof, c.f. [5], page 52-53).

Now we can give a proof of theorem 4.1.

Proof of theorem 4.1. Suppose M is a compact hyperbolic manifold. Take $X = T_1M$, and μ to be the Liouville measure on X . Then the geodesic flow on X satisfies the conditions of theorem 4.5. We continue to use the notations in theorem 4.5. By proposition 4.4, we only need to show that F consists of a.e. constant functions. Since continuous functions are dense in $L^2(X)$, we only need to show that for $\forall f \in C(X)$, $P(f)$ is a.e. constant.

By theorem 4.5, for $\forall f \in C(X)$,

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \int_0^N t \cdot f \, dt = Pf$$

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \int_{-N}^0 t \cdot f \, dt = Pf$$

where the limits are taken under the L^2 norm.

By a standard theorem in real analysis, if we fix a representative function of Pf (notice that Pf is an element in $L^2(X, \mu)$, thus it is an equivalent class of measurable functions on X), there exists a sequence of positive real numbers N_i , such that $N_i \rightarrow \infty$, and

$$\lim_{i \rightarrow +\infty} \frac{1}{N_i} \int_0^{N_i} t \cdot f(x) \, dt = Pf(x) \quad (11)$$

$$\lim_{i \rightarrow +\infty} \frac{1}{N_i} \int_{-N_i}^0 t \cdot f(x) \, dt = Pf(x) \quad (12)$$

for a.e. $x \in X$.

Since $T_1\mathbb{H}^n$ is a covering of X , we can lift f to a function \tilde{f} on $T_1\mathbb{H}^n$. For $\forall(x, v) \in T_1\mathbb{H}^n$, let $\gamma_{x,v}$ be the geodesic of \mathbb{H}^n starting at x with initial velocity v . Define

$$\begin{aligned}\tilde{f}^+((x, v)) &= \lim_{i \rightarrow \infty} \frac{1}{N_i} \int_0^{N_i} \tilde{f}((\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t))) dt \\ \tilde{f}^-((x, v)) &= \lim_{i \rightarrow \infty} \frac{1}{N_i} \int_{N_i}^0 \tilde{f}((\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t))) dt\end{aligned}$$

By (11), (12), \tilde{f}^+ and \tilde{f}^- are defined on almost every point of $T_1\mathbb{H}^n$, and $\tilde{f}^+ = \tilde{f}^-$ a.e..

Let D be the subset of $T_1\mathbb{H}^n$ consists of all the $(x, v) \in T_1\mathbb{H}^n$ such that $\tilde{f}^+((x, v))$, $\tilde{f}^-((x, v))$ are both defined and $\tilde{f}^+((x, v)) = \tilde{f}^-((x, v))$. Then $T_1\mathbb{H}^n - D$ has measure 0. If $(x, v) \in D$, then for $\forall t$,

$$(\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t)) \in D, (\gamma_{x,v}(t), -\dot{\gamma}_{x,v}(t)) \in D$$

and we have

$$\begin{aligned}\tilde{f}^+(\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t)) &= \tilde{f}^-(\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t)) = \tilde{f}^+(\gamma_{x,v}(t), -\dot{\gamma}_{x,v}(t)) \\ &= \tilde{f}^-(\gamma_{x,v}(t), -\dot{\gamma}_{x,v}(t)) = \tilde{f}^+(x, v) = \tilde{f}^-(x, v)\end{aligned}$$

Notice that \tilde{f} is uniformly continuous under the Liouville metric. Suppose $\gamma_1(t)$ and $\gamma_2(t)$ are two geodesics, such that when $t \rightarrow +\infty$, $\gamma_1(t)$ and $\gamma_2(t)$ tends to the same end at the infinity boundary. Then we can re-parametrize the two geodesics such that

$$\lim_{t \rightarrow +\infty} d(\gamma_1(t), \gamma_2(t)) = 0$$

and

$$|\dot{\gamma}_1(t)| = |\dot{\gamma}_2(t)| = 1$$

(One way to see this is to take the \mathbb{H}^n model and take the common end of γ_1 and γ_2 to be ∞ .) Therefore, if the velocity vectors of γ_1 and γ_2 are both in D , the value of \tilde{f}^+ and \tilde{f}^- on these vectors are the same.

Since the measure of $T_1\mathbb{H}^n - D$ is 0, D contains the unit tangent vectors of almost every geodesic in \mathbb{H}^n . By Fubini's theorem, there exists a $p \in \partial\mathbb{H}^n$, such that for almost every $q \in \partial\mathbb{H}^n$, the following two conditions holds:

- (1) The unit tangent vectors of the geodesic determined by p, q is contained in D .

- (2) For almost every point $r \in \partial\mathbb{H}^n$, the unit tangent vectors of the geodesic determined by q, r is contained in D . (The set of such r may depend on q)

By what we have proved before, this shows that \tilde{f}^+ and \tilde{f}^- are constant a.e., which implies that Pf is constant a.e., thus the geodesic flow on T_1M is ergodic. \square

Remark. The proof of theorem 4.1 given here follows the arguments in [6]. [6] only proved in the 2 dimensional case, but the argument works exactly the same for higher dimensions.

Remark. A theorem by Anosov and Sinai states that the geodesic flow for any compact Riemannian manifold with negative sectional curvature is ergodic. This is much harder than the hyperbolic case.

5 Proof of Mostow's theorem (compact case)

In this section we will present Mostow's proof of theorem 1.1 for the compact case:

Theorem 5.1 (Mostow). *Suppose M, N are compact hyperbolic n -manifolds, $n \geq 3$. Suppose $f : M \rightarrow N$ is a homotopy equivalence. Then f is homotopic to an isometry from M to N .*

Our exposition follows the outline given in [8]. The proof is divided into two steps. First, we lift f to a map \tilde{f} between the universal covers of M and N . We will prove that \tilde{f} can be continuously extended to the boundary at infinity. Then in the second step we will prove that the extension of \tilde{f} on the boundary is a conformal homeomorphism. This is the heart of the proof. Since the boundary map is conformal, by section 2.2 there is an isometry between the universal coverings of M and N which has the same boundary extension. We will show that this isometry induces an isometry from M to N which is homotopic to f .

We will present the first step in section 5.1, and the second step in section 5.2.

Remark. In 1982 Gromov gave another proof using the Gromov norm. We will not present the proof here. For a detailed exposition of Gromov's proof, the reader may refer to [4], chapter C. The main difference

between Gromov's proof and Mostow's proof is on how to show the boundary map is conformal.

5.1 Extension to the boundary

Suppose f, M, N satisfy the conditions in theorem 5.1. Let g be the homotopy inverse of f . By a well known result in differential topology, f and g can be homotoped to smooth maps, thus without loss of generality we may assume that f and g are both smooth.

f can be lifted to a map \tilde{f} between the universal covers of M and N . We have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{D}^n & \xrightarrow{\tilde{f}} & \mathbb{D}^n \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & N \end{array}$$

where the vertical maps are covering maps. Similarly, g can be lift to a map on the universal covers, we denote it by \tilde{g} .

The purpose of this section is to prove the following result:

Theorem 5.2. *Under the notations above, \tilde{f} can be continuously extended to $\overline{\mathbb{D}^n}$, and the restriction of such extension to $\partial\mathbb{D}^n$ is a homeomorphism.*

First we need a lemma:

Lemma 5.3. *There are positive numbers c_1 and c_2 , such that*

$$\frac{1}{c_1} \cdot d(x_1, x_2) - c_2 \leq d(\tilde{f}(x_1), \tilde{f}(x_2)) \leq c_1 \cdot d(x_1, x_2) \quad \forall x_1, x_2 \in X \quad (13)$$

Proof. Since f is smooth, the norm of df is bounded, so does $d\tilde{f}$. Thus \tilde{f} is Lipschitz. Similarly, \tilde{g} is Lipschitz. Thus there exists a positive number c such that

$$\begin{aligned} d(\tilde{f}(x_1), \tilde{f}(x_2)) &\leq c \cdot d(x_1, x_2) \\ d(\tilde{g}(x_1), \tilde{g}(x_2)) &\leq c \cdot d(x_1, x_2) \end{aligned}$$

By our assumptions, $g \circ f$ is homotopic to id_M . By a standard result in differential topology, $g \circ f$ and id_M can be homotoped by a smooth homotopy F . If we choose \tilde{g} suitably, then F can be lift to

a homotopy \tilde{F} between $\tilde{g} \circ \tilde{f}$ and $id_{\mathbb{D}^n}$. Since $|dF|$ is bounded, so is $|d\tilde{F}|$, thus there exists a positive number c' such that

$$d(\tilde{g} \circ \tilde{f}(x), x) \leq c'$$

Now we have

$$\begin{aligned} d(x_1, x_2) &\leq d(\tilde{g} \circ \tilde{f}(x_1), \tilde{g} \circ \tilde{f}(x_2)) + 2 \cdot c' \\ &\leq c \cdot d(\tilde{f}(x_1), \tilde{f}(x_2)) + 2 \cdot c' \end{aligned}$$

Thus (13) holds if we take $c_1 = c$, $c_2 = \frac{2 \cdot c'}{c}$. \square

From now on for any geodesic γ in \mathbb{D}^n we denote the orthogonal projection of \mathbb{D}^n to γ by π_γ .

Lemma 5.4. *Let γ be a geodesic in \mathbb{D}^n , $s > 0$. Let*

$$N_s(\gamma) = \{x \in \mathbb{D}^n \mid d(x, \gamma) < s\}$$

Let $N_s(\gamma)^c$ to be the complement of $N_s(\gamma)$ in \mathbb{D}^n . Then there exists a constant $c(s)$ determined by s , such that $\lim_{s \rightarrow +\infty} c(s) = +\infty$, and that for any two points $p, q \in \mathbb{D}^n$ lie at the same distance s from γ , the following inequality holds:

$$d_{N_s(\gamma)^c}(p, q) \geq c(s) \cdot d(\pi_\gamma(p), \pi_\gamma(q))$$

Here $d_{N_s(\gamma)^c}$ denotes the distance function on $N_s(\gamma)^c$ under the hyperbolic metric.

Proof. We switch to the \mathbb{H}^n model, and take γ to be the geodesic determined by 0 and ∞ . Then $N_s(\gamma)$ is a cone in the Euclidean space, as shown in figure 2. Take l_0 to be the straight (in the sense of the Euclidean metric) line segment from p to q . Take l to be an arbitrary smooth curve from p to q in $N_s(\gamma)^c$.

Suppose l is parametrized by $[0, 1]$, given by its n coordinate functions

$$l : [0, 1] \rightarrow N_s(\gamma)^c, t \mapsto (l_1(t), \dots, l_n(t))$$

with $l(0) = p$ and $l(1) = q$. Suppose the slope of a generator of the boundary cone of $N_s(\gamma)$ is λ . Then

$$\begin{aligned} \text{length}(l) &= \int_0^1 \frac{|dl(t)|}{l_n(t)} \\ &\geq \int_0^1 \frac{|dl(t)|}{\frac{\lambda}{\sqrt{1+\lambda^2}}|l(t)|} \\ &= \frac{\sqrt{1+\lambda^2}}{\lambda} \cdot |\ln |p| - \ln |q|| \end{aligned}$$

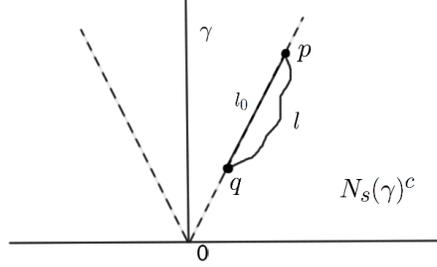


Figure 2

The equality holds if and only if l is a re-parametrization of l_0 .

Therefore, $\text{length}(l) \geq \text{length}(l_0)$ and the lemma follows easily. \square

Lemma 5.5. *Let $\varphi : \mathbb{D}^n \rightarrow \mathbb{D}^n$ be a map. Suppose there are two positive numbers c_1, c_2 such that*

$$\frac{1}{c_1} \cdot d(x_1, x_2) - c_2 \leq d(\varphi(x_1), \varphi(x_2)) \leq c_1 \cdot d(x_1, x_2) \quad \forall x_1, x_2 \in X \quad (14)$$

Then for any geodesic γ in \mathbb{D}^n , there is a unique geodesic $\tilde{\gamma}$ in \mathbb{D} such that the Hausdorff distance between $\varphi(\gamma)$ and $\tilde{\gamma}$ (under the hyperbolic metric) is finite. The distance is bounded by c_1 and c_2 . Moreover, if two geodesics γ and γ' have a same end, then $\tilde{\gamma}$ and $\tilde{\gamma}'$ have a same end.

Proof. Since any two different geodesics have infinite Hausdorff distance, the uniqueness is obvious.

For any two points x, y in \mathbb{D}^n , denote \overline{xy} to be the geodesic segment connecting them, denote $l_{x,y}$ to be the geodesic line passing through x and y . For $s > 0$, denote $N_s(l_{x,y})$ to be the open s -neighborhood of $l_{x,y}$.

First, we claim that there exists an t depending only on c_1, c_2 , such that for any $x, y \in \gamma$, $\varphi(\overline{xy}) \subseteq N_t(l_{\varphi(x), \varphi(y)})$.

Choose any $s > 0$ such that $c(s) > c_1$, where $c(s)$ is given by lemma 5.4. If \overline{xy} is not mapped into $N_s(l_{\varphi(x), \varphi(y)})$, choose any $a, b \in \overline{xy}$ such that $\varphi(a), \varphi(b)$ are on the boundary of $N_s(l_{\varphi(x), \varphi(y)})$ and $\varphi(\overline{ab})$ is outside $N_s(l_{\varphi(x), \varphi(y)})$. By lemma 5.4, we have

$$\begin{aligned} \text{length}(\varphi(\overline{ab})) &\geq c(s) \cdot d(\pi_{l_{\varphi(x), \varphi(y)}}(\varphi(a)), \pi_{l_{\varphi(x), \varphi(y)}}(\varphi(b))) \\ &\geq c(s) \cdot (d(\varphi(a), \varphi(b)) - 2s) \end{aligned}$$

By (14), we have

$$\text{length}(\varphi(\overline{ab})) \leq c_1 \cdot d(a, b)$$

Thus

$$d(\varphi(a), \varphi(b)) \leq \frac{2s \cdot c_1}{c(s) - c_1}$$

Therefore, for any point $p \in \overline{ab}$,

$$\begin{aligned} d(\varphi(p), l_{\varphi(x), \varphi(y)}) &\leq d(\varphi(p), \varphi(a)) + s \\ &\leq c_1 \cdot d(p, a) + s \\ &\leq c_1 \cdot d(a, b) + s \\ &\leq c_1(c_1 \cdot d(\varphi(a), \varphi(b)) + c_2) + s \\ &\leq c_1(c_1 \cdot \frac{2sc_1}{c(s) - c_1} + c_2) + s \end{aligned}$$

and the claim is proved for $t = c_1(c_1 \cdot \frac{2sc_1}{c(s) - c_1} + c_2) + s + 1$.

Now suppose the geodesic γ is given by a normalized parametrization $\gamma : \mathbb{R} \rightarrow \mathbb{D}^n$. By (14), the map γ is proper. Hence when $x \rightarrow +\infty$, the cluster points (under the Euclidean metric) of $\gamma(x)$ are all on $\partial\mathbb{D}^n$. Suppose there are two different cluster points. Let x_n and x'_n be two sequence tend to $+\infty$ such that the sequences $\{\varphi(x_n)\}$ and $\{\varphi(x'_n)\}$ tend to different limits on $\partial\mathbb{D}$. Then for any fixed $y \in \gamma$, the intersection

$$N_t(l_{\varphi(y), \varphi(x_n)}) \cap N_t(l_{\varphi(y), \varphi(x'_n)})$$

is bounded uniformly (under the hyperbolic metric) with regard to n . However, by our claim this set contains the image of $\overline{yx_n} \cap \overline{yx'_n}$ for any n . This is contradictory to the fact that φ is proper.

Therefore, $\varphi(x)$ converges in the Euclidean space as $x \rightarrow +\infty$. By the same reason, $\varphi(x)$ converges as $x \rightarrow -\infty$. By our claim,

$$\varphi(\gamma) \subseteq \limsup_{x \rightarrow +\infty} N_t(l_{\varphi(-x), \varphi(x)})$$

Thus $\lim_{x \rightarrow +\infty} \varphi(x)$ and $\lim_{x \rightarrow -\infty} \varphi(x)$ are two different points on $\partial\mathbb{D}^n$, and the geodesic defined by this two points has bounded Hausdorff distance with $\varphi(\gamma)$. Therefore, this geodesic is the desired $\tilde{\gamma}$.

Suppose γ' is another geodesic which shares an end with γ . Then we can re-parametrize γ' so that $d(\gamma(x), \gamma'(x)) \rightarrow 0$ when $x \rightarrow +\infty$ (the distance is taken under the hyperbolic metric). Thus by (14) we have $d(\varphi(\gamma(x)), \varphi(\gamma'(x))) \rightarrow 0$ as $x \rightarrow +\infty$. Therefore, $\tilde{\gamma}$ and $\tilde{\gamma}'$ have a same end. \square

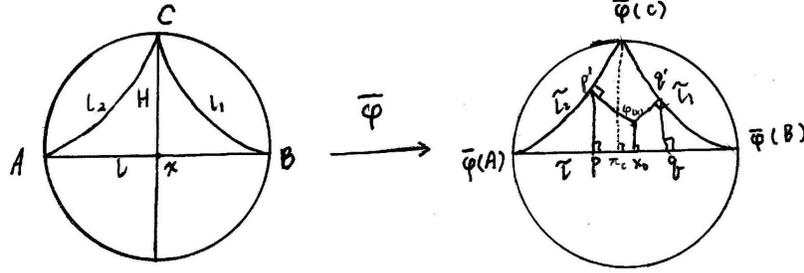


Figure 3

Suppose φ satisfies the conditions in lemma 5.5. We can extend φ to $\partial\mathbb{D}^n$ as follows: for a point $p \in \partial\mathbb{D}^n$, pick any geodesic γ in \mathbb{D}^n with p as one of its ends. Then define the image of p to be the corresponding end of $\tilde{\gamma}$. By lemma 5.5, this map is well defined. Thus we have extended the domain of φ to $\overline{\mathbb{D}^n}$. Denote the extended map by $\bar{\varphi}$.

We will prove that $\bar{\varphi}$ defined above is continuous. In order to prove that, we need a lemma:

Lemma 5.6. *Let φ be as in lemma 5.5. Suppose H is a totally geodesic $(n - 1)$ -hyperplane of \mathbb{D}^n , l is a geodesic in \mathbb{D}^n perpendicular to H . Then there is a c depending only on c_1 and c_2 such that*

$$\text{diam}(\pi_{\tilde{l}}(\varphi(H))) \leq c$$

Here \tilde{l} denotes the geodesic that remains a finite distance from $\varphi(l)$.

Proof. Suppose A and B are the two ends of l . Take any point C on the infinite boundary of H . Let l_1 be the geodesic with ends B, C , l_2 be the geodesic with ends A, C . Let \tilde{l}_1, \tilde{l}_2 be the geodesics that are of finite distance from $\varphi(l_1)$ and $\varphi(l_2)$. Let x_0 be the projection of $\varphi(x)$ to \tilde{l} . (See figure 3.)

The distance from x to l_1 is a constant d (actually it is $\text{arccosh}\sqrt{2}$). Thus the distance from $\varphi(x)$ to $\varphi(l_1)$ is no more than $c_1 d$. By lemma 5.5, there exists a constant t only depending on c_1 and c_2 so that

$$\begin{aligned} d(\varphi(x), x_0) &\leq t \\ d(\tilde{l}_1, \varphi(l_1)) &\leq t \end{aligned}$$

The second d above is the Hausdorff distance. Therefore,

$$d(x, \tilde{l}_1) \leq c_1 d + 2t$$

By the same reason,

$$d(x, \tilde{l}_2) \leq c_1 d + 2t$$

Let p' and q' be the projections from $\varphi(x)$ to l_2 and l_1 . Let p and q be the projections from p' and q' to \tilde{l} . Let π_C be the projection from $\bar{\varphi}(C)$ to \tilde{l} . From figure 3, we see that π_C and x_0 lies in the segment between p and q . Therefore

$$\begin{aligned} d(x_0, \pi_C) &\leq \max(d(x_0, p), d(x_0, q)) \\ &\leq \max(d(\varphi(x), p'), d(\varphi(x), q')) \\ &\leq c_1 d + 2t \end{aligned}$$

Since C can be taken to be any point on the infinite boundary of H , this means that the distance from any point in $\pi_{\tilde{l}}(H)$ to x_0 is no more than $c_1 d + 2t$. Hence the lemma is proved. \square

Now we prove

Lemma 5.7. *Let φ be as in lemma 5.5. $\bar{\varphi}$ be the extension of φ defined after lemma 5.5. Then $\bar{\varphi}$ is continuous.*

Proof. Let P be a point on $\partial\mathbb{D}^n$. We only need to prove that $\bar{\varphi}$ is continuous at P . Choose a geodesic line l with end P , \tilde{l} be the geodesic that remains bounded distance with $\varphi(l)$. Suppose x is a point on l . Let x' be the projection of $\varphi(x)$ to \tilde{l} . Let c be the constant given in lemma 5.6. Let I_x be an interval on \tilde{l} centered at x with radius c . (See figure 4.)

By lemma 5.6, $\phi(H) \subseteq \pi_{\tilde{l}}^{-1}(I_x)$. By the definition of $\bar{\varphi}$, $x' \rightarrow \bar{\varphi}(P)$ as $x \rightarrow P$. Since c is fixed, when x' tends to $\bar{\varphi}(P)$ the points in the set $\pi_{\tilde{l}}^{-1}(I_x)$ tend uniformly to $\bar{\varphi}(P)$ under the Euclidean metric. Therefore $\bar{\varphi}$ is continuous at P , and the lemma is proved. \square

Finally we are prepared to prove theorem 5.2.

Proof of theorem 5.2. By lemma 5.3, \tilde{f} satisfies all the conditions in lemma 5.5, thus lemma 5.7 shows that \tilde{f} can be continuously extended to the boundary. We only need to show that such extension is a homeomorphism on the boundary.

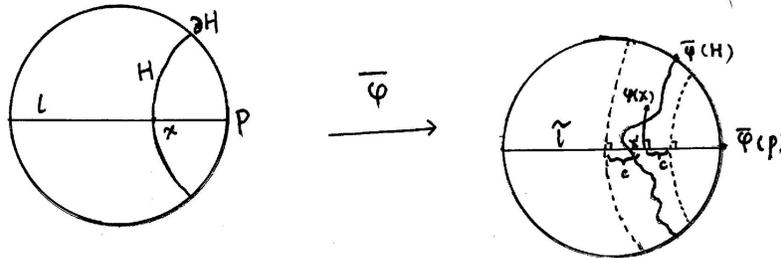


Figure 4

Notice that \tilde{g} also satisfies the conditions of lemma 5.5, thus it can be continuously extended to the boundary as well. Denote the extension of \tilde{f} and \tilde{g} by \bar{f} and \bar{g} . In the proof of lemma 5.3, we showed that for a suitably chosen \tilde{g} , $d(x, \tilde{g} \circ \tilde{f}(x))$ is bounded (the distance is taken under the hyperbolic metric). Thus the continuous extension of $\tilde{g} \circ \tilde{f}$ and $id_{\mathbb{D}^n}$ are the same on the boundary. Therefore, the restriction of \bar{f} on the boundary has a continuous right inverse. For the same reason, it has a continuous left inverse, hence it is a homeomorphism. \square

In the proof of theorem 5.2 we didn't use the fact that M and N are of the same dimension. Therefore by the invariance of domain, we have:

Corollary 5.8. *Suppose M and N are two compact hyperbolic manifolds. If M and N are homotopic, then $\dim M = \dim N$.*

5.2 Conclusion of the proof

In this subsection we finish the proof of 5.1. First we prove that the boundary map constructed in section 5.1 is conformal. As in the last subsection, let \bar{f} be the continuous extension of \tilde{f} to $\overline{\mathbb{D}^n}$.

We have

Lemma 5.9. *The restriction of \bar{f} to the boundary of \mathbb{D}^n is quasi-conformal.*

Proof. By theorem 3.5, we need to prove that $H(P, \bar{f})$ is bounded for all $P \in \partial\mathbb{D}^n$.

Switch to the \mathbb{H}^n model, and take P to be different from ∞ . Let l be the geodesic determined by P and ∞ . Without loss of generality,

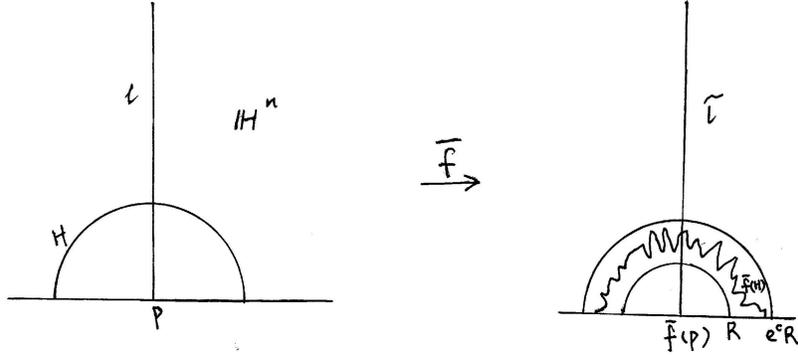


Figure 5

we may assume that $\bar{f}(\infty) = \infty$. Let \tilde{l} be the geodesic that has finite Hausdorff distance with $f(l)$. Then the two ends of \tilde{l} are $\bar{f}(P)$ and ∞ . (See figure 5.)

Let H be a hyperplane orthogonal to l . By lemma 5.6, the projection of $\bar{f}(H)$ to \tilde{l} has diameter no more than c , where c is a constant determined by \bar{f} . Therefore, there exists a positive number r such that $\bar{f}(H)$ is confined between two spheres centering at $\bar{f}(P)$ with radius r and $e^c r$. Under this coordinate of $\partial\mathbb{H}^n$, we have:

$$\frac{\sup_{|P'-P|=r} |\bar{f}(P') - \bar{f}(P)|}{\inf_{|P'-P|=r} |\bar{f}(P') - \bar{f}(P)|} \leq e^c, \quad \forall r > 0$$

Since $H(P, \bar{f})$ is invariant under conformal coordinate transformations, this proves that $H(P, \bar{f}) \leq e^c$ for all P on the boundary under any conformal coordinate near P , hence \bar{f} is quasi-conformal on the boundary \square

Next we use the ergodicity of the geodesic flow of M to prove that $\bar{f}|_{\partial\mathbb{D}^n}$ is in fact 1-quasi-conformal. We only prove it for $n = 3$. A proof for any dimension is given in [7], page 97-101. The proof given there uses deeper properties of quasi-conformal maps.

In the case of $n = 3$, Recall that \mathbb{D}^3 is a covering space of the compact manifold M . Let G be the deck transformation group, then \bar{f} is G -equivariant. Since G acts isometrically on \mathbb{D}^3 , the action of G can be extended to $\overline{\mathbb{D}^3}$. Let S_∞^2 be the boundary of \mathbb{D}^3 . There is an

action of G on $S_\infty^2 \times S_\infty^2$ defined by

$$\begin{aligned} G \times (S_\infty^2 \times S_\infty^2) &\rightarrow S_\infty^2 \times S_\infty^2 \\ (g, (x_1, x_2)) &\mapsto (g \cdot x_1, g \cdot x_2) \end{aligned}$$

We have

Lemma 5.10. *The action of G on $S_\infty^2 \times S_\infty^2$ is ergodic.*

Proof. Suppose this action is not ergodic, then there is a subset A of $S_\infty^2 \times S_\infty^2$ so that both A and its complement have positive measure. Consider the set of geodesics:

$$L = \{\text{geodesic } l \text{ in } \mathbb{D}^3 \mid \exists (p, q) \in A \text{ s.t. } p, q \text{ are the two ends of } l\}$$

Now consider

$$F = \{(x, v) \in T_1M \mid \exists l \in L, \text{ s.t. the preimage of } (x, v) \text{ in } T_1\mathbb{D}^3 \text{ is a tangent vector of } l\}$$

Then F is a subset of T_1M invariant under the geodesic flow. Since A is invariant under G , A and the complement of A both have positive measure, we have that F and the complement of F both have positive measure. Which is contradictory to theorem 4.1. \square

Notice that for a geodesic l in \mathbb{D}^3 , the tangent spaces on the points of l can be identified by parallel translations. Therefore, for two points $x, y \in l$ and two non-zero vectors $v_x \in T_x\mathbb{D}^3, v_y \in T_y\mathbb{D}^3$, we can define the angle between v_x and v_y by parallel translate v_x to y and take its angle with v_y . Such definition can be continuously extended to the tangent spaces of the two ends of l .

Suppose we have two points $p, q \in S_\infty^2$. Let $v_p \in T_p S_\infty^2$ and $v_q \in T_q S_\infty^2$ be two non-zero vectors. There is a unique geodesic with ends p, q , and we can define the angle between v_p and v_q to be their angle defined along this geodesic.

Since \bar{f} is quasi-conformal on S_∞^2 , by our definition given in 3.4 it is a.e. differentiable. For any $p \in S_\infty^2$ at which $\bar{f}|_{S_\infty^2}$ is differentiable, define the subspace

$$V_p = \{v \in T_p S_\infty^2 \mid |d(\bar{f}|_{S_\infty^2})(v)| = |d(\bar{f}|_{S_\infty^2})| \cdot |v|\}$$

Since \bar{f} is equivariant with G , $\{V_p\}$ is invariant under G . If \bar{f} is not 1-quasi-conformal, then the dimension of V_p is 1 almost everywhere,

and for any two points $p, q \in S_\infty^2$, the angle between V_p and V_q is almost everywhere constant. Use stereographic projection to project S_∞^2 onto \mathbb{R}^2 , $\{V_p\}$ gives a 1-dimensional distribution $\{V'_p\}_{p \in \mathbb{R}^2}$ defined on almost everywhere \mathbb{R}^2 . The angle condition of $\{V_p\}$ is translated to $\{V'_p\}$ as follows: for $\forall p, q \in \mathbb{R}^2$ where V' is defined, let $r_{p,q}$ be the unique reflection of \mathbb{R}^2 interchanging p and q , then the angle between V'_p and $r_{p,q}(V'_q)$ is almost everywhere constant. This is easily seen to be impossible.

Therefore, \bar{f} is 1-quasi-conformal on the boundary. By theorem 3.6, \bar{f} is conformal on the boundary. By the discussion in section 2.2, there is an isometry h of \mathbb{D}^n which has the same boundary extension as \bar{f} . Let G be the deck transformation group of the covering map $\mathbb{D}^n \rightarrow M$, then \bar{f} is G -equivariant, thus h is also G equivariant. Consider the homotopy from \bar{f} to h by geodesics (that is to say, the trajectory of any point under the homotopy is a normalized geodesic). Then this homotopy is also G -equivariant and hence can be reduced to M . Therefore f is homotopic to the isometry reduced from h and theorem 5.1 is proved.

6 Applications

In this section we present some applications of Mosow's theorem. We will first present some quick and interesting applications of theorem 5.1. Then we will discuss the hyperbolic invariants of knots and links. We will need the non-compact case of Mostow's theorem for that.

6.1 Some quick applications

In this section we present some quick applications of theorem 5.1.

Suppose M is a compact topological manifold. M is called hyperbolic if it is homeomorphic to a hyperbolic manifold. In this case, we define the hyperbolic volume of M to be the volume of this hyperbolic manifold. Then by Mostow's theorem we have:

Corollary 6.1. *The hyperbolic volume of a compact manifold of dimension at least 3 is a topological invariant.* \square

Remark. There is a more direct proof of this result. For compact hyperbolic manifolds the volume is proportional to a topological invariant called the *Gromov Norm*, hence the volume is a topological

invariant. This fact is used in Gromov's proof of Mostow's rigidity theorem. For details, see [4] Sections C.3-C.4.

We also have the following result:

Proposition 6.2. *Let f be as in theorem 5.1. Then the isometry homotopic to f is unique.*

Proof. By theorem 5.1, M and N are isometric, hence we may assume that f is a map from a compact hyperbolic manifold M to itself. Suppose h and h' are two isometries homotopic to f . Suppose $F : I \times M \rightarrow M$ is the homotopy from h to h' . Lift F to the universal cover, denote the lifting map by \tilde{F} . Let \tilde{h} and \tilde{h}' be $\tilde{F}(0, \cdot)$ and $\tilde{F}(1, \cdot)$ respectively. Then \tilde{h} and \tilde{h}' are the liftings of h and h' . Since M is compact, there exists a constant c such that:

$$d(\tilde{h}(x), \tilde{h}'(x)) < c, \quad \forall x \in \mathbb{D}^n$$

where d is the hyperbolic distance. Therefore, the continuous extensions of \tilde{h} and \tilde{h}' to \mathbb{D}^n are the same on the boundary. Hence $\tilde{h} = \tilde{h}'$, which implies $h = h'$. \square

Before we give the next application of theorem 5.1, we need a lemma from algebraic topology.

Suppose G is a group, (X, x_0) be a $K(G, 1)$ space where $x_0 \in X$ is the base point. Denote $[X, X]$ to be the semi-group of homotopy classes of maps from X to X , and denote $[X, X]^*$ to be the units in $[X, X]$.

For any homotopy equivalence $f : X \rightarrow X$ and any fixed path γ from x_0 to x , consider the map:

$$\begin{aligned} f_{\gamma*} : \pi_1(X, x_0) &\rightarrow \pi_1(X, x_0) \\ [\alpha] &\mapsto [\gamma + f \circ \alpha + \gamma^{-1}] \end{aligned}$$

here γ^{-1} denotes the reverse path of γ . $f_{\gamma*}$ depends on γ . If we change f by a homotopy or choose a different γ , $f_{\gamma*}$ will change by an composition of inner automorphism. Thus we have a well-defined map

$$\begin{aligned} \Phi : [X, X]^* &\rightarrow \text{Aut}(\pi_1(X, x_0))/\text{Inn}(\pi_1(X, x_0)) \\ [f] &\mapsto \text{the coset of } f_{\gamma*} \text{ for some } \gamma \end{aligned}$$

Lemma 6.3. *Let G, X, Φ be as above. Then Φ is an isomorphism.*

Proof. Since X is an Eilenberg-MacLane space, any endomorphism A of $\pi_1(X, x_0)$ can be realized by a map from (X, x_0) to itself. If A is an isomorphism, then the corresponding map is a homotopy equivalence. Thus Φ is surjective.

On the other hand, suppose $f : (X, x_0) \rightarrow (X, x_0)$ represents an element in $\ker(\Phi)$. By homotoping f we may assume that $f(x_0) = x_0$. Then there exists a loop γ with base point x such that

$$f_*([\alpha]) = [\gamma]^{-1} \cdot [\alpha] \cdot [\gamma], \quad \forall [\alpha] \in \pi_1(X, x_0) \quad (15)$$

By the homotopy extension theorem, we can homotope f to f' so that the trajectory of x_0 along the homotopy is the path γ . By (15), f' induces the identity map on $\pi_1(X, x_0)$, hence f' is homotopic to the identity. Therefore, f is homotopic to the identity and hence $\ker(\Phi)$ is trivial. \square

Now we have

Theorem 6.4. *Let M be a compact hyperbolic manifold, Γ be its fundamental group. Let $\text{Iso}(M)$ be the isometry group of M . Then*

$$\text{Iso}(M) \cong \text{Aut}(\Gamma)/\text{Inn}(\Gamma) \quad (16)$$

Moreover, this group is finite.

Proof. By theorem 5.1 and proposition 6.2,

$$\text{Iso}(M) \cong [M, M]^*$$

By lemma 6.3,

$$[M, M]^* \cong \text{Aut}(\Gamma)/\text{Inn}(\Gamma)$$

Hence (16) is proved.

Now we prove that $\text{Iso}(M)$ is finite. Notice that any isometry of M is determined by its tangent map at a fixed point. Since M is compact, the set $\text{Iso}(M)$ is compact under the C^0 topology. Suppose there are infinitely many elements in $\text{Iso}(M)$, then there exist two different elements $h, h' \in \text{Iso}(M)$ so that for any $x \in M$, $d(h(x), h'(x))$ is less than the injective radius of M . Therefore, h and h' are homotopic, and by proposition 6.2 we have $h = h'$, which is a contradiction. \square

6.2 Hyperbolic invariants of knots and links

This section discusses the application of theorem 1.1 to knots and links. Our discussion will be informal and we only present some basic ideas on this topic.

Suppose L is a link in S^3 . Then $S^3 - L$ is a topological 3-manifold. By theorem 1.1, if $S^3 - L$ can be endowed with a complete hyperbolic structure of finite volume, this hyperbolic structure is a topological invariant. Therefore, all the numerical invariants of the hyperbolic structure are numerical invariants of the link L . Among such invariants the volume of the hyperbolic structure is the simplest one, and it is called the hyperbolic volume. In this subsection we will describe a method to construct the hyperbolic structure on link complements. We will see that the hyperbolic volume can be easily calculated from this construction.

Suppose X is a connected simplicial 3-complex obtained by linearly gluing the faces of finitely many simplices:

$$\Delta_1 \cdots \Delta_N$$

Let X^i be the i dimensional skeleton of X ($i = 0, \dots, 3$).

Suppose X satisfies the following conditions:

- (1) $\{\Delta_i\}$ are oriented, thus the faces of Δ_i have induced orientations.
- (2) Every face in X has exactly two preimages among the faces of $\{\Delta_i\}$, and the two faces are glued in a orientation-reversing way.
- (3) Every face in $\{\Delta_i\}$ is glued with another face.

Then for any edge l in X , the preimages of l among the edges of $\{\Delta_i\}$ can be labeled properly as l_1, \dots, l_k , such that if Δ_{n_i} is the simplex containing l_i , then Δ_{n_i} and $\Delta_{n_{i+1}}$ are glued together along two faces containing respectively l_i and l_{i+1} , for $i = 1, \dots, k$. Here we understand l_{k+1} and n_{k+1} as l_1 and n_1 . Therefore $X - X^0$ is a oriented topological 3-manifold.

There is a way to endow $X - X^0$ with a hyperbolic structure. Identify Δ_i with an ideal simplex T_i of \mathbb{D}^3 , so that the orientation of Δ_i and T_i agree. Different T_i may have different shapes, and we will determine their shapes later. Here we make T_i not contain the vertices. Since any two totally geodesic ideal triangles in \mathbb{D}^3 can be identified by a unique orientation-reversing isometry of \mathbb{D}^3 , the combinatorial definition of X gives a way of gluing the faces of $\{T_i\}$. Denote the

space obtained from such gluing by M . Then M is homeomorphic to $X - X^0$.

The hyperbolic structures of T_i give a hyperbolic structure on the open 3-cells of M . Since the faces of T_i are totally geodesic planes, such structure can be extended to the interior points of the faces of M .

Suppose l is an edge of M . Let l_1, \dots, l_k be the preimages of l which are also edges in $\{T_i\}$. Let T_{n_i} be the simplex containing l_i . Denote α_i to be the angle of T_{n_i} at the edge l_i . Then, if

$$\sum_i \alpha_i = 2\pi \tag{17}$$

the hyperbolic structures of T_i can be extended to points on the interior of l .

Therefore, if the angles of T_i satisfies (17) for the set of $\{\alpha_i\}$ defined by every edge of M , the hyperbolic structures on T_i would give a hyperbolic structure on M . In this case, we say that the hyperbolic structures on $\{T_i\}$ can be spliced together.

This spliced hyperbolic structure does not have to be complete. The next lemma gives a criterion for the completeness of this hyperbolic structure. Recall that for any vertex p of T_i , the intersection of T_i and a horosphere centering at p is a Euclidean triangle, provided that the horosphere is close enough to p . We call these Euclidean triangles ‘‘horo-triangles’’.

Lemma 6.5. *Let M be as above, suppose the hyperbolic structures of T_i can be spliced together to give a hyperbolic structure to M . Then this hyperbolic structure is complete if and only if for any vertex of T_i , we can choose a suitable horo-triangle, so that under the gluing of $\{T_i\}$, the horo-triangles are glued together to form closed 2-manifolds.*

For a proof of this lemma, see [8], page 40-42.

As we have seen, the condition that T_i can be spliced together and that the resulting hyperbolic structure is complete can be expressed by explicit equations on the angles of T_i . Thus given the simplicial structure of X , we can try to endow a complete hyperbolic structure to $X - X^0$ by try to solve the equations. Since the ideal simplices have finite volume, such a complete hyperbolic structure (if it exists) has finite volume.

However, these equations do not always have solutions. For example, by lemma 6.5, we see that if a complete hyperbolic structure

can be constructed as above, the neighborhood of any vertex in X has to be homeomorphic to a cone on a surface S , where S is an oriented closed 2-manifold with a Euclidean structure. By Gauss-Bonnet formula, $S \cong \mathbb{T}^2$. Thus there are some necessary conditions on the topology of X for our construction to be possible (A more careful argument would show that this condition can be deduced from equations (17) without assuming the completeness).

Now we use what we have discussed above to give hyperbolic structures to knot complements. For many links L there is a simplicial 3-simplex X satisfying $S^3 - L \cong X - X^0$. Such X (if exists) can be canonically constructed from the link diagram. (For a detailed introduction to this construction, the reader may refer to [4], page 210-222. In the construction, vertices of X are in one-one correspondence with components of L . There is a two dimensional sub-complex Y of X , so that $X - Y$ is homeomorphic to two open 3-balls. The number of edges of Y equals the number of crossing of the link diagram.)

Suppose such X exists. Since $S^3 - L \cong X - X^0$, we can try to give it a complete hyperbolic structure by using what we have shown in the earlier part of this subsection. The construction of X is mechanical, and the procedure of looking for a complete hyperbolic structure on X is equivalent to solving a system of equations which can be explicitly written down. Therefore the whole procedure can be done by computers. The resulting hyperbolic structures obtained from this procedure are expressed by angles of ideal simplices, and the hyperbolic invariants can be computed from this data. For example, the hyperbolic volume is given by formula (6).

For some simple cases the hyperbolic structure can be calculated without resorting to computers. We take the figure-eight knot as an example.

According to the procedure given in [4], the corresponding simplicial complex X for the figure-eight knot is given by taking two 3-simplices $ABCD$ and $A'B'C'D'$, and glue the following pairs of faces together (keeping the order of vertices):

$$\begin{aligned} ABC &\sim C'B'A' \\ ABD &\sim D'A'C' \\ ACD &\sim D'A'B' \\ BCD &\sim B'D'C' \end{aligned}$$

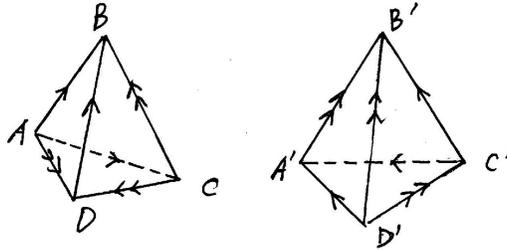


Figure 6

As shown in figure 6.

Notice that there are 2 edges in X , each has 6 preimages. Therefore, if we take all the angles of $ABCD$ and $A'B'C'D'$ to be $\frac{\pi}{3}$ (by section 2.3, this is possible), their hyperbolic structures would be spliced together. Besides, it is easy to check that the condition in lemma 6.5 is satisfied, thus this hyperbolic structure is complete. By (6), the hyperbolic volume of the figure-eight knot is

$$6 \cdot \Lambda\left(\frac{\pi}{3}\right) \approx 2.02988$$

A Flexibility of hyperbolic structures on 2-manifolds

In the appendix we give an informal discussion about the flexibility of hyperbolic structures on 2-manifolds.

By Mostow's theorem the hyperbolic structure on a compact manifold of dimension at least three is "rigid" (i.e., unique up to isometry if it exists). However, the same result does not hold for 2 dimensional manifolds. In fact, when $g \geq 2$, the hyperbolic structures on a closed, oriented, genus g surface form a moduli space of real dimension $(6g - 6)$. Therefore, unlike the case of dimension at least 3, the hyperbolic structures on 2-manifolds are "flexible".

One way to see this is as follows: every complete hyperbolic surface is isomorphic to a quotient of the Poincaré disc \mathbb{D}^2 by an isometric, properly discontinuous, free group action (c.f. section 2.1). By Schwartz's lemma, the orientation-preserving isometry group of \mathbb{D}^2 is equal to the group of holomorphic homeomorphisms from \mathbb{D}^2 to itself. By Riemann uniformization theorem, the quotients of \mathbb{D}^2 by holomor-

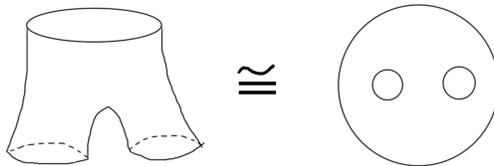


Figure 7: A trouser

phic deck transformation groups is in one-one correspondence with the set of 1 dimensional complex manifolds except for sphere, tori, and the cylinder defined as \mathbb{C} modulo a translation. Therefore, the hyperbolic structures on a closed oriented genus g surface, when $g \geq 2$, is in one-one correspondence with the complex structures on the same differential manifold. By a well-known result, such structures are not unique and they form a moduli space of complex dimension $3g - 3$.

Another way to see this is by the trouser decomposition of hyperbolic surfaces. By definition, a Riemannian 2-manifold T with boundary is called a trouser if (1) T is diffeomorphic to a sphere with three open discs removed, and (2) the Riemannian metric of T can be extended to its collar neighborhood so that the extended metric is hyperbolic and the boundaries of T are closed geodesics under this metric. Figure 7 shows the topology of a trouser. By a result in 2-dimensional hyperbolic geometry, for any given positive real numbers a, b, c there is a unique trouser with boundary lengths a, b , and c . Suppose M is a compact hyperbolic surface of genus g . By the trouser decomposition theorem, M can be decomposed into $(2g - 2)$ trousers, i.e. M is the result of pasting $(2g - 2)$ trousers along their boundaries. Figure 8 shows a closed surface of genus 2 being decomposed into 2 trousers.

Given a trouser decomposition of M , we can change the way that the trousers are pasted and obtain new hyperbolic structures on the same differentiable manifold. The procedure is described as follows:

Pastings of trousers are defined by isometries between the boundaries. The domain and range of such isometries are diffeomorphic to S^1 . Suppose $f : S^1 \rightarrow S^1$ is an isometry, then the composition of f and an arbitrary rotation is also a rotation. Thus we can rotate the pasting maps to get new hyperbolic manifolds. For each pair of pasted boundary components, such rotations are parametrized by S^1 .

We can also change the shapes of the trousers, keeping the lengths

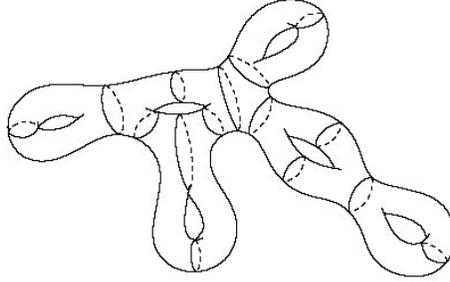


Figure 8: A surface decomposed into trousers

of glued boundary components the same. For each pair of boundary components, the possible length of them is parametrized by \mathbb{R}^+ .

Noticed that there are $(3g - 3)$ pairs of trouser boundaries in the trouser decomposition of M . Therefore, we get a family parametrized by $(S^1)^{3g-3} \times (\mathbb{R}^+)^{3g-3}$ of genus g hyperbolic manifolds. Hence we have a map:

$$(S^1)^{3g-3} \times (\mathbb{R}^+)^{3g-3} \rightarrow \{\text{the set of hyperbolic structures on } M\}$$

It can be proved that this map is surjective and has discrete fibers, thus the set of hyperbolic structures form a space of real dimension $(6g - 6)$. In this way we can actually “see” how the hyperbolic structures on a 2-manifold are perturbed to different hyperbolic structures.

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