MATH 141 - Exam 2, Problem 5, Solution

1. (20 points) Determine whether the improper integral $\int_{-3}^{2} \frac{x+1}{x^2+x-6} dx$ diverges or converges. If it converges, determine its value; if it diverges, give the reason.

Solution: First, find the antiderivative. We compute it using partial fractions.

$$\frac{x+1}{x^2+x-6} = \frac{x+1}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3} = \frac{(A+B)x + (3A-2B)}{(x-2)(x+3)}$$

So, we have $A + B = 1$ and $3A - 2B = 1$. It leads to $A = 3/5$, $B = 2/5$. (5 pts)

$$\int \frac{x+1}{x^2+x-6} dx = \int (\frac{3}{5}\frac{1}{x-2} + \frac{2}{5}\frac{1}{x+3}) dx = \frac{3}{5} \ln|x-2| + \frac{2}{5} \ln|x+3| + C$$

Note that the absolute value signs are essential in this problem because without them you end up trying to take the natural log of negative numbers which we cannot do. (10 pts)

The next step is to evaluate the improper integral. Note that the integrand is continuous over $(-5, 2)$ but is unbounded near both ends of the integral. To proceed then we must pick some $-5 < c < 2$ and try evaluating the integrals from -5 to c and from c to 2.

WARNING: THIS IS AN IMPORTANT STEP IN THE PROBLEM AND NOT SIMPLY A FORMALITY!

We treat these types of integrals in this way because it avoids things like $\infty - \infty$ which are meaningless (and definitely not 0) and give us no information.

We pick $c = 0$ for convenience and try evaluating

$$\int_{-3}^{0} \frac{x+1}{x^2+x-6} dx \quad \text{and} \quad \int_{0}^{2} \frac{x+1}{x^2+x-6} dx$$

If either of these diverge then we say that the entire integral diverges. If both converge then the integral converges and the sum of their values is the value of the entire integral. (15 pts)

Starting with the first, we have

$$\begin{aligned}
\lim_{d \to -3^+} \int_{d}^{0} \frac{x+1}{x^2+x-6} dx &= \lim_{d \to -3^+} [\frac{3}{5} \ln|x-2| + \frac{2}{5} \ln|x+3|]|_{d}^{0} \\
&= \frac{3}{5} \ln 2 + \frac{2}{5} \ln 3 - \frac{3}{5} \ln 5 - \frac{2}{5} \lim_{d \to -3^+} \ln|d+3| \\
&= \infty
\end{aligned}$$

Therefore, the integral diverges. (20 pts)

Another common mistake was to try the following:

$$\int_{-3}^{2} \frac{x+1}{x^2+x-6} dx = \frac{3}{5} \int_{-3}^{2} \frac{1}{x-2} dx + \frac{2}{5} \int_{-3}^{2} \frac{1}{x+3} dx$$

However this too results in $\infty - \infty$ is not allowed. In face, the linearity that we have for definite integrals no longer holds for improper integrals (see for example problem 61 from section 8.7 in your text).