Solutions to MATH141 Quiz 15

November 19, 2009

12 PM

Problem 1

Use exponentials and logarithms to rewrite the limit. Then use continuity to introduce the limit inside the argument of the exponential

$$\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^n = \lim_{n\to\infty} e^{\ln\left(1+\frac{1}{2n}\right)^n} = e^{\lim_{n\to\infty} n\ln\left(1+\frac{1}{2n}\right)}$$

Apply L'Hôpital's rule to evaluate the limit

$$\lim_{n\to\infty} n\ln\left(1+\frac{1}{2n}\right) = \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{2x}\right)}{\frac{1}{x}} = \lim_{x\to\infty}\frac{\left(\frac{1}{1+1/(2x)}\right)\left(-\frac{1}{2x^2}\right)}{-\frac{1}{x^2}}$$
$$= \lim_{x\to\infty}\left(\frac{1/2}{1+1/(2x)}\right) = \frac{1}{2}.$$

Substituting back this limit in the exponential,

$$\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^n = e^{\frac{1}{2}}.$$

Problem 2

Use exponentials and logarithms to rewrite the limit. Then use continuity to introduce the limit inside the argument of the exponential

$$\lim_{n\to\infty}\sqrt[2n]{n} = \lim_{n\to\infty} e^{\ln\left(\sqrt[2n]{n}\right)} = e^{\lim_{n\to\infty}\frac{\ln(n)}{2n}}$$

Apply L'Hôpital's rule to evaluate the limit

$$\lim_{n\to\infty}\frac{\ln(n)}{2n} = \lim_{x\to\infty}\frac{\ln(x)}{2x} = \lim_{x\to\infty}\frac{1/x}{2} = 0.$$

Substituting back this limit in the exponential,

$$\lim_{n\to\infty}\sqrt[2n]{n} = e^0 = 1.$$

Problem 3

The sum is a geometric series with ratio $r = 3$

$$\sum_{n=1}^{\infty} \frac{4}{3^n} = 4\sum_{1}^{\infty} \frac{1}{3^n} = \frac{4 \cdot \frac{1}{3}}{1 - \frac{1}{3}} = \frac{\frac{4}{3}}{\frac{2}{3}} = \frac{4}{2} = 2.$$

Problem 4

The first part of the problem consists in comparing the series to the series $\sum_{n=2}^{\infty} \frac{1}{n^2}$. Either using
COMPARISON TEST:

$$\frac{n^2}{n^4+4} \le \frac{1}{n^2} \iff n^4 \le n^4 + 4 \quad \text{which is true for all } n \ge 0.$$

This means that convergence of $\sum_{n=2}^{\infty} \frac{1}{n^2}$ implies convergence of $\sum_{n=2}^{\infty} \frac{n^2}{n^4+4}$. Alternatively, one can use
LIMIT COMPARISON TEST:

$$\lim_{n\to\infty} \frac{\frac{n^2}{n^4+4}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^4}{n^4+4} = 1.$$

This implies that the convergence of the two series are equivalent, so it remains to establish the convergence of $\sum_{n=2}^{\infty} \frac{1}{n^2}$. This follows from the p-Series Theorem, for $p = 2$. Since $p > 1$, we have convergence. You can also do this directly by doing the Integral Test.

1 PM

Problem 1

Use exponentials and logarithms to rewrite the limit. Then use continuity to introduce the limit inside the argument of the exponential

$$\lim_{n\to\infty} \left(1 + \frac{2}{n}\right)^n = \lim_{n\to\infty} e^{\ln\left(1+\frac{2}{n}\right)^n} = e^{\lim_{n\to\infty} n \ln\left(1+\frac{2}{n}\right)}$$

Apply L'Hôpital's rule to evaluate the limit

$$\lim_{n\to\infty} n \ln\left(1 + \frac{2}{n}\right) = \lim_{x\to\infty} \frac{\ln\left(1 + \frac{2}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\left(\frac{1}{1+2/x}\right)\left(-\frac{2}{x^2}\right)}{-\frac{1}{x^2}}$$
$$= \lim_{x\to\infty} \left(\frac{1/2}{1+2/x}\right) = 2.$$

Substituting back this limit in the exponential,

$$\lim_{n\to\infty} \left(1 + \frac{2}{n}\right)^n = e^2.$$

Problem 2

Use exponentials and logarithms to rewrite the limit. Then use continuity to introduce the limit inside the argument of the exponential

$$\lim_{n\to\infty} \sqrt[n]{2n} = \lim_{n\to\infty} e^{\ln\left(\sqrt[n]{2n}\right)} = e^{\lim_{n\to\infty} \frac{\ln(2n)}{n}}$$

Apply L'Hôpital's rule to evaluate the limit

$$\lim_{n\to\infty} \frac{\ln(2n)}{n} = \lim_{x\to\infty} \frac{\ln(2x)}{x} = \lim_{x\to\infty} \frac{1}{2x} \cdot 2 = 0.$$

Substituting back this limit in the exponential,

$$\lim_{n\to\infty} \sqrt[2n]{n} = e^0 = 1.$$

Problem 3

The sum is a geometric series with ratio $r = 3$

$$\sum_{n=3}^{\infty} \frac{1}{3^n} = \sum_{3}^{\infty} \frac{1}{3^n} = \frac{\left(\frac{1}{3}\right)^3}{1 - \frac{1}{3}} = \frac{\frac{1}{27}}{\frac{2}{3}} = \frac{1}{18}.$$

Problem 4

The first part of the problem consists in comparing the series to the series $\sum_{n=2}^{\infty} \frac{1}{n^2}$. Either using
Comparison Test:

$$\frac{n}{n^3+4} \le \frac{1}{n^2} \iff n^3 \le n^3 + 4 \quad \text{which is true for all } n \ge 0.$$

This means that convergence of $\sum_{n=2}^{\infty} \frac{1}{n^2}$ implies convergence of $\sum_{n=2}^{\infty} \frac{n}{n^3+4}$. Alternatively, one can use
Limit Comparison Test:

$$\lim_{n\to\infty} \frac{\frac{n}{n^3+4}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^3}{n^3+4} = 1.$$

This implies that the convergence of the two series are equivalent, so it remains to establish the convergence of $\sum_{n=2}^{\infty} \frac{1}{n^2}$. This follows from the p-Series Theorem, for $p = 2$. Since $p > 1$, we have convergence. You can also do this directly by doing the Integral Test.

2 PM

Problem 1

For all $n \ge 0$, $\sin(2\pi n) = 0$. Then

$$\lim_{n\to\infty} \sin^n(2\pi n) = \lim_{n\to\infty} 0^n = 0.$$

Problem 2

Use exponentials and logarithms to rewrite the limit. Then use continuity to introduce the limit inside the argument of the exponential

$$\lim_{n\to\infty}\sqrt[n]{\frac{1}{n}} = \frac{1}{\lim_{n\to\infty}\sqrt[n]{n}} = \frac{1}{\lim_{n\to\infty} e^{\ln(\sqrt[n]{n})}} = \frac{1}{e^{\lim_{n\to\infty}\frac{\ln(n)}{n}}}$$

Apply L'Hôpital's rule to evaluate the limit

$$\lim_{n\to\infty}\frac{\ln(n)}{n} = \lim_{x\to\infty}\frac{\ln(x)}{x} = \lim_{x\to\infty}\frac{1}{x} = 0.$$

Substituting back this limit in the exponential,

$$\lim_{n\to\infty}\sqrt[n]{\frac{1}{n}} = e^0 = 1.$$

Problem 3

Compute the N-th partial sum

$$S_N = \sum_{n=1}^{\infty}\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) = \left(1 - \frac{1}{2^2}\right) + \left(\frac{1}{2^2} - \frac{1}{3^2}\right) + \ldots + \left(\frac{1}{N^2} - \frac{1}{(N+1)^2}\right).$$

This is a telescoping sum, after some cancelling we're left with

$$S_N = 1 - \frac{1}{(N+1)^2}$$

The sum of a series equals the limit of its partial sums

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty}\left(1 - \frac{1}{(N+1)^2}\right) = 1 - \lim_{N\to\infty}\frac{1}{(N+1)^2} = 1.$$

Problem 4

Notice that the cosine is a bounded function

$$\cos^2(n) \leq 1$$

By the Comparison Test

$$\sum_{n=2}^{\infty}\frac{\cos^2(n)}{n^{3/2}} \quad \text{converges if} \quad \sum_{n=2}^{\infty}\frac{1}{n^{3/2}} \quad \text{converges.}$$

The latter converges by the p-Series Theorem, for $p = 3/2 > 1$.

3 PM

Problem 1

For all $n \geq 0$, $\cos(2\pi n) = 1$. Then

$$\lim_{n\to\infty}\cos^n(2\pi n) = \lim_{n\to\infty} 1^n = 1.$$

Problem 2

Use exponentials and logarithms to rewrite the limit. Then use continuity to introduce the limit inside the argument of the exponential

$$\lim_{n\to\infty} \sqrt[2n]{\frac{1}{2n}} = \frac{1}{\lim_{n\to\infty} \sqrt[2n]{2n}} = \frac{1}{\lim_{n\to\infty} e^{\ln\left(\sqrt[2n]{2n}\right)}} = \frac{1}{e^{\lim_{n\to\infty} \frac{\ln(2n)}{2n}}}$$

Apply L'Hôpital's rule to evaluate the limit

$$\lim_{n\to\infty} \frac{\ln(2n)}{2n} = \lim_{x\to\infty} \frac{\ln(2x)}{2x} = \lim_{x\to\infty} \frac{\frac{1}{2x}\cdot 2}{2} = 0.$$

Substituting back this limit in the exponential,

$$\lim_{n\to\infty} \sqrt[2n]{\frac{1}{2n}} = e^0 = 1.$$

Problem 3

Compute the N-th partial sum

$$S_N = \sum_{n=3}^{\infty} \left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) = \left(\frac{1}{3^2} - \frac{1}{4^2}\right) + \left(\frac{1}{4^2} - \frac{1}{5^2}\right) + \ldots + \left(\frac{1}{N^2} - \frac{1}{(N+1)^2}\right).$$

This is a telescoping sum, after some cancelling we're left with

$$S_N = \frac{1}{3^2} - \frac{1}{(N+1)^2}$$

The sum of a series equals the limit of its partial sums

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(\frac{1}{3^2} - \frac{1}{(N+1)^2}\right) = \frac{1}{9} - \lim_{N\to\infty} \frac{1}{(N+1)^2} = \frac{1}{9}.$$

Problem 4

Notice that the sine is a bounded function

$$\sin^2(n) \leq 1$$

By the Comparison Test

$$\sum_{n=2}^{\infty} \frac{\sin^2(n)}{n^3} \quad \text{converges if} \quad \sum_{n=2}^{\infty} \frac{1}{n^3} \quad \text{converges.}$$

The latter converges by the p-Series Theorem, for $p = 3 > 1$.