Solutions for quiz #4.

SECTIONS AT 12PM

Problem 1. Find the integral
$$\int \frac{dx}{(x^2+1)^{3/2}}$$

Solution
$$\int \frac{dx}{(x^2+1)^{3/2}} =$$
substituting $x = \tan y$, $dx = \sec^2 y dy$ where we assume $y \in (-\pi/2, \pi/2)$, we get
$$= \int \frac{\sec^2 y dy}{(1+\tan^2 y)^{3/2}} = \int \frac{dy}{\cos^2 y((\sin^2 y + \cos^2 y)/\cos^2 y)^{3/2}} =$$
$$= \int \frac{dy}{\cos^2 y(1/\cos^2 y)^{3/2}} = \int \frac{dy}{\cos^2 y |\cos y|^{-3}} = \int |\cos y| dy =$$
since we assumed $y \in (-\pi/2, \pi/2)$, then $\cos y > 0$, hence
$$= \int \cos y dy = \sin y + C = \sin(\arctan x) + C = \frac{x}{\sqrt{1+x^2}} + C$$

Problem 2. Find the integral
$$\int \frac{x^2}{x^2+1} dx$$

Solution
$$\int \frac{x^2}{x^2+1} dx = \int \frac{(x^2+1)-1}{x^2+1} dx = \int \left(1 - \frac{1}{x^2+1}\right) dx =$$
$$= x - \arctan x + C$$

Problem 3. Find the integral
$$\int \frac{1}{1-x^2} dx$$

Solution
$$\int \frac{1}{1-x^2} dx = -\int \frac{1}{(x-1)(x+1)} dx$$
using the partial fractions method, let us represent the function under the integral in the following form:
$$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}.$$
Then for any x
$$\frac{1}{(x-1)(x+1)} = \frac{A(x+1)+B(x-1)}{(x+1)(x-1)} = \frac{(A+B)x+(A-B)}{(x+1)(x-1)},$$

therefore $1 = (A+B)x + (A-B)$ for any x, and hence $A+B=0, A-B=1$, i.e. $A=1/2$, $B=-1/2$. Coming back to computing the integral,

$$\int \frac{1}{1-x^2}dx = -\int \frac{1}{(x-1)(x+1)}dx = -\int \left(\frac{1/2}{x-1} - \frac{1/2}{x+1}\right)dx =$$

$$= -1/2\int \frac{dx}{x-1} + 1/2\int \frac{dx}{x+1} = -1/2\ln|x-1| + 1/2\ln|x+1| + C$$

Problem 4 Determine the smallest value of n that guarantees an error of no more than 0.1 in the approximation by the Trapezoidal Rule of

$$\int_{\pi}^{3\pi} \sin(x)dx.$$

Solution:
Applying the estimate for the error in computing the integral $\int_{\pi}^{3\pi}\sin(x)dx$ by the trapezoidal method, we have:

$$Error \le \frac{K_T(3\pi-\pi)^3}{12n^2}, \quad K_T = \max_{x\in[\pi,3\pi]} |(\sin x)''|$$

Since $(\sin x)'' = -\sin x$, $K_T = \max_{x\in[\pi,3\pi]} |\sin x| = 1$, so, for the error not to exceed 0.1 it suffices to have

$$\frac{(2\pi)^3}{12n^2} \le 0.1 \ \Leftrightarrow\ \frac{8\pi^3}{12n^2} \le 0.1 \ \Leftrightarrow\ \frac{2\pi^3}{3n^2} \le 0.1 \ \Leftrightarrow$$

$$3n^2 \ge 20\pi^3 \ \Leftrightarrow\ n^2 \ge \frac{20\pi^3}{3} \ \Leftrightarrow\ n \ge \sqrt{\frac{20\pi^3}{3}}$$

Now we need to find the smallest integer n satisfying this inequality. Assuming $\pi \approx 3.14$, we get

$$\frac{20\pi^3}{3} \approx 206.394$$

The first complete square of an integer that is greater than 206.394 equals $225 = 15^2$. Hence, the answer is $n = 15$.

Sections at 1pm

Problem 1. Find the integral

$$\int \frac{dx}{(x^2+1)^{3/2}}$$

Solution

$$\int \frac{dx}{(x^2+1)^{3/2}} =$$

substituting $x = \tan y$, $dx = \sec^2 y dy$ where we assume $y \in (-\pi/2, \pi/2)$, we get

$$= \int \frac{\sec^2 y dy}{(1+\tan^2 y)^{3/2}} = \int \frac{dy}{\cos^2 y((\sin^2 y + \cos^2 y)/\cos^2 y)^{3/2}} =$$

$$= \int \frac{dy}{\cos^2 y(1/\cos^2 y)^{3/2}} = \int \frac{dy}{\cos^2 y|\cos y|^{-3}} = \int |\cos y| dy =$$

since we assumed $y \in (-\pi/2, \pi/2)$, $\cos y > 0$, and hence

$$= \int \cos y dy = \sin y + C = \sin(\arctan x) + C = \frac{x}{\sqrt{1+x^2}} + C$$

Problem 2. Find the integral

$$\int \frac{x^2}{x^2-1} dx$$

Solution

$$\int \frac{x^2}{x^2-1} dx = \int \frac{(x^2-1)+1}{x^2-1} dx = \int \left(1 + \frac{1}{x^2-1}\right) dx =$$

$$= x + \int \frac{1}{(x-1)(x+1)} dx \qquad (*)$$

let us represent the function under the integral in the following form:

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}.$$

Then for any x

$$\frac{1}{(x-1)(x+1)} = \frac{A(x+1)+B(x-1)}{(x+1)(x-1)} = \frac{(A+B)x+(A-B)}{(x+1)(x-1)},$$

therefore $1 = (A+B)x + (A-B)$ for any x, and hence $A+B=0$, $A-B=1$, i.e. $A = 1/2$, $B = -1/2$. Coming back to computing the integral in $(*)$,

$$\int \frac{x^2}{x^2-1} dx = x + \int \frac{1}{(x-1)(x+1)} dx = x + \int \left(\frac{1/2}{x-1} - \frac{1/2}{x+1}\right) dx =$$

$$= x + 1/2 \int \frac{dx}{x-1} - 1/2 \int \frac{dx}{x+1} = x + 1/2 \ln|x-1| - 1/2 \ln|x+1| + C$$

Problem 3. Find the integral

$$\int \frac{1}{4-x^2} dx$$

Solution

$$\int \frac{1}{4-x^2} dx = -\int \frac{1}{(x-2)(x+2)} dx$$

let us represent the function under the integral in the following form:

$$\frac{1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}.$$

Then for any x

$$\frac{1}{(x-2)(x+2)} = \frac{A(x+2)+B(x-2)}{(x+2)(x-2)} = \frac{(A+B)x+2(A-B)}{(x+2)(x-2)},$$

therefore $1 = (A+B)x + 2(A-B)$ for any x, and hence $A+B=0,\ 2(A-B)=1$, i.e. $A = 1/4$, $B = -1/4$. Hence,

$$\int \frac{1}{4-x^2}dx = -\int \frac{1}{(x-2)(x+2)}dx = -\int \left(\frac{1/4}{x-2} - \frac{1/4}{x+2}\right) dx =$$

$$= -1/4 \int \frac{dx}{x-2} + 1/4 \int \frac{dx}{x+2} = -1/4 \ln|x-2| + 1/4 \ln|x+2| + C$$

Problem 4 Determine the smallest value of n that guarantees an error of no more than 0.1 in the approximation by the Trapezoidal Rule of

$$\int_0^{4\pi} \cos x dx.$$

Solution:
Applying the estimate for the error in computing the integral $\int_0^{4\pi} \cos x dx$ by the trapezoidal method, we have:

$$Error \leq \frac{K_T(4\pi)^3}{12n^2}, \quad K_T = \max_{x\in[0,4\pi]} |(\cos x)''|$$

Since $(\cos x)'' = -\cos x$, $K_T = \max_{x\in[0,4\pi]} |\cos x| = 1$, so, for the error not to exceed 0.1 it suffices to have

$$\frac{(4\pi)^3}{12n^2} \leq 0.1 \ \Leftrightarrow\ \frac{64\pi^3}{12n^2} \leq 0.1 \ \Leftrightarrow\ \frac{16\pi^3}{3n^2} \leq 0.1 \ \Leftrightarrow$$

$$3n^2 \geq 160\pi^3 \ \Leftrightarrow\ n^2 \geq \frac{160\pi^3}{3} \ \Leftrightarrow\ n \geq \sqrt{\frac{160\pi^3}{3}}$$

Now we need to find the smallest integer n satisfying this inequality. Assuming $\pi \approx 3.14$, we get

$$\frac{160\pi^3}{3} \approx 1651.154$$

The first complete square of an integer that is greater than 1651.154 equals $1681 = 41^2$ (since $40^2 = 1600$). Hence, the answer is $n = 41$.

Sections at 2pm

Problem 1. Find the integral

$$\int \frac{dx}{x^2\sqrt{x^2+1}}$$

Solution

$$\int \frac{dx}{x^2\sqrt{x^2+1}} =$$

substituting $x = \tan y$, $dx = \sec^2 y dy$ where we assume $y \in (-\pi/2, \pi/2)$, we get

$$= \int \frac{\sec^2 y dy}{\tan^2 y(1+\tan^2 y)^{1/2}} = \int \frac{dy}{\sin^2 y((\sin^2 y + \cos^2 y)/\cos^2 y)^{1/2}} =$$

$$= \int \frac{dy}{\sin^2 y (1/\cos^2 y)^{1/2}} = \int \frac{dy}{\sin^2 y |\cos y|^{-1}} = \int \frac{|\cos y|}{\sin^2 y} dy =$$

since we assumed $y \in (-\pi/2, \pi/2)$, then $\cos y > 0$, hence

$$= \int \frac{\cos y}{\sin^2 y} dy dy =$$

using substitution $t = \sin y$, $dt = \cos y dy$, we get

$$\int \frac{dt}{t^2} = -\frac{1}{t} + C = -\frac{1}{\sin y} + C = -\frac{1}{\sin \arctan(x)} + C = -\frac{\sqrt{1+x^2}}{x} + C$$

Problem 2. Find the integral

$$\int \frac{x-1}{x+1} dx$$

Solution

$$\int \frac{x-1}{x+1} dx = \int \frac{(x+1)-2}{x+1} dx = \int \left(1 - \frac{2}{x+1}\right) dx =$$

$$= x - 2 \int \frac{dx}{x+1} = x - 2 \ln|x+1| + C$$

Problem 3. Find the integral

$$\int \frac{3}{x^2-1} dx$$

Solution

$$\int \frac{3}{x^2-1} dx = \int \frac{3}{(x-1)(x+1)} dx$$

let us represent the function under the integral in the following form:

$$\frac{3}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}.$$

Then for any x

$$\frac{3}{(x-1)(x+1)} = \frac{A(x+1) + B(x-1)}{(x+1)(x-1)} = \frac{(A+B)x + (A-B)}{(x+1)(x-1)},$$

therefore $3 = (A+B)x + (A-B)$ for any x, and hence $A + B = 0$, $(A - B) = 3$, i.e. $A = 3/2$, $B = -3/2$. Hence,

$$\int \frac{3}{x^2-1} dx = \int \frac{3}{(x-1)(x+1)} dx = \int \left(\frac{3/2}{x-1} - \frac{3/2}{x+1}\right) dx =$$

$$= 3/2 \int \frac{dx}{x-1} - 3/2 \int \frac{dx}{x+1} = 3/2 \ln|x-1| - 3/2 \ln|x+1| + C$$

Problem 4 Determine the smallest value of n that guarantees an error of no more than 0.1 in the approximation by the Trapezoidal Rule of

$$\int_0^1 e^x dx.$$

Solution:
Applying the estimate for the error in computing the integral $\int_0^1 e^x dx$ by the trapezoidal method, we have:

$$Error \leq \frac{K_T(1-0)^3}{12n^2}, \quad K_T = \max_{x\in[0,1]} |(e^x)''|$$

Since $(e^x)'' = e^x$, $K_T = \max_{x\in[0,1]} |e^x| = e$, so, for the error not to exceed 0.1 it suffices to have

$$\frac{e}{12n^2} \leq 0.1 \;\Leftrightarrow\; 12n^2 \geq 10e \;\Leftrightarrow\; n^2 \geq \frac{5e}{6} \;\Leftrightarrow\; n \geq \sqrt{\frac{5e}{6}}$$

Now we need to find the smallest integer n satisfying this inequality. Assuming $e \approx 2.72$, we get

$$\frac{5e}{6} \approx 2.27$$

The first complete square of an integer that is greater than 2.27 equals $4 = 2^2$. Hence, the answer is $n = 2$.

SECTIONS AT 3PM

Problem 1. Find the integral

$$\int \frac{2dx}{x^2\sqrt{x^2+1}}$$

Solution

$$\int \frac{2dx}{x^2\sqrt{x^2+1}} =$$

substituting $x = \tan y$, $dx = \sec^2 y dy$ where we assume $y \in (-\pi/2, \pi/2)$, we get

$$= 2\int \frac{\sec^2 y dy}{\tan^2 y(1+\tan^2 y)^{1/2}} = 2\int \frac{dy}{\sin^2 y((\sin^2 y + \cos^2 y)/\cos^2 y)^{1/2}} =$$

$$= 2\int \frac{dy}{\sin^2 y(1/\cos^2 y)^{1/2}} = 2\int \frac{dy}{\sin^2 y|\cos y|^{-1}} = 2\int \frac{|\cos y|}{\sin^2 y} dy =$$

since we assumed $y \in (-\pi/2, \pi/2)$, then $\cos y > 0$, hence

$$= 2\int \frac{\cos y}{\sin^2 y} dydy =$$

using substitution $t = \sin y$, $dt = \cos y dy$, we get

$$2\int \frac{dt}{t^2} = -2\frac{1}{t} + C = -2\frac{1}{\sin y} + C = -2\frac{1}{\sin\arctan(x)} + C = -2\frac{\sqrt{1+x^2}}{x} + C$$

Problem 2. Find the integral

$$\int \frac{x}{x+1} dx$$

Solution

$$\int \frac{x}{x+1}dx = \int \frac{(x+1)-1}{x+1}dx = \int \left(1 - \frac{1}{x+1}\right) dx =$$

$$= x - \int \frac{dx}{x+1} = x - \ln|x+1| + C$$

Problem 3. Find the integral

$$\int \frac{1}{3x^2-1}dx$$

Solution

$$\int \frac{1}{3x^2-1}dx = \frac{1}{3}\int \frac{1}{(x-1/\sqrt{3})(x+1/\sqrt{3})}dx$$

let us represent the function under the integral in the following form:

$$\frac{1}{(x-1/\sqrt{3})(x+1/\sqrt{3})} = \frac{A}{x-1/\sqrt{3}} + \frac{B}{x+1/\sqrt{3}}.$$

Then for any x

$$\frac{1}{(x-1/\sqrt{3})(x+1/\sqrt{3})} = \frac{A(x+1/\sqrt{3}) + B(x-1/\sqrt{3})}{(x+1/\sqrt{3})(x-1/\sqrt{3})} = \frac{(A+B)x + (A-B)/\sqrt{3}}{(x+1/\sqrt{3})(x-1/\sqrt{3})},$$

therefore $1 = (A+B)x + (A-B)$ for any x, and hence $A+B=0$, $(A-B)/\sqrt{3} = 1$, i.e. $A = \sqrt{3}/2$, $B = -\sqrt{3}/2$. Hence,

$$\frac{1}{3}\int \frac{1}{x^2-1/3}dx = \frac{1}{3}\int \frac{1}{(x-1/\sqrt{3})(x+1/\sqrt{3})}dx = \int \left(\frac{\sqrt{3}/2}{x-1\sqrt{3}} - \frac{\sqrt{3}/2}{x+1\sqrt{3}}\right) dx =$$

$$= \sqrt{3}/2 \int \frac{dx}{x-1\sqrt{3}} - \sqrt{3}/2 \int \frac{dx}{x+1\sqrt{3}} = 3/2 \ln|x-1\sqrt{3}| - 3/2 \ln|x+1\sqrt{3}| + C$$

Problem 4 Determine the smallest value of n that guarantees an error of no more than 0.1 in the approximation by the Trapezoidal Rule of

$$\int_1^2 x^2 dx.$$

Solution:
Applying the estimate for the error in computing the integral $\int_1^2 x^2 dx$ by the trapezoidal method, we have:

$$Error \leq \frac{K_T(2-1)^3}{12n^2}, \quad K_T = \max_{x\in[0,1]} |(x^2)''|$$

Since $(x^2)'' = 2$, $K_T = 2$, so, for the error not to exceed 0.1 it suffices to have

$$\frac{2}{12n^2} \leq 0.1 \;\Leftrightarrow\; 12n^2 \geq 20 \;\Leftrightarrow\; n^2 \geq \frac{5}{3} \;\Leftrightarrow\; n \geq \sqrt{\frac{5}{3}}$$

Now we need to find the smallest integer n satisfying this inequality. The first complete square of an integer that is greater than 5/3 equals $4 = 2^2$. Hence, the answer is $n = 2$.