Outline

1. Lecture 2: Role of Directionality
Harmonic analysis decomposes signals into simpler elements called *analyzing functions*.

Classical HA methods include Fourier series and aforementioned wavelets. These have proven extremely influential and quite effective for many applications.

However, they are fundamentally isotropic, meaning they decompose signals without considering how the signal varies directionally.

Wavelets decompose an image signal with respect to translation and scale. Since the early 2000s, there have been several attempts to incorporate directionality into the wavelet construction.
Early attempts to make wavelets more sensitive to directionality included appropriate filter design, anisotropic scaling, steerable filters, and similar techniques.

**Directional wavelets:** J.-P. Antoine, R. Murenzi, P. Vandergheynst, and S. Ali introduced more complicated group actions for parametrization of 2-dimensional wavelet transforms, including rotations or similitude group. These results were later generalized to construct wavelets on sphere and other manifolds.


Subsequently **Radon transform** has been introduced in combination with wavelet transforms to replace the angular parametrization; This results in systems such as **ridgelets** (E. Candès and D. Donoho) or **Gabor ridge functions** (L. Grafakos and C. Sansing)

**Contourlets:** M. Do and M. Vetterli constructed a discrete-domain multiresolution and multidirection expansion using non-separable filter banks, in much the same way that wavelets were derived from filter banks.
Multiscale Directional Representations

- **Curvelets**: E. Candès and D. L. Donoho introduced the curvelets as an efficient tool to extract directional information from images. Curvelets consist of translations and rotations of a sequence of basic functions depending on a parabolic scaling parameter. The curvelet transform is first developed in the continuous domain and then discretized for sampled data.

- **Wavelets with Composite Dilations**: K. Guo, D. Labate, W.-Q. Lim, B. Manning, G. Weiss, and E. Wilson studied affine systems built by using a composition of two sets of matrices as the dilation.

- **Shearlets**: D. Labate, K. Guo, G. Kutyniok, and G. Weiss introduced a special example of the Composite Dilation Wavelets.

- **Surfacelets** (Do, Lu), **bandlets** (Le Pennec, Mallat), **brushlets** (Meyer, Coifman), **wedgelets** (Donoho), **phaselets** (Gopinath), **complex wavelets** (Daubechies), **surflets** (Baraniuk), etc etc...
These constructions incorporate directionality in a variety of ways.

To summarize, some of the major constructions include:

- Ridgelets.


- Curvelets.


- Contourlets.


- Shearlets.


- Wavelets, ridgelets, curvelets, and shearlets are surprisingly related, as they all are special cases of the recently introduced $\alpha$—molecules.

Many of the aforementioned representations were designed specifically for dealing with images, i.e., for the case of 2-dimensional Euclidean space.

Multiscale directional representations can also be constructed analogously for higher dimensional spaces, as well as for some manifolds.

A different approach is needed to deal with discrete structures, such as graphs, networks, or point clouds. R. Coifman and M. Maggioni proposed to use diffusion processes on such structures to introduce the notion of scale and certain directions.

A useful model for real images is the class of cartoon-like images, $\mathcal{E}^2(\mathbb{R}^2)$. Roughly, they are functions that are smooth away from a smooth curve of discontinuity.

**Definition**

Cartoon-like image functions Let $f \in L^2(\mathbb{R}^2)$ be a function with support contained in the closed unit square $[0, 1]^2$ and such that $f$ can be written as

$$f = f_0 + 1_B f_1,$$

for some $B \subset [0, 1]^2$ with a closed $C^2$ boundary. If $f_0 \in C^2(\mathbb{R}^2)$ and $f_1 \in C^2(\mathbb{R}^2)$, then we say that $f \in \mathcal{E}^2(\mathbb{R}^2)$. 
Let $f \in \mathcal{E}^2(\mathbb{R}^2)$ and let $f_N$ be its best $N$-term approximation with respect to a set of analyzing functions. The optimal asymptotic decay rate of $\|f - f_N\|^2_2$ is $O(N^{-2})$, $N \to \infty$, achieved adaptively.


**Definition**

Let $\{\psi_i : i \in I\} \subset L^2(\mathbb{R}^2)$ be a normalized frame for $L^2(\mathbb{R}^2)$. Then, we say that $\{\psi_i : i \in I\}$ provides optimally sparse approximation for $\mathcal{E}^2(\mathbb{R}^2)$ if the best $N$-term nonlinear approximation error in $L^2(\mathbb{R}^2)$:

$$\|f - f_N\|^2_2 = \left\| f - \sum_{i \in I_N} \langle f, \psi_i \rangle \psi_i \right\|^2_2,$$

where $\langle f, \psi_i \rangle, i \in I_N$, are the $N$ largest coefficients in magnitude, satisfies

$$\|f - f_N\|^2_2 \leq CN^{-2},$$

as $N \to \infty$. 
Up to a log factor, curvelets, contourlets, and shearlets satisfy this optimal decay rate (ridgelets are only optimal for linear boundaries). Hence, these analyzing functions are essentially optimally sparse for cartoon-like images. Wavelets can only achieve $O(N^{-1})$. Fourier series are even worse with $O(N^{-1/2})$.

We focus on shearlets since they have multiple, efficient numerical implementations.

Shearlets’ optimality is nearly ideal for sufficiently chosen shearlet frames:

$$
\|f - f_N\|_2^2 \leq CN^{-2} \log^3(N), \quad \text{as} \quad N \to \infty.
$$

Continuous shearlets in $\mathbb{R}^2$ depend on three parameters: the scaling parameter $a > 0$, the shear parameter $s \in \mathbb{R}$, and the translation parameter $t \in \mathbb{R}^2$, and they are defined as follows:

We define the *parabolic scaling matrices*

$$A_a = \begin{pmatrix} a & 0 \\ 0 & a^{1/2} \end{pmatrix}, \quad a > 0$$

and the *shearing matrices*

$$S_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \quad s \in \mathbb{R}.$$

Also, let $D_M$ be the dilation operator defined by

$$D_M \psi = \left| \det M \right|^{-1/2} \psi(\mathbf{M}^{-1} \cdot), \quad M \in GL_2(\mathbb{R})$$

and $T_t$ the translation operator defined by

$$T_t \psi = \psi(\cdot - t), \quad t \in \mathbb{R}^2.$$
Let $\psi \in L^2(\mathbb{R}^2)$. The *Continuous Shearlet Transform* of $f \in L^2(\mathbb{R}^2)$ is

$$f \mapsto \mathcal{SH}_\psi f(a, s, t) = \langle f, T_tD_{a} D_{s} \psi \rangle, \quad a > 0, \; s \in \mathbb{R}, \; t \in \mathbb{R}^2.$$  

- Parabolic scaling allows for directional sensitivity.
- Shearing allows us to change this direction.
- By carefully choosing $\psi$ and discretizing the parameter space, we can decompose $f \in L^2(\mathbb{R}^2)$ into a Parseval frame.
Shearlets

- It’s generally assumed that $\hat{\psi}$ splits as $\hat{\psi}(\xi_1, \xi_2) = \hat{\psi}_1(\xi_1)\hat{\psi}_2(\xi_2/\xi_1)$.
- The basic shearlet $\psi$ is only used in a horizontal cone, while the reflection of $\psi$ across the line $\xi_2 = \xi_1$ is used in a vertical cone. A scaling function $\phi$ is used for the low-pass region. This construction is known as cone-adapted shearlets.

**Figure**: Frequency tiling for cone-adapted shearlets.

Shearlets have several efficient numerical implementations in MATLAB that are freely available.

- 2D Shearlet Toolbox (Easley, Labate, and Lim). ¹
- Shearlab (Kutyniok, Shahram, Zhuang et al.). ²
- Fast Finite Shearlet Transform (Häuser and Steidl).³

We used the last option (FFST) here, which is in many ways the most intuitive of the implementations.

¹http://www.math.uh.edu/~dlabate/software.html
²http://www.shearlab.org/
³http://www.mathematik.uni-kl.de/imagepro/software/ffst/
Consider an $M \times N$ image. Define $j_0 := \lfloor \log_2 \max\{M, N\} \rfloor$. We discretize the parameters as follows:

$$a_j := 2^{-2j} = \frac{1}{4^j}, \quad j = 0, \ldots, j_0 - 1,$$

$$s_{j,k} := k2^{-j}, \quad -2^j \leq k \leq 2^j,$$

$$t_m := \left( \frac{m_1}{M}, \frac{m_2}{N} \right), \quad m_1 = 0, \ldots, M - 1, \ m_2 = 0, \ldots, N - 1.$$

Note that the shears vary from $-1$ to $1$. To fill out the remaining directions, we also shear with respect to the $y$-axis.

Shearlets whose supports overlap are “glued” together.

The transform is computed through the 2D FFT and iFFT.
Fast Finite Shearlet Transform

Figure: Frequency tiling for FFST.

Figure: $\hat{\psi}_1$ and $\hat{\psi}_2$ for the FFST.

ibid.
Figure: Demonstration of output from the FFST on the cameraman image. The shearlet coefficients are from scale 3 (out of 4) in the direction of slope 4.
We have covered some of the multiscale directional representations systems in use today.

These systems are equipped with good approximation properties, exceeding in certain aspects what wavelet theory provides us with, and they have fast implementations.

But we have to address the question of determining the specific set of directions/parameters needed for any given data of interest.