MATH 416, Spring 10, Midterm 1 Review

1. We say that a collection of vectors $\{e_1, \ldots, e_n\} \subset \mathbb{R}^d$, $n \geq d$ is a *spanning set* for $\mathbb{R}^d$ if every vector in $\mathbb{R}^d$ can be represented as a linear combination of vectors $\{e_1, \ldots, e_n\}$. We say that a collection of vectors $\{f_1, \ldots, f_n\} \subset \mathbb{R}^d$, $n \geq d$ is a *finite frame* for $\mathbb{R}^d$ if there exist constants $A, B > 0$ $(A < B)$ such that for every vector $x \in \mathbb{R}^d$ the following holds:

$$A\|x\|^2 \leq \sum_{k=1}^{n} |\langle x, f_k \rangle|^2 \leq B\|x\|^2.$$

Show that every finite spaning set for $\mathbb{R}^d$ is a frame for $\mathbb{R}^d$.

2. With the definitions of Problem 1, show that every finite frame $\{f_1, \ldots, f_n\} \subset \mathbb{R}^d$, for $\mathbb{R}^d$, $n \geq d$, is a spanning set for $\mathbb{R}^d$.

3. Prove that $\|x - y\| \geq |\|x\| - \|y\||$ for any vectors $x, y$ in a normed vector space.

4. Find the 1-periodization of the function $f(x) = e^{-|x|}$.

5. For real $\epsilon > 0$ and $\alpha$, define the dilation operator $D_\epsilon$ and the translation operator $T_\alpha$, which act on functions $f = f(t)$ of one real variable as follows:

$$T_\alpha(u)(t) = u(t - \alpha) \quad D_\epsilon(u)(t) = \epsilon^{-1/2} u(t/\epsilon).$$

a) Show that these are linear transformations with inverses $T_\alpha^{-1} = T_{-\alpha}$ and $D_\epsilon^{-1} = D_{1/\epsilon}$

b) Compute the composition $T_\alpha(D_\epsilon(F))$ for a function $F = F(x)$.

6. Show that the set of functions $\{1, \sqrt{2}\sin(2\pi nt), \sqrt{2}\cos(2\pi nt) : n = 1, 2, 2 \ldots\}$ is orthonormal with respect to the Hermitean inner product.

7. Show that the set of functions $\{\sqrt{2}\sin(2\pi nt) : n = 1, 2, 3, \ldots\}$ is orthonormal with respect to the real inner product.

8. Show that the set of functions $\{1, \sqrt{2}\cos(2\pi nt) : n = 1, 2, 3, \ldots\}$ is orthonormal with respect to the real inner product.

9. Compute the sine-cosine Fourier series of the 1-periodic function $f(x) = \cos^2(2\pi x)$.

10. Compute the complex exponential Fourier series of the 1-periodic function $\sin(2\pi kt - d)$, where $d$ is a constant real number, and $k$ is an integer.