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Electromagnetic fields in air of traveling-wave currents above the earth

Dionisios Margetis[a)]
Gordon McKay Laboratory, Harvard University, Cambridge, Massachusetts 02138-2901

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The problem of the electromagnetic field created by a thin, straight conductor of infinite length carrying a forward traveling-wave current with a complex propagation constant γ above a homogeneous and isotropic planar earth of wave number k_1 is formulated in terms of contour integrals. In the limit where γ becomes equal to the free-space wave number k_0, the component of the magnetic field in air normal to both conductor and interface is evaluated in closed form in terms of known special functions while the remaining components of the field in air are expressed as series expansions in $\delta = k_0^2(k_1^4 - k_0^4)^{-1/2}$ via the application of a contour integration technique. The new analytical formulas involve familiar transcendental functions and are valid at any distance from the source. The analysis sheds light on the intricate nature of low-frequency electromagnetic fields generated by transmission lines in the presence of a conducting or a dielectric half-space. © *1998 American Institute of Physics.* [S0022-2488(98)02811-4]

I. INTRODUCTION

The determination of the propagation modes for the current and evaluation of the ensuing electromagnetic field of a thin, infinitely long straight wire immersed in a stratified medium is a fundamental and fairly old problem in electromagnetic theory. Of particular interest is the case in which the conductor is placed parallel to an isotropic, homogeneous half-space. In the thin-wire approximation, the field is computed with the assumption that the current is axial and concentrated in a line at the center of the cross section of the wire. Early investigations[1,2] employed approximate transmission-line theory and were essentially limited to a relatively dense neighboring medium and to distances from the source that are short compared to the medium's wavelength. In later analyses[3–6] the field components of a conductor carrying an exponential current were expressed in terms of Fourier integrals in the direction normal to the conductor and parallel to the interface, with the tacit or explicit assumption that the field tends to zero when the distance perpendicular to the wire approaches infinity. In these studies, it has not been possible to evaluate all requisite integrals of the so-called Sommerfeld type, even for special values of the propagation constant γ for the current.

Coleman[7] was probably the first to point out that in the limit $a/d \to 0$, where a is the radius of the wire and d is the distance from the interface, the modal equation yields a solution for γ which approaches the wave number k_0 of the ambient medium. Approximate analytical derivations through disparate mathematical methods by Chang and Wait[4] and by King *et al.*[8] with the condition $0 < k_0 d \ll 1$ and for a relatively dense neighboring medium have corroborated and refined Coleman's[7] findings when $a \ll d$. For the air-earth configuration, deviations from the value $\gamma = k_0$ result then in negligible deviations of the fields in air for all distances of interest at extremely low frequencies. In the case of two or more thin, parallel conductors the condition of a vanishing total current reinforces the reasoning for the assumption $\gamma = k_0$ because of the confinement of the field closer to the source.[9,10] Notably, γ changes drastically when the wire approaches the vicinity of the boundary ($d \sim a$) because of the proximity effect of the earth's surface.[4,7,9] Investigation of the modal equation for a wide range of frequencies is a formidable task.[11,12]

In a recent paper by King and Wu[10] simple approximate formulas for the field were derived under the condition $k_0^2 \ll |k_1^2|$ via integration of the approximate lateral-wave formulas for the field

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[a)]Electronic mail: margetis@huhepl.harvard.edu of a horizontal electric dipole,[13] where k_1 is the wave number of the earth. However, as pointed out in Ref. 10, the violation of the requisite condition $|k_1 r'|>1$ at extremely low frequencies, where r' is the distance from the dipole, introduced an inaccuracy for the axial component of the electric field.

This paper has a twofold purpose. The first is to outline a procedure for obtaining integral representations for all components of the electromagnetic field under the thin-wire approximation in the general case when the current in the x-directed conductor at a fixed frequency ω is of the form $e^{i\gamma x - i\omega t}$, where Re $\gamma>0$, Im $\gamma\geqslant 0$, by invoking the condition that the field be described as a superposition of outgoing waves in the direction normal to both conductor and interface. This in turn leads to a radiation field which is exponentially increasing in the direction normal to the wire when Im $\gamma>0$ and the surrounding medium is a perfect dielectric, such as air, in agreement with the studies on the theory of the microstrip.[14–17] This point is discussed further in Sec. IV.

The second purpose is to evaluate exactly the requisite integrals for the field in air at any distance from the source in the limit $\gamma\to k_0$ by relaxing the condition $k_0^2 \ll |k_1^2|$, where k_1 is the wave number of the adjacent medium. This is achieved by applying a contour integration technique. The new integrated formulas involve series expansions in δ^2 for $|\delta|<1$, where δ is expressed as $\delta=\eta_S/k$, η_S being the limit as $\gamma\to k_0$ of the Sommerfeld pole and $k=\sqrt{k_1^2-k_0^2}$, with series coefficients that depend on known special functions. These series are shown to converge uniformly in distance. In certain limiting cases the formulas in question reduce to simplified expressions, such as the well known Carson's series,[1] that have been previously derived by different approximate means. It is emphasized that the proposed treatment is not intended to serve as a substitute for previous simple approximate results; it rather aims at demonstrating rigorously exact solubility of the model in question with the removal of certain restrictions on the physical parameters. Consequently, stringent conditions for the validity of practically appealing simplifications can follow.

Compared with actual current-carrying wires the main idealizations involved here are (1) the current-carrying conductor in air is infinitely long and thin $(a \ll d)$; (2) the air-earth interface is planar; (3) the source current has the dependence $e^{i\gamma x}$ with the distance x along the wire. The limit $\gamma\to k_0$ corresponds to the transverse electromagnetic (TEM) mode in air in the absence of the earth. This propagation constant reasonably describes the slow-wave currents in actual multiphase transmission lines above the earth.[9,10] The $e^{-i\omega t}$ time dependence is suppressed throughout the analysis.

II. FORMULATION

The geometry of the problem is shown in Fig. 1. It consists of an infinitely long x-directed conductor lying in the vertical plane $y=0$ in the air (region 0, $z>0$) at height d above the surface $(z=0)$ of an isotropic, homogeneous and nonmagnetic earth (region 1, $z<0$). The associated current density is assumed to be

$$\mathbf{J}(\mathbf{r}) = e^{i\gamma x}\,\delta(y)\,\delta(z-d)\,\hat{\mathbf{x}}, \quad \text{Re } \gamma>0, \quad \text{Im } \gamma>0, \tag{1}$$

where $\mathbf{r}=(x,y,z)$. At this point, it is not advisable to employ the usual spatial Fourier transform in y of the field since the requisite integrability condition for $|y|\to\infty$ is not necessarily met. A remedy to this problem is to seek a meaningful integral representation of the form

$$\mathbf{F}_j(x,y,z) = \frac{1}{2\pi}\int_\Gamma d\eta\, \overline{\mathbf{F}}_j(x,\eta,z)\, e^{i\eta y}, \tag{2}$$

where $\mathbf{F}_j=\mathbf{E}_j$, $\mathbf{B}_j$ $(j=0,1)$ denote the electric and magnetic field, respectively, in each region 0, 1 and Γ is a properly chosen infinite integration path with horizontal asymptotes in the η-plane. Specifically, when a scalar integral coefficient becomes 1, the respective integral yields the Dirac delta function $\delta(y)$. Conventional inversion of the integral formula of Eq. (2) is not generally possible.

Formally, with a dependence of the form

$$\mathbf{F}_j(\mathbf{r}) = e^{i\gamma x}\mathbf{f}_j(y,z), \tag{3}$$

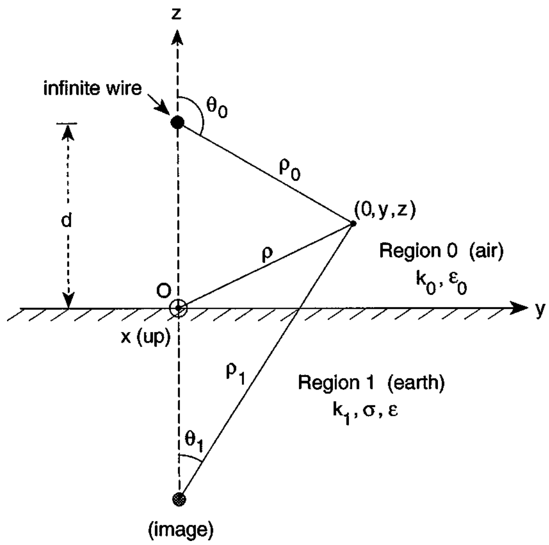

FIG. 1. Cross section of an infinitely thin, infinitely long conductor above the earth.

$\mathbf{E}_j$ and $\mathbf{B}_j$ satisfy Maxwell's equations if the coefficients $\overline{\mathbf{f}}_j = \overline{\mathbf{e}}_j$, $\overline{\mathbf{b}}_j$ $(j=0,1)$ obey the following equations:

$$\overline{e}_{jy} = \frac{i}{k_j^2-\gamma^2}\left[i\eta\gamma\overline{e}_{jx} + \omega\frac{\partial \overline{b}_{jx}}{\partial z}\right], \tag{4}$$

$$\overline{e}_{jz} = \frac{i}{k_j^2-\gamma^2}\left[\gamma\frac{\partial \overline{e}_{jx}}{\partial z} - i\eta\omega\overline{b}_{jx}\right], \tag{5}$$

$$\overline{b}_{jy} = -\frac{1}{k_j^2-\gamma^2}\left[\eta\gamma\overline{b}_{jx} + \frac{ik_j^2}{\omega}\frac{\partial \overline{e}_{jx}}{\partial z}\right], \tag{6}$$

$$\overline{b}_{jz} = \frac{1}{k_j^2-\gamma^2}\left[i\gamma\frac{\partial \overline{b}_{jx}}{\partial z} - \frac{\eta k_j^2}{\omega}\overline{e}_{jx}\right], \tag{7}$$

$$\frac{\partial^2 \overline{e}_{jx}}{\partial z^2} + \gamma_j^2\overline{e}_{jx} = -\frac{i\omega\mu_0}{k_j^2}(k_j^2-\gamma^2)\,\delta(z-d), \tag{8}$$

$$\frac{\partial^2 \overline{b}_{jx}}{\partial z^2} + \gamma_j^2\overline{b}_{jx} = 0, \tag{9}$$

where

$$\gamma_j^2 = k_j^2 - \gamma^2 - \eta^2. \tag{10}$$

The η-Riemann surface for $\overline{\mathbf{e}}_j$, $\overline{\mathbf{b}}_j$ consists of four sheets. For definiteness, the first Riemann sheet $\mathscr{R}_1$ is defined as

$$\forall\,\eta\in\mathscr{R}_1, \quad -\pi/2 < \arg\,\gamma_j \leqslant \pi/2. \tag{11}$$

The corresponding branch cuts are parts of hyperbolas, as depicted in Fig. 2. Final expressions for $\overline{\mathbf{e}}_j$, $\overline{\mathbf{b}}_j$ are uniquely determined by imposition of:

(1) Outgoing waves in z; this excludes solutions of the form $e^{-i\gamma_0 z}$ for $z>d$ and $e^{i\gamma_1 z}$ for $z<0$ in Eqs. (8) and (9).

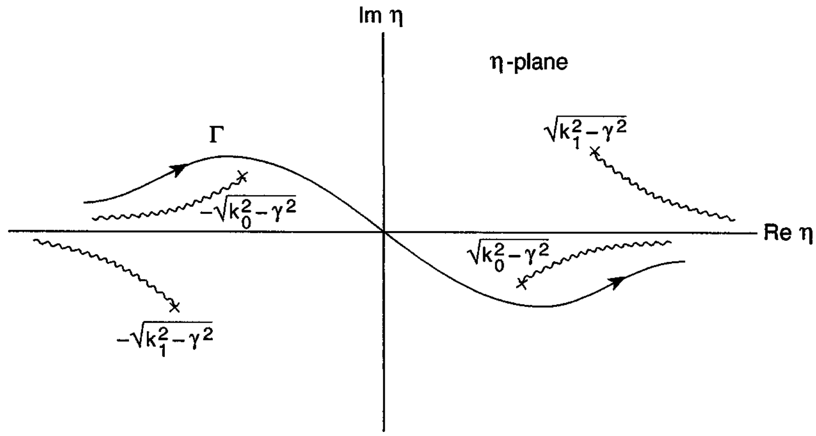

FIG. 2. Integration path Γ in the complex η-plane for the electric and magnetic field of an infinitely long wire in air (wave number k_0) carrying an $e^{i\gamma x}$, Re $\gamma>0$, Im $\gamma>0$, current over the earth (wave number k_1).

(2) Continuity of $\overline{e}_{jx}$, $\overline{b}_{jx}$, $\overline{e}_{jy}$, $\overline{b}_{jy}$ at $z=0$.
Accordingly, for $z>0$ the axial components read

$$\overline{e}_{0x}=-\omega\mu_0\frac{\gamma_1(k_0^2-\gamma^2)+\gamma_0(k_1^2-\gamma^2)}{K(\eta)L(\eta)}e^{i\gamma_0(z+d)}+\frac{i\omega\mu_0}{\gamma_0 k_0^2}(k_0^2-\gamma^2)e^{i\gamma_0 z_>}\sin(\gamma_0 z_<), \tag{12}$$

$$\overline{b}_{0x}=\mu_0\frac{\eta\gamma(k_1^2-k_0^2)}{K(\eta)L(\eta)}e^{i\gamma_0(z+d)}, \tag{13}$$

while for $z<0$

$$\overline{e}_{1x}=-\omega\mu_0\frac{\gamma_1(k_0^2-\gamma^2)+\gamma_0(k_1^2-\gamma^2)}{K(\eta)L(\eta)}e^{i\gamma_0 d}e^{-i\gamma_1 z}, \tag{14}$$

$$\overline{b}_{1x}=\mu_0\frac{\eta\gamma(k_1^2-k_0^2)}{K(\eta)L(\eta)}e^{i\gamma_0 d}e^{-i\gamma_1 z}, \tag{15}$$

where

$$K(\eta)=\gamma_0+\gamma_1, \tag{16a}$$

$$L(\eta)=k_1^2\gamma_0+k_0^2\gamma_1. \tag{16b}$$

In the above, $z_>$ is the larger one of z and d and $z_<$ is the smaller one. Similar formulas for the remaining components are obtained directly from Eqs. (4)–(7). If γ is considered to be real, $\overline{\mathbf{e}}_j$ and $\overline{\mathbf{b}}_j$ designate the two-dimensional spatial Fourier transforms in x,y of the projections in x of the tensor Green's functions for the electric and magnetic field. From Eq. (16b) it is inferred that when $\gamma\neq k_0k_1/\sqrt{k_0^2+k_1^2}$, $L(\eta)$ has four simple zeros in the η-Riemann surface which are the Sommerfeld poles:

$$\eta=\pm\eta_S=\pm\sqrt{\frac{k_0^2k_1^2}{k_0^2+k_1^2}-\gamma^2}. \tag{17}$$

In contrast, $K(\eta)$ is free of zeros in any finite region of this surface. It is noteworthy that the $\overline{b}_{jz}$ are free of the $L(\eta)$ denominator.

An integration path Γ which is symmetric under inversion through the origin is subsequently chosen so that: (1) It lies entirely in $\mathscr{R}_1$. (2) All integrals are absolutely convergent. (3) The positive and negative real axes are asymptotes of the path. Among these requirements, (1) and (2) suffice to determine the field. The chosen contour is shown in Fig. 2. Integral representations for the field follow from Eq. (2). These satisfy Maxwell's equations in each region 0, 1 along with the prescribed boundary conditions at $z=0$ and are expressed as a Fourier superposition of outgoing waves in z.

In the limit $z\to\infty$, the principal contribution to integration arises from the vicinity of $\eta=0$, yielding a leading term that is exponentially growing in z. Similar asymptotic behavior can be found via the method of steepest descents[18] when $\sqrt{y^2+z^2}\to\infty$ for $z>0$, as outlined in Appendix A.

Evidently, the integral representations may be continued to $\gamma=k_0$. The respective field tends to zero when $\sqrt{y^2+z^2}\to\infty$.

III. THE CASE $\gamma=k_0$

A. Integral representations for the field in air

In the limit $\gamma\to k_0$, the integration path Γ in Eq. (2) can be deformed to coincide with the entire real axis. By use of Eqs. (2)–(7), (12) and (13), and evaluation of the elementary integrals, it is straightforward to arrive at the following expressions for the fields $\mathbf{E}_0$, $\mathbf{B}_0$:

$$E_{0x}=-\frac{\omega\mu_0}{2\pi}\,e^{ik_0x}\int_{-\infty}^{\infty}d\eta\, e^{i\eta y}e^{-|\eta|(z+d)}\left[\frac{1}{i|\eta|+\sqrt{k^2-\eta^2}}-\frac{k_0^2}{k_0^2\sqrt{k^2-\eta^2}+ik_1^2|\eta|}\right] \tag{18a}$$

$$=\frac{i\omega\mu_0}{2\pi}\,e^{ik_0x}[\Phi_+(y,z;1)-(k_0^2/k_1^2)\Phi_+(y,z;k_0^2/k_1^2)], \tag{18b}$$

$$E_{0y}=\frac{\omega\mu_0k_0}{2\pi}\,e^{ik_0x}\int_{-\infty}^{\infty}d\eta\, e^{i\eta y}e^{-|\eta|d}\,\eta\left[\frac{-i\,\sinh(\eta z)}{\eta k_0^2}+\frac{e^{-|\eta|z}}{ik_1^2|\eta|+k_0^2\sqrt{k^2-\eta^2}}\right],\quad 0<z<d \tag{19a}$$

$$=\frac{\mu_0c}{2\pi}\,e^{ik_0x}\left[\left(\frac{y}{\rho_0^2}-\frac{y}{\rho_1^2}\right)+k\,\frac{k_0^2}{k_1^2}\,\Psi_-(y,z;k_0^2/k_1^2)\right], \tag{19b}$$

$$E_{0z}=\frac{\omega\mu_0k_0}{2\pi}\,e^{ik_0x}\int_{-\infty}^{\infty}d\eta\, e^{i\eta y}e^{-|\eta|d}\left[-\frac{\cosh(\eta z)}{k_0^2}+\frac{\sqrt{k^2-\eta^2}}{ik_1^2|\eta|+k_0^2\sqrt{k^2-\eta^2}}\,e^{-|\eta|z}\right],\quad 0<z<d \tag{20a}$$

$$=\frac{\mu_0c}{2\pi}\,e^{ik_0x}\left[\left(\frac{z-d}{\rho_0^2}-\frac{z+d}{\rho_1^2}\right)-ik\,\frac{k_0^2}{k_1^2}\,U(y,z;k_0^2/k_1^2)\right], \tag{20b}$$

$$B_{0x}=-i\,\frac{\mu_0k_0}{2\pi}\,e^{ik_0x}\int_{-\infty}^{\infty}d\eta\, e^{i\eta y}e^{-|\eta|(z+d)}\,\mathrm{sgn}(\eta)\left[\frac{1}{\sqrt{k^2-\eta^2}+i|\eta|}-\frac{k_0^2}{ik_1^2|\eta|+k_0^2\sqrt{k^2-\eta^2}}\right] \tag{21a}$$

$$=\frac{\mu_0\omega}{2\pi c}\,e^{ik_0x}[\Phi_-(y,z;1)-(k_0^2/k_1^2)\Phi_-(y,z;k_0^2/k_1^2)], \tag{21b}$$

$$B_{0y}=\frac{\mu_0}{2\pi}\,e^{ik_0x}\int_{-\infty}^{\infty}d\eta\, e^{i\eta y}e^{-|\eta|d}\Bigg[\cosh(\eta z)+\frac{k_1^2\eta^2-k_0^2(k^2+i|\eta|\sqrt{k^2-\eta^2})}{(i|\eta|+\sqrt{k^2-\eta^2})(ik_1^2|\eta|+k_0^2\sqrt{k^2-\eta^2})}\,e^{-|\eta|z}\Bigg],\quad 0<z<d \tag{22a}$$

$$= -\frac{\mu_0}{2\pi} e^{ik_0x}\left[\left(\frac{z-d}{\rho_0^2} - \frac{z+d}{\rho_1^2}\right) - ikW(y,z;k_0^2/k_1^2)\right], \tag{22b}$$

$$B_{0z} = \frac{\mu_0}{2\pi} e^{ik_0x}\int_{-\infty}^{\infty} d\eta e^{i\eta y}e^{-|\eta|d}\eta\left[\frac{-i\ \sinh(\eta z)}{\eta} + \frac{e^{-|\eta|z}}{i|\eta| + \sqrt{k^2-\eta^2}}\right], \quad 0<z<d \tag{23a}$$

$$= \frac{\mu_0}{2\pi} e^{ik_0x}\left[\left(\frac{y}{\rho_0^2} - \frac{y}{\rho_1^2}\right) + k\Psi_-(y,z;1)\right], \tag{23b}$$

where

$$k = \sqrt{k_1^2 - k_0^2}, \quad \text{Im } k \geqslant 0, \tag{24}$$

$c = (\epsilon_0\mu_0)^{-1/2}$ is the velocity of light, sgn(η) is the sign function

$$\text{sgn}(\eta) = \begin{cases} 1, & \eta>0 \\ 0, & \eta=0 \\ -1, & \eta<0, \end{cases}$$

and, with the introduction of a new variable $s = ik^{-1}|\eta|$,

$$\Phi_\pm(y,z;\beta) = \varphi(\alpha_+;\beta) \pm \varphi(\alpha_-;\beta), \quad \Psi_\pm(y,z;\beta) = \psi(\alpha_+;\beta) \pm \psi(\alpha_-;\beta), \tag{25}$$

$$U(y,z;\beta) = u(\alpha_+;\beta) + u(\alpha_-;\beta) = \frac{1}{\beta}\left(\frac{1}{\alpha_+} + \frac{1}{\alpha_-}\right) - \frac{1}{\beta}\Psi_+(y,z;\beta), \tag{26}$$

$$W(y,z;\beta) = w(\alpha_+;\beta) + w(\alpha_-;\beta) = \Psi_+(y,z;1) - (1+\beta)\Psi_+(y,z;\beta) + \left(\frac{1}{\alpha_+} + \frac{1}{\alpha_-}\right), \tag{27}$$

$$\varphi(\alpha;\beta) = \int_0^{\infty e^{i\tau}} ds e^{-\alpha s} \frac{1}{s + \beta\sqrt{s^2+1}}, \tag{28}$$

$$\psi(\alpha;\beta) = \int_0^{\infty e^{i\tau}} ds e^{-\alpha s} \frac{s}{s + \beta\sqrt{s^2+1}} = -\frac{\partial}{\partial\alpha}\varphi(\alpha;\beta), \tag{29}$$

$$u(\alpha;\beta) = \int_0^{\infty e^{i\tau}} ds e^{-\alpha s} \frac{\sqrt{s^2+1}}{s + \beta\sqrt{s^2+1}} = \frac{1}{\beta}\left[\frac{1}{\alpha} - \psi(\alpha;\beta)\right], \tag{30}$$

$$w(\alpha;\beta) = \int_0^{\infty e^{i\tau}} ds e^{-\alpha s} \frac{s^2 + \beta(1 + s\sqrt{1+s^2})}{(s+\sqrt{1+s^2})(s+\beta\sqrt{1+s^2})} = \psi(\alpha;1) - (1+\beta)\psi(\alpha;\beta) + \frac{1}{\alpha}, \tag{31}$$

$$\alpha_\pm = -ik(z+d\pm iy) = -ik\rho_1 e^{\pm i\theta_1}, \quad -\pi/2<\theta_1<\pi/2,$$

$$\rho_0 = \sqrt{(z-d)^2+y^2}, \quad \rho_1 = \sqrt{(z+d)^2+y^2}, \quad \sin\ \theta_{0,1} = y/\rho_{0,1}, \quad z>0, \tag{32}$$

$$\tau = \frac{\pi}{2} - \arg(k). \tag{33}$$

The integral formulas in the first expressions, Eqs. (18a)–(23a), are invertible.[19] The restriction $0<z<d$ has been removed in the second expressions, Eqs. (18b)–(23b), since the singular term $1/\rho_0^2$ is extracted by direct integration and the ensuing representations may be continued to $z \geqslant d$. In Eq. (29), the interchange of the order of differentiation and integration is legitimate because the integrals are absolutely and uniformly convergent.

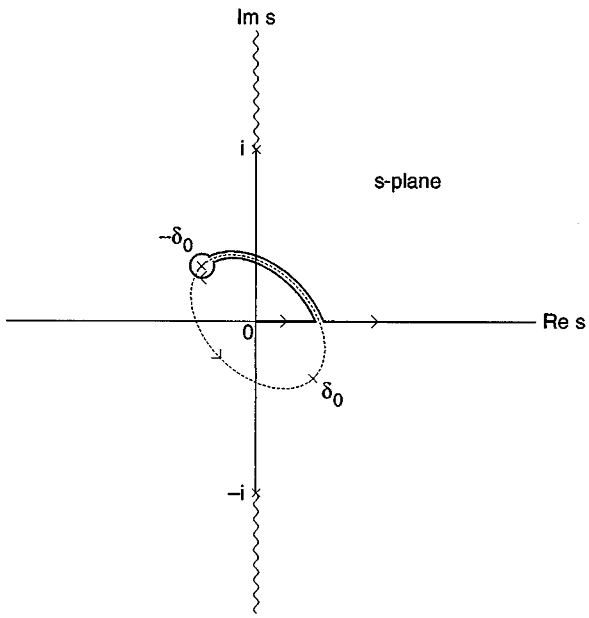

FIG. 3. Branch-cut configuration in the s-plane for the integral of Eq. (34). The modified path serves the analytic continuation of $\varphi(\alpha;\beta)$ along the respective contour enclosing the origin in the β-plane.

The quantities $\Phi_\pm$, $\Psi_\pm$, U, W introduced above designate solely the corrections to the solution for a perfectly conducting earth ($k_1 \to \infty$). The form of the solution above verifies that for a current supporting a TEM mode in air the x-components of the field satisfy Laplace's equation in y,z in the simply connected region $\{(y,z):z>0\}$ whereas the rest of the components satisfy the Poisson equation in the same region.

An examination of Eqs. (25)–(31) indicates that only the integrations for $\varphi(\alpha;\beta)$, where $\beta = 1$ and k_0^2/k_1^2, need to be carried out.

B. Evaluation of φ, ψ, u, w

For mathematical convenience it is assumed that $\alpha>0$. The extension over the range of complex values of α for which $|\arg \alpha|<\pi$ is attained via analytic continuation. The principal integral reads

$$\varphi(\alpha;\beta)=\int_0^{+\infty} ds e^{-\alpha s} \frac{1}{s+\beta\sqrt{s^2+1}}, \tag{34}$$

where the integration path coincides with the entire positive real axis. The integrand in Eq. (34) has two branch points at $s=\pm i$. A branch of the square root is chosen such that $\sqrt{1+s^2}>0$ for $s>0$. The first Riemann sheet is subsequently defined by taking the branch cuts along the positive and negative imaginary axis, as shown in Fig. 3. A study of the cases $\beta=1$ and $\beta=k_0^2/k_1^2$ essentially evinces the differences introduced by the denominators $K(\eta)$ and $L(\eta)$ of Eqs. (16a) and (16b), respectively. These two possibilities need to be considered separately.

(i) $\beta=1$. In this case, $\varphi(\alpha;\beta)$ is cast in the form

$$\varphi(\alpha;1)=\int_0^{+\infty} ds e^{-\alpha s}(\sqrt{1+s^2}-s)=\frac{1}{\alpha} S_{1,1}(\alpha)-\frac{1}{\alpha^2}. \tag{35}$$

In the above, $S_{\nu,\nu}(\alpha)$ ($\nu=1$) denotes Lommel's function

$$S_{\nu,\nu}(\alpha)=2^{\nu-1}\pi^{1/2}\Gamma\left(\nu+\frac{1}{2}\right)[\mathbf{H}_\nu(\alpha)-Y_\nu(\alpha)], \tag{36}$$

where $\mathbf{H}_\nu(\alpha)$ and $Y_\nu(\alpha)$ are Struve's function and Neumann's function, respectively, from p. 18 of Ref. 20.

(ii) $\beta = k_0^2/k_1^2$. In this case, it has not been possible to obtain similar expressions in closed form. In particular, in the limit $\beta \to 0$ the integral of Eq. (34) becomes divergent. For $\beta \neq 0$, the denominator of the integrand vanishes to first order at

$$s = \pm\delta, \quad \delta = \beta/(1-\beta^2)^{1/2} = k_0^2/(k_1^4 - k_0^4)^{1/2}, \tag{37}$$

which implies the existence of two simple poles in the s-Riemann surface. For definiteness, it is assumed that $\delta > 0$ for $0 < \beta < 1$. Accordingly, the pole $s = -\delta$ alone is present in the selected Riemann sheet.

The analytic continuation of $\varphi(\alpha;\beta)$, qua function of β, along any simple closed curve $\beta = \beta(t)$ enclosing the origin in the β-plane is carried out through deformation of the integration path in the s-plane in order that the pole $s = -\delta(\beta(t))$ does not cross the modified path. In particular, when the initial point is $\beta = \beta_0$ and the contour is described once in the counterclockwise sense in the β-plane, the s-integration picks up the residue at $s = -\delta(\beta_0) \equiv -\delta_0$, as shown in Fig. 3:

$$\varphi(\alpha;\beta_0 e^{i2\pi}) - \varphi(\alpha;\beta_0) = -2\pi i \operatorname{Res}_{s=-\delta_0}\left[\frac{e^{-\alpha s}}{s + \beta_0\sqrt{s^2+1}}\right] = -2\pi i(1+\delta_0^2)e^{\alpha\delta_0}. \tag{38}$$

This calculation indicates the existence of a logarithmic singularity at $\beta = 0$. Accordingly, the task is assigned of expanding the regular part of $\varphi(\alpha;\beta(\delta))$ in powers of δ in the unit disc $\mathscr{D} = \{\delta \in \mathbb{C} : |\delta| < 1\}$.

In order to extract the singular contribution, it is desirable to write

$$\frac{1}{s + \beta\sqrt{s^2+1}} = \frac{1+\delta^2}{s+\delta} - \delta\sqrt{1+\delta^2}\,\frac{1}{\sqrt{s^2+1} + \sqrt{1+\delta^2}}, \tag{39}$$

which in turn leads to

$$\varphi(\alpha;\beta) = -(1+\delta^2)e^{\delta\alpha}\operatorname{Ei}(-\delta\alpha) - \delta\sqrt{1+\delta^2}\,\varphi_1(\alpha;\delta), \tag{40}$$

where

$$\varphi_1(\alpha;\delta) = \int_0^\infty ds\, e^{-\alpha s}\,\frac{1}{\sqrt{s^2+1} + \sqrt{1+\delta^2}}, \tag{41}$$

and $\operatorname{Ei}(-z)$ is the exponential integral defined on p. 267 of Ref. 21 as

$$\operatorname{Ei}(-z) = -\int_z^\infty dt\, e^{-t} t^{-1}. \tag{42}$$

The integrand in Eq. (41) no longer admits any pole in the first Riemann sheet. By inspection, $\varphi_1(\alpha;\delta)$ is holomorphic in $\mathscr{D}$. It is noted in passing that

$$e^z \operatorname{Ei}(-z) = \sum_{n=0}^{N-1} \frac{(-1)^{n+1}}{n!} z^{-1-n} + O(|z|^{-N-1}) \quad \text{for } z \to \infty, \quad N = 1,2,\ldots, |\arg(z)| < 3\pi/2, \tag{43}$$

$$\operatorname{Ei}(-z) = \gamma + \ln z + \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n!\,n}, \quad \operatorname{Ei}(-z e^{i2\pi}) = \operatorname{Ei}(-z) + 2\pi i, \tag{44}$$

where $\gamma = 0.5772\ldots$ is Euler's constant.

For the purpose of obtaining a series expansion of $\varphi_1(\alpha;\delta)$ in δ^2, $\varphi_1(\alpha;\delta)$ is expressed as a contour integral, namely,

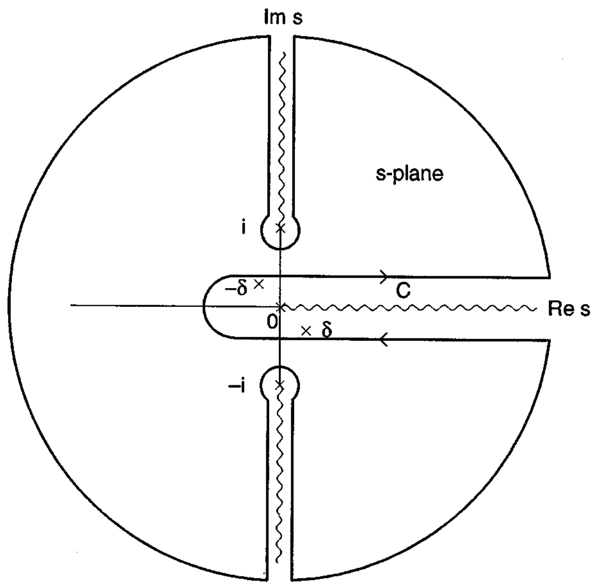

FIG. 4. Branch-cut configuration and contour of integration C in the s-plane for the integral of Eq. (45).

$$\varphi_1(\alpha;\delta) = -\frac{1}{2\pi i}\int_C ds e^{-\alpha s}\mathrm{Ei}(-\alpha s e^{-i\pi})\,\frac{\sqrt{s^2+1}-\sqrt{1+\delta^2}}{s^2-\delta^2}, \tag{45}$$

in view of Eq. (44). A branch of $\mathrm{Ei}(-\alpha s e^{-i\pi})$ is chosen such that $0\leq \arg(s) < 2\pi$. Contour C is traversed in the clockwise sense and its interior contains the points $s=\pm\delta$, as depicted in Fig. 4. By virtue of

$$\frac{1}{s^2-\delta^2} = \sum_{l=0}^{L-1}\delta^{2l}s^{-2l-2} + \delta^{2L}\,\frac{s^{-2L}}{s^2-\delta^2}, \tag{46}$$

where L is any fixed positive integer, Eq. (45) is rearranged to give

$$\begin{aligned}\varphi_1(\alpha;\delta) = {} & \sum_{l=0}^{L-1}\delta^{2l}\left(-\frac{1}{2\pi i}\right)\int_C ds e^{-\alpha s}\mathrm{Ei}(-\alpha s e^{-i\pi}) s^{-2l-2}\left[\sqrt{1+s^2} - \sum_{m=0}^{l}\binom{\frac{1}{2}}{m}s^{2m}\right] \\ & + \sum_{l=0}^{L-1}\sum_{m=0}^{l}\binom{\frac{1}{2}}{m}\delta^{2l}\left(-\frac{1}{2\pi i}\right)\int_C ds e^{-\alpha s}\mathrm{Ei}(-\alpha s e^{-i\pi}) s^{2m-2l-2} \\ & + \sqrt{1+\delta^2}\,\frac{1}{2\pi i}\int_C ds e^{-\alpha s}\mathrm{Ei}(-\alpha s e^{-i\pi})\,\frac{1}{s^2-\delta^2} \\ & - \delta^{2L}\,\frac{1}{2\pi i}\int_C ds e^{-\alpha s}\mathrm{Ei}(-\alpha s e^{-i\pi}) s^{-2L}\,\frac{\sqrt{s^2+1}}{s^2-\delta^2}. \end{aligned} \tag{47}$$

In regard to the first sum of the preceding expression, the integration may be performed using the positive real axis since the integrands are all integrable at $s=0,\infty$. The double sum and the third term are zero because the respective integrals can be indented at infinity and no singularity lies within the resulting contour. The fourth term represents the remainder of the summation and is evaluated by folding the contour around the branch cuts in the imaginary axis, as shown in Fig. 4. Accordingly, $\varphi_1(\alpha;\delta)$ reads

$$\varphi_1(\alpha;\delta) = \sum_{l=0}^{L-1}\delta^{2l}q_{2l+2}(\alpha) + R_L(\alpha;\delta), \tag{48}$$

where

$$q_{2l+2}(\alpha)=\int_0^\infty ds e^{-\alpha s}s^{-2l-2}\left[\sqrt{1+s^2}-\sum_{m=0}^{l}\binom{\frac{1}{2}}{m}s^{2m}\right] \tag{49a}$$

$$=(-1)^l\frac{1}{i\pi}\int_1^\infty d\xi[e^{i\alpha\xi}\mathrm{Ei}(-i\alpha\xi)-e^{-i\alpha\xi}\mathrm{Ei}(i\alpha\xi)]\xi^{-2l-2}\sqrt{\xi^2-1}, \tag{49b}$$

and

$$R_L(\alpha;\delta)=(-1)^L\delta^{2L}\frac{1}{i\pi}\int_1^\infty d\xi[e^{i\alpha\xi}\mathrm{Ei}(-i\alpha\xi)-e^{-i\alpha\xi}\mathrm{Ei}(i\alpha\xi)]\xi^{-2L}\frac{\sqrt{\xi^2-1}}{\xi^2+\delta^2}. \tag{50a}$$

When $2L\gg 1$,

$$R_L(\alpha;\delta)=(-1)^L\delta^{2L}\frac{\pi^{-1/2}}{4i}\left\{\frac{1}{1+\delta^2}[e^{i\alpha}\mathrm{Ei}(-i\alpha)-e^{-i\alpha}\mathrm{Ei}(i\alpha)]L^{-3/2}+\mathrm{O}(L^{-5/2})\right\} \tag{50b}$$

for any α, δ ($|\delta|<1$). In the limit $L\to\infty$, $R_L(\alpha;\delta)\to 0$ uniformly in α, δ. Furthermore,

$$\frac{q_{2l+4}(\alpha)}{q_{2l+2}(\alpha)}=-\frac{\int_1^\infty d\xi[e^{i\alpha\xi}\mathrm{Ei}(-i\alpha\xi)-e^{-i\alpha\xi}\mathrm{Ei}(i\alpha\xi)]\xi^{-2l-4}\sqrt{\xi^2-1}}{\int_1^\infty d\xi[e^{i\alpha\xi}\mathrm{Ei}(-i\alpha\xi)-e^{-i\alpha\xi}\mathrm{Ei}(i\alpha\xi)]\xi^{-2l-2}\sqrt{\xi^2-1}}=\mathrm{O}(1) \tag{51a}$$

to all orders in α. Specifically,

$$\frac{q_{2l+4}(\alpha)}{q_{2l+2}(\alpha)}\sim-\frac{\int_1^\infty d\xi\xi^{-2l-4}\sqrt{\xi^2-1}}{\int_1^\infty d\xi\xi^{-2l-2}\sqrt{\xi^2-1}}=-\frac{2l}{2l+3}\quad(2l\gg 1) \tag{51b}$$

uniformly in α. The preceding expressions are suggestive of the efficiency of the expansion in question when $|\delta|\ll 1$.

The final formula for $\varphi(\alpha;\beta)$ is

$$\varphi(\alpha;\beta(\delta))=-(1+\delta^2)e^{\delta\alpha}\mathrm{Ei}(-\delta\alpha)-\delta\sqrt{1+\delta^2}\sum_{l=0}^\infty\delta^{2l}q_{2l+2}(\alpha),\quad \text{for } |\delta|<1, \tag{52}$$

where the $q_{2l+2}(\alpha)$ are given by Eqs. (49a) and (49b) and can be expressed in terms of known special functions for any finite l. An outline of the procedure along with the first six coefficients are provided in Appendix B. A straightforward, yet tedious, alternative derivation of Eq. (52) is carried out in Appendix C by invoking the Mellin transform technique. A strong indication of the validity of Eq. (52) is provided in Tables I and II which display values of the principal quantity $\varphi_1(\alpha;\delta)$ obtained both numerically through the integral of Eq. (41) and from the series of Eq. (48), for selected positive values of δ and α as well as for negative imaginary values of α. As verified analytically, the rate of convergence of the series is essentially independent of α.

Once $\varphi(\alpha;\beta)$ is evaluated, $\psi(\alpha;\beta)$ is obtained by direct differentiation. In particular, for $|\delta|<1$, term-by-term differentiation of the right-hand side of Eq. (52) is justified on grounds of uniform convergence of the resulting series when $|\alpha|>0$. Consequently,

$$\psi(\alpha;1)=-\frac{2}{\alpha^3}-\frac{1}{3}+\frac{1}{3\alpha}S_{2,2}(\alpha), \tag{53}$$

$$\psi(\alpha;\beta(\delta))=\frac{1+\delta^2}{\alpha}+\delta(1+\delta^2)e^{\delta\alpha}\mathrm{Ei}(-\delta\alpha)-\delta\sqrt{1+\delta^2}\sum_{l=0}^\infty\delta^{2l}q_{2l+1}(\alpha),\quad \text{for } |\delta|<1, \tag{54}$$

TABLE I. Values for $\varphi_1(\alpha;\delta)$ obtained for positive α, δ ($\delta<1$) by numerical integration according to Eq. (41) and by evaluation of series on right-hand side of Eq. (48). The coefficients $q_n(\alpha), n=2, 4, 6, 8$, are evaluated both by numerical integration from Eq. (49a) and by use of the results of Appendix B.

		Series from Eq. (48)			
α	From Eq. (41)	$L=1$	$L=2$	$L=3$	$L=4$
		Case A: $\varphi_1(\alpha;\delta=0.8)$			
10^{-15}	33.481 87	33.654 71	33.441 37	33.495 99	33.476 015
10^{-5}	10.456 17	10.628 98	10.415 68	10.470 29	10.450 32
10^{-2}	3.612 42	3.774 62	3.573 72	3.625 98	3.606 79
1	0.373 44	0.419 04	0.360 50	0.378 28	0.371 36
10	4.3665×10^{-2}	4.9763×10^{-2}	4.1838×10^{-2}	4.4368×10^{-2}	4.3357×10^{-2}
100	4.3846×10^{-3}	$4.999\,75\times10^{-3}$	4.1998×10^{-3}	4.4558×10^{-3}	4.3534×10^{-3}
10^4	4.3848×10^{-5}	5.0000×10^{-5}	4.2000×10^{-5}	4.4560×10^{-5}	4.3536×10^{-5}
		Case B: $\varphi_1(\alpha;\delta=0.1)$			
10^{-15}	33.651 39	33.654 71	33.651 37	33.651 39	33.651 39
10^{-5}	10.625 66	10.628 98	10.625 65	10.625 66	10.625 66
10^{-2}	3.771 49	3.774 62	3.771 48	3.771 49	3.771 49
1	0.418 13	0.419 04	0.418 13	0.418 13	0.418 13
10	4.9640×10^{-2}	4.9763×10^{-2}	4.9639×10^{-2}	4.9640×10^{-2}	4.9640×10^{-2}
100	4.9873×10^{-3}	$4.999\,75\times10^{-3}$	$4.987\,25\times10^{-3}$	4.9873×10^{-3}	4.9873×10^{-3}
10^4	4.9876×10^{-5}	5.0000×10^{-5}	4.9875×10^{-5}	4.9876×10^{-5}	4.9876×10^{-5}

where

$$q_{2l+1}(\alpha)=-\frac{d}{d\alpha}\,q_{2l+2}(\alpha)=\int_0^\infty ds e^{-\alpha s}s^{-2l-1}\left[\sqrt{1+s^2}-\sum_{m=0}^{l}\binom{\frac{1}{2}}{m}s^{2m}\right] \tag{55a}$$

$$=(-1)^{l+1}\,\frac{1}{\pi}\int_1^\infty d\xi[e^{i\alpha\xi}\mathrm{Ei}(-i\alpha\xi)+e^{-i\alpha\xi}\mathrm{Ei}(i\alpha\xi)]\xi^{-2l-1}\sqrt{\xi^2-1}. \tag{55b}$$

The first six coefficients $q_{2l+1}(\alpha)$ of the above series are tabulated in Appendix B. The remainder of this expansion when L terms are summed equals

$$Q_L(\alpha;\delta)=-\frac{\partial}{\partial\alpha}\,R_L(\alpha;\delta)=(-1)^{L+1}\delta^{2L}\,\frac{1}{\pi}\int_1^\infty d\xi[e^{i\alpha\xi}\mathrm{Ei}(-i\alpha\xi)+e^{-i\alpha\xi}\mathrm{Ei}(i\alpha\xi)]\xi^{-2L+1}\,\frac{\sqrt{\xi^2-1}}{\xi^2+\delta^2}, \tag{56}$$

which is O(ln α) as $\alpha\to0$ and O($1/\alpha^2$) as $\alpha\to\infty$, i.e., integrable at $\alpha=0,\infty$. In the limit $L\to\infty$, $Q_L(\alpha;\delta)\to0$ uniformly in $|\delta|<1$, $|\alpha|>0$. The preceding expansion preserves the same interesting features as those implied by Eqs. (51a) and (51b).

Substitution for $\psi(\alpha;\beta)$ into Eqs. (30) and (31) yields

$$u(\alpha;\beta(\delta))=-(1+\delta^2)^{3/2}e^{\delta\alpha}\mathrm{Ei}(-\delta\alpha)-\frac{\delta\sqrt{1+\delta^2}}{\alpha}+(1+\delta^2)\sum_{l=0}^{\infty}\delta^{2l}q_{2l+1}(\alpha), \tag{57}$$

TABLE II. Complex values of $\varphi_1(\alpha;\delta)$ obtained for positive δ and pure imaginary α by numerical integration according to Eq. (41) and by evaluation of series on right-hand side of Eq. (48). These integrals arise, for example, over a very dry earth (conductivity $\sigma\simeq 0$) at $y=0$.

		Series from Eq. (48)		
$i\alpha$	From Eq. (41)	$L=1$	$L=3$	$L=4$
		Case A: $\varphi_1(\alpha;\delta=0.8)$		
10^{-15}	$33.481\,87+i1.570\,80$	$33.654\,71+i1.570\,80$	$33.495\,99+i1.570\,80$	$33.476\,01+i1.570\,80$
10^{-5}	$10.456\,035+i1.570\,64$	$10.628\,87+i1.570\,67$	$10.470\,16+i1.570\,64$	$10.450\,18+i1.570\,64$
10^{-2}	$3.568\,05+i1.506\,88$	$3.736\,64+i1.517\,41$	$3.581\,98+i1.507\,44$	$3.562\,27+i1.506\,66$
1	$0.118\,17+i0.441\,83$	$0.147\,40+i0.496\,82$	$0.122\,12+i0.447\,35$	$0.116\,42+i0.439\,50$
10	$9.5813\times10^{-7}+i4.4054\times10^{-2}$	$14.4412\times10^{-7}+i5.0269\times10^{-2}$	$10.9325\times10^{-7}+i4.4777\times10^{-2}$	$8.8476\times10^{-7}+i4.3737\times10^{-2}$
100	$i4.3850\times10^{-3}$	$i5.000\,25\times10^{-3}$	$i4.4562\times10^{-3}$	$i4.3538\times10^{-3}$
10^4	$i4.3848\times10^{-5}$	$i5.0000\times10^{-5}$	$i4.4560\times10^{-5}$	$i4.3536\times10^{-5}$
		Case B: $\varphi_1(\alpha;\delta=0.1)$		
10^{-15}	$33.651\,39+i1.570\,80$	$33.654\,71+i1.570\,80$	$33.651\,39+i1.570\,80$	$33.651\,39+i1.570\,80$
10^{-5}	$10.625\,55+i1.570\,67$	$10.628\,87+i1.570\,67$	$10.625\,55+i1.570\,67$	$10.625\,55+i1.570\,67$
10^{-2}	$3.733\,40+i1.517\,22$	$3.736\,64+i1.517\,41$	$3.733\,40+i1.517\,22$	$3.733\,40+i1.517\,22$
1	$0.146\,80+i0.495\,73$	$0.147\,40+i0.496\,82$	$0.146\,80+i0.495\,73$	$0.146\,80+i0.495\,73$
10	$1.4327\times10^{-6}+i5.0144\times10^{-2}$	$1.4441\times10^{-6}+i5.0269\times10^{-2}$	$1.4327\times10^{-6}+i5.0144\times10^{-2}$	$1.4327\times10^{-6}+i5.0144\times10^{-2}$
100	$i4.9878\times10^{-3}$	$i5.000\,25\times10^{-3}$	$i4.9878\times10^{-3}$	$i4.9878\times10^{-3}$
10^4	$i4.9876\times10^{-5}$	$i5.0000\times10^{-5}$	$i4.9876\times10^{-5}$	$i4.9876\times10^{-5}$

$$
\begin{aligned}
w(\alpha;\beta(\delta)) = {} & -\frac{2}{\alpha^3}-\frac{1}{3}+\frac{1}{3\alpha}\,S_{2,2}(\alpha)-\frac{\delta[\delta+(1+\delta^2)^{1/2}]}{\alpha} \\
& -\delta(1+\delta^2)^{1/2}[\delta+(1+\delta^2)^{1/2}]e^{\delta\alpha}\mathrm{Ei}(-\delta\alpha) \\
& +\delta[\delta+(1+\delta^2)^{1/2}]\sum_{l=0}^{\infty}\delta^{2l}q_{2l+1}(\alpha), \quad \text{for } |\delta|<1. \qquad (58)
\end{aligned}
$$

The analytic continuation to complex values of α can be carried out through the appropriate rotation of both contour C and the enclosed branch cut in the right half s-plane. The condition $|\delta|<1$ holds when the surrounding medium is air and the adjacent medium is earth, being dictated on mathematical grounds by the use of a power series representation in δ along with the existence of branch points at $\delta=\pm i$. A physical situation corresponding to these points arises when the ambient medium becomes perfectly conducting and the field is identically zero.

C. The z-component of the magnetic field in air

The field is easily evaluated with Eqs. (18b)–(27). Only the z-component of the magnetic field B_{0z} is calculable in simple closed form:

$$
\begin{aligned}
B_{0z} = {} & \frac{\mu_0}{2\pi}\,e^{ik_0x}\left[y\left(\frac{1}{\rho_0^2}-\frac{1}{\rho_1^2}\right)-\frac{4y}{k^2}\,\frac{y^2-3(z+d)^2}{\rho_1^6}+\frac{1}{3i(z+d-iy)}\,S_{2,2}(-ky-ik(z+d))\right. \\
& \left.-\frac{1}{3i(z+d+iy)}\,S_{2,2}(ky-ik(z+d))\right]. \qquad (59)
\end{aligned}
$$

This relatively simple structure is not surprising since it originates from the absence of any pole in the integrand of Eq. (29) when $\beta=1$.

D. Approximate expressions for the field when $k_0^2 \ll |k_1^2|$, $|k_0 k_1^{-1}(k_0\rho_1)| \ll 1$

Clearly, the critical parameters of the problem are $\alpha_{\pm} = -ik(z+d\pm iy)$ and k_0^2/k_1^2. When $k_0^2 \ll |k_1^2|$, at most only the first term of the associated series expansions needs to be retained essentially regardless of the magnitude of $|\alpha_{\pm}|$, while the condition $|\delta\alpha_{\pm}| \sim |k_0 k_1^{-1}(k_0\rho_1)| \ll 1$ enables approximation of the exponential integral $\mathrm{Ei}(-\alpha_{\pm}\delta)$ according to Eq. (44).

In particular, the exact integrated formulas may be greatly simplified under the condition $|k_1\rho_1|^2 \ll 1$ as, for instance, when the earth is highly conducting and the vertical distance from the image is much shorter than the wavelength in earth. Indeed, then $|\alpha_{\pm}| \ll 1$ in Eqs. (25)–(27) and small-argument approximations become effective. The resulting approximate expressions are

$$E_{0x} \sim -\frac{i\omega\mu_0}{2\pi} e^{ik_0x}\left\{ \gamma - \frac{1}{2} - \frac{i\pi}{2} + \frac{2i}{3} k(z+d) + \left[1+\frac{1}{8}k^2(z+d)^2 - \frac{1}{8}k^2y^2\right]\ln\left(\frac{1}{2}k\rho_1\right) \right.$$
$$- \frac{1}{32}(5-4\gamma+2\pi i)k^2[(z+d)^2-y^2] - \frac{1}{4}k^2(z+d)y\,\tan^{-1}\left(\frac{y}{z+d}\right)$$
$$\left. - \frac{2k_0^2}{k_1^2}\left[\gamma - \frac{i\pi}{2} + \ln\left(\frac{k_0^2}{k_1^2}k\rho_1\right)\right]\right\}, \tag{60}$$

$$E_{0y} \sim \frac{\mu_0 c}{2\pi} e^{ik_0x} y\left[\frac{1}{\rho_0^2} - \frac{1-2k_0^2/k_1^2}{\rho_1^2}\right], \tag{61}$$

$$E_{0z} \sim \frac{\mu_0 c}{2\pi} e^{ik_0x}\left[\frac{z-d}{\rho_0^2} - \frac{(1-2k_0^2/k_1^2)(z+d)}{\rho_1^2}\right], \tag{62}$$

$$B_{0x} \sim -i\frac{\mu_0\omega}{2\pi c} e^{ik_0x}\left\{\left[1+\frac{1}{8}k^2(z+d)^2 - \frac{1}{8}k^2y^2\right]\tan^{-1}\left(\frac{y}{z+d}\right) + \frac{2i}{3}ky \right.$$
$$\left. + \frac{1}{4}k^2(z+d)y\,\ln\left(\frac{1}{2}k\rho_1\right) - \frac{1}{16}(5-4\gamma+2\pi i)k^2(z+d)y - \frac{k_0^2}{k_1^2}\tan^{-1}\left(\frac{y}{z+d}\right)\right\}, \tag{63}$$

$$B_{0y} \sim -\frac{\mu_0}{2\pi} e^{ik_0x}\left\{\frac{z-d}{\rho_0^2} + \frac{1}{4}k^2(z+d)\ln\left(\frac{1}{2}k\rho_1\right) + \frac{2ik}{3} - \frac{1}{16}(3-4\gamma+2\pi i)k^2(z+d) \right.$$
$$\left. - \frac{1}{4}k^2y\,\tan^{-1}\left(\frac{y}{z+d}\right) + \frac{2i}{15}k^3[(z+d)^2-y^2] - 2ik\frac{k_0^2}{k_1^2}\left[\ln\left(\frac{2k_1^2}{k_0^2}\right) - 1\right]\right\}, \tag{64}$$

$$B_{0z} \sim \frac{\mu_0}{2\pi} e^{ik_0x}\left\{ y\left[\frac{1}{\rho_0^2} + \frac{1}{16}(3-4\gamma+2\pi i)k^2 - \frac{1}{4}k^2\,\ln\left(\frac{1}{2}k\rho_1\right) - \frac{4}{15}ik^3(z+d)\right] \right.$$
$$\left. - \frac{1}{4}k^2(z+d)\tan^{-1}\left(\frac{y}{z+d}\right)\right\}. \tag{65}$$

Equation (60) gives the first few terms of Carson's series[1,3,9] if k is approximated by k_1, with the exception of the correction term proportional to k_0^2/k_1^2. This discrepancy is due to the neglect of the Sommerfeld-pole contribution in Carson's formulation. Equations (60)–(65) agree with the findings of image theory in the static limit $\omega \to 0^+$. In this limit, the exact integrated formulas (18)–(23) reduce to the results of image theory without any restriction on k_0 and k_1. In particular, when the earth has zero conductivity and $\omega \to 0^+$, only the first term in the series expansion of Eq. (54) is nonzero. Only the approximate formula in Eq. (61) and the zeroth-order terms in Eq. (62) for the y- and z-components of the electric field are in accord with the application of the lateral-wave formulas of Ref. 10. No $1/\rho_1^2$ term of the reflected field is involved in the y- and z-components of the magnetic field because the earth was assumed to be nonmagnetic.

Another limiting case involves $|k_1\rho_1| \sim |\alpha_{\pm}| \gg 1$. Then large-argument approximations apply and the integrated formulas simplify to

$$E_{0x}\sim-\frac{\omega\mu_0}{\pi}\,e^{ik_0x}\left\{\frac{z+d}{k\rho_1^2}-i\,\frac{(z+d)^2-y^2}{k^2\rho_1^4}-i\,\frac{k_0^2}{k_1^2}\left[\gamma-\frac{i\pi}{2}+\ln\left(\frac{k_0^2}{k_1^2}\,k\rho_1\right)\right]\right\}, \tag{66}$$

$$E_{0y}\sim\frac{\mu_0c}{2\pi}\,e^{ik_0x}y\left\{\frac{1}{\rho_0^2}-\frac{1-2k_0^2/k_1^2}{\rho_1^2}\right\}, \tag{67}$$

$$E_{0z}\sim\frac{\mu_0c}{2\pi}\,e^{ik_0x}\left\{\frac{z-d}{\rho_0^2}-\frac{z+d}{\rho_1^2}+i\,\frac{k_0^2}{k_1^2}\,\frac{(z+d)^2-y^2}{k\rho_1^4}+2ik\,\frac{k_0^2}{k_1^2}\left[\gamma-\frac{i\pi}{2}+\ln\left(\frac{k_0^2}{k_1^2}\,k\rho_1\right)\right]\right\}, \tag{68}$$

$$B_{0x}\sim\frac{\mu_0\omega}{\pi c}\,e^{ik_0x}\left\{\frac{y}{k\rho_1^2}-\frac{2iy(z+d)}{k^2\rho_1^4}+i\,\frac{k_0^2}{k_1^2}\tan^{-1}\left(\frac{y}{z+d}\right)\right\}, \tag{69}$$

$$B_{0y}\sim-\frac{\mu_0}{2\pi}\,e^{ik_0x}\left\{\frac{z-d}{\rho_0^2}-\frac{(1+2k_0^2/k_1^2)(z+d)}{\rho_1^2}+2ik\,\frac{k_0^2}{k_1^2}\left[\gamma-\frac{i\pi}{2}+\ln\left(\frac{k_0^2}{k_1^2}\,k\rho_1\right)\right]\right\}, \tag{70}$$

$$B_{0z}\sim\frac{\mu_0}{2\pi}\,e^{ik_0x}y\left\{\frac{1}{\rho_0^2}-\frac{1}{\rho_1^2}+\frac{4i(z+d)}{k\rho_1^4}\right\}. \tag{71}$$

Only the y-component of the electric field preserves its static character to first order in k_0^2/k_1^2. The correction terms $O(k_0^2/k_1^2)$ become significant over a very dry earth. These either stem from the vicinity of the Sommerfeld pole (logarithm and inverse-tangent terms) or are due to secondary imperfect reflections from the image inside the earth (terms proportional to $1/\rho_1^2$, $1/\rho_1^4$).

IV. CONCLUSIONS AND DISCUSSION

The problem of the field created by an infinitely long and infinitely thin conductor carrying a traveling-wave current above a planar earth is revisited, but with a perspective different from previous analyses. Integral representations for the field in terms of contour integrals are derived in a physically meaningful way. It is shown that, as a direct consequence of this formulation, an exponentially decreasing outgoing current along the conductor (in the positive x direction) causes exponential increase of the radiation field in any direction perpendicular to the line source. This result is in accord with preliminary studies on the theory of the microstrip[14] that seem to have escaped attention in a considerable part of the recent literature.

In the limit where the propagation constant γ for the current becomes equal to the free-space wave number, all integrals for the field in air are, for the first time, evaluated *exactly* in terms of series expansions in the Sommerfeld pole. A prominent feature of the new series representations is the essential independence of their rates of convergence from distance. These series describe corrective contributions for the reflected (secondary) field to take account of the fact that it is not actually the field of an ideal image, and involve coefficients of nontrivial, yet calculable, form in terms of incomplete integrals of cylindrical functions. Finally, the component of the magnetic field normal to both conductor and interface is evaluated in simple closed form in terms of Lommel functions.

It should be borne in mind that the method of solution here crucially depends on the fact that the radiative structure is planar, and therefore is restricted in its applicability. Nevertheless, the present exposition for $\gamma=k_0$ reveals the nature of the field produced by multiphase transmission lines operating at sufficiently low frequencies above the earth, where the total field, trivially obtained via superposition, is spatially confined near the wires.

In the case of a complex propagation constant γ with Im $\gamma>0$, the usual radiation condition in three dimensions[22,23] is not explicitly invoked because the line has infinite length. This point is also discussed in Ref. 14. Heuristically speaking, the field growth in the radial distance $\sqrt{y^2+z^2}$ characterizes an asymptotic region in space where the assumed exponential increase along the *negative* x direction dominates over the free-space radiation decrease inversely proportional with distance. This region extends to infinity, in some sense, when the line becomes infinitely long.

Application of a limiting procedure that, starting with a localized source and the usual boundary condition at infinity, may lead to Eq. (2) and the accompanying assumptions in full mathematical rigor remains an open question for future work.

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APPENDIX A: ON THE ASYMPTOTIC BEHAVIOR OF THE RADIATION FIELD IN AIR

For definiteness, consider $y \geqslant 0$. The phase associated with Eqs. (12) and (13) reads

$$\Phi_{\mp}(\eta) = \sqrt{k_0^2 - \gamma^2 - \eta^2}(z \mp d) + \eta y, \quad z > d. \tag{A1}$$

The corresponding saddle point, $\eta = \eta_{\mp}^{(\mathrm{sp})}$, is

$$\eta_{\mp}^{(\mathrm{sp})} = \sqrt{k_0^2 - \gamma^2} \sin \theta_{0,1}, \quad 0 \leqslant \theta_{0,1} < \pi/2, \tag{A2}$$

where $-\pi/2 < \arg\sqrt{k_0^2 - \gamma^2} \leqslant 0$, yielding

$$\Phi_{\mp}(\eta^{(\mathrm{sp})}) = \sqrt{k_0^2 - \gamma^2}\rho_{0,1}, \quad \mathrm{Im}\, \Phi_{\mp}(\eta^{(\mathrm{sp})}) < 0, \tag{A3}$$

$$\left.\frac{d^2\Phi_{\mp}}{d\eta^2}\right|_{\eta = \eta^{(\mathrm{sp})}} = -\frac{\rho_1^3}{(z \mp d)^2}(k_0^2 - \gamma^2)^{-1/2}. \tag{A4}$$

$\rho_{0,1}$ and $\theta_{0,1}$ are defined by Eqs. (32). For $|\sqrt{k_0^2 - \gamma^2}|\rho_{0,1} \geqslant 1$ and fixed $\theta_{0,1}$, the steepest descents path is determined by the conditions

$$\mathrm{Re}[\Phi_{\mp}(\eta) - \Phi_{\mp}(\eta^{(\mathrm{sp})})] = 0, \tag{A5}$$

$$\mathrm{Im}[\Phi_{\mp}(\eta) - \Phi_{\mp}(\eta^{(\mathrm{sp})})] > 0. \tag{A6}$$

Once the integration path Γ is properly deformed to pick up the saddle-point contribution, the exponential growth in $\sqrt{y^2 + z^2}$ of the field follows easily from (A3).

Under the usual substitution

$$\eta = \eta(\chi) = \sqrt{k_0^2 - \gamma^2} \sin \chi, \tag{A7}$$

each $\Phi_{\mp}(\eta)$ is recast in the form

$$\Phi_{\mp}(\eta(\chi)) = \sqrt{k_0^2 - \gamma^2}\rho_{0,1} \cos(\theta_{0,1} - \chi). \tag{A8}$$

A branch of $\chi = \chi(\eta)$ can be chosen so that $\chi(0) = 0$. In the χ-plane, the saddle point is $\chi_{\mp}^{(\mathrm{sp})} = \theta_{0,1}$, while the steepest descents path is now described by the relations

$$A \cos(\chi_r - \theta_{0,1})\cosh \chi_i - D \sinh \chi_i \sin(\chi_r - \theta_{0,1}) = A, \tag{A9}$$

$$A \sinh \chi_i \sin(\chi_r - \theta_{0,1}) + D \cos(\chi_r - \theta_{0,1})\cosh \chi_i < D, \tag{A10}$$

where $A = \mathrm{Re}\sqrt{k_0^2 - \gamma^2} > 0$, $D = -\mathrm{Im}\sqrt{k_0^2 - \gamma^2} > 0$, and $\chi_r = \mathrm{Re}\, \chi$, $\chi_i = \mathrm{Im}\, \chi$. The path configuration in the χ-plane is depicted in Fig. 5. Note that

$$\tan \overline{\theta} = A/D. \tag{A11}$$

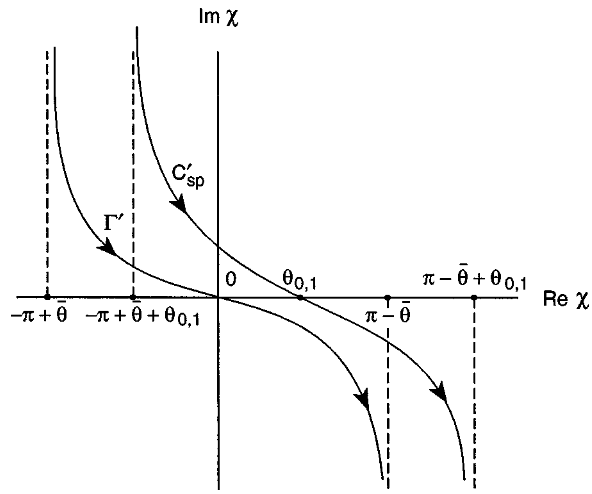

FIG. 5. A sketch of the integration-path configuration in the χ-plane under (A7). Γ' is the image of Γ of Fig. 2 and $C'_{\rm sp}$ is the steepest descents path. For simplicity, the branch points corresponding to $\eta=\pm\sqrt{k_1^2-\gamma^2}$ are not shown here. The dashed lines are vertical asymptotes ($\bar{\theta}=\pi/2-\arg\sqrt{k_0^2-\gamma^2}$).

Possible contribution from a Sommerfeld pole is recognized as exponentially recessive.

APPENDIX B: EVALUATION OF $q_n(\alpha)$

From Eqs. (49a) and (55a), the $q_n(\alpha)$ read

$$q_n(\alpha)=\int_0^{+\infty}dse^{-\alpha s}s^{-n}\left[\sqrt{1+s^2}-\sum_{m=0}^{[(n-1)/2]}\binom{\frac{1}{2}}{m}s^{2m}\right], \tag{B1}$$

where $\alpha>0$, $n=0,1,\ldots$, and $[x]$ denotes the integral part of the real number x. These satisfy the following equations:

$$\frac{d}{d\alpha}\,q_{2l+2}(\alpha)=-q_{2l+1}(\alpha), \tag{B2}$$

$$\frac{d}{d\alpha}\left[q_{2l+1}(\alpha)-\binom{\frac{1}{2}}{l}\ln\,\alpha\right]=-q_{2l}(\alpha), \tag{B3}$$

where $l\geqslant 0$ and

$$q_0(\alpha)=\int_0^{+\infty}dse^{-\alpha s}\sqrt{1+s^2}=\frac{1}{\alpha}\,S_{1,1}(\alpha). \tag{B4}$$

$S_{\nu,\nu}(\alpha)$ is given by Eq. (36).

The application of Lebesgue's dominated convergence theorem to Eq. (49b) when $\alpha\to 0$, $l\geqslant 1$ entails:

$$c_l\equiv\lim_{\alpha\to 0}\,q_{2l+2}(\alpha)=(-1)^l\int_1^{\infty}d\xi\xi^{-2l-2}\sqrt{\xi^2-1}=\frac{(-1)^l}{2}\,\frac{\Gamma(3/2)\Gamma(l)}{\Gamma((3/2)+l)}. \tag{B5}$$

For $l\geqslant 1$, the integral of Eq. (55b) is recast in the form

$$q_{2l+1}(\alpha)-\binom{\frac{1}{2}}{l}\ln\,\alpha=\frac{(-1)^{l-1}}{\pi}\int_1^{\infty}d\xi[e^{i\alpha\xi}\mathrm{Ei}(-i\alpha\xi)+e^{-i\alpha\xi}\mathrm{Ei}(i\alpha\xi)-2\ln\,\alpha]\xi^{-2l-1}\sqrt{\xi^2-1}. \tag{B6}$$

For $0<\alpha\ll1$, the preceding integral may be rewritten as

$$\int_1^\infty=\int_1^{O(\alpha^{-\lambda})}+\int_{O(\alpha^{-\lambda})}^\infty,$$

where $0<\lambda<1$. In the limit $\alpha\to0$, the second term tends to zero because absolute convergence obtains. In regard to the first term, the integrand in brackets is properly expanded according to Eq. (44). This in turn leads to

$$d_l\equiv\lim_{\alpha\to0}\left\{q_{2l+1}(\alpha)-\binom{\frac{1}{2}}{l}\ln\alpha\right\}=2\,\frac{(-1)^{l-1}}{\pi}\int_1^\infty d\xi(\gamma+\ln\xi)\xi^{-2l-1}\sqrt{\xi^2-1}$$
$$=2^{-1}\binom{\frac{1}{2}}{l}\left[2\gamma+\psi(1+l)-\psi\left(-\frac{1}{2}+l\right)\right],\tag{B7}$$

where $l\geqslant1$ and

$$\psi(z)=\frac{d}{dz}\ln\Gamma(z),\quad \psi(1)=-\gamma,\quad \psi(\tfrac{1}{2})=-\gamma-2\ln 2,$$

$$\psi(z+l)=\frac{1}{z}+\cdots+\frac{1}{z+l-1}+\psi(z).$$

Note that $q_2(\alpha)$ and $q_1(\alpha)-\ln\alpha$ exhibit different behavior from that described in Eqs. (B5) and (B7), respectively. By use of Eq. (B1),

$$q_1(\alpha)=\ln\alpha+\frac{1}{\alpha}+\gamma+\ln 2-1+O(\alpha\ln\alpha)\quad\text{for }\alpha\to0.\tag{B8}$$

Likewise,

$$q_2(\alpha)=-\ln\alpha-\gamma+\ln 2-1+O(\alpha\ln\alpha).\tag{B9}$$

Equations (B2)–(B5) and (B7)–(B9) suffice to determine $q_n(\alpha)$ for any $n\geqslant1$.

In consideration of the following formulas:[20]

$$\frac{d}{dt}[tS_{1,1}(t)]=tS_{0,0}(t),\quad S_{1,1}(t)=1-\frac{d}{dt}S_{0,0}(t),\tag{B10}$$

along with Eqs. (B8) and (B9), Eqs. (B3) and (B2) are properly integrated for $l=0$ to give

$$q_1(\alpha)=\frac{1}{\alpha}+\ln\alpha+(\gamma+\ln 2-1)+\lim_{\delta_1\to0^+}\int_{\delta_1}^{\alpha}dt\left\{\frac{1}{t^2}-\frac{1}{t}S_{1,1}(t)\right\}$$
$$=-1+\gamma+\ln(2\alpha)-D(\alpha)+S_{1,1}(\alpha),\tag{B11}$$

$$q_2(\alpha)=-\ln\alpha-\gamma+\ln 2-1-\lim_{\delta_1\to0^+}\int_{\delta_1}^{\alpha}dt\left\{\ln t+\gamma+\ln 2-1+S_{1,1}(t)-D(t)-\frac{1}{t}\right\}$$
$$=-\alpha[-1+\gamma+\ln(2\alpha)-D(\alpha)]+S_{0,0}(\alpha)-\alpha S_{1,1}(\alpha).\tag{B12}$$

In the above, use is made of the small-argument approximations for $\mathbf{H}_{0,1}(\alpha)$ and $Y_{0,1}(\alpha)$, and $D(\alpha)$ is defined as

$$D(\alpha)=\int_0^\alpha dtS_{0,0}(t).\tag{B13}$$

This function was initially studied in connection with a problem in acoustic radiation.[24] Despite the fact that[20]

$$\int_0^\alpha dt Y_0(t)=\frac{\pi\alpha}{2}[Y_0(\alpha)\mathbf{H}_{-1}(\alpha)+\mathbf{H}_0(\alpha)Y_1(\alpha)],\quad \mathbf{H}_{-1}(\alpha)=\frac{2}{\pi}-\mathbf{H}_1(\alpha), \tag{B14}$$

$$\int_0^\alpha dt\mathbf{H}_0(t)=\frac{\alpha^2}{\pi}\,{}_2F_3(1,1;\tfrac{3}{2},\tfrac{3}{2},2;-\alpha^2/4), \tag{B15}$$

the form of Eq. (B13) is more advantageous to employ here. $D(\alpha)$ cannot be expressed in terms of simpler transcendental functions. It is noted in passing that the integral in Eq. (B15) is a special case of the incomplete Lipschitz-Hankel integral of the Struve type.[25]

For $l\geqslant 1$, it is permissible to integrate Eqs. (B2) and (B3) explicitly over the range $[0,\alpha]$ subject to the conditions of Eqs. (B5) and (B7):

$$q_{2l+2}(\alpha)=c_l-\int_0^\alpha dt q_{2l+1}(t),\quad q_{2l+1}(\alpha)=\binom{\frac{1}{2}}{l}\ln\,\alpha+d_l-\int_0^\alpha dt q_{2l}(t). \tag{B16}$$

Successive integrations by parts according to the recursive scheme

$$\int^t dt t^\nu D(t)=\frac{t^{\nu+1}}{\nu+1}D(t)-\frac{1}{\nu+1}\int^t dt t^{\nu+1}S_{0,0}(t), \tag{B17}$$

$$\int^t dt t^\nu S_{0,0}(t)=t^\nu S_{1,1}(t)-(\nu-1)\int^t dt t^{\nu-1}S_{1,1}(t), \tag{B18}$$

$$\int^t dt t^\nu S_{1,1}(t)=\frac{t^{\nu+1}}{\nu+1}-t^\nu S_{0,0}(t)+\nu\int^t dt t^{\nu-1}S_{0,0}(t) \tag{B19}$$

provide a routine procedure for the evaluation of $q_n(\alpha)$ for any finite $n\geqslant 3$. For all reasonable purposes, the first 13 integrals $q_n(\alpha)$ $(n=0,1,\dots,12)$ are sufficient. It is readily found that $q_n(\alpha)$ $(n\geqslant 3)$ can be expressed as

$$q_n(\alpha)=\frac{(-1)^{n-1}}{(n-1)!}\{\mathscr{P}^{(1)}_{n-1}(\alpha)[\gamma+\ln(2\alpha)-D(\alpha)] \\ -\mathscr{P}^{(2)}_{n-1}(\alpha)-\mathscr{P}^{(3)}_{n-2}(\alpha)S_{0,0}(\alpha)+\mathscr{P}^{(4)}_{n-1}(\alpha)S_{1,1}(\alpha)\}, \tag{B20}$$

where the $\mathscr{P}^{(j)}_m(\alpha)$ $(j=1,2,3,4)$ denote even or odd mth-degree polynomials in α, $\mathscr{P}^{(j)}_m(-\alpha)=(-1)^m\mathscr{P}^{(j)}_m(\alpha)$. For $n=3,\dots,12$, the $\mathscr{P}^{(j=1,2,4)}_{n-1}(\alpha)$ and $\mathscr{P}^{(j=3)}_{n-2}(\alpha)$ read

$$\mathscr{P}^{(1)}_2(\alpha)=\alpha^2+1,\quad \mathscr{P}^{(1)}_3(\alpha)=\alpha^3+3\alpha,\quad \mathscr{P}^{(1)}_4(\alpha)=\alpha^4+6\alpha^2-3,$$
$$\mathscr{P}^{(1)}_5(\alpha)=\alpha^5+10\alpha^3-15\alpha,\quad \mathscr{P}^{(1)}_6(\alpha)=\alpha^6+15\alpha^4-45\alpha^2+45,$$
$$\mathscr{P}^{(1)}_7(\alpha)=\alpha^7+21\alpha^5-105\alpha^3+315\alpha,$$
$$\mathscr{P}^{(1)}_8(\alpha)=\alpha^8+28\alpha^6-210\alpha^4+1260\alpha^2-1575, \tag{B21}$$
$$\mathscr{P}^{(1)}_9(\alpha)=\alpha^9+36\alpha^7-378\alpha^5+3780\alpha^3-14\,175\alpha,$$
$$\mathscr{P}^{(1)}_{10}(\alpha)=\alpha^{10}+45\alpha^8-630\alpha^6+9450\alpha^4-70\,875\alpha^2+99\,225,$$
$$\mathscr{P}^{(1)}_{11}(\alpha)=\alpha^{11}+55\alpha^9-990\alpha^7+20\,790\alpha^5-259\,875\alpha^3+1\,091\,475\alpha;$$

$$\mathscr{P}_2^{(2)}(\alpha)=\alpha^2-\frac{1}{2},\quad \mathscr{P}_3^{(2)}(\alpha)=\alpha^3+\frac{3}{2}\alpha,\quad \mathscr{P}_4^{(2)}(\alpha)=\alpha^4+\frac{9}{2}\alpha^2-\frac{3}{4},$$

$$\mathscr{P}_5^{(2)}(\alpha)=\alpha^5+\frac{17}{2}\alpha^3-\frac{75}{4}\alpha,\quad \mathscr{P}_6^{(2)}(\alpha)=\alpha^6+\frac{27}{2}\alpha^4-\frac{225}{4}\alpha^2+\frac{75}{4},$$

$$\mathscr{P}_7^{(2)}(\alpha)=\alpha^7+\frac{39}{2}\alpha^5-\frac{501}{4}\alpha^3+\frac{1785}{4}\alpha,$$

$$\mathscr{P}_8^{(2)}(\alpha)=\alpha^8+\frac{53}{2}\alpha^6-\frac{963}{4}\alpha^4+\frac{3255}{2}\alpha^2-\frac{6195}{8}, \tag{B22}$$

$$\mathscr{P}_9^{(2)}(\alpha)=\alpha^9+\frac{69}{2}\alpha^7-\frac{1683}{4}\alpha^5+\frac{8979}{2}\alpha^3-\frac{169\,155}{8}\alpha,$$

$$\mathscr{P}_{10}^{(2)}(\alpha)=\alpha^{10}+\frac{87}{2}\alpha^8-\frac{2745}{4}\alpha^6+\frac{42\,555}{4}\alpha^4-\frac{732\,375}{8}\alpha^2+\frac{424\,305}{8},$$

$$\mathscr{P}_{11}^{(2)}(\alpha)=\alpha^{11}+\frac{107}{2}\alpha^9-\frac{4245}{4}\alpha^7+\frac{90\,525}{4}\alpha^5-\frac{2\,433\,855}{8}\alpha^3+\frac{13\,399\,155}{8}\alpha;$$

$$\mathscr{P}_1^{(3)}(\alpha)=\alpha,\quad \mathscr{P}_2^{(3)}(\alpha)=\alpha^2,\quad \mathscr{P}_3^{(3)}(\alpha)=\alpha^3+3\alpha,\quad \mathscr{P}_4^{(3)}(\alpha)=\alpha^4+7\alpha^2,$$

$$\mathscr{P}_5^{(3)}(\alpha)=\alpha^5+12\alpha^3-45\alpha,\quad \mathscr{P}_6^{(3)}(\alpha)=\alpha^6+18\alpha^4-123\alpha^2,$$

$$\mathscr{P}_7^{(3)}(\alpha)=\alpha^7+25\alpha^5-249\alpha^3+1575\alpha,$$

$$\mathscr{P}_8^{(3)}(\alpha)=\alpha^8+33\alpha^6-441\alpha^4+4959\alpha^2, \tag{B23}$$

$$\mathscr{P}_9^{(3)}(\alpha)=\alpha^9+42\alpha^7-720\alpha^5+11\,790\alpha^3-99\,225\alpha,$$

$$\mathscr{P}_{10}^{(3)}(\alpha)=\alpha^{10}+52\alpha^8-1110\alpha^6+24\,660\alpha^4-354\,195\alpha^2;$$

$$\mathscr{P}_2^{(4)}(\alpha)=\alpha^2,\quad \mathscr{P}_3^{(4)}(\alpha)=\alpha^3+2\alpha,\quad \mathscr{P}_4^{(4)}(\alpha)=\alpha^4+5\alpha^2,$$

$$\mathscr{P}_5^{(4)}(\alpha)=\alpha^5+9\alpha^3-16\alpha,\quad \mathscr{P}_6^{(4)}(\alpha)=\alpha^6+14\alpha^4-51\alpha^2,$$

$$\mathscr{P}_7^{(4)}(\alpha)=\alpha^7+20\alpha^5-117\alpha^3+384\alpha,$$

$$\mathscr{P}_8^{(4)}(\alpha)=\alpha^8+27\alpha^6-229\alpha^4+1497\alpha^2, \tag{B24}$$

$$\mathscr{P}_9^{(4)}(\alpha)=\alpha^9+35\alpha^7-405\alpha^5+4257\alpha^3-18\,432\alpha,$$

$$\mathscr{P}_{10}^{(4)}(\alpha)=\alpha^{10}+44\alpha^8-666\alpha^6+10\,260\alpha^4-85\,095\alpha^2,$$

$$\mathscr{P}_{11}^{(4)}(\alpha)=\alpha^{11}+54\alpha^9-1036\alpha^7+22\,050\alpha^5-290\,925\alpha^3+1\,474\,560\alpha.$$

As explained in Sec. III, the final formulas for $q_n(\alpha)$ may be extended over complex values of α for which $|\arg\alpha|<\pi$. Small-argument approximations for $q_n(\alpha)$ follow from Eqs. (B11)–(B15) and (B20)–(B24) if the series expansions in α for $\mathbf{H}_{0,1}(\alpha)$, $Y_{0,1}(\alpha)$, and ${}_2F_3(1,1;\frac{3}{2},\frac{3}{2},2;-\alpha^2/4)$ are taken into account. Large-argument approximations when $|\alpha|\to\infty$, $|\arg\alpha|<\pi$ can be readily obtained either from Eq. (B1) via Laplace's method[18] or by use of the asymptotic expansions of the related transcendental functions.[20,26]

APPENDIX C: EVALUATION OF $\varphi(\alpha;\beta)$ BY THE MELLIN TRANSFORM TECHNIQUE

In this appendix, Eq. (52) is derived formally via application of the Mellin transform technique.[27,28] For mathematical convenience, let $\beta=i\tilde{\beta}$ and take $\tilde{\beta}>0$. Then,

$$\varphi(\alpha;\beta(\delta))=\int_0^\infty ds\,\frac{e^{-\alpha s}}{s+i\tilde{\beta}\sqrt{1+s^2}}=(1-\tilde{\delta}^2)I_1(\alpha;\tilde{\delta})-i\tilde{\delta}\sqrt{1-\tilde{\delta}^2}I_2(\alpha;\tilde{\delta}), \tag{C1}$$

where

$$\tilde{\delta}=\frac{\tilde{\beta}}{\sqrt{1+\tilde{\beta}^2}}=-i\delta, \quad \delta=\frac{\beta}{\sqrt{1-\beta^2}}, \tag{C2}$$

$$I_1(\alpha;\tilde{\delta})=\int_0^\infty ds\,\frac{se^{-\alpha s}}{s^2+\tilde{\delta}^2}=-\frac{1}{2}\,[e^{i\tilde{\delta}\alpha}\mathrm{Ei}(-i\tilde{\delta}\alpha)+e^{-i\tilde{\delta}\alpha}\mathrm{Ei}(i\tilde{\delta}\alpha)], \tag{C3}$$

$$I_2(\alpha;\tilde{\delta})=\int_0^\infty ds\,\frac{\sqrt{1+s^2}}{s^2+\tilde{\delta}^2}\,e^{-\alpha s}. \tag{C4}$$

When $\alpha=O(1)>0$, Eq. (38) implies that an expansion of $I_2(\alpha;\tilde{\delta})$ in powers of $\tilde{\delta}$ can be obtained quite straightforwardly by invoking its Mellin transform $\overline{I_2}(\alpha;\zeta)$. Avoiding elaboration on mathematical subtleties, such as issues related to convergence of any of the resulting series, consider

$$\overline{I_2}(\alpha;\zeta)=\int_0^\infty d\tilde{\delta}^2(\tilde{\delta}^2)^{-\zeta}I_2(\alpha;\tilde{\delta})=\frac{\pi}{\sin\,\pi\zeta}\int_0^\infty ds e^{-\alpha s}s^{-2\zeta}\sqrt{1+s^2}, \quad 0<\mathrm{Re}\ \zeta<\frac{1}{2}, \tag{C5}$$

by interchanging the order of integration. Inversion of the above formula is studied initially according to the formula

$$I_2(\alpha;\tilde{\delta})=\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty}d\zeta\overline{I_2}(\alpha;\zeta)(\tilde{\delta}^2)^{\zeta-1}, \quad 0<c'<\frac{1}{2}. \tag{C6}$$

Of course, $\overline{I_2}(\alpha;\zeta)$ defined by (C5) can be analytically continued into Re $\zeta\leqslant 0$, Re $\zeta\geqslant 1/2$, as shown below. It is noteworthy that

$$\begin{aligned}\overline{I_2}(\alpha;\zeta)&\sim\frac{\pi}{\sin\,\pi\zeta}\int_0^\infty ds e^{-\alpha s}s^{-2\zeta}\\&=\frac{\pi}{\sin\,\pi\zeta}\,\alpha^{2\zeta-1}\Gamma(1-2\zeta)\\&=\frac{\pi^2}{\sin\,\pi\zeta\,\sin 2\pi\zeta}\,\frac{\alpha^{2\zeta-1}}{\Gamma(2\zeta)}\\&\sim-(2\pi)^{3/2}(2\zeta)^{1/2}e^{-2\zeta\,\ln 2\zeta+(2\pm 3i\pi)\zeta+(2\zeta-1)\ln\,\alpha},\\&\qquad |\zeta|\to\infty, \quad 0<|\arg\,\zeta|<\pi/2,\end{aligned} \tag{C7}$$

where the upper sign holds for Re $\zeta>0$ and the lower sign for Re $\zeta<0$, and use was made of the Stirling formula

$$\Gamma(z)\sim\sqrt{2\pi}e^{-z+(z-1/2)\ln z}, \quad |z|\to\infty, \quad |\arg z|<\pi.$$

In view of

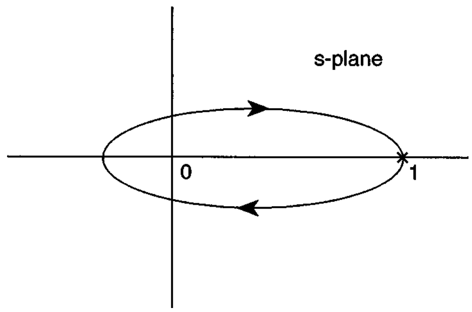

FIG. 6. Contour C_0 that serves the analytic continuation of the integral formula (C5) into the right and left ζ-plane according to (C8).

$$\int_0^1 ds e^{-\alpha s} s^{-2\zeta} \sqrt{1+s^2} = \frac{e^{i2\pi\zeta}}{2i\,\sin 2\pi\zeta} \int_{C_0} ds e^{-\alpha s} s^{-2\zeta} \sqrt{1+s^2}, \tag{C8}$$

where the upper limit $s=1$ is chosen arbitrarily and C_0 is a closed contour passing through 1 and encircling 0 clockwise without enclosing any of the branch points $s=\pm i$ (as shown in Fig. 6), it is not difficult to deduce that $\overline{I_2}(\alpha;\zeta)$ is a meromorphic function of ζ. For Re $\zeta>0$, its poles are located at $\zeta=l$ and $\zeta=l-1/2$, where l is any positive integer. By writing

$$\overline{I_2}(\alpha;\zeta) = \frac{\pi}{\sin \pi\zeta} \left\{ \int_0^\infty ds\ e^{-\alpha s} s^{-2\zeta} \sum_{m=0}^{l-1} \binom{\frac{1}{2}}{m} s^{2m} + \int_0^\infty ds e^{-\alpha s} s^{-2\zeta} \times \left[\sqrt{1+s^2} - \sum_{m=0}^{l-1} \binom{\frac{1}{2}}{m} s^{2m} \right] \right\}, \tag{C9}$$

the singular behavior of $\overline{I_2}(\alpha;\zeta)(\tilde{\delta}^2)^{\zeta-1}$ in the vicinity of $\zeta=l$ is singled out as

$$\begin{aligned} \overline{I_2}(\alpha;\zeta)(\tilde{\delta}^2)^{\zeta-1} \sim (\tilde{\delta}^2)^{l-1} \Bigg\{ & \frac{(-1)^{l+1}}{2(\zeta-l)^2} \sum_{m=0}^{l-1} \binom{\frac{1}{2}}{m} \frac{\alpha^{2(l-m)-1}}{(2l-2m-1)!} \\ & + \frac{(-1)^{l+1}}{\zeta-l} \sum_{m=0}^{l-1} \binom{\frac{1}{2}}{m} \frac{\alpha^{2(l-m)-1}}{(2l-2m-1)!} [\gamma - h_{2l-2m-1} + \ln(\tilde{\delta}\alpha)] \\ & + \frac{(-1)^{l+1}}{\zeta-l} \int_0^\infty ds e^{-\alpha s} s^{-2l} \left[\sqrt{1+s^2} - \sum_{m=0}^{l-1} \binom{\frac{1}{2}}{m} s^{2m} \right] \Bigg\} \quad (l=1,2,\dots), \end{aligned} \tag{C10}$$

where

$$h_n = 1 + 2^{-1} + \cdots + n^{-1}, \quad n \geq 1. \tag{C11}$$

In the vicinity of $\zeta=l-1/2$,

$$\overline{I_2}(\alpha;\zeta)(\tilde{\delta}^2)^{\zeta-1} \sim \frac{\pi}{2} \frac{(-1)^l}{\zeta-l+1/2} \tilde{\delta}^{2l-3} \sum_{m=0}^{l-1} \binom{\frac{1}{2}}{m} \frac{\alpha^{2(l-m-1)}}{(2l-2m-2)!}. \tag{C12}$$

Equation (C7) suggests that deformation of the original inversion path $\mathscr{C}_1$ so that it finally coincides with $\mathscr{C}_2$, which wraps around the positive real axis (see Fig. 7), is legitimate. Subsequent summation of the residues from all pole contributions there yields

$$I_2(\alpha;\tilde{\delta}) = \frac{1}{2\pi i}\oint_{\mathscr{C}_2} d\zeta \overline{I_2}(\alpha;\zeta)(\overline{\delta^2})^{\zeta-1}$$
$$= \frac{\pi}{2}\mathscr{S}_{21}(\alpha;\tilde{\delta}) + [\gamma + \ln(\tilde{\delta}\alpha)]\mathscr{S}_{22}(\alpha;\tilde{\delta}) - \mathscr{S}_{23}(\alpha;\tilde{\delta}) + \lim_{L\to\infty}\sum_{l=0}^{L}(-1)^l\tilde{\delta}^{2l}q_{2l+2}(\alpha), \tag{C13}$$

where $\tilde{\delta}$ is sufficiently small, $q_{2l+2}(\alpha)$ is given by Eq. (49a), and

$$\mathscr{S}_{21}(\alpha;\tilde{\delta}) \equiv \sum_{l=0}^{\infty}(-1)^l\tilde{\delta}^{2l-1}\sum_{m=0}^{l}\binom{\frac{1}{2}}{m}\frac{\alpha^{2(l-m)}}{(2l-2m)!}$$
$$= \sum_{m=0}^{\infty}\sum_{p=0}^{\infty}(-1)^{p+m}\binom{\frac{1}{2}}{m}\tilde{\delta}^{2(p+m)-1}\frac{\alpha^{2p}}{(2p)!}$$
$$= \frac{1}{\tilde{\delta}}\sqrt{1-\tilde{\delta}^2}\cos\tilde{\delta}\alpha, \tag{C14}$$

$$\mathscr{S}_{22}(\alpha;\tilde{\delta}) \equiv \sum_{l=0}^{\infty}(-1)^l\tilde{\delta}^{2l}\sum_{m=0}^{l}\binom{\frac{1}{2}}{m}\frac{\alpha^{2(l-m)+1}}{(2l-2m+1)!}$$
$$= \sum_{m=0}^{\infty}\sum_{p=0}^{\infty}(-1)^{p+m}\binom{\frac{1}{2}}{m}\tilde{\delta}^{2(p+m)}\frac{\alpha^{2p+1}}{(2p+1)!}$$
$$= \frac{\sqrt{1-\tilde{\delta}^2}}{\tilde{\delta}}\sin\tilde{\delta}\alpha, \tag{C15}$$

$$\mathscr{S}_{23}(\alpha;\tilde{\delta}) \equiv \sum_{l=0}^{\infty}(-1)^l\tilde{\delta}^{2l}\sum_{m=0}^{l}\binom{\frac{1}{2}}{m}\frac{\alpha^{2(l-m)+1}}{(2l-2m+1)!}h_{2l-2m+1}$$
$$= \sum_{p=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{p+m}\binom{\frac{1}{2}}{m}\tilde{\delta}^{2(p+m)}\frac{\alpha^{2p+1}}{(2p+1)!}h_{2p+1}$$
$$= \frac{1}{\tilde{\delta}}\sqrt{1-\tilde{\delta}^2}\sum_{p=0}^{\infty}(-1)^p\frac{(\tilde{\delta}\alpha)^{2p+1}}{(2p+1)!}h_{2p+1}. \tag{C16}$$

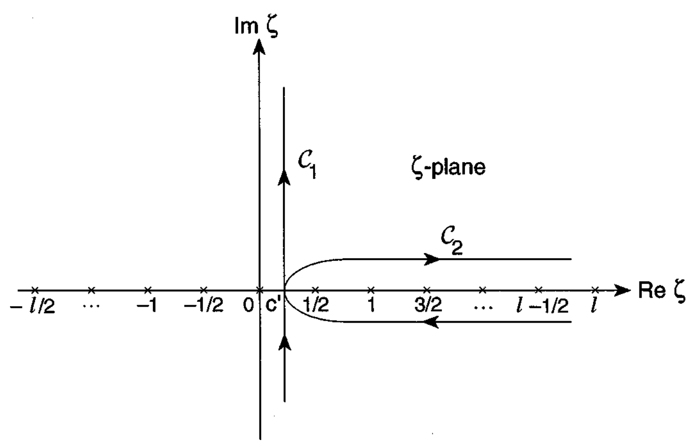

FIG. 7. Inversion paths for the Mellin transform $\overline{I_2}(\alpha;\zeta)$ of (C5). $\mathscr{C}_1$ is the original inversion path and $\mathscr{C}_2$ serves the expansion of $I_2(\alpha;\delta)$ in powers of δ. l is any positive integer.

The last series is easily evaluated by noting that

$$h_m = -\sum_{l=1}^{m} \frac{(-1)^l}{l} \binom{m}{l}. \tag{C17}$$

Therefore

$$\begin{aligned}\sum_{p=0}^{\infty} (-1)^p \frac{z^{2p+1}}{(2p+1)!} h_{2p+1} &= \sum_{p=0}^{\infty} \sum_{l=0}^{2p} (-1)^{p+l} \frac{z^{2p+1}}{(l+1)!(l+1)} \frac{1}{(2p-l)!} \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^{m+n} \frac{z^{2m+2n+1}}{(2n+1)!(2n+1)} \frac{1}{(2m)!} \\ &\quad + \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (-1)^{m+n} \frac{z^{2m+2n+3}}{(2n+2)!(2n+2)} \frac{1}{(2m+1)!} \\ &= z \cos z\, F(z) + \sin z\, G(z),\end{aligned} \tag{C18}$$

where

$$F(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!(2n+1)}, \tag{C19}$$

$$G(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+2}}{(2n+2)!(2n+2)}. \tag{C20}$$

$F(z)$ satisfies

$$\frac{d}{dz}(zF) = \frac{1}{z} \sin z, \quad F(0)=1, \tag{C21}$$

with the solution

$$F(z) = \frac{1}{z} \int_0^z dt\, \frac{\sin t}{t} = \frac{\pi}{2z} - \frac{1}{2iz} [\mathrm{Ei}(-iz) - \mathrm{Ei}(iz)]. \tag{C22}$$

By inspection of (C20),

$$\begin{aligned}G(z) &= \int_0^z dt\, \frac{1-\cos t}{t} = \int_0^{\infty} \frac{dt}{t} \left(\frac{1}{1+t^2} - \cos t \right) + \int_0^z dt\, \frac{t}{1+t^2} - \int_z^{\infty} \frac{dt}{t(1+t^2)} + \int_z^{\infty} dt\, \frac{\cos t}{t} \\ &= \gamma + \ln z - \tfrac{1}{2} [\mathrm{Ei}(-iz) + \mathrm{Ei}(iz)].\end{aligned} \tag{C23}$$

Equations (C18), (C22), and (C23), along with (C16), lead to

$$\begin{aligned}\mathscr{S}_{23}(\alpha;\tilde{\delta}) = \tilde{\delta}^{-1} \sqrt{1-\tilde{\delta}^2} \Bigg\{ & \frac{\pi}{2} \cos \tilde{\delta}\alpha + [\gamma + \ln(\tilde{\delta}\alpha)] \sin \tilde{\delta}\alpha \\ & + \frac{i}{2} [e^{i\tilde{\delta}\alpha} \mathrm{Ei}(-i\tilde{\delta}\alpha) - e^{-i\tilde{\delta}\alpha} \mathrm{Ei}(i\tilde{\delta}\alpha)] \Bigg\}.\end{aligned} \tag{C24}$$

Substitution of (C14), (C15), and (C24) into (C13) yields

$$I_2(\alpha;\tilde{\delta}) = \frac{\sqrt{1-\tilde{\delta}^2}}{2i\tilde{\delta}} [e^{i\tilde{\delta}\alpha} \mathrm{Ei}(-i\tilde{\delta}\alpha) - e^{-i\tilde{\delta}\alpha} \mathrm{Ei}(i\tilde{\delta}\alpha)] + \sum_{l=0}^{\infty} (-1)^l \tilde{\delta}^{2l} q_{2l+2}(\alpha). \tag{C25}$$

By virtue of (C1) and (C3),

$$\varphi(\alpha;\beta)=-(1-\tilde{\delta}^2)e^{i\tilde{\delta}\alpha}\mathrm{Ei}(-i\tilde{\delta}\alpha)-i\tilde{\delta}\sqrt{1-\tilde{\delta}^2}\sum_{l=0}^{\infty}(-1)^l\tilde{\delta}^{2l}q_{2l+2}(\alpha), \tag{C26}$$

which is Eq. (52). This result can be analytically continued to complex values of $\tilde{\delta}=-i\delta$ and α ($|\arg\alpha|<\pi$).

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