A note on the hypoellipticity of Folland-Stein heat operators

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Abstract

This note contains a proof for the hypoellipticity of admissible Folland-Stein heat operators. The techniques presented here were first used by Greiner [3] for different problems, and the main result of this article is implicitly contained in [1]. The purpose of this note is to give a more straightforward and self-contained presentation.

1 Introduction

Let n be a positive integer. A partial differential operator $\mathcal{P}$ defined on $\mathbb{R}^n$ is called hypoelliptic, if for every distribution μ defined on an open subset U of $\mathbb{R}^n$ such that $P\mu$ is smooth, μ must also be smooth.

Now consider differential operators on $\mathbb{R}^{2n+1}$. Let x_1, x_2, $\cdots$, x_n, y_1, y_2, $\cdots$, y_n, t be the coordinate functions. Define the following vectors fields:

$$\begin{aligned} X_j &= \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \qquad j = 1, 2, \cdots, n \\ Y_j &= \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \qquad j = 1, 2, \cdots, n \\ T &= \frac{\partial}{\partial t}. \end{aligned}$$

Let

$$\begin{aligned} Z_j &= \frac{1}{2}(X_j - iY_j), \\ \bar{Z}_j &= \frac{1}{2}(X_j + iY_j), \end{aligned}$$

then $[Z_j, Z_k] = [\bar{Z}_j, \bar{Z}_k] = 0$, and $[Z_j, \bar{Z}_k] = -2i\,\delta_{jk}T$. The Folland-Stein operator on $\mathbb{R}^{2n+1}$ is defined as

$$\mathscr{L}_\alpha = -\frac{1}{2}\sum_{j=1}^{n}(Z_j\bar{Z}_j + \bar{Z}_j Z_j) + i\alpha T,$$

where $\alpha \in \mathbb{R}$ is a fixed constant. The parameter α is called admissible if $\pm\alpha \neq n, n+2, n+4, \cdots$.

The key property of the Folland-Stein operator is the following:

Theorem 1.1 (Folland-Stein [2]). *The operator $\mathscr{L}_\alpha$ is hypoelliptic if and only if α is admissible.*

It is natural to ask about the hypoellipticity of the heat operator of $\mathscr{L}_\alpha$ as well. In fact, one has the following result.

Theorem 1.2 (Greiner [3], Calin-Chang-Furutani-Iwasaki [1]). *Let m be a positive integer, then the operator*

$$\frac{\partial}{\partial s} + (\mathscr{L}_\alpha)^m$$

is hypoelliptic on $\mathbb{R}^{2n+2}$ if and only if α is admissible.

The "only if" part of theorem 1.1 and 1.2 can be proved by a direct construction. When α is not admissible, Folland and Stein [2] constructed a locally integrable function φ_α on $\mathbb{R}^{2n+1}$ such that φ_α has a singularity at zero but $\mathscr{L}_\alpha(\varphi_\alpha) = 0$ in the sense of distributions. Therefore neither $\mathscr{L}_\alpha$ nor $\frac{\partial}{\partial s} + (\mathscr{L}_\alpha)^m$ is hypoelliptic.

In the following sections we will give a proof for the hypoellipticity of $\frac{\partial}{\partial s} + (\mathscr{L}_\alpha)^m$ when α is admissible. The techniques presented here were first used by Greiner [3] to study left-invariant operators on Heisenberg groups. In [1] the idea was applied to the heat operator $\frac{\partial}{\partial s} + (\mathscr{L}_\alpha)^m$, and a formula for the heat kernel was given. Although it is possible to prove hypoellipticity from properties of the heat kernel, the formula given in [1] is quite involved and the result does not follow in an immediate way. This note presents a more straightforward and self-contained proof for the result. Of course, the hypoellipticity of the operator $\mathscr{L}_\alpha$ is implied by the hypoellipticity of $\frac{\partial}{\partial s} + (\mathscr{L}_\alpha)^m$.

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2 Fourier transform with respect to t

In cartesian coordinates, the operator $\mathscr{L}_\alpha$ is written as

$$\mathscr{L}_\alpha = -\frac{1}{4}\Big(\sum_j \frac{\partial^2}{\partial x_j} + \sum_j \frac{\partial^2}{\partial y_j^2}\Big) + \sum_j \Big(x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j}\Big)\frac{\partial}{\partial t} - \sum_j (x_j^2 + y_j^2)\frac{\partial^2}{\partial t^2} + i\alpha\frac{\partial}{\partial t}.$$

Let $\mathscr{S}(\mathbb{R}^{2n+1})$ be the Schwartz space on $\mathbb{R}^{2n+1}$. Taking the Fourier transform with respect to the t variable, every $f \in \mathscr{S}(\mathbb{R}^{2n+1})$ can be written as

$$f(x_1, \cdots, x_n, y_1, \cdots, y_n, t) = \int_{-\infty}^{\infty} \hat{f}_\xi(x_1, \cdots, x_n, y_1, \cdots, y_n) e^{i\xi t} d\xi.$$

The Plancherel identity states that

$$\|f\|^2_{L^2(\mathbb{R}^{2n+1})} = 2\pi \int_{-\infty}^{\infty} \|\hat{f}_\xi\|^2_{L^2(\mathbb{R}^{2n})}\, d\xi.$$

The Fourier transform of the operator $\mathscr{L}_\alpha$ reads as

$$(\mathscr{L}_\alpha f)^\wedge_\xi = \Big[-\frac{1}{4}\Big(\sum_j \frac{\partial^2}{\partial x_j} + \sum_j \frac{\partial^2}{\partial y_j^2}\Big) + i\xi \cdot \sum_j \Big(x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j}\Big) + \sum_j (x_j^2 + y_j^2)\xi^2 - \alpha\xi\Big](\hat{f}_\xi). \quad (1)$$

Let $(\mathscr{L}_\alpha)^\wedge_\xi$ be the operator acting on $\hat{f}_\xi$ in right hand side of (1). This section finds a diagonalization for the operator $(\mathscr{L}_\alpha)^\wedge_\xi$ on $L^2(\mathbb{R}^{2n})$ when

ξ is nonzero. For the rest of this section ξ will be considered as a fixed nonzero constant.

Define the following differential operators on $\mathbb{R}^{2n}$:

$$\begin{aligned} a^\dagger_{x_j} &= |\xi| x_j - \frac{1}{2}\frac{\partial}{\partial x_j}, \\ a_{x_j} &= |\xi| x_j + \frac{1}{2}\frac{\partial}{\partial x_j}, \\ a^\dagger_{y_j} &= |\xi| y_j - \frac{1}{2}\frac{\partial}{\partial y_j}, \\ a_{y_j} &= |\xi| y_j + \frac{1}{2}\frac{\partial}{\partial y_j}. \end{aligned}$$

Consider the function $\hat{e}_0 = \exp\big(-|\xi| \cdot (\sum_j x_j^2 + \sum_j y_j^2)\big)$ and define $e_0 = \hat{e}_0/\|\hat{e}_0\|_{L^2}$. We have $a_{x_j}(e_0) = a_{y_j}(e_0) = 0$.

Now define

$$\begin{aligned} u_j &= a_{x_j} - i a_{y_j}, \\ u^\dagger_j &= a^\dagger_{x_j} + i a^\dagger_{y_j}, \\ v_j &= a_{x_j} + i a_{y_j}, \\ v^\dagger_j &= a^\dagger_{x_j} - i a^\dagger_{y_j}. \end{aligned}$$

Then

$$\begin{aligned} [u_j, u^\dagger_k] &= 2\delta_{jk}|\xi|, \\ [u^\dagger_j, u_k] &= -2\delta_{jk}|\xi|, \\ [v_j, v^\dagger_k] &= 2\delta_{jk}|\xi|, \\ [v^\dagger_j, v_k] &= -2\delta_{jk}|\xi|, \end{aligned}$$

and all the other Lie brackets among u_j, $u^\dagger_j$, v_j, $v^\dagger_j$ are zero.

For a $(2n)$-tuple of nonnegative integers $I = (a_1, \cdots, a_n, b_1, \cdots, b_n)$, let $a = \sum_j a_j$, $b = \sum_j b_j$, $a! = \prod_j a_j!$, $b! = \prod_j b_j!$. Define

$$e_I = \frac{(u^\dagger_1)^{a_1} \cdots (u^\dagger_n)^{a_n} (v^\dagger_1)^{b_1} \cdots (v^\dagger_n)^{b_n}(e_0)}{[(2|\xi|)^{a+b}\, a!\, b!]^{1/2}}.$$

Since $u_j(e_0) = v_j(e_0) = 0$, we have

$$\begin{aligned} &\langle (u^\dagger_1)^{a_1} \cdots (u^\dagger_n)^{a_n} (v^\dagger_1)^{b_1} \cdots (v^\dagger_n)^{b_n}(e_0),\ (u^\dagger_1)^{a_1} \cdots (u^\dagger_n)^{a_n} (v^\dagger_1)^{b_1} \cdots (v^\dagger_n)^{b_n}(e_0)\rangle \\ =&\langle e_0,\ (v_n)^{b_n} \cdots (v_1)^{b_1} (u_n)^{a_n} \cdots (u_1)^{a_1} (u^\dagger_1)^{a_1} \cdots (u^\dagger_n)^{a_n} (v^\dagger_1)^{b_1} \cdots (v^\dagger_n)^{b_n}(e_0)\rangle \\ =&\langle e_0,\ (2|\xi|)^{a+b} a! b!\,(e_0)\rangle \\ =&(2|\xi|)^{a+b} a! b!, \end{aligned}$$

therefore $\|e_I\|_{L^2} = 1$. Similarly, if $I \neq I' = (a'_1, \cdots, a'_n, b'_1, \cdots, b'_n)$, without loss of generality we may assume $a'_1 > a_1$, then

$$\begin{aligned} &\langle (u^\dagger_1)^{a'_1} \cdots (u^\dagger_n)^{a'_n} (v^\dagger_1)^{b'_1} \cdots (v^\dagger_n)^{b'_n}(e_0),\ (u^\dagger_1)^{a_1} \cdots (u^\dagger_n)^{a_n} (v^\dagger_1)^{b_1} \cdots (v^\dagger_n)^{b_n}(e_0)\rangle \\ =&\langle e_0,\ (v_n)^{b'_n} \cdots (v_1)^{b'_1} (u_n)^{a'_n} \cdots (u_1)^{a'_1} (u^\dagger_1)^{a_1} \cdots (u^\dagger_n)^{a_n} (v^\dagger_1)^{b_1} \cdots (v^\dagger_n)^{b_n}(e_0)\rangle \\ =&\langle e_0, 0\rangle \\ =&0, \end{aligned}$$

therefore $\langle e_I, e_{I'}\rangle = 0$. Thus we have shown that the vectors in the set $\{e_I\}_{I\in\mathbb{Z}^{2n}_{\geq 0}}$ are orthonormal to each other in $L^2(\mathbb{R}^{2n})$. A well known property about quantum harmonic oscillators states that vectors of the form

$$(a^\dagger_{x_1})^{\alpha_1}\cdots(a^\dagger_{x_n})^{\alpha_n}(b^\dagger_{y_1})^{\beta_1}\cdots(a^\dagger_{y_n})^{\beta_n}(e_0)$$

give a complete basis for $L^2(\mathbb{R}^{2n})$. Therefore the set $\{e_I\}_{I\in\mathbb{Z}^{2n}_{\geq 0}}$ is a complete orthonormal basis of $L^2(\mathbb{R}^{2n})$.

Now we compute the operator $(\mathscr{L}_\alpha)^\wedge_\xi$ in terms of the basis $\{e_I\}$. Notice that

$$\begin{aligned}
&[i(y_j\frac{\partial}{\partial_{x_j}} - x_j\frac{\partial}{\partial_{y_j}}), u^\dagger_k] = \delta_{jk}u^\dagger_k,\\
&[i(y_j\frac{\partial}{\partial_{x_j}} - x_j\frac{\partial}{\partial_{y_j}}), v^\dagger_k] = -\,\delta_{jk}v^\dagger_k,
\end{aligned}$$

and

$$i(y_j\frac{\partial}{\partial_{x_j}} - x_j\frac{\partial}{\partial_{y_j}})(e_0) = 0,$$

therefore when $I = (a_1,\cdots,a_n,b_1,\cdots,b_n)$, we have

$$i(y_j\frac{\partial}{\partial_{x_j}} - x_j\frac{\partial}{\partial_{y_j}})(e_I) = (a-b)e_I.$$

On the other hand,

$$\begin{aligned}
&-\frac{1}{4}(\sum_j\frac{\partial^2}{\partial x_j^2} + \sum\frac{\partial^2}{\partial y_j^2}) + \sum_j(x_j^2+y_j^2)\xi^2\\
=&\frac{1}{2}(\sum_j u^\dagger_j u_j + \sum_j v^\dagger_j v_j) + n|\xi|.
\end{aligned}$$

Therefore

$$\begin{aligned}
&(\mathscr{L}_\alpha)^\wedge_\xi(e_I)\\
=&\big[-\frac{1}{4}(\sum_j\frac{\partial^2}{\partial x_j} + \sum_j\frac{\partial^2}{\partial y_j^2}) + i\xi\sum_j(x_j\frac{\partial}{\partial y_j} - y_j\frac{\partial}{\partial x_j}) +\\
&\qquad\sum_j(x_j^2+y_j^2)\xi^2 - \alpha\xi\big](e_I)\\
=&\big[\frac{1}{2}(\sum_j u^\dagger_j u_j + \sum_j v^\dagger_j v_j) + n|\xi|\big](e_I) + \xi(b-a) - \alpha\xi\big](e_I)\\
=&\big[|\xi|(a+b) + \xi(b-a) + n|\xi| - \alpha\xi\big](e_I).
\end{aligned}$$

In other words,

$$(\mathscr{L}_\alpha)^\wedge_\xi(e_I) = \begin{cases}(2b+n-\alpha)|\xi|\,e_I & \text{if } \xi>0,\\ (2a+n+\alpha)|\xi|\,e_I & \text{if } \xi<0.\end{cases} \tag{2}$$

Equation (2) shows that $(\mathscr{L}_\alpha)^\wedge_\xi$ is diagonalized under the basis $\{e_I\}_{I\in\mathbb{Z}^{2n}_{\geq 0}}$. Let λ_I be the eigenvalue of $(\mathscr{L}_\alpha)^\wedge_\xi$ associated to e_I. When α is admissible, it follows from equation (2) that there is a positive constant C, depending only on n and α, such that

$$\begin{cases}C\cdot b|\xi| \geq |\lambda_I| \geq \frac{1}{C}\cdot b|\xi| & \text{if } \xi>0\\ C\cdot a|\xi| \geq |\lambda_I| \geq \frac{1}{C}\cdot a|\xi| & \text{if } \xi<0\end{cases} \tag{3}$$

holds for every I.

We can also compute the Fourier transforms of the derivative operators along Z_j and $\bar{Z}_j$. When $\xi > 0$, the Fourier transforms of the operators are

$$(Z_j)^\wedge_\xi = \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}) + i|\xi|(y_j + ix_j) = v_j^\dagger,$$
$$(\bar{Z}_j)^\wedge_\xi = \frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}) + i|\xi|(y_j - ix_j) = -v_j,$$

and when $\xi < 0$, the Fourier transforms of the operators are

$$(Z_j)^\wedge_\xi = \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}) - i|\xi|(y_j + ix_j) = u_j,$$
$$(\bar{Z}_j)^\wedge_\xi = \frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}) - i|\xi|(y_j - ix_j) = -u_j^\dagger.$$

For $I = (a_1, \cdots, a_n, b_1, \cdots, b_n)$, let

$$I^+_{a_j} = (a_1, \cdots, a_{j-1}, a_j + 1, a_{j+1}, \cdots, a_n, \cdots, b_n),$$
$$I^+_{b_j} = (a_1, \cdots, a_n, b_1, \cdots, b_{j-1}, b_j + 1, b_{j+1} \cdots, b_n),$$
$$I^-_{a_j} = (a_1, \cdots, a_{j-1}, a_j - 1, a_{j+1}, \cdots, a_n, \cdots, b_n),$$
$$I^-_{b_j} = (a_1, \cdots, a_n, b_1, \cdots, b_{j-1}, b_j - 1, b_{j+1} \cdots, b_n).$$

Then, when $\xi > 0$ we have

$$(Z_j)^\wedge_\xi(e_I) = (2|\xi|(b_j + 1))^{1/2} \cdot e_{I^+_{b_j}}, \tag{4}$$

and

$$(\bar{Z}_j)^\wedge_\xi(e_I) = \begin{cases} -(2|\xi|b_j)^{1/2} \cdot e_{I^-_{b_j}} & \text{if } b_j \geq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

When $\xi < 0$ we have

$$(Z_j)^\wedge_\xi(e_I) = \begin{cases} (2|\xi|a_j)^{1/2} \cdot e_{I^-_{a_j}} & \text{if } a_j \geq 1, \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

and

$$(\bar{Z}_j)^\wedge_\xi(e_I) = -(2|\xi|(a_j + 1))^{1/2} \cdot e_{I^+_{a_j}}. \tag{7}$$

3 Hypoellipticity of $\frac{\partial}{\partial s} + (\mathscr{L}_\alpha)^m$

Let m be a positive integer, and $\alpha \in \mathbb{R}$ be a fixed admissible constant. This section proves the hypoellipticity of $\frac{\partial}{\partial s} + (\mathscr{L}_\alpha)^m$. Taking the Fourier transform with respect to the t and s variables, we can write every function $f \in \mathscr{S}(\mathbb{R}^{2n+2})$ as

$$f(x_1, \cdots, x_n, y_1, \cdots, y_n, t, s) = \int_{\xi,\eta} e^{i\xi t} e^{i\eta s} \hat{f}_{\xi,\eta}(x_1, \cdots, x_n, y_1, \cdots, y_n)\, d\xi d\eta.$$

The Fourier transform of the heat operator then becomes

$$\left(\frac{\partial}{\partial s} + (\mathscr{L}_\alpha)^m\right)^\wedge_{\xi,\eta} = \left[(\mathscr{L}_\alpha)^\wedge_\xi\right]^m + i\eta. \tag{8}$$

The previous section constructed a set of functions $e_I \in L^2(\mathbb{R}^{2n})$, with $I \in \mathbb{Z}^{2n}_{\geq 0}$, such that the set $\{e_I\}$ is a complete orthonormal basis of $L^2(\mathbb{R}^{2n})$ and it diagonalizes the operator $(\mathscr{L}_\alpha)^\wedge_\xi$. Since in this section we will consider different values for ξ, we will use $\{e_{I,\xi}\}$ instead of $\{e_I\}$ to denote the basis corresponding to $(\mathscr{L}_\alpha)^\wedge_\xi$. Use $\lambda_{I,\xi}$ to denote the eigenvalue of $(\mathscr{L}_\alpha)^\wedge_\xi$ with respect to $e_{I,\xi}$. Then equation (8) can be written as

$$\left(\frac{\partial}{\partial s} + (\mathscr{L}_\alpha)^m\right)^\wedge_{\xi,\eta}(e_{I,\xi}) = (\lambda_I^m + i\eta)e_{I,\xi}. \tag{9}$$

Now for any $l \in \mathbb{Z}$, define a norm $\|\cdot\|_l$ on $\mathscr{S}(\mathbb{R}^{2n+2})$ by

$$\|f\|_l^2 := \int_{\xi,\eta} \sum_I (1 + |\lambda_{I,\xi}| + |\eta|^{1/m})^l |\langle \hat{f}_{\xi,\eta}, e_{I,\xi}\rangle|^2 \, d\xi d\eta.$$

Before we proceed further, we need to make a convention about the order of certain differential operators on $\mathbb{R}^{2n+2}$. Consider the scaling map $\Delta_c : \mathbb{R}^{2n+2} \to \mathbb{R}^{2n+2}$ which sends $(x_1, \cdots, x_n, y_1, \cdots, y_n, t, s)$ to $(cx_1, \cdots, cx_n, cy_1, \cdots, cy_n, c^2 t, c^{2m} s)$. A differential operator $\mathcal{P}$ on $\mathbb{R}^{2n+2}$ is called homogeneous of order k if

$$\Delta_c^*(\mathcal{P}(f)) = c^{-1}\mathcal{P}\Delta_c^*(f)$$

holds for every pair of $c \in \mathbb{R}^+$ and $f \in C^\infty(\mathbb{R}^{2n+2})$. The derivatives with respect to Z_j and $\bar{Z}_j$ are then homogeneous of order 1, and $\frac{\partial}{\partial s}$ is homogeneous of order $2m$. For an integer $k \geq 1$, let $\mathcal{D}_k$ be the finite set of homogeneous differential operators of order k consisting of compositions of operators among $Z_1, \cdots, Z_n, \bar{Z}_1, \cdots \bar{Z}_n, \frac{\partial}{\partial s}$. For example, we have $Z_j \circ \bar{Z}_k \in \mathcal{D}_2$, but $(Z_j + Z_k) \notin \mathcal{D}_1$. For $l \geq 0$, define another norm $\|\cdot\|'_l$ on $\mathscr{S}(\mathbb{R}^{2n+2})$ as

$$\|f\|'_l := \sum_{0 \leq k \leq l} \sum_{\mathcal{P} \in \mathcal{D}_k} \|\mathcal{P}f\|_{L^2(\mathbb{R}^{2n+2})}.$$

The Plancherel identity, the inequality (3), and equations (4) to (7) then imply that the norms $\|\cdot\|_l$ and $\|\cdot\|'_l$ are equivalent when $l \geq 0$.

Now consider a function $\chi \in C_0^\infty(\mathbb{R}^{2n+2})$. It follows immediately from the definition that the multiplication map

$$m_\chi : \mathscr{S}(\mathbb{R}^{2n+2}) \to \mathscr{S}(\mathbb{R}^{2n+2})$$
$$f \mapsto \chi f$$

is a bounded linear operator in the $\|\cdot\|'_l$ norm. Therefore, when $l \geq 0$, m_χ is also bounded in the $\|\cdot\|_l$ norm. In fact, the same result holds for negative l as well, as we have the following

Lemma 3.1. *Let $\chi \in C_0^\infty(\mathbb{R}^{2n+2})$, then for every $l \in \mathbb{Z}$, the map m_χ is bounded in the $\|\cdot\|_l$ norm.*

Proof. If $l \geq 0$, the result is already proved in the previous paragraph.

If $l < 0$, for every $f \in \mathscr{S}(\mathbb{R}^{2n+2})$ consider the linear functional

$$T_f : \mathscr{S}(\mathbb{R}^{2n+2}) \to \mathbb{C}$$
$$g \mapsto \int fg.$$

If we endow $\mathscr{S}(\mathbb{R}^{2n+2})$ with the $\|\cdot\|_{-l}$ norm, then $\|f\|_l$ equals the operator norm of T_f. Notice that $T_{(m_\chi f)} = T_f \circ m_\chi$. Since $-l \geq 0$, the operator m_χ is bounded in the $\|\cdot\|_{-l}$ norm, hence the operator norm of $\|T_{(m_\chi f)}\|$ is bounded by a constant times the operator norm of $\|Tf\|$. Therefore m_χ is bounded in the $\|\cdot\|_l$ norm. □

Let S_l be the completion of $\mathscr{S}(\mathbb{R}^{2n+2})$ under the norm $\|\cdot\|_l$. Then S_l is a Hilbert space, and S_l is natually isomorphic to the dual of S_{-l}. When $l \geq l'$, we have $S_l \subset S_{l'}$. By lemma 3.1, the map m_χ extends to bounded linear maps on S_l.

Every element in $\mu \in S_l$ defines a distribution on $\mathbb{R}^{2n+2}$. In fact, when $l \geq 0$, we have $S_l \subset L^2(\mathbb{R}^{2n+2})$. When $l < 0$, consider a testing function $f \in \mathscr{S}(\mathbb{R}^{2n+2})$. The operator T_f defined in the proof of lemma 3.1 extends to a bounded operator on S_l. Define $(\mu, f) := T_f(\mu)$, then (μ, f) is continuous with respect to f in the $\|\cdot\|_{-l}$ norm, hence such an assignment makes μ a distribution on $\mathbb{R}^{2n+2}$.

Conversely, we have the following lemma:

Lemma 3.2. *Let μ be a compactly supported distribution on $\mathbb{R}^{2n+2}$, then there exists $l \in \mathbb{Z}$ such that $\mu \in S_l$.*

Proof. The distribution μ defines a linear functional from $\mathscr{S}(\mathbb{R}^{2n+2})$ to $\mathbb{C}$. Since μ is compactly supported, there exists an integer $N > 0$ such that μ is bounded in the C^N norm. By the Sobolev embedding theorem, there exists $M > 0$ such that μ is bounded in the L^2_M norm. Notice that $\frac{\partial}{\partial t} = \frac{i}{2}[Z_j, \bar{Z}_j]$. Therefore, if we define $\widehat{M} = M \cdot \max\{2, m\}$, then for any bounded open set $U \subset \mathbb{R}^{2n+2}$, there exists a constant C depending on U and M such that

$$C\,\|f\|'_{\widehat{M}} \geq \|f\|_{L^2_M}, \qquad \forall f \in C_0^\infty(U).$$

Since μ is compactly supported, this implies that μ is bounded in the $\|\cdot\|'_{\widehat{M}}$ norm. Recall that $\|\cdot\|'_{\widehat{M}}$ and $\|\cdot\|_{\widehat{M}}$ are equivalent norms, therefore μ defines a bounded linear functional on $S_{\widehat{M}}$. Since the dual space of $S_{\widehat{M}}$ is canonically isomorphic to $S_{-\widehat{M}}$, there exists an element $\mu' \in S_{-\widehat{M}}$ such that

$$\langle \mu', f\rangle_{L^2} = \mu(f), \qquad \forall f \in S_{\widehat{N}}.$$

The equation above implies that μ' equals μ when viewed as distributions on $\mathbb{R}^{2n+2}$, hence the result is proved. ☐

We also have the following regularity result:

Lemma 3.3. *Let μ be a distribution on $\mathbb{R}^{2n+2}$. If $\mu \in S_l$ for every l, then μ must be smooth.*

Proof. We only need to prove that $\chi\mu$ is smooth for every $\chi \in C_0^\infty(\mathbb{R}^{2n+2})$. Since the multiplication by χ gives a bounded linear operator on S_l, the assumption on μ implies that $\chi\mu \in S_l$ for every l.

Let $U \subset \mathbb{R}^{2n+2}$ be a bounded open set such that $\operatorname{supp}\chi \subset U$. For any $N > 0$, there exists a constant C_1 and a positive integer M such that

$$C_1\,\|f\|_{L^2_M} \geq \|f\|_{C^N}.$$

Let $\widehat{M} = M \cdot \max\{2, m\}$, then there is a constant C_2 depending on U and M such that

$$C_2\,\|f\|_{\widehat{M}} \geq \|f\|_{L^2_M}.$$

Combining the two inequalities above, we conclude that $\chi\mu$ has bounded C^N norm. Since N is arbitrary, this implies that $\chi\mu$ is smooth, hence the result is proved. ☐

With the preparations above, we can now present the proof of the main theorem

Theorem 3.4 (Greiner [3], Calin-Chang-Furutani-Iwasaki [1]). *Suppose α is admissible, then the operator $\frac{\partial}{\partial s} + (\mathscr{L}_\alpha)^m$ is hypoelliptic.*

Proof. Let U be an open subset of $\mathbb{R}^{2n+2}$. Let μ be a distribution on $\mathbb{R}^{2n+2}$ such that $\left[\frac{\partial}{\partial s} + (\mathscr{L}_\alpha)^m\right](\mu)|_U$ is smooth. We need to prove that μ is smooth on U. Take an arbitrary open subset $U' \subset U$ such that $\overline{U'} \subset U$, we only need to prove that μ is smooth on U'. To simplify notations, let $\mathcal{D} = \frac{\partial}{\partial s} + (\mathscr{L}_\alpha)^m$.

Notice that equation (9) has the following consequence: for any $f \in S_l$, if $\mathcal{D}f \in S_l$, then $f \in S_{l+2m}$. Moreover, we have the following Garding-type inequality:

$$C(\|\mathcal{D}(f)\|_l + \|f\|_l) \geq \|f\|_{l+2m}, \tag{10}$$

where C is a constant depending on n, α and l.

By lemma 3.2, we may assume that $\mu \in S_l$ for some integer l. Lemma 3.1 then implies $\chi\mu \in S_l$.

Recall that previously we have defined $\mathcal{D}_k$ to be the finite set of homogeneous differential operators of order k consisting of compositions of operators among $Z_1, \cdots, Z_n, \bar{Z}_1, \cdots \bar{Z}_n, \frac{\partial}{\partial s}$. For an element $f \in S_l$, use $\nabla^k f$ to denote the tuple $(\mathcal{P}f)_{\mathcal{P}\in\mathcal{D}_k}$. Inequality (3) and equations (4) to (7) then imply that when $f \in S_l$, every entry of $\nabla^j f$ is an element of S_{l-j}.

Notice that

$$\mathcal{D}(\chi\mu) = \chi\mathcal{D}(\mu) + \sum_{j=0}^{2m-1} \nabla^{m-j}\chi \boxtimes \nabla^j \mu,$$

where $\boxtimes$ are some bilinear pairings depending smoothly on the coordinates $(x_1, \cdots, x_n, y_1, \cdots, y_n, s, t)$. By the assumption, $\chi\mathcal{D}(\mu)$ is smooth and compactly supported. Therefore by lemma 3.1, the distribution $\mathcal{D}(\chi\mu)$ is an element of S_{l-2m+1}.

Now we have $\chi\mu \in S_l \subset S_{l-2m+1}$, and $\mathcal{D}(\chi\mu) \in S_{l-2m+1}$. Therefore, by the previous discussion, the distribution $\chi\mu$ is an element of S_{l+1}. The smoothness of μ on U' then follows from a standard bootstrap argument and lemma 3.3. □

References

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