

On solutions of Maxwell's equations with dipole sources over a thin conducting film

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We derive and interpret solutions of time-harmonic Maxwell's equations with a vertical and a horizontal electric dipole near a planar, thin conducting film, e.g., graphene sheet, lying between two unbounded isotropic and non-magnetic media. Exact expressions for all field components are extracted in terms of rapidly convergent series of known transcendental functions when the ambient media have equal permittivities and both the dipole and observation point lie on the plane of the film. These solutions are simplified for all distances from the source when the film surface resistivity is large in magnitude compared to the intrinsic impedance of the ambient space. The formulas reveal the analytical structure of two types of waves that can possibly be excited by the dipoles and propagate on the film. One of these waves is intimately related to the surface plasmon-polariton of transverse-magnetic polarization of plane waves. © 2016 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4945083>]

I. INTRODUCTION

In the last decade, rapid advances have been made in the design and fabrication of two-dimensional materials¹ which can be used to manipulate light at small scales.² The highly active field of plasmonics focuses on the interaction of electromagnetic radiation at the mid- and near-infrared spectrum with the collective motion of electrons in conducting interfaces and nanostructures.³ A goal is to generate electromagnetic waves that propagate with relatively small energy loss close to the interface between a conducting material, e.g., graphene, and a dielectric.^{4–9} In plasmonics, such *lateral waves* should decay fast enough away from the interface, while, on the other hand, they should attenuate slowly enough in the direction of propagation along the interface. A wave that has attracted much attention in this context is the surface plasmon-polariton,^{3,10,11} with a variety of reported applications^{12,13} including invisibility cloaking,¹⁴ photovoltaics,¹⁵ and nanolithography.¹⁶

Most recently, direct experimental evidence was provided for generating surface plasmons by placing a receiving resonant antenna near a graphene sheet.¹⁷ Motivated by this advance, our goal with this paper is to analytically study the generation of surface plasmons by current-carrying sources via a solvable model for a fundamental setting. To this end, we formulate and solve exactly boundary value problems for the time-harmonic Maxwell equations in the presence of vertical and horizontal electric (Hertzian) dipoles near an isotropic and homogeneous conducting sheet between two isotropic and non-magnetic unbounded media.^{3,6}

The underlying theme, wave propagation near boundaries, has been the subject of important studies for over a century;^{18–26} see particularly the systematic and extensive treatment of Ref. 26. In these works, approximation techniques are developed for radiowave propagation; these have offered valuable insights into the lateral electromagnetic waves traveling near the boundaries between

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media of very different indices of refraction.²⁶ In the frequency band of plasmonics,² however, additional considerations have emerged because of properties of novel two-dimensional materials used in microscale applications. For instance, a thin layer of graphene has a complex surface conductivity, with a positive or negative imaginary part depending on frequency and doping, and introduces a jump in the tangential component of the magnetic field across the interface. The analytical consequences of this discontinuity are largely unexplored.^{7,8,27} This view suggests that the associated lateral electromagnetic waves be studied in detail in the near- and mid-infrared spectrum.

In this paper, we analytically address aspects of the following question. What is the structure of the waves generated by dipole sources on a thin conducting film? Our goal is to single out fundamental attributes of the field which are intimately related to the film by analyzing a minimal solvable model. Our tasks can be summarized as follows.

- We explicitly represent all field components in terms of one-dimensional Fourier-Bessel (Sommerfeld-type) integrals,³⁷ in the spirit of Ref. 26. Our derivations, focusing on the electromagnetic field itself, differ from the use of the Hertzian potential invoked, e.g., in Ref. 7, by which the field components are derived via successive differentiations.
- By a generalized Schwinger-Feynman representation for a class of integrals,³⁰ we compute all field components via fast convergent series of known functions such as the Fresnel integrals, when both the dipole and observation point lie on the plane of the film and the ambient media have equal permittivities. Our model is thus simplified, yet without obscuring the goal to analytically understand the role of the interface.
- In accord with applications in plasmonics,⁶ we further simplify the exact solutions when the surface resistivity (inverse of conductivity) of the thin layer is much larger in magnitude than the intrinsic impedance of the ambient space. Then, a few terms are retained in the series expansions for the fields yielding simple approximate formulas for all distances from the source.

Our approach is based on systematically solving Maxwell's equations in the spirit of Refs. 26 and 30. Thus, we avoid any *a priori* plane-wave approximations. We recognize a specific type of lateral wave as intimately related to the surface plasmon-polariton of transverse-magnetic (TM) polarization of plane waves^{3,11} via the contribution of a certain pole in the complex plane of the dual (Fourier) variable, see also the treatments of Refs. 7, 8, and 27. For a horizontal dipole on a thin film in free space, when contributions related to TM polarization, in principle, may co-exist with contributions of transverse-electric (TE) polarization, our analysis reveals that the TM surface plasmon-polariton, when present, is accompanied by a wave expressed by Fresnel integrals.²⁶

The analysis presented here, with focus on explicit, physically transparent expressions for the electromagnetic field in terms of known functions, differs in methodology from previous studies of waves in similar settings.^{7,8,11,27} For example, in Ref. 11, the authors review dispersion relations for plane waves in plasmonics for a variety of experimentally relevant geometries, without discussing effects of point sources. In Ref. 7, exact integrals are formulated for the Hertzian potential produced by dipoles in the presence of a graphene sheet; the electromagnetic field is then computed by numerical evaluation of integrals. In Ref. 8, a similar task is carried out more extensively, with numerical evidence that the surface plasmon-polariton of TM polarization, recognized as a discrete spectral contribution, may dominate wave propagation under certain conditions on the surface conductivity. In Ref. 27, the authors numerically describe the field produced by dipoles near a graphene sheet, distinguishing a "core region," where the electric field can be much larger than its values in free space, from an "outer region," where the field approaches its values in free space.

Our work expands previous numerical approaches^{7,8,27} in the following sense. By focusing on a simple yet nontrivial model with a conducting thin film, we are able to derive closed-form expressions which explicitly separate the primary field of the dipole, produced in the absence of the layer, from the scattered field which is sensitive to the film conductivity, for all distances from the point source. This approach singles out analytic aspects of the wave produced by the point source that are intimately connected to the surface conductivity of the film, thus showing *how* the surface plasmon related to TM polarization can dominate propagation in cases of physical and practical

interest. Our analysis lacks generality, since we restrict attention to the case where the source and the observation point both lie on the plane of the thin film; nonetheless, we view our treatment as a step necessary for tackling the problem of radiation by a realistic current-carrying source (rather than an incident plane wave) placed on the material surface.¹⁷ The character of the wave produced by the source of course becomes important at short distances or high enough frequencies. Our work aims at illuminating the complicated structure of the field in a simple nontrivial setting.

The present work illustrates analytic aspects of the tensor (or, dyadic) Green function for the geometry of a thin conducting film at fixed frequency. Naturally, the response to any imposed current-carrying source can then be carried out by superposition; depending on the source, this superposition can be achieved numerically or analytically. This task lies beyond our present scope. Furthermore, we do not pursue numerical computations of the fields. The explicit computation by asymptotic methods²⁶ of field components when the observation point or the dipole is away from the layer is left for future work.

The remainder of this paper is organized as follows. In Section II, we describe the boundary value problem for Maxwell's equations. Section III focuses on the derivation of Fourier-Bessel representations, related to the Hankel transform,²⁸ for the electromagnetic field. In Section IV, we evaluate exactly the requisite integrals when the dipole and the observation point are on the film, which is placed in a homogeneous space. Section V discusses the field resulting from our computations for distances from the source that are comparable to the wavelength in free space. Section VI concludes our paper with an outline of open problems.

Notation and terminology. The $e^{-i\omega t}$ time dependence is assumed throughout, where ω is the angular frequency. \mathbb{R} is the set of reals, \mathbb{Z} is the set of integers, and boldface symbols denote vectors in \mathbb{R}^3 . We write $f = \mathcal{O}(g)$ ($f = o(g)$) to mean that $|f/g|$ is bounded by a nonzero constant (approaches zero) in a prescribed limit. $f \sim g$ implies $f - g = o(g)$ in a prescribed limit. $\Re w$ ($\Im w$) denotes the real (imaginary) part of complex w . The term “sheet” has a two-fold meaning as either a material thin film or, as a “Riemann sheet,” a particular branch of a multiple-value function of a complex variable; and the terms “top Riemann sheet” and “first Riemann sheet” are used interchangeably. We use the terms “TM polarization” and “TE polarization” in the context of waves produced by dipoles to indicate the presence of certain denominators, denoted by \mathcal{P} and \mathcal{Q} in the main text, respectively, in the Fourier representations of the corresponding fields; see (12) and (23). Each denominator, \mathcal{P} or \mathcal{Q} , appears in the reflection coefficient for the TM- or TE-polarized plane wave incident upon the thin film. The terms “surface plasmon-polariton”³ and “surface plasmon” are used interchangeably.

II. BOUNDARY VALUE PROBLEM

In this section, we formulate the boundary value problem for Maxwell's equations. The current density of the vertical unit electric dipole, shown in Fig. 1, is

$$\mathbf{J}(x, y, z) = \mathbf{J}^v(x, y, z) = \mathbf{e}_z \delta_{\mathbf{r}_0}, \quad \mathbf{r}_0 = (0, 0, a), \quad (1)$$

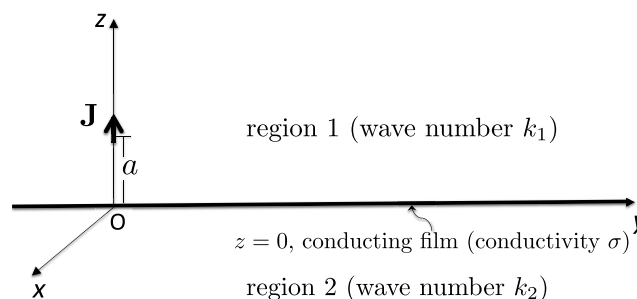


FIG. 1. Vertical unit electric dipole at distance a from planar thin conducting film. The infinitely thin film lies in the plane $z = 0$, between region 1 (half-space $\{z > 0\}$ with wave number k_1) and region 2 ($\{z < 0\}$ with wave number k_2) and has surface conductivity σ .

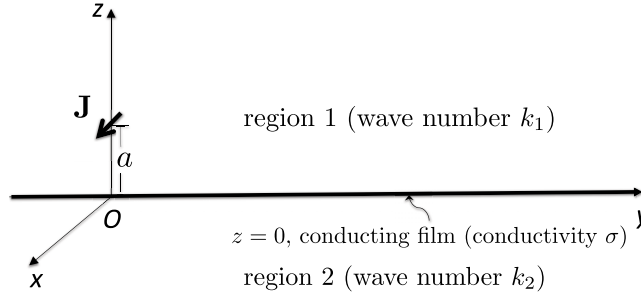


FIG. 2. Horizontal unit electric dipole at distance a from planar thin conducting film. The film lies in the plane $z=0$, separating region 1 from region 2 and has surface conductivity σ .

where a is the distance of the dipole from the layer, $a > 0$, \mathbf{e}_z is the unit vector along the z -axis, and $\delta_{\mathbf{r}}$ denotes the Dirac mass at point \mathbf{r} . For the horizontal unit electric dipole, shown in Fig. 2, the current density reads

$$\mathbf{J}(x, y, z) = \mathbf{J}^h(x, y, z) = \mathbf{e}_x \delta_{\mathbf{r}_0}. \quad (2)$$

The film has infinitesimal thickness and scalar surface conductivity σ , which is, in principle, complex and ω -dependent.⁷ The film lies in the plane $z=0$ which separates region 1, the upper half-space $\{z > 0\}$ with wave number k_1 , from region 2, the lower half-space $\{z < 0\}$ with wave number k_2 (Figs. 1 and 2). We assume that $\Re k_j > 0$ and $\Im k_j > 0$ ($j = 1, 2$), i.e., a lossy medium j , including the case with $\Im k_j \ll \Re k_j$. Note that $k_j^2 = \omega^2 \mu_0 \tilde{\epsilon}_j$, where μ_0 is the magnetic permeability of free space, since the media are assumed non-magnetic, and $\tilde{\epsilon}_j$ is the complex permittivity of medium j ; in practically appealing situations, this $\tilde{\epsilon}_j$ has a small imaginary part.

The time-harmonic Maxwell equations for the field $(\mathbf{E}_j, \mathbf{B}_j)$ in region j dictate that

$$\nabla \times \mathbf{E}_j = i\omega \mathbf{B}_j, \quad (3a)$$

$$\nabla \times \mathbf{B}_j = -i(k_j^2/\omega)\mathbf{E}_j + \mu_0 \mathbf{J}. \quad (3b)$$

By Gauss' law, $\nabla \cdot \mathbf{B}_j = 0$ and $\nabla \cdot \mathbf{E}_j = 0$ if $(x, y, z) \neq \mathbf{r}_0$; however, these equations are not independent from (3) in the time-harmonic case and, therefore, are not utilized here. Equations (3) are supplemented with boundary conditions for the tangential components, viz.,^{7,11}

$$\mathbf{e}_z \times (\mathbf{E}_1 - \mathbf{E}_2)|_{z=0} = 0, \quad (4a)$$

$$\mathbf{e}_z \times (\mathbf{B}_1 - \mathbf{B}_2)|_{z=0} = \mu_0 \sigma \mathbf{E}_{\parallel}, \quad (4b)$$

where $\mathbf{E}_{\parallel} := \{\mathbf{E}_1 - (\mathbf{e}_z \cdot \mathbf{E}_1)\mathbf{e}_z\}|_{z=0} = \{\mathbf{E}_2 - (\mathbf{e}_z \cdot \mathbf{E}_2)\mathbf{e}_z\}|_{z=0}$ denotes the (continuous) tangential electric field at $z=0$. Notably, condition (4b) expresses the physical property that the thin conducting film amounts to an effective surface current of density $\mathbf{J}_s = \sigma \mathbf{E}_{\parallel}$ at $z=0$; this \mathbf{J}_s is viewed as a free current density for the field outside the film. The boundary conditions for the normal components of (\mathbf{E}, \mathbf{B}) are redundant for the derivation of a solution; see Appendix A. In addition to (4), we impose the Sommerfeld radiation condition, viz.,²⁹

$$\left(\frac{\partial}{\partial r} - ik_j\right)\tilde{\mathfrak{F}}_{js} = o\left(\frac{1}{r}\right) \quad \text{as } r \rightarrow \infty \quad (r = \sqrt{x^2 + y^2 + z^2}), \quad (5)$$

uniformly in \mathbf{r}/r if $z \neq 0$, for each scalar component $\tilde{\mathfrak{F}}_{js}$ ($s = x, y, z$) of the vector-valued field $\tilde{\mathfrak{F}}_j$ ($\tilde{\mathfrak{F}}_j = \mathbf{E}_j, \mathbf{B}_j$); $\mathbf{r} = x \mathbf{e}_x + y \mathbf{e}_y + z \mathbf{e}_z$ is the position vector in Cartesian coordinates. Equations (3)–(5) with (1) or (2) constitute the desired boundary value problem.

III. FOURIER REPRESENTATION OF SOLUTION

In this section, we derive one-dimensional integral representations for $(\mathbf{E}_j, \mathbf{B}_j)$ for a vertical and a horizontal electric dipole. The starting point is the Fourier transform with respect to (x, y) of (3)

and (4). Accordingly, let

$$\mathfrak{F}_j(x, y, z) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} d\eta d\xi \widehat{\mathfrak{F}}_j(\xi, \eta, z) e^{i(\xi x + \eta y)}, \tag{6}$$

where $\mathfrak{F}_j = \mathbf{E}_j, \mathbf{B}_j$ and $\widehat{\mathfrak{F}}_j$ is the Fourier transform of \mathfrak{F}_j in (x, y) , assuming that the integral converges in an appropriate sense. Consequently, by (3) the transformed variables obey

$$\begin{aligned} i\eta \widehat{E}_{jz} - \frac{\partial}{\partial z} \widehat{E}_{jy} &= i\omega \widehat{B}_{jx}, \\ -i\xi \widehat{E}_{jz} + \frac{\partial}{\partial z} \widehat{E}_{jx} &= i\omega \widehat{B}_{jy}, \\ i\xi \widehat{E}_{jy} - i\eta \widehat{E}_{jx} &= i\omega \widehat{B}_{jz}, \end{aligned} \tag{7a}$$

and

$$\begin{aligned} i\eta \widehat{B}_{jz} - \frac{\partial}{\partial z} \widehat{B}_{jy} &= -i(k_j^2/\omega) \widehat{E}_{jx} + \mu_0 \widehat{J}_x, \\ \frac{\partial}{\partial z} \widehat{B}_{jx} - i\xi \widehat{B}_{jz} &= -i(k_j^2/\omega) \widehat{E}_{jy} + \mu_0 \widehat{J}_y, \\ i\xi \widehat{B}_{jy} - i\eta \widehat{B}_{jx} &= -i(k_j^2/\omega) \widehat{E}_{jz} + \mu_0 \widehat{J}_z, \end{aligned} \tag{7b}$$

where $(\widehat{J}_x, \widehat{J}_y, \widehat{J}_z) = (0, 0, \delta(z - a))$ or $(\delta(z - a), 0, 0)$ is the Fourier transform of \mathbf{J}^v or \mathbf{J}^h . Equations (7) are complemented with the transformation of boundary conditions (4) and radiation condition (5).

A. Vertical dipole

Consider Fig. 1. In this case, by symmetry, we have $B_{jz} \equiv 0$ for $j = 1, 2$. The remaining field components can be expressed in terms of \widehat{B}_{jy} , as shown below. Equations (7) combined yield the differential equation²⁶

$$\left(\frac{\partial^2}{\partial z^2} + \beta_j^2 \right) \widehat{B}_{jy} = i\xi \mu_0 \delta(z - a) \quad z \in \mathbb{R} \setminus \{0\},$$

where

$$\beta_j := (k_j^2 - \xi^2 - \eta^2)^{1/2}, \quad \Im \beta_j > 0 \quad (j = 1, 2).$$

In compliance with radiation condition (5), we write

$$\widehat{B}_{1y}(\xi, \eta, z) = C_{>} e^{i\beta_1 z} + \frac{\xi \mu_0}{2\beta_1} e^{i\beta_1 |z - a|} \quad (z > 0), \quad \widehat{B}_{2y}(\xi, \eta, z) = C_{<} e^{-i\beta_2 z} \quad (z < 0), \tag{8}$$

where $C_{>}$ and $C_{<}$ are integration constants to be determined. In view of (7), the remaining field components are given in terms of \widehat{B}_{jy} by the relations²⁶

$$\begin{aligned} \widehat{B}_{jx} &= -\frac{\eta}{\xi} \widehat{B}_{jy}, \tag{9} \\ \widehat{E}_{jx} &= -\frac{i\omega}{k_j^2} \frac{\partial \widehat{B}_{jy}}{\partial z}, \quad \widehat{E}_{jy} = -\frac{i\omega \eta}{k_j^2 \xi} \frac{\partial \widehat{B}_{jy}}{\partial z}, \quad \widehat{E}_{jz} = -\frac{\omega}{k_j^2} \frac{\xi^2 + \eta^2}{\xi} \widehat{B}_{jy} \quad (z \neq 0, a). \end{aligned} \tag{10}$$

To determine $C_{>}$ and $C_{<}$, we resort to conditions (4), by which $\widehat{E}_{1x} = \widehat{E}_{2x}, \widehat{E}_{1y} = \widehat{E}_{2y}, \widehat{B}_{1x} - \widehat{B}_{2x} = \mu_0 \sigma \widehat{E}_{2y}$, and $\widehat{B}_{1y} - \widehat{B}_{2y} = -\mu_0 \sigma \widehat{E}_{2x}$ at $z = 0$. In fact, we need to only apply the first and third (or, second and fourth) of these conditions since we have set $\widehat{B}_{jz} \equiv 0$ *ab initio*; the other two

conditions are then satisfied. Thus, we obtain

$$C_{>} = -\frac{\mu_0 \xi}{2\beta_1} \mathcal{R}_m e^{i\beta_1 a},$$

$$C_{<} = \mu_0 k_2^2 \xi \frac{e^{i\beta_1 a}}{\mathcal{P}},$$

where the factor \mathcal{R}_m is defined as

$$\mathcal{R}_m = \frac{k_1^2 \beta_2 - k_2^2 \beta_1 - \omega \mu_0 \sigma \beta_1 \beta_2}{\mathcal{P}} \tag{11}$$

which is associated with the reflection of TM-polarized plane waves from the thin layer (the subscript ‘‘m’’ stands for ‘‘magnetic,’’ implying TM polarization);²⁶ and the corresponding denominator, \mathcal{P} , is^{3,7}

$$\mathcal{P} = k_1^2 \beta_2 + k_2^2 \beta_1 + \omega \mu_0 \sigma \beta_1 \beta_2. \tag{12}$$

The two-dimensional Fourier integrals for (\mathbf{E}, \mathbf{B}) then follow from (6) with (8)–(10).

To reduce representation (6) to one-dimensional integrals, we resort to the cylindrical coordinates (ρ, ϕ, z) , where $x = \rho \cos \phi$ and $y = \rho \sin \phi$ ($0 \leq \phi < 2\pi$), following Ref. 26. The cylindrical components of the field \mathfrak{F}_j ($\mathfrak{F}_j = \mathbf{E}_j, \mathbf{B}_j$) include $\mathfrak{F}_{j\rho} = \mathfrak{F}_{jx} \cos \phi + \mathfrak{F}_{jy} \sin \phi$ and $\mathfrak{F}_{j\phi} = -\mathfrak{F}_{jx} \sin \phi + \mathfrak{F}_{jy} \cos \phi$. Accordingly, let $(\xi, \eta) \mapsto (\lambda, \phi')$ with $(\xi, \eta) = (\lambda \cos \phi', \lambda \sin \phi')$, where $\lambda \geq 0$ and $0 \leq \phi' < 2\pi$; thus, $\xi x + \eta y = \lambda \rho \cos(\phi - \phi')$ and $d\xi d\eta = \lambda d\phi' d\lambda$. By direct integration in ϕ' , we find

$$E_{j\phi} \equiv 0 \quad \text{and} \quad B_{j\rho} \equiv 0,$$

as expected by symmetry because of the dipole orientation. For the remaining components, we invoke the known formula³¹

$$e^{in\phi} J_n(w) = \frac{i^{-n}}{2\pi} \int_0^{2\pi} d\phi' e^{iw \cos(\phi - \phi') + in\phi'} \tag{13}$$

for $n = 0$ and 1, where J_n is the Bessel function of n th order, and $n \in \mathbb{Z}$.

Consequently, after some algebra, we find the following integral representations:

for $z > 0$ (region 1) with $(x, y, z) \neq (0, 0, a)$,

$$E_{1\rho}(\rho, \phi, z) = \frac{i\omega\mu_0}{4\pi k_1^2} \int_0^\infty d\lambda \lambda^2 J_1(\lambda\rho) [\text{sgn}(z - a) e^{i\beta_1|z-a|} - \mathcal{R}_m e^{i\beta_1(z+a)}], \tag{14a}$$

$$E_{1z}(\rho, \phi, z) = -\frac{\omega\mu_0}{4\pi k_1^2} \int_0^\infty d\lambda \frac{\lambda^3}{\beta_1} J_0(\lambda\rho) [e^{i\beta_1|z-a|} - \mathcal{R}_m e^{i\beta_1(z+a)}], \tag{14b}$$

$$B_{1\phi}(\rho, \phi, z) = \frac{i\mu_0}{4\pi} \int_0^\infty d\lambda \frac{\lambda^2}{\beta_1} J_1(\lambda\rho) [e^{i\beta_1|z-a|} - \mathcal{R}_m e^{i\beta_1(z+a)}], \tag{15}$$

and for $z < 0$ (region 2),

$$E_{2\rho}(\rho, \phi, z) = -\frac{i\omega\mu_0}{2\pi} \int_0^\infty d\lambda \lambda^2 J_1(\lambda\rho) \frac{\beta_2}{\mathcal{P}} e^{-i\beta_2 z + i\beta_1 a}, \tag{16a}$$

$$E_{2z}(\rho, \phi, z) = -\frac{\omega\mu_0}{2\pi} \int_0^\infty d\lambda \lambda^3 J_0(\lambda\rho) \frac{1}{\mathcal{P}} e^{-i\beta_2 z + i\beta_1 a}, \tag{16b}$$

$$B_{2\phi}(\rho, \phi, z) = \frac{i\mu_0 k_2^2}{2\pi} \int_0^\infty d\lambda \lambda^2 J_1(\lambda\rho) \frac{1}{\mathcal{P}} e^{-i\beta_2 z + i\beta_1 a}. \tag{17}$$

In the above, $\text{sgn}(z) = 1$ if $z > 0$ and $\text{sgn}(z) = -1$ if $z < 0$; and, by (11) and (12), \mathcal{R}_m and \mathcal{P} are functions of λ with

$$\beta_j(\lambda) = (k_j^2 - \lambda^2)^{1/2} \quad (j = 1, 2). \tag{18a}$$

In accord with the Sommerfeld radiation condition, the top (physical) Riemann sheet in the four-sheeted λ -Riemann surface for the field components is fixed by imposition of

$$\Im \beta_j(\lambda) > 0, \tag{18b}$$

for each $j = 1, 2$; thus, $\beta_j(\lambda) = \beta_j(-\lambda)$ in this Riemann sheet. We note in passing that in the appropriate limit where $a \downarrow 0$ and $z \rightarrow 0$, integrals (14)–(17) should remain physically acceptable descriptions although they become divergent in the conventional mathematical sense. The resulting integrals are meaningful as finite via the methodology of Section IV.³⁰

B. Horizontal dipole

Next, we focus on the geometry of Fig. 2. In this case, all cylindrical field components are nonzero. By (7), \widehat{E}_{jx} and \widehat{B}_{jx} satisfy²⁶

$$\begin{aligned} \left(\frac{\partial^2}{\partial z^2} + \beta_j^2\right)\widehat{E}_{jx} &= -i\frac{\omega\mu_0(k_1^2 - \xi^2)}{k_1^2}\delta(z - a), \\ \left(\frac{\partial^2}{\partial z^2} + \beta_j^2\right)\widehat{B}_{jx} &= 0, \end{aligned}$$

where $z \neq 0$ and $\beta_j(\lambda)$ is defined in (18), with admissible solutions

$$\widehat{E}_{1x} = -\frac{\omega\mu_0(k_1^2 - \xi^2)}{2\beta_1 k_1^2}e^{i\beta_1|z-a|} + K_>e^{i\beta_1 z} \quad (z > 0), \quad \widehat{E}_{2x} = K_<e^{-i\beta_2 z} \quad (z < 0), \quad (19)$$

$$\widehat{B}_{1x} = C_>e^{i\beta_1 z} \quad (z > 0), \quad \widehat{B}_{2x} = C_<e^{-i\beta_2 z} \quad (z < 0), \quad (20)$$

consistent with radiation condition (5). Note that $C_> \neq C_<$ here because, in view of condition (4b), the tangential component of the **B** field is not continuous across $z = 0$, in contrast to the case with the earth-air interface which is examined in Ref. 26. The integration constants $C_<$, $C_>$, $K_<$, and $K_>$ are determined through boundary conditions (4). By transformed Maxwell equations (7), the remaining field components can be expressed in terms of \widehat{E}_{jx} and \widehat{B}_{jx} as

$$\widehat{B}_{jy} = -\frac{1}{k_j^2 - \xi^2}\left(i\frac{k_j^2}{\omega}\frac{\partial\widehat{E}_{jx}}{\partial z} + \eta\xi\widehat{B}_{jx}\right), \quad \widehat{B}_{jz} = \frac{1}{k_j^2 - \xi^2}\left(-\eta\frac{k_j^2}{\omega}\widehat{E}_{jx} + i\xi\frac{\partial\widehat{B}_{jx}}{\partial z}\right), \quad (21)$$

$$\widehat{E}_{jy} = \frac{i}{k_j^2 - \xi^2}\left(i\eta\xi\widehat{E}_{jx} + \omega\frac{\partial\widehat{B}_{jx}}{\partial z}\right), \quad \widehat{E}_{jz} = \frac{i}{k_j^2 - \xi^2}\left(\xi\frac{\partial\widehat{E}_{jx}}{\partial z} - i\eta\omega\widehat{B}_{jx}\right) \quad (z \neq 0, a). \quad (22)$$

We proceed to compute the fields. By (4a), we impose $\widehat{E}_{1x} = \widehat{E}_{2x}$ at $z = 0$; thus,

$$K_> = K_< + \frac{\omega\mu_0(k_1^2 - \xi^2)}{2\beta_1 k_1^2}.$$

We henceforth set $K = K_<$ for ease of notation. By (4b), we enforce $\widehat{B}_{1x} - \widehat{B}_{2x} = \mu_0\sigma\widehat{E}_{2y}$ at $z = 0$ which yields

$$C_> = -\frac{\mu_0\sigma\eta\xi}{k_2^2 - \xi^2}K + \left(1 + \frac{\omega\mu_0\sigma\beta_2}{k_2^2 - \xi^2}\right)C_<.$$

Now introduce $C = C_<$. Thus, it suffices to determine the integration constants K and C .

By (4), the continuity condition $\widehat{E}_{1y} = \widehat{E}_{2y}$ and the jump condition $\widehat{B}_{1y} - \widehat{B}_{2y} = -\mu_0\sigma\widehat{E}_{2x}$ at $z = 0$ lead to the system

$$\begin{aligned} \eta\xi\left[\frac{1}{k_1^2 - \xi^2}\left(1 - \frac{\omega\mu_0\sigma\beta_1}{k_2^2 - \xi^2}\right) - \frac{1}{k_2^2 - \xi^2}\right]K + \omega\left[\frac{\beta_1}{k_1^2 - \xi^2}\left(1 + \frac{\omega\mu_0\sigma\beta_2}{k_2^2 - \xi^2}\right) + \frac{\beta_2}{k_2^2 - \xi^2}\right]C &= 0, \\ -\left\{\frac{k_1^2\beta_1}{k_1^2 - \xi^2} + \frac{k_2^2\beta_2}{k_2^2 - \xi^2} + \omega\mu_0\sigma\left[1 + \frac{\eta^2\xi^2}{(k_1^2 - \xi^2)(k_2^2 - \xi^2)}\right]\right\}K \\ + \omega\eta\xi\left[\frac{1}{k_1^2 - \xi^2}\left(1 + \frac{\omega\mu_0\sigma\beta_2}{k_2^2 - \xi^2}\right) - \frac{1}{k_2^2 - \xi^2}\right]C &= \omega\mu_0e^{i\beta_1 a}. \end{aligned}$$

After some algebra, the determinant for this system is found to be

$$D = \frac{\mathcal{P} Q}{(k_1^2 - \xi^2)(k_2^2 - \xi^2)},$$

where \mathcal{P} is defined by (12), and

$$Q = \beta_1 + \beta_2 + \omega \mu_0 \sigma \tag{23}$$

corresponds to TE polarization.⁷ It follows that

$$C = \mu_0 \eta \xi \frac{k_2^2 - k_1^2 - \omega \mu_0 \sigma \beta_1}{\mathcal{P} Q} e^{i\beta_1 a},$$

$$K = -\omega \mu_0 \frac{\beta_1(k_2^2 - \xi^2) + \beta_2(k_1^2 - \xi^2) + \omega \mu_0 \sigma \beta_1 \beta_2}{\mathcal{P} Q} e^{i\beta_1 a}.$$

Accordingly, all field components are now obtained via (19)–(22) by use of Fourier integral (6). By representing the resulting two-dimensional integrals in cylindrical coordinates (ρ, ϕ, z) and using (13),²⁶ we obtain the following expressions: for $z > 0$ (region 1),

$$E_{1\rho}(\rho, \phi, z) = -\frac{\omega \mu_0}{4\pi} \cos \phi \left(\frac{1}{2} \int_0^\infty d\lambda \lambda \left\{ [J_0(\lambda \rho) - J_2(\lambda \rho)] \frac{\beta_1}{k_1^2} \mathcal{R}_m - [J_0(\lambda \rho) + J_2(\lambda \rho)] \frac{1}{\beta_1} \mathcal{R}_e \right\} e^{i\beta_1(z+a)} + \int_0^\infty d\lambda \lambda \left\{ J_0(\lambda \rho) - \frac{\lambda^2}{2k_1^2} [J_0(\lambda \rho) - J_2(\lambda \rho)] \right\} \frac{1}{\beta_1} e^{i\beta_1|z-a|} \right), \tag{24a}$$

$$E_{1\phi}(\rho, \phi, z) = \frac{\omega \mu_0}{8\pi k_1^2} \sin \phi \left(\int_0^\infty d\lambda \lambda \left\{ -[J_0(\lambda \rho) - J_2(\lambda \rho)] \frac{k_1^2}{\beta_1} \mathcal{R}_e + [J_0(\lambda \rho) + J_2(\lambda \rho)] \beta_1 \mathcal{R}_m \right\} e^{i\beta_1(z+a)} + \int_0^\infty d\lambda \lambda \left\{ k_1^2 J_0(\lambda \rho) - \frac{\lambda^2}{2} [J_0(\lambda \rho) + J_2(\lambda \rho)] \right\} \frac{2}{\beta_1} e^{i\beta_1|z-a|} \right), \tag{24b}$$

$$E_{1z}(\rho, \phi, z) = \frac{i\omega \mu_0}{4\pi k_1^2} \cos \phi \int_0^\infty d\lambda \lambda^2 J_1(\lambda \rho) [\text{sgn}(z - a) e^{i\beta_1|z-a|} + \mathcal{R}_m e^{i\beta_1(z+a)}], \tag{24c}$$

$$B_{1\rho}(\rho, \phi, z) = -\frac{\mu_0}{8\pi} \sin \phi \left(\int_0^\infty d\lambda \lambda \left\{ [J_0(\lambda \rho) + J_2(\lambda \rho)] \mathcal{R}_m - [J_0(\lambda \rho) - J_2(\lambda \rho)] \mathcal{R}_e \right\} e^{i\beta_1(z+a)} + 2 \text{sgn}(z - a) \int_0^\infty d\lambda \lambda J_0(\lambda \rho) e^{i\beta_1|z-a|} \right), \tag{25a}$$

$$B_{1\phi}(\rho, \phi, z) = -\frac{\mu_0}{8\pi} \cos \phi \left(\int_0^\infty d\lambda \lambda \left\{ [J_0(\lambda \rho) - J_2(\lambda \rho)] \mathcal{R}_m - [J_0(\lambda \rho) + J_2(\lambda \rho)] \mathcal{R}_e \right\} e^{i\beta_1(z+a)} + 2 \text{sgn}(z - a) \int_0^\infty d\lambda \lambda J_0(\lambda \rho) e^{i\beta_1|z-a|} \right), \tag{25b}$$

$$B_{1z}(\rho, \phi, z) = -\frac{i\mu_0}{4\pi} \sin \phi \int_0^\infty d\lambda \lambda^2 J_1(\lambda \rho) \frac{1}{\beta_1} [\mathcal{R}_e e^{i\beta_1(z+a)} - e^{i\beta_1|z-a|}], \tag{25c}$$

and for $z < 0$ (region 2),

$$E_{2\rho}(\rho, \phi, z) = -\frac{\omega\mu_0}{4\pi} \cos \phi \int_0^\infty d\lambda \lambda \left\{ [J_0(\lambda\rho) + J_2(\lambda\rho)] \frac{1}{\mathcal{Q}} + [J_0(\lambda\rho) - J_2(\lambda\rho)] \frac{\beta_1\beta_2}{\mathcal{P}} \right\} e^{i\beta_1 a - i\beta_2 z}, \quad (26a)$$

$$E_{2\phi}(\rho, \phi, z) = \frac{\omega\mu_0}{4\pi} \sin \phi \int_0^\infty d\lambda \lambda \left\{ [J_0(\lambda\rho) - J_2(\lambda\rho)] \frac{1}{\mathcal{Q}} + [J_0(\lambda\rho) + J_2(\lambda\rho)] \frac{\beta_1\beta_2}{\mathcal{P}} \right\} e^{i\beta_1 a - i\beta_2 z}, \quad (26b)$$

$$E_{2z}(\rho, \phi, z) = -\frac{i\omega\mu_0}{2\pi} \cos \phi \int_0^\infty d\lambda \lambda^2 J_1(\lambda\rho) \frac{\beta_1}{\mathcal{P}} e^{i\beta_1 a - i\beta_2 z}, \quad (26c)$$

$$B_{2\rho}(\rho, \phi, z) = \frac{\mu_0}{4\pi} \sin \phi \int_0^\infty d\lambda \lambda \left\{ [J_0(\lambda\rho) + J_2(\lambda\rho)] \frac{k_2^2 \beta_1}{\mathcal{P}} + [J_0(\lambda\rho) - J_2(\lambda\rho)] \frac{\beta_2}{\mathcal{Q}} \right\} e^{i\beta_1 a - i\beta_2 z}, \quad (27a)$$

$$B_{2\phi}(\rho, \phi, z) = \frac{\mu_0}{4\pi} \cos \phi \int_0^\infty d\lambda \lambda \left\{ [J_0(\lambda\rho) - J_2(\lambda\rho)] \frac{k_2^2 \beta_1}{\mathcal{P}} + [J_0(\lambda\rho) + J_2(\lambda\rho)] \frac{\beta_2}{\mathcal{Q}} \right\} e^{i\beta_1 a - i\beta_2 z}, \quad (27b)$$

$$B_{2z}(\rho, \phi, z) = \frac{i\mu_0}{2\pi} \sin \phi \int_0^\infty d\lambda \lambda^2 J_1(\lambda\rho) \frac{1}{\mathcal{Q}} e^{i\beta_1 a - i\beta_2 z}. \quad (27c)$$

In the above, the factor

$$\mathcal{R}_e = \frac{\beta_2 - \beta_1 + \omega\mu_0\sigma}{\mathcal{Q}} \quad (28)$$

is associated with the reflection of plane waves with TE polarization from a thin layer (the subscript “e” stands for “electric,” implying TE polarization);²⁶ also, recall (11), (12), and (23) for \mathcal{R}_m , \mathcal{P} , and \mathcal{Q} , respectively. Similarly to the vertical-dipole case (Section III A), the top Riemann sheet is defined by $\Im\beta_j(\lambda) > 0$, as in (18b). In the appropriate limit with $a \downarrow 0$ and $z \rightarrow 0$, integrals (24)–(27) appear divergent in the conventional sense yet should remain physically admissible descriptions. These integrals are meaningful as finite via the methodology of Section IV.³⁰

C. Poles in λ -Riemann surface

We now discuss the role of singularities in evaluating integrals (14)–(17) and (24)–(27). The singularities of the integrands include branch points at $\lambda = k_1, k_2$ because of the multiple-valued $\beta_j(\lambda)$ of (18a) and simple poles identified with zeros of $\mathcal{P}(\lambda)$ and $\mathcal{Q}(\lambda)$ by (12) and (23). The λ -Riemann surface associated with each integrand consists of four Riemann sheets if $k_1 \neq k_2$; the physically relevant (top) one is fixed by $\Im\beta_j(\lambda) > 0$. The location of the poles in this Riemann surface is closely related to the complex surface conductivity, σ .⁷

1. Denominator $\mathcal{P}(\lambda)$

By (12), the zeros of $\mathcal{P}(\lambda)$ satisfy

$$\frac{k_1^2}{(k_1^2 - \lambda^2)^{1/2}} + \frac{k_2^2}{(k_2^2 - \lambda^2)^{1/2}} + \omega\mu_0\sigma = 0. \quad (29)$$

Equation (29) is identified with the dispersion relation for the surface plasmon in the context of TM polarization for plane waves³ if λ is viewed as the component of the vector wave number tangential to the interface. For nearly lossless ambient media ($\Im k_j \simeq 0$), roots of (29) lie in the first Riemann sheet if

$$\Im\sigma > 0. \quad (30)$$

For a graphene layer, this is consistent with the Kubo formula in the far-infrared regime.^{6,7,32,33}

Consider the special case with $k_1 = k_2 =: k$ (Section IV) in which the λ -Riemann surface for the fields has two sheets. Then, $\mathcal{P}(\lambda) = (2k^2 + \omega\mu_0\sigma\sqrt{k^2 - \lambda^2})\sqrt{k^2 - \lambda^2}$ and (29) reduces to $\sqrt{k^2 - \lambda^2} = -2k^2/(\omega\mu_0\sigma)$, which is solved in the first Riemann sheet if $-\pi < \arg k^2 - \arg \sigma < 0$. Equation (29) has two solutions, $\lambda = \pm k_m$, where

$$k_m = \sqrt{k^2 - \frac{4k^4}{(\omega\mu_0\sigma)^2}}, \tag{31}$$

and the branch of the square root is chosen so that $\Im k_m > 0$. These $\pm k_m$ are simple poles of the corresponding integrands for the fields. In the “nonretarded frequency regime,”⁶ one imposes $|\omega\mu_0\sigma| \ll |k|$ by which $1/|\sigma| \gg \sqrt{\mu_0/|\epsilon|}$, where $\sqrt{\mu_0/|\epsilon|}$ is the magnitude of the intrinsic impedance of the adjacent homogeneous space; thus, we obtain $k_m \sim i2k^2/(\omega\mu_0\sigma)$. For $\arg \sigma = \pi/2 - \delta$, $0 < \delta < \pi$, the analytically continued square-root function yields $(k^2 - k_m^2)^{1/2} = ie^{i\delta}2k^2/(\omega\mu_0\sigma)$ and k_m lies in the first Riemann sheet for the appropriate range of δ and phase of k , as outlined above; this k_m reads

$$k_m \sim \frac{2k^2}{\omega\mu_0\sigma} e^{i\delta}, \quad |\omega\mu_0\sigma| \ll |k|. \tag{32}$$

If $\Im\sqrt{k^2 - k_m^2} < 0$, then k_m is in the second Riemann sheet.

We adhere to the following definition.

Definition 1 (TM surface plasmon). The **TM surface plasmon** corresponds to the residue contribution to the electromagnetic field from the pole $\lambda = k_m$, provided this pole lies in the first Riemann sheet.

We note in passing that in the more general setting with $k_1 \neq k_2$, one finds the simple poles $\pm k_m \sim \pm i(k_1^2 + k_2^2)/(\omega\mu_0\sigma)$ if $|(k_1^2 + k_2^2)/(\omega\mu_0\sigma)| \gg \max\{|k_1|, |k_2|\}$.

2. Denominator $Q(\lambda)$

By (23), the zeros of $Q(\lambda)$ obey

$$(k_1^2 - \lambda^2)^{1/2} + (k_2^2 - \lambda^2)^{1/2} + \omega\mu_0\sigma = 0, \tag{33}$$

which has two roots, $\pm k_e$, in the λ -Riemann surface, where^{7,8}

$$k_e = \frac{1}{2} \sqrt{-\frac{(k_1^2 - k_2^2)^2}{(\omega\mu_0\sigma)^2} + 2(k_1^2 + k_2^2) - (\omega\mu_0\sigma)^2}.$$

For definiteness, the branch of the square root is chosen so that $k_e \rightarrow k$ if one sets $k_1 = k_2 = k$ and then lets $\omega\mu_0\sigma \rightarrow 0$. The $\pm k_e$ are simple poles of the corresponding integrands for the fields. By (33) and for nearly lossless ambient media ($\Im k_j \simeq 0$), $\pm k_e$ lie in the top Riemann sheet if

$$\Im \sigma < 0, \tag{34}$$

in contrast to (30). Thus, for the appealing case with nonzero $\Im \sigma$ and an ambient lossless dielectric,^{6,7} only one of k_m and k_e lies in the top Riemann sheet.⁷ If $|\omega\mu_0\sigma|^2 \ll |k_1^2 + k_2^2|$, then $\pm k_e \sim \pm(i/2)(k_1^2 - k_2^2)/(\omega\mu_0\sigma)$ for $k_1 \neq k_2$; hence, $|k_e|$ may take a wide range of values.

In the special case with $k_1 = k_2 = k$ (Section IV), (33) reduces to $\sqrt{k^2 - \lambda^2} = -\omega\mu_0\sigma/2$, which is solved in the first Riemann sheet if $-\pi < \arg \sigma < 0$. We find

$$k_e = \sqrt{k^2 - \frac{(\omega\mu_0\sigma)^2}{4}}. \tag{35}$$

If $|\omega\mu_0\sigma| \ll |k|$ then $k_e \sim k - (\omega\mu_0\sigma)^2/(8k) \sim k$. By setting $\arg \sigma = \pi/2 - \delta$, we find

$$k_e - k \sim \frac{(\omega\mu_0\sigma)^2}{8k} e^{-i2\delta}, \quad |\omega\mu_0\sigma| \ll |k| \quad (0 < \delta < \pi). \tag{36}$$

The above discussion leads to the following remark.

Remark 1 (On the co-existence of poles k_m and k_e in top Riemann sheet). Consider the case with $k_1 = k_2 = k$. We henceforth assume that $0 \leq \arg k^2 < \pi/2$ and $\Re \sigma > 0$ with $\arg \sigma = \pi/2 - \delta$ ($0 < \delta < \pi$). Thus, at most one of the poles at k_m and k_e is present in the top λ -Riemann sheet.

Specifically, if $\Im \sigma > 0$ with $0 < \delta < \pi/2 - \arg k^2$, then k_m lies in the top Riemann sheet, and if $\Im \sigma < 0$ ($\pi/2 < \delta < \pi$), then k_e lies in the top Riemann sheet.

IV. EXACT EVALUATION OF FIELD ON PLANE OF FILM

In this section, we analytically evaluate the integrals for the electromagnetic field (Sections III A and III B) on one side of the layer by setting $k_1 = k_2 = k$ and allowing the dipole and observation point (x, y, z) to approach $z = 0$.

A. Formalism

For the vertical dipole (Fig. 1), let $a \downarrow 0$ with $0 < z < a$, so that (ρ, ϕ, z) appropriately approaches the film from region 1. Equations (14) and (15) reduce to the following expressions:

$$E_{1\rho}^{(vd)} = -\frac{i\omega\mu_0}{2\pi} \frac{1}{(\omega\mu_0\sigma)^2} \left[\frac{2k^2}{\rho} I_1 - \frac{\omega\mu_0\sigma}{\rho^2} I_2 + 2k^2 k_m^2 \rho I_3(k_m\rho) - \omega\mu_0\sigma k_m^2 I_4(k_m\rho) \right], \quad (37)$$

$$B_{1\phi}^{(vd)} = \frac{i\mu_0}{2\pi} \left\{ \frac{k^2}{\omega\mu_0\sigma} \frac{1}{\rho} I_1 - \frac{1}{\rho^2} I_2 + \left[k^2 - \frac{2k^4}{(\omega\mu_0\sigma)^2} \right] \check{I}_2 + \frac{k^2 k_m^2 \rho}{\omega\mu_0\sigma} I_3(k_m\rho) - \frac{2k^4 k_m^2 \rho^2}{(\omega\mu_0\sigma)^2} \check{I}_4(k_m\rho) \right\}, \quad (38)$$

$$E_{1z}^{(vd)} = \frac{i\omega}{k^2} \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho B_{1\phi}^{(vd)}) \\ = -\frac{\omega\mu_0}{2\pi k^2} \frac{1}{\rho} \frac{\partial}{\partial \rho} \left\{ \frac{k^2}{\omega\mu_0\sigma} \frac{1}{\rho} I_1 - \frac{1}{\rho^2} I_2 + \left[k^2 - \frac{2k^4}{(\omega\mu_0\sigma)^2} \right] \check{I}_2 + \frac{k^2 k_m^2 \rho}{\omega\mu_0\sigma} I_3(k_m\rho) - \frac{2k^4 k_m^2 \rho^2}{(\omega\mu_0\sigma)^2} \check{I}_4(k_m\rho) \right\}, \quad (39)$$

where the requisite integrals read

$$I_1 = \int_0^\infty d\lambda' J_1(\lambda') = -\int_0^\infty d\lambda' \frac{d}{d\lambda'} J_0(\lambda') = J_0(0) = 1, \quad (40a)$$

$$I_2 = \int_0^\infty d\lambda' \sqrt{(k\rho)^2 - \lambda'^2} J_1(\lambda'), \quad \check{I}_2 = \int_0^\infty d\lambda' \frac{J_1(\lambda')}{\sqrt{(k\rho)^2 - \lambda'^2}}, \quad (40b)$$

$$I_3(\varpi) = \int_0^\infty d\lambda' \frac{J_1(\lambda')}{\lambda'^2 - \varpi^2}, \quad (40c)$$

$$I_4(\varpi) = \int_0^\infty d\lambda' \frac{\sqrt{(k\rho)^2 - \lambda'^2}}{\lambda'^2 - \varpi^2} J_1(\lambda'), \quad \check{I}_4(\varpi) = \int_0^\infty d\lambda' \frac{1}{\sqrt{(k\rho)^2 - \lambda'^2}} \frac{J_1(\lambda')}{\lambda'^2 - \varpi^2}, \quad (40d)$$

with $\check{I}_4(\varpi) = -[\varpi^2 - (k\rho)^2]^{-1} [\check{I}_2 + I_4(\varpi)]$. Thus, we only need to compute integrals $I_2, \check{I}_2, I_3,$ and I_4 .

For a horizontal dipole (Fig. 2), we initially choose to evaluate the field on the side of the layer that faces the source-free region. The fields on the other side can be found through the imposed boundary conditions (Section II and Appendix A). Let $a \downarrow 0$ and $z \uparrow 0$ so that the observation point approaches the plane $z = 0$ from region 2. By (26) and (27), the ensuing representations for the field components at $z = 0$ in region 2 read

$$E_{2z}^{(\text{hd})} = -\frac{i\omega\mu_0}{2\pi} \frac{1}{(\omega\mu_0\sigma)^2} \cos\phi \left[\frac{2k^2}{\rho} I_1 - \frac{\omega\mu_0\sigma}{\rho^2} I_2 + 2k^2 k_m^2 \rho I_3(k_m\rho) - \omega\mu_0\sigma k_m^2 I_4(k_m\rho) \right], \quad (41)$$

$$E_{2\rho}^{(\text{hd})} = -\frac{\omega\mu_0}{4\pi} \cos\phi \left\{ \frac{1}{2\rho} [\omega\mu_0\sigma\rho I_3(k_e\rho) - 2I_4(k_e\rho)] + \frac{2}{(\omega\mu_0\sigma)^2} \frac{d}{d\rho} \left[\frac{\omega\mu_0\sigma}{\rho} I_1 - \frac{4k^4\rho}{\omega\mu_0\sigma} I_3(k_m\rho) + 2k^2 I_4(k_m\rho) \right] \right\}, \quad (42)$$

$$E_{2\phi}^{(\text{hd})} = \frac{\omega\mu_0}{4\pi} \sin\phi \left\{ \frac{1}{2} \frac{d}{d\rho} [\omega\mu_0\sigma\rho I_3(k_e\rho) - 2I_4(k_e\rho)] + \frac{2}{(\omega\mu_0\sigma)^2} \frac{1}{\rho} \left[\frac{\omega\mu_0\sigma}{\rho} I_1 - \frac{4k^4\rho}{\omega\mu_0\sigma} I_3(k_m\rho) + 2k^2 I_4(k_m\rho) \right] \right\}, \quad (43)$$

$$B_{2z}^{(\text{hd})} = \frac{i\mu_0}{8\pi} \sin\phi \left[\frac{\omega\mu_0\sigma}{\rho} I_1 - \frac{2}{\rho^2} I_2 + \omega\mu_0\sigma k_e^2 \rho I_3(k_e\rho) - 2k_e^2 I_4(k_e\rho) \right], \quad (44)$$

$$B_{2\rho}^{(\text{hd})} = \frac{\mu_0}{4\pi} \sin\phi \left\{ \frac{2k^2}{(\omega\mu_0\sigma)^2} \frac{1}{\rho} [2k^2\rho I_3(k_m\rho) - \omega\mu_0\sigma I_4(k_m\rho)] + \frac{1}{2} \frac{d}{d\rho} \left[\frac{2}{\rho} I_1 - \frac{(\omega\mu_0\sigma)^2}{2} \rho I_3(k_e\rho) + \omega\mu_0\sigma I_4(k_e\rho) \right] \right\}, \quad (45)$$

$$B_{2\phi}^{(\text{hd})} = \frac{\mu_0}{4\pi} \cos\phi \left\{ \frac{2k^2}{(\omega\mu_0\sigma)^2} \frac{d}{d\rho} [2k^2\rho I_3(k_m\rho) - \omega\mu_0\sigma I_4(k_m\rho)] + \frac{1}{2\rho} \left[\frac{2}{\rho} I_1 - \frac{(\omega\mu_0\sigma)^2}{2} \rho I_3(k_e\rho) + \omega\mu_0\sigma I_4(k_e\rho) \right] \right\}. \quad (46)$$

The integrals $I_3(\varpi)$, $I_4(\varpi)$, and $\check{I}_4(\varpi)$ are evaluated in Section IV B. The reader who wishes to skip technical derivations may proceed directly to Section IV C.

The field of the vertical dipole bears TM polarization, as indicated by (37)–(39) with (40). On the other hand, the field of the horizontal dipole carries the signatures of TM and TE polarizations.

B. Key integrals and generalized Schwinger-Feynman representation

Next, we provide derivations for requisite integrals (40b)–(40d) in terms of known transcendental functions. This section forms a crucial part of our formal analysis. We follow the formalism of Ref. 30, elements of which we repeat here for the sake of completeness. This procedure should clarify the role of certain singularities related to material properties of the film.

For later algebraic convenience, set $k\rho =: iq$ and $\varpi =: i\beta$ recalling that $\varpi = k_m\rho$ or $k_e\rho$, and treat q and β as positive for the purpose of computing the integrals. The extension of the requisite integrals to the actual, physically relevant values of these parameters will be carried out once the integrals are evaluated. Accordingly, set $\sqrt{(k\rho)^2 - \lambda'^2} = i\sqrt{\lambda'^2 + q^2}$ and treat $\sqrt{\lambda'^2 + q^2}$ as positive if λ' is real. First, we directly find that^{30,34}

$$I_2 = i(q + e^{-q}) = k\rho + ie^{ik\rho}, \quad \check{I}_2 = -iq^{-1}(1 - e^{-q}) = \frac{1 - e^{ik\rho}}{k\rho}. \quad (47)$$

1. Integral $I_3(\varpi)$

By (40c), we resort to writing $J_1(\lambda) = (1/2)[H_1^{(1)}(\lambda) + H_1^{(2)}(\lambda)]$, where $H_\nu^{(1,2)}$ is the (first or second) Hankel function of order ν .³¹ By the identity $H_1^{(2)}(\lambda e^{-i\pi}) = H_1^{(1)}(\lambda)$, valid for $\lambda \neq 0$, (40c) becomes

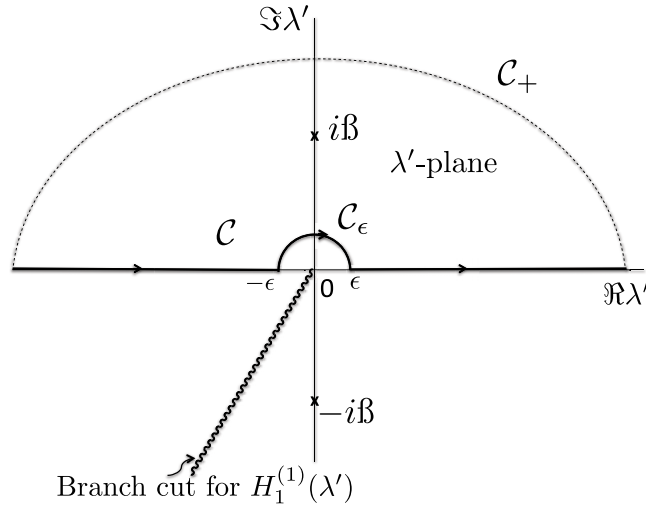


FIG. 3. Branch-cut configuration and contour in the λ' -plane for the integral $I_3(\varpi)$, see (48). The integrand has simple poles at $\pm i\beta$ ($\beta = -i\varpi$ is treated as positive). Path $C = (-\infty, -\epsilon) \cup C_\epsilon \cup (\epsilon, \infty)$ (solid line with arrows) is identified with the real axis except for a small indentation, C_ϵ , of radius ϵ around 0 in the upper half-plane, $\epsilon \downarrow 0$. Semicircle C_+ (dashed line) in the upper half-plane closes the path; the contour $C \cup C_+$ picks up the residue at $\lambda = i\beta$.

$$\begin{aligned}
 I_3(\varpi) &= \frac{1}{2} \int_C d\lambda' \frac{H_1^{(1)}(\lambda')}{\lambda'^2 + \beta^2} - \frac{1}{2} \lim_{\epsilon \downarrow 0} \int_{C_\epsilon} d\lambda' \frac{H_1^{(1)}(\lambda')}{\lambda'^2 + \beta^2} = \frac{\pi}{2\beta} H_1^{(1)}(i\beta) + \beta^{-2} \\
 &= \frac{i\pi}{2\varpi} H_1^{(1)}(\varpi) - \frac{1}{\varpi^2}, \tag{48}
 \end{aligned}$$

by contour integration, where the contours C and C_ϵ are depicted in Fig. 3; cf. Ref. 35. The integrals in the first line of (48) provide the Cauchy principal value of $\frac{1}{2} \int_{-\infty}^{\infty} d\lambda' H_1^{(1)}(\lambda') (\lambda'^2 + \beta^2)^{-1}$ which accounts for the singularity of $H_1^{(1)}(\lambda')$ at $\lambda' = 0$.

2. Integrals $I_4(\varpi)$ and $\check{I}_4(\varpi)$

We seek exact formulas for (40d) with $\varpi = k_m \rho$ and $\varpi = k_e \rho$. Such formulas should be particularly useful for $|\omega \mu_0 \sigma| \ll |k|$, by which $|k_e| \sim |k| \ll |k_m|$, cf. (31) and (35). Thus, we view as *small* the parameters: $|k/k_m|$, with $\varpi = k_m \rho$; and $|(k_e - k)/k_e|$, with $\varpi = k_e \rho$. Our derivation is tailored to the case where k_m lies in the first Riemann sheet (see Remark 1); the extension to other values of the physical parameters is discussed in the end of this section.

The first step is to convert $I_4(\varpi)$ in (40d) to an integral of an elementary function. In the spirit of the Schwinger-Feynman approach, consider the generalized representation³⁰

$$\Re^{-\nu} \mathfrak{C}^{-1} = \frac{1}{\Gamma(\nu)} \int_0^\infty \int_0^\infty ds_1 ds_2 s_1^{\nu-1} e^{-(\Re s_1 + \mathfrak{C} s_2)} = \nu \int_0^1 du u^{-1+\nu} [(\Re - \mathfrak{C})u + \mathfrak{C}]^{-1-\nu}, \quad \Re \nu > 0,$$

via the change of variable $(s_1, s_2) \mapsto (u, v)$ with $(s_1, s_2) = (uv, (1-u)v)$ and integration in v ; $\Gamma(z)$ is the Gamma function. We assume that $0 < \mathfrak{C} < \Re$, a condition to be relaxed later; here, $\Re = \lambda'^2 + q^2$, $\mathfrak{C} = \lambda'^2 + \beta^2$. By using the above representation for $\Re^{-\nu} \mathfrak{C}^{-1}$, we obtain

$$\begin{aligned}
 \int_0^\infty d\lambda' \frac{(\lambda'^2 + q^2)^{-\nu}}{\lambda'^2 + \beta^2} J_1(\lambda') &= \frac{\nu}{1 - e^{i2\pi\nu}} \oint_{C_0} du u^{-1+\nu} [q^2 u + \beta^2(1-u)]^{-(1+\nu)} \left\{ 1 - \frac{2^{-\nu}}{\Gamma(1+\nu)} \right. \\
 &\quad \left. \times [q^2 u + \beta^2(1-u)]^{\frac{1+\nu}{2}} K_{1+\nu}((q^2 u + \beta^2(1-u))^{1/2}) \right\}. \tag{49}
 \end{aligned}$$

The contour C_0 serves the analytic continuation of the integral on the left-hand side to values of ν with $\Re \nu < 0$ (see Fig. 4). To derive (49), we interchanged the order of integration (in λ' and u) and made use of a result from Ref. 34 to carry out the integration in λ' ; K_ν is the modified Bessel function of order ν , with $K_{1/2}(\zeta) = \sqrt{\pi/(2\zeta)} e^{-\zeta}$.³¹ By the change of variable $u \mapsto \zeta$ with $\zeta^2 =$

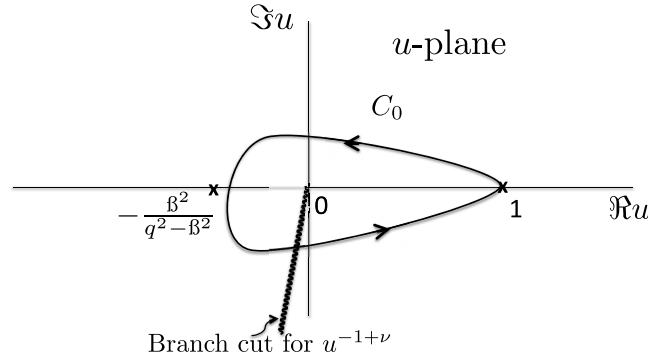


FIG. 4. Contour C_0 of integration for generalized integral on right-hand side of (49) pertaining to $I_4(\varpi)$; $0 < \beta^2 < q^2$. The integrand has branch points at $u = 0, -\beta^2/(q^2 - \beta^2)$; only the cut emanating from 0 is shown here, indicating the passage of the integrand from the first Riemann sheet for $u^{-1+\nu}$, where $u^{-1+\nu} > 0$ if $u > 0$, to an adjacent Riemann sheet.

$q^2u + \beta^2(1 - u)$, integration by parts via the identity $\frac{d}{d\zeta}[\zeta^{-2\nu}(\zeta^2 - \beta^2)^\nu] = 2\beta^2\nu\zeta^{-1-2\nu}(\zeta^2 - \beta^2)^{\nu-1}$, and the subsequent substitution $\nu = -1/2$, we find

$$\begin{aligned} I_4(\varpi) &= i \int_0^\infty d\lambda' \frac{\sqrt{\lambda'^2 + q^2}}{\lambda'^2 + \beta^2} J_1(\lambda') \\ &= \frac{iq}{\beta^2} (1 - e^{-q}) - i \frac{\sqrt{q^2 - \beta^2}}{\beta^2} \int_B^q d\zeta \frac{\zeta}{\sqrt{\zeta^2 - \beta^2}} e^{-\zeta} \\ &= -\frac{k\rho}{\varpi^2} (1 - e^{ik\rho}) + i \frac{\sqrt{\varpi^2 - (k\rho)^2}}{\varpi} \int_{k\rho/\varpi}^1 d\eta \frac{\eta}{\sqrt{1 - \eta^2}} e^{i\varpi\eta}, \end{aligned} \tag{50}$$

which provides $I_4(\varpi)$ in terms of an integral of an elementary function.

We proceed to simplify (50). For this purpose, define

$$W(\ell, \varsigma; \varpi) := \int_\ell^\varsigma d\eta \frac{\eta}{\sqrt{1 - \eta^2}} e^{i\varpi\eta}. \tag{51}$$

We seek appropriate expansions for this W if $\varsigma = 1$ with (i) $\ell = k/k_m$ and $\varpi = k_m\rho$, where $|\ell| < 1$, and (ii) $\ell = k/k_e$ and $\varpi = k_e\rho$, where $|1 - \ell| < 1$. $|\varpi|$ should be unrestricted in each case.

(i) Case $\ell = k/k_m, \varpi = k_m\rho$. By writing

$$W(k/k_m, 1; k_m\rho) = W(0, 1; k_m\rho) - W(0, k/k_m; k_m\rho) \tag{52}$$

and resorting to Ref. 31, we obtain the formula

$$W(0, 1; k_m\rho) = 1 - \frac{\pi}{2} [\mathbb{H}_1(k_m\rho) - Y_1(k_m\rho)] + \frac{i\pi}{2} H_1^{(1)}(k_m\rho), \tag{53}$$

where \mathbb{H}_ν (Y_ν) is the Struve (Neumann) function of order ν . In (53), the last term corresponds to the contribution to $I_4(k_m\rho)$ from the residue at $\lambda' = k_m\rho$. For the remaining terms, note the large- $|\zeta|$ expansion³¹

$$\mathbb{H}_\nu(\zeta) - Y_\nu(\zeta) = \pi^{-1} \sum_{n=0}^{M-1} \frac{\Gamma(\frac{1}{2} + n)}{\Gamma(\nu + \frac{1}{2} - n)} \left(\frac{2}{\zeta}\right)^{2n-\nu+1} + \mathcal{O}(|\zeta|^{-2M+\nu-1}), \quad |\arg \zeta| < \pi. \tag{54}$$

Now consider the term $W(0, k/k_m; k_m\rho)$ in (52) by inspection of (51). By expanding $\eta(1 - \eta^2)^{-1/2}$ at $\eta = 0$ with $|\eta| < 1$ and integrating term by term, we derive the geometrically convergent series

$$W(0, k/k_m; k_m\rho) = \sum_{l=0}^\infty \left(\frac{1}{2}\right)_l \frac{(-1)^{l+1}}{l!} \left(\frac{k^2}{k_m^2}\right)^{l+1} \frac{d^{2l+1}}{d\zeta^{2l+1}} \left(\frac{1 - e^{i\zeta}}{\zeta}\right) \Big|_{\zeta=k\rho}, \quad \left|\frac{k^2}{k_m^2}\right| < 1, \tag{55}$$

where $(b)_l$ is Pochhammer's symbol; $(b)_l = \Gamma(l + b)/\Gamma(b)$. Formula (55) concludes the evaluation of $W(k/k_m, 1; k_m\rho)$ by (52).

(ii) Case $\ell = k/k_e, \varpi = k_e\rho$. By changing integration variable from η to $v = 1 - \eta$ and expanding $\eta(1 - \eta^2)^{-1/2} = (1 - v)v^{-1/2}(2 - v)^{-1/2}$ at $v = 0$, we recast (51) to

$$W(k/k_e, 1; k_e\rho) = \frac{e^{ik\rho}}{\sqrt{2k_e\rho}} \sum_{l=0}^{\infty} \left(\frac{1}{2}\right)_l \frac{1 + 2l(2k_e\rho)^{-l}}{1 - 2l} \frac{1}{l!} f_l((k_e - k)\rho),$$

where

$$f_l(z) = \int_0^z dv v^{-1/2+l} e^{i(z-v)}.$$

This $f_l(z)$ has generating function

$$\mathfrak{G}(\zeta; z) := \sum_{l=0}^{\infty} \frac{f_l(z)}{l!} (-i\zeta)^l = \sqrt{2\pi} e^{iz} (1 + \zeta)^{-1/2} \mathcal{F}_0((1 + \zeta)z),$$

where \mathcal{F}_0 is given in terms of the Fresnel integrals $C(z)$ and $S(z)$,³⁶ viz.,

$$\begin{aligned} \mathcal{F}_0(z) &= \int_0^z d\tau \frac{e^{-i\tau}}{\sqrt{2\pi\tau}} = C(z) - iS(z) \\ &= \frac{1}{\sqrt{2}} e^{-i\pi/4} - e^{-iz} \mathcal{F}(z), \quad \mathcal{F}(z) := e^{iz} \int_z^{\infty} d\tau \frac{e^{-i\tau}}{\sqrt{2\pi\tau}}. \end{aligned} \tag{56a}$$

In particular, note the asymptotic expansion

$$\mathcal{F}(z) = -\frac{i}{\sqrt{2\pi z}} \left[1 + \frac{i}{2z} + O(z^{-2}) \right] \quad \text{as } |z| \rightarrow \infty. \tag{56b}$$

Thus, by use of the generating function \mathfrak{G} , we write

$$f_l(z) = \sqrt{2\pi} i^l z^{l+1/2} e^{iz} \frac{d^l}{dz^l} [z^{-1/2} \mathcal{F}_0(z)]$$

and derive the desired expansion for W ,

$$W(k/k_e, 1; k_e\rho) = \sqrt{\pi} e^{ik\rho} \sum_{l=0}^{\infty} \left(\frac{1}{2}\right)_l \frac{1 + 2l}{1 - 2l} \frac{(i/2)^l}{l!} \left(\frac{k_e - k}{k_e}\right)^{l+1/2} e^{i\varphi} \frac{d^l}{dz^l} [z^{-1/2} \mathcal{F}_0(z)] \Big|_{z=\varphi}. \tag{57}$$

This series exhibits geometric rate of convergence for $|(k_e - k)/k_e| < 1$ and all φ , where

$$\varphi := (k_e - k)\rho \sim e^{-i2\delta} \frac{(\omega\mu_0|\sigma|)^2\rho}{8k}, \quad |\omega\mu_0\sigma| \ll |k|, \tag{58}$$

with $\arg \sigma = \pi/2 - \delta$ ($0 < \delta < \pi$). This φ is analogous to Sommerfeld’s “numerical distance,”³⁷ cf. (36).

Thus, by (50), integral $I_4(\varpi)$ reads

$$I_4(\varpi) = -\frac{k\rho}{\varpi^2} (1 - e^{ik\rho}) + i \frac{\sqrt{\varpi^2 - (k\rho)^2}}{\varpi} W(k\rho/\varpi, 1; \varpi), \tag{59}$$

where W is provided by (52) with (53) and (55) for $\varpi = k_m\rho$; and by (57) for $\varpi = k_e\rho$. By (40d), $\check{I}_4(\varpi)$ equals

$$\check{I}_4(\varpi) = \frac{-1}{\varpi^2 - (k\rho)^2} \left[\left(\frac{1}{k\rho} - \frac{k\rho}{\varpi^2}\right) (1 - e^{ik\rho}) + i \frac{\sqrt{\varpi^2 - (k\rho)^2}}{\varpi} W(k\rho/\varpi, 1; \varpi) \right]. \tag{60}$$

In the above, $\Re\sqrt{\varpi^2 - (k\rho)^2} = \Im\sqrt{(k\rho)^2 - \varpi^2} > 0$, in accord with the definition of the top λ -Riemann sheet, and k_m is assumed to lie in this Riemann sheet; see (32).

We now discuss formulas (59) and (60) for a wider range of parameters.

Remark 2 (On the values of I_4 and \check{I}_4). Equations (59) and (60) can be extended to all physically relevant values of k_m (and k_e) with $\Re\sigma > 0$ (Remark 1), by computing $(1/\varpi)\sqrt{\varpi^2 - (k\rho)^2} W(k\rho/\varpi, 1; \varpi)$ according to the following rules. These rules are consistent with the starting formulation of the Sommerfeld-type integrals (Section III).

- For $\varpi = k_m \rho$: In regard to the residue contribution in $W(0, 1; k_m \rho)$, set

$$\frac{\sqrt{(k_m \rho)^2 - (k \rho)^2}}{k_m \rho} H_1^{(1)}(k_m \rho) = \frac{\sqrt{k_m^2 - k^2}}{k_m} H_1^{(1)}(k_m \rho) = \pm \frac{i 2k^2}{\omega \mu_0 \sigma k_m} H_1^{(1)}(k_m \rho), \quad (61a)$$

where $\Im k_m > 0$ and the upper (lower) sign holds if the pole at k_m lies in the first (second) Riemann sheet; see (32). For the remaining terms in W , set $\sqrt{(k_m \rho)^2 - (k \rho)^2} / (k_m \rho) = \sqrt{1 - k^2/k_m^2}$ and choose the branch so that $\lim_{k_m \rightarrow \infty} \sqrt{1 - k^2/k_m^2} = 1$ for fixed k ; thus,

$$\alpha_m := \sqrt{1 - k^2/k_m^2} = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)_n \frac{(k^2/k_m^2)^n}{n!}, \quad \text{if } |k^2/k_m^2| < 1. \quad (61b)$$

- For $\varpi = k_e \rho$: Set

$$\sqrt{(k_e \rho)^2 - (k \rho)^2} = -i \sqrt{(k \rho)^2 - (k_e \rho)^2} = \pm i \omega \mu_0 \sigma \rho / 2, \quad (61c)$$

the upper (lower) sign holds if the pole at k_e lies in the first (second) λ -Riemann sheet.

The sign in (61a) controls the appearance or absence of the TM surface plasmon (Definition 1). If the pole at $\lambda = k_m$ does not lie on the top Riemann sheet, e.g., when $k > 0$ and $\Im \sigma < 0$, contribution (61a) causes cancellation of the residue from $I_3(k_m \rho)$ in each relevant field component; then, the TM surface plasmon is absent. An analogous effect comes from (61c) in regard to TE polarization. Equation (61b) warrants that the primary field (in the absence of the layer) of the dipole is recovered as $\sigma \rightarrow 0$ regardless of the location of the pole at k_m . See Section IV C for consequences of this solution.

C. Field components: Evaluation

The field components are computed by substitution of the formulas for integrals I_n ($n = 1, 2, 3, 4$) and \check{I}_n ($n = 1, 2$) from Section IV B into (37)–(39) and (41)–(46), bearing in mind Remarks 1 and 2. To render the resulting expressions physically transparent, we subsequently apply the condition $|\omega \mu_0 \sigma| \ll |k|$ which warrants that $|k/k_m| \ll 1$ and $|(k_e - k)/k_e| \ll 1$; the requisite series expansions can be approximated by their first few terms for all distances ρ from the source. In Appendix B, we provide alternate evaluations of the field components for sufficiently small and large distance, ρ , from the source.

Remark 3 (On the pole at k_m). We assume that the parameter σ is chosen so that the pole at $\lambda = k_m$ lies in the top λ -Riemann sheet (see Remark 1), unless we state otherwise.

1. Vertical dipole

The ρ -component of the electric field deserves special attention, since it vanishes at $z = 0$ in the absence of the layer.²⁶ By using this component, we illustrate the role of the location of the pole at k_m in the Riemann surface associated with the dual (λ) variable. By (37), we write

$$\begin{aligned} E_{1\rho}^{(\text{vd})} = & \frac{i}{2\pi\sigma} \left\{ \left(\frac{i}{\rho^2} + \frac{k}{\rho} \right) e^{ik\rho} - \frac{i\pi}{2} k_m \left(\frac{2k^2}{\omega \mu_0 \sigma} - i \sqrt{k_m^2 - k^2} \right) H_1^{(1)}(k_m \rho) \right. \\ & + ik_m^2 \alpha_m \left[1 - \frac{\pi}{2} (\mathbb{H}_1(k_m \rho) - Y_1(k_m \rho)) \right. \\ & \left. \left. + \sum_{l=0}^{\infty} \left(\frac{1}{2} \right)_l \frac{(-1)^{l+1}}{l!} \left(\frac{k^2}{k_m^2} \right)^{l+1} \frac{d^{2l+1}}{d\zeta^{2l+1}} \left(\frac{1 - e^{i\zeta}}{\zeta} \right) \Big|_{\zeta=k\rho} \right] \right\}. \quad (62a) \end{aligned}$$

Notice the term proportional to $[2k^2/(\omega \mu_0 \sigma) - i \sqrt{k_m^2 - k^2}] H_1^{(1)}(k_m \rho)$, which expresses the TM surface plasmon; see Remark 2. By (61a), in this term $\sqrt{k_m^2 - k^2} = \pm i 2k^2/(\omega \mu_0 \sigma)$, depending on the location of the associated pole. On the other hand, in regard to the second line of (62a), $k_m^2 \alpha_m \sim k_m^2 [1 - k^2/(2k_m^2)] \sim k_m^2$ if $|k_m|^2 \gg |k|^2$. Suppose that the pole at k_m is present in the top

λ -Riemann sheet, if, say, $\Im\sigma > 0$ and $k > 0$ as outlined in Section III C. By retaining *two terms* in the series expansion of (62a) and enforcing $|\omega\mu_0\sigma| \ll |k|$, we obtain

$$E_{1\rho}^{(\text{vd})} \sim \frac{\omega\mu_0}{2\pi} \left\{ \frac{\omega\mu_0\sigma}{4} \left(\frac{3}{k^4\rho^4} - \frac{3i}{k^3\rho^3} - \frac{1}{k^2\rho^2} \right) e^{ik\rho} + i\pi \frac{4k^4}{(\omega\mu_0\sigma)^3} H_1^{(1)}(k_m\rho) + \frac{4k^4}{(\omega\mu_0\sigma)^3} \right. \\ \left. \times \left(1 + \frac{(\omega\mu_0\sigma)^2}{8k^2} \right) \left[1 - \frac{\pi}{2} (\mathbb{H}_1(k_m\rho) - Y_1(k_m\rho)) + \frac{1}{(k_m\rho)^2} - \frac{3}{(k_m\rho)^4} \right] \right\}. \quad (62b)$$

The TM surface plasmon contribution has *survived* here as the term proportional to $H_1^{(1)}(k_m\rho)$; see also Appendix B. In contrast, if k_m lies in the second Riemann sheet, if, for example, $\Im\sigma < 0$ and $k > 0$, the above residue contribution disappears (see Remark 2), viz.,

$$E_{1\rho}^{(\text{vd})} \sim \frac{\omega\mu_0}{2\pi} \left\{ \frac{\omega\mu_0\sigma}{4} \left(\frac{3}{k^4\rho^4} - \frac{3i}{k^3\rho^3} - \frac{1}{k^2\rho^2} \right) e^{ik\rho} + \frac{4k^4}{(\omega\mu_0\sigma)^3} \left(1 + \frac{(\omega\mu_0\sigma)^2}{8k^2} \right) \right. \\ \left. \times \left[1 - \frac{\pi}{2} (\mathbb{H}_1(k_m\rho) - Y_1(k_m\rho)) + \frac{1}{(k_m\rho)^2} - \frac{3}{(k_m\rho)^4} \right] \right\}, \quad |\omega\mu_0\sigma| \ll |k|. \quad (62c)$$

A similar calculation can be carried out for the other field components of the vertical dipole. We henceforth emphasize main consequences of Remark 3.

Next, we turn our attention to the magnetic field. By (38) and Remark 3, we compute

$$B_{1\phi}^{(\text{vd})} = -\frac{i\mu_0}{4\pi} \left\{ 2 \left(\frac{i}{\rho^2} + \frac{k}{\rho} \right) e^{ik\rho} - i\pi \frac{k^2 k_m}{\omega\mu_0\sigma} \left(1 + \frac{2ik^2}{\omega\mu_0\sigma} \frac{1}{\sqrt{k_m^2 - k^2}} \right) H_1^{(1)}(k_m\rho) \right. \\ \left. - \frac{4ik^4}{(\omega\mu_0\sigma)^2} \frac{1}{\alpha_m} \left[1 - \frac{\pi}{2} (\mathbb{H}_1(k_m\rho) - Y_1(k_m\rho)) \right. \right. \\ \left. \left. + \sum_{l=0}^{\infty} \left(\frac{1}{2} \right)_l \frac{(-1)^{l+1}}{l!} \left(\frac{k^2}{k_m^2} \right)^{l+1} \frac{d^{2l+1}}{d\zeta^{2l+1}} \left(\frac{1 - e^{i\zeta}}{\zeta} \right) \Big|_{\zeta=k\rho} \right] \right\} \\ \sim -\frac{i\mu_0}{4\pi} \left\{ \left(\frac{i}{\rho^2} + \frac{k}{\rho} \right) e^{ik\rho} + 4\pi \frac{k^4}{(\omega\mu_0\sigma)^2} H_1^{(1)}(k_m\rho) - i \frac{4k^4}{(\omega\mu_0\sigma)^2} \left[1 - \frac{\pi}{2} (\mathbb{H}_1(k_m\rho) \right. \right. \\ \left. \left. - Y_1(k_m\rho)) + \frac{1}{k_m^2\rho^2} \right] \right\}, \quad |\omega\mu_0\sigma| \ll |k|. \quad (63)$$

The terms with the $e^{ik\rho}$ factor in the fourth line (but not on the first line) of (63) correspond to the primary field of a z -directed dipole in *free space*,²⁶ the remaining terms in the second equation amount to the scattered field due to the presence of the thin layer for $|\omega\mu_0\sigma| \ll |k|$. Notice the surviving residue contribution expressed by the term containing $H_1^{(1)}(k_m\rho)$; this is identified with the TM surface plasmon via Definition 1. In Appendix B, we provide alternate calculations for $B_{1\phi}^{(\text{vd})}$ in the disparate cases with $|k\rho| \gg 1$ and $|k_m\rho| \ll 1$.

The z -component of the electric field is likewise computed by (39) under Remark 3,

$$E_{1z}^{(\text{vd})} = -\frac{\omega\mu_0}{4\pi k^2} \left\{ 2 \left(\frac{i}{\rho^3} + \frac{k}{\rho^2} - \frac{ik^2}{\rho} \right) e^{ik\rho} + i\pi \frac{k^2 k_m^2}{\omega\mu_0\sigma} \left(1 + i \frac{2k^2}{\omega\mu_0\sigma} \frac{1}{\sqrt{k_m^2 - k^2}} \right) H_0^{(1)}(k_m\rho) \right. \\ \left. + i \frac{4k^4}{(\omega\mu_0\sigma)^2} \frac{1}{\alpha_m} \frac{1}{\rho} \left[1 - \frac{\pi}{2} k_m\rho (\mathbb{H}_0(k_m\rho) - Y_0(k_m\rho)) \right. \right. \\ \left. \left. + \sum_{l=0}^{\infty} \left(\frac{1}{2} \right)_l \frac{(-1)^{l+1}}{l!} \left(\frac{k^2}{k_m^2} \right)^{l+1} \frac{d}{d\zeta} \left\{ \zeta \frac{d^{2l+1}}{d\zeta^{2l+1}} \left(\frac{1 - e^{i\zeta}}{\zeta} \right) \right\} \Big|_{\zeta=k\rho} \right] \right\} \\ \sim -\frac{\omega\mu_0}{4\pi} \left\{ \left(\frac{i}{k^2\rho^3} + \frac{1}{k\rho^2} - \frac{i}{\rho} \right) e^{ik\rho} - 8\pi i \frac{k^4}{(\omega\mu_0\sigma)^3} H_0^{(1)}(k_m\rho) + i \frac{4k^2}{(\omega\mu_0\sigma)^2} \frac{1}{\rho} \left[1 - \frac{\pi}{2} k_m\rho \right. \right. \\ \left. \left. \times (\mathbb{H}_0(k_m\rho) - Y_0(k_m\rho)) - \frac{1}{k_m^2\rho^2} \right] \right\}, \quad |\omega\mu_0\sigma| \ll |k|. \quad (64)$$

The terms proportional to $e^{ik\rho}$ in the fourth line of (64) amount to the primary field of the dipole.²⁶ The TM surface plasmon contribution is expressed by the term proportional to $H_0^{(1)}(k_m\rho)$.

If $|\omega\mu_0\sigma| \ll |k|$ and $|k_m\rho| = O(1)$, the primary field is subdominant to this contribution provided $|k\rho| \gg |(\omega\mu_0\sigma)/k|$ (in addition to having $|k\rho| \ll 1$). See [Appendix B](#) for asymptotic evaluations of $E_{1z}^{(vd)}$ in the far- and near-field regimes.

2. Horizontal dipole

Next, by (41)–(46), we compute the field of the horizontal dipole. We address the z -components first, since each of these fields involves only one type of polarization, as indicated by the presence of only one denominator (\mathcal{P} or \mathcal{Q}) in their Fourier representations for $k_1 \neq k_2$.

By (41), $E_{2z}^{(hd)}$ involves only TM polarization and is computed as

$$E_{2z}^{(hd)} = -\frac{i\omega\mu_0}{2\pi} \frac{1}{(\omega\mu_0\sigma)^2} \cos\phi \left\{ -i\omega\mu_0\sigma \left(\frac{1}{\rho^2} - \frac{ik}{\rho} \right) e^{ik\rho} + \frac{i\pi k_m}{2} \left(2k^2 - i\omega\mu_0\sigma\sqrt{k_m^2 - k^2} \right) \right. \\ \times H_1^{(1)}(k_m\rho) - i\omega\mu_0\sigma k_m^2 \alpha_m \left[1 - \frac{\pi}{2} (\mathbb{H}_1(k_m\rho) - Y_1(k_m\rho)) \right. \\ \left. \left. + \sum_{l=0}^{\infty} \left(\frac{1}{2} \right)_l \frac{(-1)^{l+1}}{l!} \left(\frac{k^2}{k_m^2} \right)^{l+1} \frac{d^{2l+1}}{d\zeta^{2l+1}} \left(\frac{1 - e^{i\zeta}}{\zeta} \right) \Big|_{\zeta=k\rho} \right] \right\}. \tag{65a}$$

Notice the term $(2k^2 - i\omega\mu_0\sigma\sqrt{k_m^2 - k^2})H_1^{(1)}(k_m\rho)$, in conjunction with Remark 2. By Remark 3 and $|\omega\mu_0\sigma| \ll |k|$ (i.e., $|k_m|^2 \gg |k|^2$), we keep two terms in the series expansion and readily compute (see also [Appendix B](#))

$$E_{2z}^{(hd)} \sim -\frac{i\omega\mu_0}{2\pi} \cos\phi \left\{ -\frac{i\omega\mu_0\sigma}{4} \left(\frac{1}{k^2\rho^2} + \frac{3i}{k^3\rho^3} - \frac{3}{k^4\rho^4} \right) e^{ik\rho} - 4\pi \frac{k^4}{(\omega\mu_0\sigma)^3} H_1^{(1)}(k_m\rho) \right. \\ \left. + \frac{2k^2 k_m}{(\omega\mu_0\sigma)^2} \left[1 - \frac{\pi}{2} (\mathbb{H}_1(k_m\rho) - Y_1(k_m\rho)) + \frac{1}{k_m^2\rho^2} - \frac{3}{k_m^4\rho^4} \right] \right\}. \tag{65b}$$

In the last formula, the TM surface plasmon is evident as the term proportional to $H_1^{(1)}(k_m\rho)$. In contrast, if k_m lies in the second Riemann sheet (see Remark 2), then

$$E_{2z}^{(hd)} \sim -\frac{i\omega\mu_0}{2\pi} \cos\phi \left\{ -\frac{i\omega\mu_0\sigma}{4} \left(\frac{1}{k^2\rho^2} + \frac{3i}{k^3\rho^3} - \frac{3}{k^4\rho^4} \right) e^{ik\rho} \right. \\ \left. + \frac{2k^2 k_m}{(\omega\mu_0\sigma)^2} \left[1 - \frac{\pi}{2} (\mathbb{H}_1(k_m\rho) - Y_1(k_m\rho)) + \frac{1}{k_m^2\rho^2} - \frac{3}{k_m^4\rho^4} \right] \right\}. \tag{65c}$$

We next compute the z -component of the magnetic field by resorting to Remark 2. Equation (44) yields

$$B_{2z}^{(hd)} = \frac{i\mu_0}{8\pi} \sin\phi \left\{ -2i \left(\frac{1}{\rho^2} - \frac{ik}{\rho} \right) e^{ik\rho} + \frac{i\pi}{2} \omega\mu_0\sigma k_e H_1^{(1)}(k_e\rho) - 2ik_e \sqrt{k_e^2 - k^2} \right. \\ \left. \times \sqrt{\pi} e^{ik\rho} \sum_{l=0}^{\infty} \left(\frac{1}{2} \right)_l \frac{1 + 2l}{1 - 2l} \frac{(i/2)^l}{l!} \left(\frac{k_e - k}{k_e} \right)^{l+1/2} e^{i\varphi} \frac{d^l}{dz^l} [z^{-1/2} \mathcal{F}_0(z)] \Big|_{z=\varphi} \right\}, \tag{66a}$$

which involves TE polarization; \mathcal{F}_0 and φ are defined by (56a) and (58). The term proportional to $e^{ik\rho}$ in the first line of (66a) corresponds to the field of the x -directed dipole in free space. Beware of the factor $\sqrt{k_e^2 - k^2}$ in (66a), see also Remark 2 with $\varpi = k_e\rho$. To simplify formula (66a), consider $|\omega\mu_0\sigma| \ll |k|$, by which $k_e \sim k$. Thus, (66a) is reduced to

$$B_{2z}^{(hd)} \sim \frac{i\mu_0}{8\pi} \sin\phi \left\{ -2i \left(\frac{1}{\rho^2} - \frac{ik}{\rho} \right) e^{ik\rho} + \frac{i\pi}{2} \omega\mu_0\sigma k \left[H_1^{(1)}(k_e\rho) - 2 \frac{\sqrt{k_e^2 - k^2}}{\omega\mu_0\sigma} \sqrt{\frac{2}{\pi k_e\rho}} e^{i(k_e\rho - \pi/4)} \right] \right. \\ \left. + 2ik \sqrt{k_e^2 - k^2} e^{ik\rho} \sqrt{\frac{\pi}{k\rho}} \mathcal{F}(\varphi) \right\}, \tag{66b}$$

where \mathcal{F} is defined in (56a). If we adhere to Remark 3, we expect that (66b) should not manifest the contribution, $H_1^{(1)}(k_e\rho)$, of the residue at the pole $\lambda = k_e$.²⁶ To demonstrate its

elimination for large enough distances in this component, we use the asymptotic expansion³¹ $H_v^{(1)}(z) \sim \sqrt{2/(\pi z)} e^{i(z-\nu\pi/2-\pi/4)}$, which provides a reasonable approximation for $H_v^{(1)}(k_e\rho)$ for all but small values of $|k_e\rho|$. Recall that $\sqrt{k_e^2 - k^2} = -i\omega\mu_0\sigma/2$ if k_e does not lie on the top Riemann sheet of $\sqrt{k^2 - \lambda^2}$. Thus, for sufficiently large $|k_e\rho| \sim |k\rho|$, (66b) becomes

$$B_{2z}^{(hd)} = \frac{i\mu_0}{8\pi} \sin \phi \left\{ -2i \left(\frac{1}{\rho^2} - \frac{ik}{\rho} \right) e^{ik\rho} + \omega\mu_0\sigma k \sqrt{\frac{\pi}{k\rho}} e^{ik\rho} \mathcal{F}(\varphi) \right\}, \tag{66c}$$

which manifests the $e^{ik\rho}$ -wave behavior in view of (56b); see Appendix B. In contrast, by the hypothetical scenario when k_e lies in the physical (top) λ -Riemann sheet, (66b) yields

$$B_{2z}^{(hd)} = \frac{i\mu_0}{8\pi} \sin \phi \left\{ -2i \left(\frac{1}{\rho^2} - \frac{ik}{\rho} \right) e^{ik\rho} + \omega\mu_0\sigma k \sqrt{\frac{2\pi}{k\rho}} e^{i(k_e\rho - \pi/4)} - \omega\mu_0\sigma k \sqrt{\frac{\pi}{k\rho}} e^{ik\rho} \mathcal{F}(\varphi) \right\}, \tag{66d}$$

in view of Remark 2.

In regard to the ρ -component of the electric field, formula (42) entails

$$\begin{aligned} E_{2\rho}^{(hd)} = & -\frac{\omega\mu_0}{4\pi} \cos \phi \left\{ \left[-i \left(1 - \frac{k^2}{k_m^2} \right) \frac{1}{\rho} + \left(1 - \frac{k^2}{k_e^2} - \frac{k^2}{k_m^2} \right) \frac{1}{k\rho^2} \right] e^{ik\rho} \right. \\ & + i\pi \frac{\omega\mu_0\sigma}{4k_e\rho} H_1^{(1)}(k_e\rho) - i \frac{\sqrt{k_e^2 - k^2}}{k_e\rho} \sqrt{\pi} e^{ik\rho} \\ & \times \sum_{l=0}^{\infty} \left(\frac{1}{2} \right)_l \frac{1 + 2l}{1 - 2l} \frac{(i/2)^l}{l!} \left(\frac{k_e - k}{k_e} \right)^{l+1/2} e^{i\varphi} \frac{d^l}{dz^l} \left[z^{-1/2} \mathcal{F}_0(z) \right] \Big|_{z=\varphi} \\ & - 2\pi i \frac{k^2}{(\omega\mu_0\sigma)^2} \left[\frac{2k^2}{\omega\mu_0\sigma} - i\sqrt{k_m^2 - k^2} \right] H_1^{(1)\prime}(k_m\rho) + i \frac{4k^2 k_m}{(\omega\mu_0\sigma)^2} \alpha_m \left[-\frac{\pi}{2} (\mathbb{H}'_1(k_m\rho) \right. \\ & \left. - Y_1'(k_m\rho)) + \frac{k}{k_m} \sum_{l=0}^{\infty} \left(\frac{1}{2} \right)_l \frac{(-1)^{l+1}}{l!} \left(\frac{k^2}{k_m^2} \right)^{l+1} \frac{d^{2l+2}}{d\zeta^{2l+2}} \left(\frac{1 - e^{i\zeta}}{\zeta} \right) \Big|_{\zeta=k\rho} \right] \Big\}, \tag{67a} \end{aligned}$$

where the prime denotes derivative with respect to the argument and $\mathbb{H}'_1(z) = \frac{1}{2} [\mathbb{H}_0(z) - \mathbb{H}_2(z)] + \Gamma(5/2)^{-1} (4\sqrt{\pi})^{-1} z$. By imposing $|\omega\mu_0\sigma| \ll |k|$ under Remark 3, we keep one term from each of the above series expansion and find

$$\begin{aligned} E_{2\rho}^{(hd)} \sim & -\frac{\omega\mu_0}{4\pi} \cos \phi \left\{ -2 \left(\frac{1}{k\rho^2} + \frac{i}{k^2\rho^3} \right) e^{ik\rho} \right. \\ & + i\pi \frac{\omega\mu_0\sigma}{4k\rho} \left[H_1^{(1)}(k_e\rho) - \sqrt{\frac{2}{\pi k_e\rho}} e^{i(k_e\rho - 3\pi/4)} \right] + \frac{\omega\mu_0\sigma}{2k\rho} \sqrt{\frac{\pi}{k\rho}} e^{ik\rho} \mathcal{F}(\varphi) \\ & \left. - i\pi \frac{8k^4}{(\omega\mu_0\sigma)^3} H_1^{(1)\prime}(k_m\rho) + \frac{4k^4}{(\omega\mu_0\sigma)^3} \left[\pi (\mathbb{H}'_1(k_m\rho) - Y_1'(k_m\rho)) + \frac{4}{(k_m\rho)^3} \right] \right\}. \tag{67b} \end{aligned}$$

The primary field of the dipole is singled out in the first line of formula (67b), while the remaining terms of this formula express the scattered wave; see Appendix B. In the scattered wave, there are two noteworthy contributions. The TM surface plasmon contribution is expressed by the term proportional to $H_1^{(1)\prime}(k_m\rho)$. The other contribution, $H_1^{(1)}(k_e\rho)$, is cancelled out for large enough $|k_e\rho|$ in a fashion similar to that shown for $B_{2z}^{(hd)}$ above.

By (43), we likewise obtain the exact formula

$$\begin{aligned}
 E_{2\phi}^{(\text{hd})} &= \frac{\omega\mu_0}{4\pi} \sin\phi \left\{ \left[\frac{k}{k_e^2} \left(1 + \frac{4k^2k_e^2}{k_m^2(\omega\mu_0\sigma)^2} \right) \frac{1}{\rho^2} - \frac{ik^2}{k_e^2} \frac{1}{\rho} \right] e^{ik\rho} \right. \\
 &\quad + \frac{i\pi}{4} \omega\mu_0\sigma H_1^{(1)\prime}(k_e\rho) + \sqrt{k_e^2 - k^2} \sqrt{\pi} e^{ik\rho} \\
 &\quad \times \sum_{l=0}^{\infty} \left(\frac{1}{2} \right)_l \frac{3 + 4l^2}{(1-2l)(3-2l)} \frac{(i/2)^l}{l!} \left(\frac{k_e - k}{k_e} \right)^{l+1/2} e^{i\varphi} \frac{d^l}{dz^l} \left[z^{-1/2} \mathcal{F}_0(z) \right] \Big|_{z=\varphi} \\
 &\quad - \frac{2\pi ik^2}{(\omega\mu_0\sigma)^2} \left(\frac{2k^2}{\omega\mu_0\sigma} - i\sqrt{k_m^2 - k^2} \right) \frac{H_1^{(1)}(k_m\rho)}{k_m\rho} + \frac{4ik^2}{(\omega\mu_0\sigma)^2} \frac{\alpha_m}{\rho} \left[1 - \frac{\pi}{2} (\mathbb{H}_1(k_m\rho)) \right. \\
 &\quad \left. - Y_1(k_m\rho) \right] + \sum_{l=0}^{\infty} \left(\frac{1}{2} \right)_l \frac{(-1)^{l+1}}{l!} \left(\frac{k^2}{k_m^2} \right)^{l+1} \frac{d^{2l+1}}{d\zeta^{2l+1}} \left(\frac{1 - e^{i\zeta}}{\zeta} \right) \Big|_{\zeta=k\rho} \left. \right\}. \tag{68a}
 \end{aligned}$$

By Remark 3 and $|\omega\mu_0\sigma| \ll |k|$, this formula is simplified to

$$\begin{aligned}
 E_{2\phi}^{(\text{hd})} &\sim \frac{\omega\mu_0}{4\pi} \sin\phi \left\{ \left(-\frac{i}{\rho} + \frac{1}{k\rho^2} + \frac{i}{k^2\rho^3} \right) e^{ik\rho} \right. \\
 &\quad + \frac{i\pi}{4} \omega\mu_0\sigma \left[H_1^{(1)\prime}(k_e\rho) - i\sqrt{\frac{2}{\pi k_e\rho}} e^{i(k_e\rho - 3\pi/4)} \right] + \frac{i\omega\mu_0\sigma}{2} e^{ik\rho} \sqrt{\frac{\pi}{k\rho}} \mathcal{F}(\varphi) \\
 &\quad - i\pi \frac{8k^4}{(\omega\mu_0\sigma)^3} \frac{1}{k_m\rho} H_1^{(1)}(k_m\rho) \\
 &\quad \left. - \frac{8k^4}{(\omega\mu_0\sigma)^3} \frac{1}{k_m\rho} \left[1 - \frac{\pi}{2} (\mathbb{H}_1(k_m\rho) - Y_1(k_m\rho)) + \frac{1}{(k_m\rho)^2} \right] \right\}. \tag{68b}
 \end{aligned}$$

In the first line of approximate formula (68b), we display the radiation field in the absence of the layer. The additional terms of this formula account for the scattered field. See Appendix B for alternate computations based on asymptotics.

By (45), the ρ -component of the magnetic field reads

$$\begin{aligned}
 B_{2\rho}^{(\text{hd})} &= \frac{\mu_0}{4\pi} \sin\phi \left\{ \left[-\left(\frac{\omega\mu_0\sigma}{2} \frac{k}{k_e^2} + \frac{2k^3}{\omega\mu_0\sigma} \frac{1}{k_m^2} \right) \frac{1}{\rho^2} + \frac{i\omega\mu_0\sigma}{2} \frac{k^2}{k_e^2} \frac{1}{\rho} \right] e^{ik\rho} \right. \\
 &\quad + i\pi \frac{2k^4}{(\omega\mu_0\sigma)^2} \frac{1}{k_m\rho} \left[1 - \frac{i\omega\mu_0\sigma}{2k^2} \sqrt{k_m^2 - k^2} \right] H_1^{(1)}(k_m\rho) - \frac{2ik^2}{\omega\mu_0\sigma} \frac{\alpha_m}{\rho} \\
 &\quad \times \left[1 - \frac{\pi}{2} (\mathbb{H}_1(k_m\rho) - Y_1(k_m\rho)) + \sum_{l=0}^{\infty} \left(\frac{1}{2} \right)_l \frac{(-1)^{l+1}}{l!} \left(\frac{k^2}{k_m^2} \right)^{l+1} \frac{d^{2l+1}}{d\zeta^{2l+1}} \left(\frac{1 - e^{i\zeta}}{\zeta} \right) \Big|_{\zeta=k\rho} \right] \\
 &\quad - \frac{i\pi}{8} (\omega\mu_0\sigma)^2 H_1^{(1)\prime}(k_e\rho) - \frac{\omega\mu_0\sigma}{2} \sqrt{k_e^2 - k^2} \sqrt{\pi} e^{ik\rho} \\
 &\quad \times \sum_{l=0}^{\infty} \left(\frac{1}{2} \right)_l \frac{3 + 4l^2}{(1-2l)(3-2l)} \frac{(i/2)^l}{l!} \left(\frac{k_e - k}{k_e} \right)^{l+1/2} e^{i\varphi} \frac{d^l}{dz^l} \left[z^{-1/2} \mathcal{F}_0(z) \right] \Big|_{z=\varphi} \left. \right\}. \tag{69a}
 \end{aligned}$$

By Remark 3 and condition $|\omega\mu_0\sigma| \ll |k|$, the last expression becomes

$$\begin{aligned}
 B_{2\rho}^{(\text{hd})} &\sim \frac{\mu_0}{4\pi} \sin\phi \left\{ \frac{\omega\mu_0\sigma}{2} \left(\frac{i}{\rho} - \frac{1}{k\rho^2} - \frac{i}{k^2\rho^3} \right) e^{ik\rho} \right. \\
 &\quad + i\pi \frac{4k^4}{(\omega\mu_0\sigma)^2} \frac{1}{k_m\rho} H_1^{(1)}(k_m\rho) - i \frac{2k^2}{\omega\mu_0\sigma} \frac{1}{\rho} \left[1 - \frac{\pi}{2} (\mathbb{H}_1(k_m\rho) - Y_1(k_m\rho)) + \frac{1}{(k_m\rho)^2} \right] \\
 &\quad \left. - \frac{i\pi}{8} (\omega\mu_0\sigma)^2 \left[H_1^{(1)\prime}(k_e\rho) - \sqrt{\frac{2}{\pi k_e\rho}} e^{i(k_e\rho - \pi/4)} \right] - \frac{i(\omega\mu_0\sigma)^2}{4} \sqrt{\frac{\pi}{k\rho}} e^{ik\rho} \mathcal{F}(\varphi) \right\}; \tag{69b}
 \end{aligned}$$

see Appendix B for alternate derivations of the far and near field.

We conclude this section with the computation of the ϕ -component of the magnetic field. By (46), we obtain

$$\begin{aligned}
 B_{2\phi}^{(\text{hd})} = & \frac{\mu_0}{4\pi} \cos \phi \left\{ \left[\left(\frac{\omega\mu_0\sigma}{2} \frac{k}{k_e^2} + \frac{2k^3}{\omega\mu_0\sigma} \frac{1}{k_m^2} \right) \frac{1}{\rho^2} - i \frac{2k^4}{\omega\mu_0\sigma} \frac{1}{k_m^2 \rho} \right] e^{ik\rho} \right. \\
 & + i\pi \frac{k^2}{\omega\mu_0\sigma} \left(\frac{2k^2}{\omega\mu_0\sigma} - i\sqrt{k_m^2 - k^2} \right) H_1^{(1)'}(k_m\rho) - i \frac{2k^2 k_m}{\omega\mu_0\sigma} \alpha_m \left[-\frac{\pi}{2} (\mathbb{H}'_1(k_m\rho)) \right. \\
 & \left. \left. - Y_1'(k_m\rho) \right) + \frac{k}{k_m} \sum_{l=0}^{\infty} \left(\frac{1}{2} \right)_l \frac{(-1)^{l+1}}{l!} \left(\frac{k^2}{k_m^2} \right)^{l+1} \frac{d^{2l+2}}{d\zeta^{2l+2}} \left(\frac{1 - e^{i\zeta}}{\zeta} \right) \Big|_{\zeta=k\rho} \right] \\
 & - \frac{i\pi}{8} \frac{(\omega\mu_0\sigma)^2}{k_e\rho} H_1^{(1)}(k_e\rho) + \frac{i\omega\mu_0\sigma}{2} \frac{\sqrt{k_e^2 - k^2}}{k_e\rho} \\
 & \left. \times \sqrt{\pi} e^{ik\rho} \sum_{l=0}^{\infty} \left(\frac{1}{2} \right)_l \frac{1 + 2l}{1 - 2l} \frac{(i/2)^l}{l!} \left(\frac{k_e - k}{k_e} \right)^{l+1/2} e^{i\varphi} \frac{d^l}{dz^l} [z^{-1/2} \mathcal{F}_0(z)] \Big|_{z=\varphi} \right\}. \quad (70a)
 \end{aligned}$$

Suppose that $|\omega\mu_0\sigma| \ll |k|$. By Remark 3, the preceding formula is reduced to

$$\begin{aligned}
 B_{2\phi}^{(\text{hd})} \sim & \frac{\mu_0}{4\pi} \cos \phi \left\{ \omega\mu_0\sigma \left(\frac{1}{k\rho^2} + \frac{i}{k^2\rho^3} \right) e^{ik\rho} \right. \\
 & + i\pi \frac{4k^4}{(\omega\mu_0\sigma)^2} H_1^{(1)'}(k_m\rho) - \frac{4k^4}{(\omega\mu_0\sigma)^2} \left[\frac{\pi}{2} (\mathbb{H}'_1(k_m\rho) - Y_1'(k_m\rho)) + \frac{2}{(k_m\rho)^3} \right] \\
 & \left. - \frac{i\pi}{8} \frac{(\omega\mu_0\sigma)^2}{k_e\rho} \left[H_1^{(1)}(k_e\rho) - \sqrt{\frac{2}{\pi k_e\rho}} e^{i(k\rho - 3\pi/4)} \right] - \frac{(\omega\mu_0\sigma)^2}{4} \frac{1}{k\rho} \sqrt{\frac{\pi}{k\rho}} e^{ik\rho} \mathcal{F}(\varphi) \right\}. \quad (70b)
 \end{aligned}$$

In Appendix B, we compute this field component by asymptotic methods, in the near- ($|k_m\rho| \ll 1$) and far-field ($|k\rho| \gg 1$) regimes for $|\omega\mu_0\sigma| \ll |k|$; the results are in agreement with (70b).

V. DISCUSSION

In this section, we discuss the derived formulas of Section IV under Remark 3 in the case with $|k_m\rho| \gg 1$ and $k\rho = O(1)$, by assuming that $|\omega\mu_0\sigma| \ll |k|$. An issue that we address is the relative strength of the TM surface plasmon contribution to the electromagnetic field. A practical consideration that motivates this study is that layers of two-dimensional materials such as graphene may sustain a TM surface plasmon wave number, k_m , with $0 < \Im k_m \ll \Re k_m$ and $\Re k_m \sim |k_m| \gg |k|$ at a certain accessible frequency range;¹⁷ thus, the wavelength of the generated surface plasmon can be much smaller than the wavelength of the incident electromagnetic field in free space. It is of interest to investigate how the TM surface plasmon can prevail over other field contributions on the layer at distances of the order of the free-space wavelength.

A. Field of vertical dipole, $|k_m\rho| \gg 1$ and $k\rho = O(1)$

By (62b), the ρ -component of the electric field reduces to

$$E_{1\rho}^{(\text{vd})} \sim \frac{\omega\mu_0}{2\pi} \left\{ \frac{\omega\mu_0\sigma}{4} \left(\frac{3}{k^4\rho^4} - \frac{3i}{k^3\rho^3} - \frac{1}{k^2\rho^2} \right) e^{ik\rho} + \frac{4k^4}{(\omega\mu_0\sigma)^3} \sqrt{\frac{2\pi}{k_m\rho}} e^{i(k_m\rho - \pi/4)} \right\}, \quad (71)$$

if $|k_m\rho| \gg 1$, by use of asymptotic expansions (54) and $H_\nu^{(1)}(z) \sim \sqrt{2/(\pi z)} e^{i(z - \nu\pi/2 - \pi/4)}$ as $z \rightarrow \infty$ with $\nu = 1$. Hence, the TM surface plasmon dominates in this regime if

$$\left| \frac{\omega\mu_0\sigma}{k} \right| \ll e^{-(2/7)\Im k_m/|k|}. \quad (72)$$

In the same vein, by (63), we compute the ϕ -component of the magnetic field,

$$B_{1\phi}^{(\text{vd})} \sim -\frac{i\mu_0 k^2}{4\pi} \left\{ \left(\frac{i}{k^2\rho^2} + \frac{1}{k\rho} \right) e^{ik\rho} - \frac{8k^2}{(\omega\mu_0\sigma)^2} \sqrt{\frac{\pi}{2k_m\rho}} e^{i(k_m\rho + \pi/4)} \right\}. \quad (73)$$

Evidently, the TM surface plasmon contribution dominates in this component if

$$\left| \frac{\omega \mu_0 \sigma}{k} \right| \ll e^{-(2/3)\Im k_m/|k|}. \tag{74}$$

Equation (64) for the z -component of the electric field yields

$$E_{1z}^{(vd)} \sim -\frac{\omega \mu_0 k}{4\pi} \left\{ \left(\frac{i}{k^3 \rho^3} + \frac{1}{k^2 \rho^2} - \frac{i}{k \rho} \right) e^{ik\rho} - \frac{8k^3}{(\omega \mu_0 \sigma)^3} \sqrt{\frac{2\pi}{k_m \rho}} e^{i(k_m \rho + \pi/4)} \right\}, \tag{75}$$

which implies that the TM surface plasmon is dominant if

$$\left| \frac{\omega \mu_0 \sigma}{k} \right| \ll e^{-(2/5)\Im k_m/|k|}. \tag{76}$$

Thus, for $k\rho = O(1)$, the TM surface plasmon dominates in all field components of the vertical dipole if (74) holds.

B. Field of horizontal dipole, $|k_m \rho| \gg 1$ and $k\rho = O(1)$

First, we address the z -components. By (65b), the z -component of the electric field reads

$$E_{2z}^{(hd)} \sim \frac{\omega \mu_0}{2\pi} \cos \phi \left\{ -\frac{\omega \mu_0 \sigma}{4} \left(\frac{1}{k^2 \rho^2} + \frac{3i}{k^3 \rho^3} - \frac{3}{k^4 \rho^4} \right) e^{ik\rho} + \frac{4k^4}{(\omega \mu_0 \sigma)^3} \sqrt{\frac{2\pi}{k_m \rho}} e^{i(k_m \rho - \pi/4)} \right\}. \tag{77}$$

For $k\rho = O(1)$, the pole contribution dominates under condition (72). In contrast, by (66) the z -component of the magnetic field does not carry any TM surface plasmon contribution; thus, we omit its evaluation from this discussion.

The remaining components are more richly structured. By (67b), we have

$$E_{2\rho}^{(hd)} \sim -\frac{\omega \mu_0 k}{4\pi} \cos \phi \left\{ -2 \left(\frac{1}{k^2 \rho^2} + \frac{i}{k^3 \rho^3} \right) e^{ik\rho} - \frac{8k^3}{(\omega \mu_0 \sigma)^3} \sqrt{\frac{2\pi}{k_m \rho}} e^{i(k_m \rho + \pi/4)} \right. \\ \left. + \frac{i\pi}{4} \frac{\omega \mu_0 \sigma}{k} \frac{1}{k\rho} \left[H_1^{(1)}(k_e \rho) - \sqrt{\frac{2}{\pi k_e \rho}} e^{i(k_e \rho - 3\pi/4)} \right] + \frac{\omega \mu_0 \sigma}{2k} \frac{1}{k\rho} \sqrt{\frac{\pi}{k\rho}} e^{ik\rho} \mathcal{F}(\varphi) \right\}. \tag{78}$$

If $k\rho = O(1)$, all terms in the second line are relatively small. The residue contribution of the pole at k_m dominates over the primary field if (76) holds.

Equation (68b) for the ϕ -component of the electric field gives

$$E_{2\phi}^{(hd)} \sim \frac{\omega \mu_0 k}{4\pi} \sin \phi \left\{ \left(-\frac{i}{k\rho} + \frac{1}{k^2 \rho^2} + \frac{i}{k^3 \rho^3} \right) e^{ik\rho} - \frac{8k^3}{(\omega \mu_0 \sigma)^3} \frac{1}{k_m \rho} \sqrt{\frac{2\pi}{k_m \rho}} e^{i(k_m \rho - \pi/4)} \right. \\ \left. + \frac{i\pi}{4} \frac{\omega \mu_0 \sigma}{k} \left[H_1^{(1)'}(k_e \rho) - i \sqrt{\frac{2}{\pi k_e \rho}} e^{i(k_e \rho - 3\pi/4)} \right] + \frac{i\omega \mu_0 \sigma}{2k} e^{ik\rho} \sqrt{\frac{\pi}{k\rho}} \mathcal{F}(\varphi) \right\}, \tag{79}$$

which is dominated by the TM surface plasmon at distances with $k\rho = O(1)$ provided (74) holds.

We now focus on the components of the magnetic field. By (69b), we compute

$$B_{2\rho}^{(hd)} \sim \frac{\mu_0 k^2}{4\pi} \sin \phi \left\{ \frac{\omega \mu_0 \sigma}{2k} \left(\frac{i}{k\rho} - \frac{1}{k^2 \rho^2} - \frac{i}{k^3 \rho^3} \right) e^{ik\rho} + \frac{4k^2}{(\omega \mu_0 \sigma)^2} \frac{1}{k_m \rho} \sqrt{\frac{2\pi}{k_m \rho}} e^{i(k_m \rho - \pi/4)} \right. \\ \left. - \frac{i\pi}{8} \frac{(\omega \mu_0 \sigma)^2}{k^2} \left[H_1^{(1)'}(k_e \rho) - \sqrt{\frac{2}{\pi k_e \rho}} e^{i(k_e \rho - \pi/4)} \right] - i \frac{(\omega \mu_0 \sigma)^2}{4k^2} \sqrt{\frac{\pi}{k\rho}} e^{ik\rho} \mathcal{F}(\varphi) \right\}. \tag{80}$$

All terms of the second line are negligible; in contrast, the TM surface plasmon can dominate over the $e^{ik\rho}$ -wave of the first line under (74). Similarly, formula (70b) is reduced to

$$B_{2\phi}^{(\text{hd})} \sim \frac{\mu_0 k^2}{4\pi} \cos \phi \left\{ \frac{\omega \mu_0 \sigma}{k} \left(\frac{1}{k^2 \rho^2} + \frac{i}{k^3 \rho^3} \right) e^{ik\rho} + \frac{4k^2}{(\omega \mu_0 \sigma)^2} \sqrt{\frac{2\pi}{k_m \rho}} e^{i(k_m \rho + \pi/4)} \right. \\ \left. - \frac{i\pi (\omega \mu_0 \sigma)^2}{8} \frac{1}{k^2} \frac{1}{k\rho} \left[H_1^{(1)}(k_e \rho) - \sqrt{\frac{2}{\pi k_e \rho}} e^{i(k_e \rho - 3\pi/4)} \right] - \frac{(\omega \mu_0 \sigma)^2}{4k^2} \frac{e^{ik\rho}}{k\rho} \sqrt{\frac{\pi}{k\rho}} \mathcal{F}(\varphi) \right\}, \quad (81)$$

which suggests the dominance of the TM surface plasmon if (76) holds.

Therefore, on the basis of the approximate formulas of Sections V A and V B, we reach the following conclusion.

Proposition 1 (On dominance of TM surface plasmon contribution). Suppose that the pole at $\lambda = k_m$ is present in the top λ -Riemann sheet; and $|\omega \mu_0 \sigma| \ll |k|$ which amounts to $\check{\zeta} := |\sigma| \sqrt{\mu_0 / |\bar{\epsilon}|} \ll 1$, where $k = \omega \sqrt{\mu_0 \bar{\epsilon}}$. Then, the TM surface plasmon is dominant at distances ρ of the order of the wavelength in free space if condition (74) holds. In particular, if $k > 0$ ($\bar{\epsilon} > 0$) and $\arg \sigma = \pi/2 - \delta$, $0 < \delta < \pi/2$, condition (74) reads

$$\check{\zeta} \ll \exp\left(-\frac{4}{3\check{\zeta}} \sin \delta\right). \quad (82)$$

VI. CONCLUSION

In this paper, we computed by analytical means the electromagnetic field of a vertical and a horizontal dipole located near a thin film with scalar, complex conductivity, σ , inside an unbounded medium of wave number k . This setting provides a minimal yet nontrivial model of wave propagation on a two-dimensional material such as a graphene sheet.

Our starting point was the Fourier-Bessel representation, or Sommerfeld-type integrals, for all field components. In the particular case when both the dipole and the observation point lie on the layer, we showed that the fields can be expressed in terms of geometrically convergent series and known transcendental functions such as the Fresnel integrals, and the Bessel and Struve functions. We simplified this result considerably when $|\omega \mu_0 \sigma| \ll |k|$ and derived analytic formulas that hold for all distances, ρ , from the source. These formulas connect smoothly the near field with the far field of the dipole, as we verified via independent asymptotic evaluation of the requisite integral representations. Our treatment provides an analytical description of a particular case of the dyadic Green function associated with the geometry of the conducting film and can expand insights obtained from previous numerical or semi-analytical approaches for related problems.^{7,8,27}

Depending on the frequency ω and the parameters σ and k , the electromagnetic field of the dipole may manifest a contribution from a pole analogous to the TM surface plasmon of plane waves.³ We demonstrated explicitly how this contribution may be manifested or suppressed in each relevant component of the electromagnetic field for all distances from the dipole source.

The present work admits several extensions and also points to pending issues. For example, we have not computed the field components at points away from the film; this task would require an elaborate asymptotic evaluation of the requisite Fourier-Bessel integrals by imposing some restriction on $|z|$. The possible anisotropy of the thin-film material, e.g., black phosphorus,³⁸ where σ is a tensor, deserves attention; this anisotropy dramatically affects the dispersion relations for surface plasmons and the analytic structure of the electromagnetic field produced by current-carrying sources. Another problem is the excitation of surface plasmons by a resonant receiving antenna placed on the material sheet,¹⁷ which requires solving an integral equation for the associated current distribution; this task is left for future work.

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APPENDIX A: ON THE NORMAL FIELD COMPONENTS

In this appendix, we discuss the conditions on the field components normal to the plane, $z = 0$, of the film. Such boundary conditions were not explicitly invoked in the derivation of the Fourier-Bessel integral representations (Section III).

First, we describe the (effective) surface charge density of the film which produces a jump of the normal component, $\tilde{\epsilon}_j E_{jz}$, of the electric displacement field $\mathbf{D}_j = \tilde{\epsilon}_j \mathbf{E}_j$. By (4b), the thin layer bears the surface *current* density $\mathbf{J}_s = \sigma \mathbf{E}_{\parallel} = \sigma \{\mathbf{E}_j - (\mathbf{E}_j \cdot \mathbf{e}_z) \mathbf{e}_z\}$ at $z = 0$, which by the continuity equation is associated with the surface *charge* density

$$\varrho_s(x, y) = -(i/\omega) \nabla_s \cdot \mathbf{J}_s|_{z=0} = -(i\sigma/\omega) \nabla_s \cdot \mathbf{E}_{\parallel}|_{z=0}, \quad \nabla_s = \nabla - (\nabla \cdot \mathbf{e}_z) \mathbf{e}_z.$$

By Gauss' law in region j , $\nabla_s \cdot \mathbf{E}_{\parallel} = -\partial E_{jz}/\partial z$ for $(x, y, z) \neq (0, 0, a)$; allowing z to approach 0 from region j , we conclude that the normal derivative, $\partial E_{jz}/\partial z$, of E_{jz} is continuous across $z = 0$. Thus,

$$\varrho_s = \frac{i\sigma}{\omega} \frac{\partial E_{1z}}{\partial z} \Big|_{z=0^+} = \frac{i\sigma}{\omega} \frac{\partial E_{2z}}{\partial z} \Big|_{z=0^-}, \quad (\text{A1})$$

where writing $z = 0^+$ ($z = 0^-$) implies that $z = 0$ is approached from above (below).

The boundary conditions for the normal field components consist of (i) the jump of the z -component of the electric displacement field due to ϱ_s and (ii) the continuity of the z -component of the magnetic field. Accordingly, we write

$$[k_j^2 E_{jz}] := (k_1^2 E_{1z} - k_2^2 E_{2z})|_{z=0} = \omega^2 \mu_0 \varrho_s = i\omega \mu_0 \sigma \frac{\partial E_{jz}}{\partial z} \Big|_{z=0}, \quad (\text{A2a})$$

$$[B_{jz}] := (B_{1z} - B_{2z})|_{z=0} = 0. \quad (\text{A2b})$$

Next, we proceed to demonstrate that (A2) are satisfied by the integral representations of Section III for an elevated dipole ($a > 0$), as expected.

1. Vertical dipole

By differentiation in z of (14b) for $0 < z < a$ and (16b) for $z < 0$, we obtain

$$\frac{\partial E_{1z}}{\partial z} \Big|_{z=0^+} = \frac{i\omega \mu_0}{2\pi} \int_0^\infty d\lambda \lambda^3 J_0(\lambda \rho) \frac{\beta_2(\lambda)}{\mathcal{P}(\lambda)} e^{i\beta_1 a} = \frac{\partial E_{2z}}{\partial z} \Big|_{z=0^-}, \quad (\text{A3})$$

which shows the continuity of the normal derivative of E_{jz} . On the other hand, we compute

$$\begin{aligned} [k_j^2 E_{jz}] &= -\frac{\omega \mu_0}{4\pi} \int_0^\infty d\lambda \frac{\lambda^3}{\beta_1} J_0(\lambda \rho) \left(1 + \frac{k_2^2 \beta_1 - k_1^2 \beta_2 + \omega \mu_0 \sigma \beta_1 \beta_2}{k_2^2 \beta_1 + k_1^2 \beta_2 + \omega \mu_0 \sigma \beta_1 \beta_2} \right) e^{i\beta_1 a} \\ &\quad + \frac{\omega \mu_0}{2\pi} \int_0^\infty d\lambda \lambda^3 J_0(\lambda \rho) \frac{k_2^2}{k_2^2 \beta_1 + k_1^2 \beta_2 + \omega \mu_0 \sigma \beta_1 \beta_2} e^{i\beta_1 a} \\ &= -\frac{\omega \mu_0}{2\pi} (\omega \mu_0 \sigma) \int_0^\infty d\lambda \lambda^3 J_0(\lambda \rho) \frac{\beta_2}{\mathcal{P}} e^{i\beta_1 a}, \end{aligned}$$

which, by (A3), is equal to $i\omega \mu_0 \sigma (\partial E_{jz}/\partial z)$ at $z = 0$. Thus, condition (A2a) is satisfied. Note that (A2b) is trivially satisfied in this case because $B_{jz} \equiv 0$.

2. Horizontal dipole

By differentiating (24c) and (26c), we verify that $\partial E_{jz}/\partial z$ is continuous across $z = 0$,

$$\frac{\partial E_{1z}}{\partial z} \Big|_{z=0^+} = -\frac{\omega\mu_0}{2\pi} \cos\phi \int_0^\infty d\lambda \lambda^2 J_1(\lambda\rho) \frac{\beta_1\beta_2}{\mathcal{P}} e^{i\beta_1 a} = \frac{\partial E_{2z}}{\partial z} \Big|_{z=0^-}. \tag{A4}$$

On the other hand, the jump of $k_j^2 E_{jz}$ is equal to

$$[k_j^2 E_{jz}] = \frac{i\omega\mu_0}{4\pi} \cos\phi \int_0^\infty d\lambda \lambda^2 J_1(\lambda\rho) \left(\frac{k_1^2\beta_2 - k_2^2\beta_1 - \omega\mu_0\sigma\beta_1\beta_2}{k_1^2\beta_2 + k_2^2\beta_1 + \omega\mu_0\sigma\beta_1\beta_2} - 1 \right) e^{i\beta_1 a} + \frac{i\omega\mu_0}{2\pi} \cos\phi \int_0^\infty d\lambda \lambda^2 J_1(\lambda\rho) \frac{k_2^2\beta_1}{k_1^2\beta_2 + k_2^2\beta_1 + \omega\mu_0\sigma\beta_1\beta_2} e^{i\beta_1 a},$$

which, after elementary algebra, is equal to $i\omega\mu_0\sigma(\partial E_{jz}/\partial z)$ at $z = 0$; cf. (A4).

We now turn our attention to B_{jz} . By (25c), as $z \downarrow 0$ (with $0 < z < a$), we obtain

$$B_{1z}|_{z=0^+} = \frac{i\mu_0}{4\pi} \sin\phi \int_0^\infty d\lambda \lambda^2 J_1(\lambda\rho) \frac{1}{\beta_1} \left(1 - \frac{\beta_2 - \beta_1 + \omega\mu_0\sigma}{\beta_2 + \beta_1 + \omega\mu_0\sigma} \right) e^{i\beta_1 a},$$

which is equal to B_{2z} at $z = 0^-$ by (27c). This shows the expected validity of (A2b).

APPENDIX B: ASYMPTOTIC EVALUATION OF SOMMERFELD-TYPE INTEGRALS

In this appendix, we derive the near- and far-field of each dipole by asymptotic evaluation of the Sommerfeld-type integrals under $|\omega\mu_0\sigma| \ll |k|$ and corresponding restrictions on ρ . The results of these calculations are compared to the respective approximate formulas of Section IV C. We assume that the pole at $\lambda = k_m$ is present in the physical Riemann sheet throughout this appendix (see Remarks 1–3).

In the following analysis, we will need the relations:³¹

$$J_0(z) + J_2(z) = \frac{2}{z} J_1(z), \quad J_0(z) - J_2(z) = 2 J_1'(z), \tag{B1}$$

along with the leading-order asymptotic formula³¹

$$J_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \cos(z - \nu\pi/2 - \pi/4) \quad \text{as } |z| \rightarrow \infty, \quad |\arg z| < \pi. \tag{B2}$$

In addition, we will make use of the following result from Ref. 26.

Lemma B.1 (On the Fresnel integrals). The integral

$$\mathfrak{S} := \int_{-\infty}^\infty d\tau \frac{e^{i(k_e-k)\rho\tau}}{\sqrt{\tau}(\omega\mu_0\sigma + i2\sqrt{2k}\sqrt{k_e - k}\sqrt{\tau})} = \frac{\pi}{\sqrt{k}\sqrt{k_e - k}} e^{i\pi/4} \mathcal{F}(\wp_0), \tag{B3}$$

where $\sqrt{\tau} > 0$ for $\tau > 0$, $\mathcal{F}(z)$ is defined in (56a), and $\wp_0 = -(\omega\mu_0\sigma)^2\rho/(8k)$ [see (36) and (58)].

Proof. By Ref. 26, first write

$$\mathfrak{S} = \int_0^\infty d\tau \frac{e^{i(k_e-k)\rho\tau}}{\sqrt{\tau}(\omega\mu_0\sigma + i2\sqrt{2k}\sqrt{k_e - k}\sqrt{\tau})} + \int_0^\infty d\tau \frac{e^{-i(k_e-k)\rho\tau}}{-i\sqrt{\tau}(\omega\mu_0\sigma + 2\sqrt{2k}\sqrt{k_e - k}\sqrt{\tau})}.$$

Then, apply $\tau \mapsto \varsigma$ with $\sqrt{\varsigma} = 2i\sqrt{2k}\sqrt{k_e - k}(\omega\mu_0\sigma)^{-1}\sqrt{\tau}$ to find

$$\mathfrak{S} = \frac{i}{2\sqrt{2k}\sqrt{k_e - k}} \mathfrak{S}_0(\wp_0),$$

where

$$\mathfrak{S}_0(z) = -\int_0^\infty d\varsigma \frac{e^{iz\varsigma}}{\sqrt{\varsigma}(\sqrt{\varsigma} + 1)} + \int_0^\infty d\varsigma \frac{e^{-iz\varsigma}}{\sqrt{\varsigma}(\sqrt{\varsigma} + i)}.$$

This $\mathfrak{I}_0(z)$ satisfies

$$\frac{d\mathfrak{I}_0}{dz} - i\mathfrak{I}_0(z) = -2e^{-i\pi/4} \sqrt{\frac{\pi}{z}}, \quad \lim_{z \rightarrow +\infty} \mathfrak{I}_0(z) = 0, \tag{B4}$$

with solution $\mathfrak{I}_0(z) = 2\pi\sqrt{2} e^{-i\pi/4} \mathcal{F}(z)$, which concludes the proof. □

1. Far field: $|k\rho| \gg 1$ with $|\omega\mu_0\sigma| \ll |k|$

In this regime, the major contribution to integration in the Sommerfeld-type integrals comes from the singularities at $\lambda = k$ and $\lambda = k_m$. The Bessel functions in all integrands are expressed in terms of $J_\nu(\lambda\rho)$ for $\nu = 0, 1$ by use of (B1); $J_\nu(\lambda\rho)$ is in turn replaced by its large-argument approximation according to (B2).

a. Field of vertical dipole on film

Taking the limit as $a \downarrow 0$ with $0 < z < a$ of (14a) and then invoking (B2) readily yield

$$E_{1\rho} = -\frac{i\omega\mu_0}{2\pi} \int_0^\infty d\lambda \lambda^2 J_1(\lambda\rho) \frac{1}{2k^2 + \omega\mu_0\sigma\sqrt{k^2 - \lambda^2}}$$

$$\stackrel{|k\rho| \gg 1}{\sim} -\frac{i\omega\mu_0}{4\pi} \int_{-\infty}^\infty d\lambda \lambda^2 \sqrt{\frac{2}{\pi\lambda\rho}} \frac{1}{2k^2 + \omega\mu_0\sigma\sqrt{k^2 - \lambda^2}} e^{i(\lambda\rho - 3\pi/4)}. \tag{B5}$$

In this integral, we single out the contributions from the simple pole at $\lambda = k_m$ and the branch point at $\lambda = k$, viz.,

$$E_{1\rho}^{(vd)} \sim -\frac{i\omega\mu_0}{4\pi} \left\{ 2\pi i k_m^2 \sqrt{\frac{2}{\pi k_m \rho}} e^{i(k_m \rho - 3\pi/4)} \lim_{\lambda \rightarrow k_m} \left[\frac{2k^2 - \omega\mu_0\sigma\sqrt{k^2 - \lambda^2}}{(\omega\mu_0\sigma)^2(2\lambda)} \right] \right.$$

$$\left. + ik^3 \int_0^\infty d\tau (1 + i\tau)^2 \left[\frac{1}{2k^2 - e^{-i\pi/4} \omega\mu_0\sigma k \sqrt{\tau} \sqrt{2 + i\tau}} - \frac{1}{2k^2 + e^{-i\pi/4} \omega\mu_0\sigma k \sqrt{\tau} \sqrt{2 + i\tau}} \right] \right.$$

$$\left. \times \sqrt{\frac{2}{\pi k \rho (1 + i\tau)}} e^{i(k\rho - 3\pi/4)} e^{-k\rho\tau} \right\},$$

where for the last integral, we appropriately deformed the integration path in the upper λ -half-plane, set $\lambda = k(1 + i\tau)$, and integrated on each side of the positive real τ -axis. The major contribution to integration comes from the vicinity of $\tau = 0$ with width $O((k\rho)^{-1})$; the corresponding integrand yields a term of the order of $\omega\mu_0\sigma/k$. By enforcing the condition $|\omega\mu_0\sigma| \ll |k|$, we reduce the preceding formula for $E_{1\rho}^{(vd)}$ to

$$E_{1\rho}^{(vd)} \sim \frac{i\omega\mu_0 k}{4\pi} \left\{ \frac{k_m^3}{k^3} \sqrt{\frac{2\pi}{k_m \rho}} e^{i(k_m \rho - \pi/4)} \right.$$

$$\left. - i \frac{\omega\mu_0\sigma}{2k} \sqrt{2} e^{-i\pi/4} \sqrt{\frac{2}{\pi k \rho}} e^{i(k\rho - 3\pi/4)} \int_0^\infty d\tau \sqrt{\tau} e^{-k\rho\tau} \right\}$$

$$= \frac{i\omega\mu_0 k}{4\pi} \left\{ \frac{k_m^3}{k^3} \sqrt{\frac{2\pi}{k_m \rho}} e^{i(k_m \rho - \pi/4)} + \frac{i\omega\mu_0\sigma}{2k} \frac{e^{ik\rho}}{k^2 \rho^2} \right\}, \tag{B6}$$

in agreement with (62b) which comes from the exact series expansion for this component.

In the same vein, by (14b) and asymptotic formula (B2) for $\nu = 0$, we have

$$E_{1z}^{(vd)} = -\frac{\omega\mu_0}{2\pi k^2} \int_0^\infty d\lambda J_0(\lambda\rho) \frac{\lambda^3}{\sqrt{k^2 - \lambda^2}} \frac{k^2 + \omega\mu_0\sigma\sqrt{k^2 - \lambda^2}}{2k^2 + \omega\mu_0\sigma\sqrt{k^2 - \lambda^2}}$$

$$\stackrel{|k\rho| \gg 1}{\sim} -\frac{\omega\mu_0}{4\pi k^2} \int_{-\infty}^\infty d\lambda \frac{\lambda^3}{\sqrt{k^2 - \lambda^2}} \frac{k^2 + \omega\mu_0\sigma\sqrt{k^2 - \lambda^2}}{2k^2 + \omega\mu_0\sigma\sqrt{k^2 - \lambda^2}} \sqrt{\frac{2}{\pi\lambda\rho}} e^{i(\lambda\rho - \pi/4)}. \tag{B7}$$

By singling out the contributions from two singular points, at $\lambda = k_m$ and $\lambda = k$, and additionally imposing $|\omega\mu_0\sigma| \ll |k|$, after some algebra, we find

$$\begin{aligned}
 E_{1z}^{(vd)} &\sim \frac{\omega\mu_0}{4\pi} \left\{ 2\pi i \frac{4k^4}{(\omega\mu_0\sigma)^3} \sqrt{\frac{2}{\pi k_m \rho}} e^{i(k_m \rho - \pi/4)} \right. \\
 &\quad - ik^2 \int_0^\infty d\tau \sqrt{\frac{2}{\pi k \rho (1+i\tau)}} (1+i\tau)^3 e^{i(k\rho - \pi/4)} e^{-(k\rho)\tau} \left[\frac{1}{-e^{-i\pi/4} k \sqrt{2+i\tau} \sqrt{\tau}} \right. \\
 &\quad \times \frac{k^2 - e^{-i\pi/4} \omega\mu_0\sigma k \sqrt{\tau} \sqrt{2+i\tau}}{2k^2 - e^{-i\pi/4} \omega\mu_0\sigma k \sqrt{\tau} \sqrt{2+i\tau}} - \frac{1}{e^{-i\pi/4} k \sqrt{2+i\tau} \sqrt{\tau}} \\
 &\quad \left. \left. \times \frac{k^2 + e^{-i\pi/4} \omega\mu_0\sigma k \sqrt{\tau} \sqrt{2+i\tau}}{2k^2 + e^{-i\pi/4} \omega\mu_0\sigma k \sqrt{\tau} \sqrt{2+i\tau}} \right] \right\} \\
 &\sim \frac{\omega\mu_0 k}{4\pi} \left\{ 8\pi i \frac{k^3}{(\omega\mu_0\sigma)^3} \sqrt{\frac{2}{\pi k_m \rho}} e^{i(k_m \rho - \pi/4)} + \frac{i}{k\rho} e^{ik\rho} \right\}, \tag{B8}
 \end{aligned}$$

by appropriately expanding the integrand at $\tau = 0$. The preceding result is in agreement with (64) of the exact solution for $|\omega\mu_0\sigma| \ll |k|$ and $|k\rho| \gg 1$. By a similar computation, (15) yields

$$\begin{aligned}
 B_{1\phi}^{(vd)} &= \frac{i\mu_0}{2\pi} \int_0^\infty d\lambda J_1(\lambda\rho) \frac{\lambda^2}{\sqrt{k^2 - \lambda^2}} \frac{k^2 + \omega\mu_0\sigma\sqrt{k^2 - \lambda^2}}{2k^2 + \omega\mu_0\sigma\sqrt{k^2 - \lambda^2}} \\
 &\stackrel{|k\rho| \gg 1}{\sim} \frac{i\mu_0}{4\pi} \int_{-\infty}^\infty d\lambda \frac{\lambda^2}{\sqrt{k^2 - \lambda^2}} \frac{k^2 + \omega\mu_0\sigma\sqrt{k^2 - \lambda^2}}{2k^2 + \omega\mu_0\sigma\sqrt{k^2 - \lambda^2}} \sqrt{\frac{2}{\pi\lambda\rho}} e^{i(\lambda\rho - 3\pi/4)} \\
 &\stackrel{|\omega\mu_0\sigma| \ll |k|}{\sim} - \frac{i\mu_0 k^2}{4\pi} \left\{ 4\pi \frac{k^2}{(\omega\mu_0\sigma)^2} \sqrt{\frac{2}{\pi k_m \rho}} e^{i(k_m \rho - 3\pi/4)} + \frac{1}{k\rho} e^{ik\rho} \right\}, \tag{B9}
 \end{aligned}$$

which, alternatively, results from (63) of the exact solution if $|\omega\mu_0\sigma| \ll |k|$ and $|k\rho| \gg 1$.

b. Field of horizontal dipole on film

In this case, the contribution from the branch point at $\lambda = k$ should be affected by the proximity of the pole at $\lambda = k_e$ if $|(k_e - k)\rho| \leq O(1)$. This effect is taken into account via the integral of Lemma B.1, where $(k_e - k)\rho \sim \wp_0$ enters as a parameter. This situation is typical in radiowave propagation on the boundary separating media of significantly different wavenumbers.²⁶

In view of (B1) and (B2), the limit as $a \downarrow 0$ and $z \uparrow 0$ of (26a) yields

$$\begin{aligned}
 E_{2\rho}^{(hd)} &= -\frac{\omega\mu_0}{2\pi} \cos\phi \int_0^\infty d\lambda \left\{ \frac{1}{\rho} J_1(\lambda\rho) \frac{1}{2\sqrt{k^2 - \lambda^2} + \omega\mu_0\sigma} + \lambda J_1'(\lambda\rho) \frac{\sqrt{k^2 - \lambda^2}}{2k^2 + \omega\mu_0\sigma\sqrt{k^2 - \lambda^2}} \right\} \\
 &\stackrel{|k\rho| \gg 1}{\sim} -\frac{\omega\mu_0}{4\pi} \cos\phi \left\{ \int_{-\infty}^\infty d\lambda \frac{1}{\rho} \sqrt{\frac{2}{\pi\lambda\rho}} e^{i(\lambda\rho - 3\pi/4)} \frac{1}{2\sqrt{k^2 - \lambda^2} + \omega\mu_0\sigma} \right. \\
 &\quad \left. + \int_{-\infty}^\infty d\lambda i\lambda \sqrt{\frac{2}{\pi\lambda\rho}} e^{i(\lambda\rho - 3\pi/4)} \frac{\sqrt{k^2 - \lambda^2}}{2k^2 + \omega\mu_0\sigma\sqrt{k^2 - \lambda^2}} \right\}. \tag{B10}
 \end{aligned}$$

There is only one residue contribution, $E_{2\rho, k_m}^{(hd)}$, to this component, which comes from the last integral and concerns the simple pole at $\lambda = k_m$,

$$E_{2\rho, k_m}^{(hd)} \sim -\frac{\omega\mu_0}{4\pi} \cos\phi 2\pi i \left\{ ik_m \sqrt{\frac{2}{\pi k_m \rho}} e^{i(k_m \rho - 3\pi/4)} \sqrt{k^2 - k_m^2} \lim_{k \rightarrow k_m} \left[\frac{2k^2 - \omega\mu_0\sigma\sqrt{k^2 - \lambda^2}}{(\omega\mu_0\sigma)^2 (2\lambda)} \right] \right\}.$$

The contribution from the branch point at $\lambda = k$ is singled out as

$$E_{2\rho,k}^{(hd)} \sim -\frac{\omega\mu_0}{4\pi} \cos\phi e^{i(k\rho-3\pi/4)} \sqrt{\frac{2}{\pi k\rho}} \left\{ \frac{k_e - k}{\rho} \int_{-\infty}^{\infty} d\tau' \left(1 + \frac{k_e - k}{k} \tau'\right)^{-1/2} e^{i(k_e - k)\rho\tau'} \right. \\ \times \frac{1}{2i\sqrt{2k}\sqrt{k_e - k}\sqrt{1 + \frac{k_e - k}{2k}\tau'}\sqrt{\tau'} + \omega\mu_0\sigma} + (ik)^2 \int_0^{\infty} d\tau \sqrt{1 + i\tau} e^{-(k\rho)\tau} \\ \left. \times \left[\frac{-e^{-i\pi/4} k\sqrt{2 + i\tau}\sqrt{\tau}}{2k^2 - e^{-i\pi/4} \omega\mu_0\sigma k\sqrt{2 + i\tau}\sqrt{\tau}} - \frac{e^{-i\pi/4} k\sqrt{2 + i\tau}\sqrt{\tau}}{2k^2 + e^{-i\pi/4} \omega\mu_0\sigma k\sqrt{2 + i\tau}\sqrt{\tau}} \right] \right\}.$$

For the first integral above, we have set $\lambda = k + (k_e - k)\tau'$ whereas for the second integral, we deformed the path in the upper λ -half-plane and then set $\lambda = k(1 + i\tau)$ as before. In each of the transformed integrals, the major contribution arises from a small neighborhood of the origin. The value of the first integral crucially depends on which term in its denominator prevails when $(k_e - k)\rho\tau' = O(1)$; by this interplay, the parameter \wp_0 emerges naturally. In contrast, in the second integral, such interplay is not present; one may simply Taylor-expand the integrand in powers of $\omega\mu_0\sigma/k$ and keep the leading-order term. To subtract the primary field contribution in the first integral, we split each term, evaluated near $\tau' = 0$, as

$$\frac{1}{2i\sqrt{2k}\sqrt{k_e - k}\sqrt{\tau'} + \omega\mu_0\sigma} = \frac{1}{2i\sqrt{2k}\sqrt{k_e - k}\sqrt{\tau'}} - \frac{\omega\mu_0\sigma}{2i\sqrt{2k}\sqrt{k_e - k}\sqrt{\tau'}(2i\sqrt{2k}\sqrt{k_e - k}\sqrt{\tau'} + \omega\mu_0\sigma)}$$

and carry out the integration without further approximations for these terms. Thus, we find

$$E_{2\rho,k}^{(hd)} \underset{|\omega\mu_0\sigma| \ll |k|}{\sim} -\frac{\omega\mu_0}{4\pi} \cos\phi \left\{ -\frac{2}{k\rho^2} e^{ik\rho} - \frac{k_e - k}{\rho} \frac{\omega\mu_0\sigma}{2i\sqrt{2k}\sqrt{k_e - k}} \sqrt{\frac{2}{\pi k\rho}} e^{i(k\rho-3\pi/4)} \wp \right\},$$

where \wp is given in Lemma B.1. Thus, by $E_{2\rho}^{(hd)} \sim E_{2\rho,k_m}^{(hd)} + E_{2\rho,k}^{(hd)}$, we compute

$$E_{2\rho}^{(hd)} \sim -\frac{\omega\mu_0}{4\pi} \cos\phi \left\{ \frac{8k^4}{(\omega\mu_0\sigma)^3} \sqrt{\frac{2\pi}{k_m\rho}} e^{i(k_m\rho-3\pi/4)} - \frac{2e^{ik\rho}}{k\rho^2} + \frac{\omega\mu_0\sigma}{2k\rho} \sqrt{\frac{\pi}{k\rho}} e^{ik\rho} \mathcal{F}(\wp_0) \right\}, \tag{B11}$$

which is in agreement with (67b) for $|k\rho| \gg 1$.

The remaining components can be approximately computed by the same methodology, although the algebraic details are different in each case. By (26b), we have

$$E_{2\phi}^{(hd)} = \frac{\omega\mu_0}{2\pi} \sin\phi \int_0^{\infty} d\lambda \left\{ \lambda J_1'(\lambda\rho) \frac{1}{2\sqrt{k^2 - \lambda^2} + \omega\mu_0\sigma} + \frac{1}{\rho} J_1(\lambda\rho) \frac{\sqrt{k^2 - \lambda^2}}{2k^2 + \omega\mu_0\sigma\sqrt{k^2 - \lambda^2}} \right\} \\ \underset{|k\rho| \gg 1}{\sim} \frac{\omega\mu_0}{4\pi} \sin\phi \left\{ \int_{-\infty}^{\infty} d\lambda i\lambda \sqrt{\frac{2}{\pi\lambda\rho}} e^{i(\lambda\rho-3\pi/4)} \frac{1}{2\sqrt{k^2 - \lambda^2} + \omega\mu_0\sigma} \right. \\ \left. + \int_{-\infty}^{\infty} d\lambda \frac{1}{\rho} \sqrt{\frac{2}{\pi\lambda\rho}} e^{i(\lambda\rho-3\pi/4)} \frac{\sqrt{k^2 - \lambda^2}}{2k^2 + \omega\mu_0\sigma\sqrt{k^2 - \lambda^2}} \right\}. \tag{B12}$$

By separating the pole and branch-point contributions, we find

$$E_{2\phi}^{(hd)} \underset{|\omega\mu_0\sigma| \ll |k|}{\sim} \frac{\omega\mu_0}{4\pi} \sin\phi \left\{ -\frac{8\pi ik^4}{(\omega\mu_0\sigma)^3} \frac{1}{k_m\rho} \sqrt{\frac{2}{\pi k_m\rho}} e^{i(k_m\rho-3\pi/4)} + i \left(-\frac{1}{\rho} + \frac{1}{k^2\rho^3} \right) e^{ik\rho} \right. \\ \left. - \frac{\omega\mu_0\sigma}{2\sqrt{2}} \sqrt{k_e - k}\sqrt{k} \sqrt{\frac{2}{\pi k\rho}} e^{i(k\rho-3\pi/4)} \wp \right\}, \tag{B13}$$

which, by Lemma B.1, is in agreement with (68b) if $|k\rho| \gg 1$. We note in passing that the $e^{ik\rho}/(k^2\rho^3)$ term in the first line of (B13) arises from the (TM-polarization) integral containing $(2k^2 + \omega\mu_0\sigma\sqrt{k^2 - \lambda^2})^{-1}$, is small compared to $e^{ik\rho}/\rho$ in this far-field regime and, thus, should be dropped to leading order in $k\rho$.

By (26c), the z -component of the electric field does not contain any TE polarization term and thus lacks the Fresnel integrals. By similar asymptotics, we reach the following result:

$$\begin{aligned}
 E_{2z}^{(\text{hd})} &= -\frac{i\omega\mu_0}{2\pi} \cos\phi \int_0^\infty d\lambda \lambda^2 J_1(\lambda\rho) \frac{1}{2k^2 + \omega\mu_0\sigma\sqrt{k^2 - \lambda^2}} \\
 &\stackrel{|k\rho| \gg 1}{\sim} -\frac{i\omega\mu_0}{4\pi} \cos\phi \int_{-\infty}^\infty d\lambda \lambda^2 \sqrt{\frac{2}{\pi\lambda\rho}} e^{i(\lambda\rho - 3\pi/4)} \frac{1}{2k^2 + \omega\mu_0\sigma\sqrt{k^2 - \lambda^2}} \\
 &\stackrel{|\omega\mu_0\sigma| \ll |k|}{\sim} -\frac{i\omega\mu_0}{4\pi} \cos\phi \left\{ 4\pi i k_m \frac{k^2}{(\omega\mu_0\sigma)^2} \sqrt{\frac{2}{\pi k_m \rho}} e^{i(k_m \rho - 3\pi/4)} - \frac{i\omega\mu_0\sigma}{2} \frac{e^{ik\rho}}{k^2 \rho^2} \right\}; \tag{B14}
 \end{aligned}$$

cf. (65) from the exact solution, recalling that $k_m \sim i2k^2/(\omega\mu_0\sigma)$.

Next, we turn attention to the components of the magnetic field. By (27a), we have

$$\begin{aligned}
 B_{2\rho}^{(\text{hd})} &= \frac{\mu_0}{2\pi} \sin\phi \int_0^\infty d\lambda \left\{ \frac{1}{\rho} J_1(\lambda\rho) \frac{k^2}{2k^2 + \omega\mu_0\sigma\sqrt{k^2 - \lambda^2}} + \lambda J_1'(\lambda\rho) \frac{\sqrt{k^2 - \lambda^2}}{2\sqrt{k^2 - \lambda^2} + \omega\mu_0\sigma} \right\} \\
 &\stackrel{|k\rho| \gg 1}{\sim} \frac{\mu_0}{4\pi} \sin\phi \left\{ \int_{-\infty}^\infty d\lambda \frac{1}{\rho} \sqrt{\frac{2}{\pi\lambda\rho}} e^{i(\lambda\rho - 3\pi/4)} \frac{k^2}{2k^2 + \omega\mu_0\sigma\sqrt{k^2 - \lambda^2}} \right. \\
 &\quad \left. - \int_{-\infty}^\infty d\lambda \frac{i\lambda}{2} \sqrt{\frac{2}{\pi\lambda\rho}} e^{i(\lambda\rho - 3\pi/4)} \frac{\omega\mu_0\sigma}{2\sqrt{k^2 - \lambda^2} + \omega\mu_0\sigma} \right\}, \tag{B15}
 \end{aligned}$$

where in the third line, we replaced $\sqrt{k^2 - \lambda^2}/(2\sqrt{k^2 - \lambda^2} + \omega\mu_0\sigma)$ by $1/2 - (\omega\mu_0\sigma/2)/(2\sqrt{k^2 - \lambda^2} + \omega\mu_0\sigma)$ and integrated out the constant. For the first integral, we deform the path in the upper λ -half-plane, picking up the residue from $\lambda = k_m$, and then evaluate the contribution from the vicinity of $\lambda = k$. For the second integral, we set $\lambda \mapsto \tau$ with $\tau = (\lambda - k)/(k_e - k)$ and evaluate the contribution from the vicinity of $\tau = 0$, ending up with Fresnel integrals via Lemma B.1. Accordingly, we derive the expression

$$\begin{aligned}
 B_{2\rho}^{(\text{hd})} &\sim \frac{\mu_0}{4\pi} \sin\phi \left\{ \frac{4\pi i k^4}{(\omega\mu_0\sigma)^2} \frac{1}{k_m \rho} \sqrt{\frac{2}{\pi k_m \rho}} e^{i(k_m \rho - 3\pi/4)} + \frac{i\omega\mu_0\sigma}{2} \left(\frac{1}{\rho} - \frac{1}{k^2 \rho^3} \right) e^{ik\rho} \right. \\
 &\quad \left. - i \frac{(\omega\mu_0\sigma)^2}{4} \sqrt{\frac{\pi}{k\rho}} e^{ik\rho} \mathcal{F}(\wp_0) \right\}, \tag{B16}
 \end{aligned}$$

where the $e^{ik\rho}/(k^2 \rho^3)$ term in the parenthesis of the first line comes from the TM-polarization integral and is neglected. The ensuing asymptotic result can also be obtained from (69b) in the far-field regime, $|k\rho| \gg 1$.

In this vein, by (27b), we compute the following expression [cf. (70b)]:

$$\begin{aligned}
 B_{2\phi}^{(\text{hd})} &= \frac{\mu_0}{2\pi} \cos\phi \int_0^\infty d\lambda \left\{ \lambda J_1'(\lambda\rho) \frac{k^2}{2k^2 + \omega\mu_0\sigma\sqrt{k^2 - \lambda^2}} + \frac{1}{\rho} J_1(\lambda\rho) \frac{\sqrt{k^2 - \lambda^2}}{2\sqrt{k^2 - \lambda^2} + \omega\mu_0\sigma} \right\} \\
 &\stackrel{|k\rho| \gg 1}{\sim} \frac{\mu_0}{4\pi} \cos\phi \left\{ \int_{-\infty}^\infty d\lambda i\lambda \sqrt{\frac{2}{\pi\lambda\rho}} e^{i(\lambda\rho - 3\pi/4)} \frac{k^2}{2k^2 + \omega\mu_0\sigma\sqrt{k^2 - \lambda^2}} \right. \\
 &\quad \left. - \int_{-\infty}^\infty d\lambda \frac{1}{2\rho} \sqrt{\frac{2}{\pi\lambda\rho}} e^{i(\lambda\rho - 3\pi/4)} \frac{\omega\mu_0\sigma}{2\sqrt{k^2 - \lambda^2} + \omega\mu_0\sigma} \right\} \\
 &\stackrel{|\omega\mu_0\sigma| \ll |k|}{\sim} \frac{\mu_0}{4\pi} \cos\phi \left\{ -\frac{4\pi k^4}{(\omega\mu_0\sigma)^2} \sqrt{\frac{2}{\pi k_m \rho}} e^{i(k_m \rho - 3\pi/4)} + \omega\mu_0\sigma \frac{e^{ik\rho}}{k\rho^2} \right. \\
 &\quad \left. - \frac{(\omega\mu_0\sigma)^2}{4} \frac{1}{k\rho} \sqrt{\frac{\pi}{k\rho}} e^{ik\rho} \mathcal{F}(\wp_0) \right\}. \tag{B17}
 \end{aligned}$$

Here, the branch-point contribution for each polarization (TE and TM) integral carries a free-space term proportional to $[\omega\mu_0\sigma/(k\rho^2)]e^{ik\rho}$, besides the Fresnel integral contribution.

We conclude this subsection with the approximate computation of the z -component of the magnetic field by similar means. This component expresses only TE polarization and, thus, lacks any pole contribution (by Remark 3). By (27c) and similar algebraic manipulations as above, we find

$$\begin{aligned}
 B_{2z}^{(\text{hd})} &= \frac{i\mu_0}{2\pi} \sin \phi \int_0^\infty d\lambda \lambda^2 J_1(\lambda\rho) \frac{1}{2\sqrt{k^2 - \lambda^2} + \omega\mu_0\sigma} \\
 &\stackrel{|k\rho| \gg 1}{\sim} \frac{i\mu_0}{4\pi} \sin \phi \int_{-\infty}^\infty d\lambda \lambda^2 \sqrt{\frac{2}{\pi\lambda\rho}} e^{i(\lambda\rho - 3\pi/4)} \frac{1}{2\sqrt{k^2 - \lambda^2} + \omega\mu_0\sigma} \\
 &\stackrel{|\omega\mu_0\sigma| \ll |k|}{\sim} \frac{i\mu_0 k^2}{4\pi} \sin \phi \left\{ -\frac{1}{k\rho} e^{ik\rho} + \frac{\omega\mu_0\sigma}{2k} \sqrt{\frac{\pi}{k\rho}} e^{ik\rho} \mathcal{F}(\varphi_0) \right\}, \quad (\text{B18})
 \end{aligned}$$

in agreement with (66c).

2. Near (static) field: $|k_m\rho| \ll 1$ with $|\omega\mu_0\sigma| \ll |k|$

In this case, the major contribution in each Sommerfeld-type integral comes from $\lambda = \mathcal{O}(1/\rho) \gg |k_m| \gg |k|$. Thus, we make the approximations

$$\sqrt{k^2 - \lambda^2} \sim i\lambda, \quad 2k^2 + \omega\mu_0\sigma\sqrt{k^2 - \lambda^2} \sim i\lambda\omega\mu_0\sigma, \quad 2\sqrt{k^2 - \lambda^2} + \omega\mu_0\sigma \sim 2i\lambda,$$

where $\lambda > 0$ and exactly evaluate the ensuing integrals containing $J_\nu(\lambda\rho)$.

a. Vertical dipole

In the limit $a \downarrow 0$ with $0 < z < a$, (14a) yields

$$E_{1\rho}^{(\text{vd})} \stackrel{|k_m\rho| \ll 1}{\sim} \frac{i\omega\mu_0}{4\pi k^2} \frac{i2k^2}{\omega\mu_0\sigma} \int_0^\infty d\lambda \lambda J_1(\lambda\rho) = -\frac{\omega\mu_0}{2\pi} \frac{1}{\omega\mu_0\sigma} \frac{1}{\rho^2}, \quad (\text{B19})$$

which agrees with (62b) in the regime where $|k\rho| \ll |k_m\rho| \ll 1$ and $|\omega\mu_0\sigma| \ll |k|$. In this regime, the factor $e^{ik\rho}$ as well as terms containing the Struve and Neumann functions must be expanded out in powers of $k\rho$ and $k_m\rho$;³¹ after several mutual cancellations, it is found that the term $(k_m\rho)^{-2}$ in the second line of (62b) provides the leading-order contribution. Likewise, (14b) furnishes the integral

$$E_{1z}^{(\text{vd})} \sim \frac{i\omega\mu_0}{2\pi k^2} \int_0^\infty d\lambda \lambda^2 J_0(\lambda\rho) = \frac{i\omega\mu_0}{2\pi k^2} \frac{1}{\rho} \int_0^\infty d\lambda \lambda \frac{d}{d\lambda} [\lambda J_1(\lambda)] = -\frac{i\omega\mu_0}{2\pi k^2} \frac{1}{\rho^3}, \quad (\text{B20})$$

via integration by parts; this result is in agreement with (64) where the leading-order contribution comes from the last term. The remaining component is provided by (15), which is reduced to the following expression [cf. (63)]:

$$B_{1\phi}^{(\text{vd})} \sim \frac{\mu_0}{2\pi} \int_0^\infty \lambda J_1(\lambda\rho) = \frac{\mu_0}{2\pi} \frac{1}{\rho^2}. \quad (\text{B21})$$

b. Horizontal dipole

In the static field of the horizontal dipole, the two types of polarization may provide contributions of different orders of magnitude. By (26a), we have

$$\begin{aligned}
 E_{2\rho}^{(\text{hd})} &\sim -\frac{\omega\mu_0}{4\pi} \cos \phi \int_0^\infty d\lambda \lambda \left\{ \frac{1}{2i\lambda} [J_0(\lambda\rho) + J_2(\lambda\rho)] + \frac{1}{\omega\mu_0\sigma} [J_0(\lambda\rho) - J_2(\lambda\rho)] \right\} \\
 &\sim -\frac{\omega\mu_0}{4\pi} \cos \phi \frac{2}{\omega\mu_0\sigma} \int_0^\infty d\lambda \frac{\partial}{\partial \rho} J_1(\lambda\rho) = \frac{\omega\mu_0}{2\pi} \cos \phi \frac{1}{\omega\mu_0\sigma} \frac{1}{\rho^2}, \quad (\text{B22})
 \end{aligned}$$

with dominance of the term pertaining to TM polarization. This result agrees with (67b), in which $H_1^{(1)'}(k_m\rho)$ and $Y_1'(k_m\rho)$ provide the leading-order contribution after cancellation of the $\mathcal{O}(1/\rho^3)$

terms. By (26b), the ϕ -component of the electric field is

$$\begin{aligned} E_{2\phi}^{(\text{hd})} &\sim \frac{\omega\mu_0}{4\pi} \sin\phi \int_0^\infty d\lambda \lambda \left\{ \frac{1}{2i\lambda} [J_0(\lambda\rho) - J_2(\lambda\rho)] + \frac{1}{\omega\mu_0\sigma} [J_0(\lambda\rho) + J_2(\lambda\rho)] \right\} \\ &\sim \frac{\omega\mu_0}{4\pi} \sin\phi \frac{1}{\omega\mu_0\sigma} \frac{2}{\rho} \int_0^\infty d\lambda J_1(\lambda\rho) = \frac{\omega\mu_0}{2\pi} \sin\phi \frac{1}{\omega\mu_0\sigma} \frac{1}{\rho^2}, \end{aligned} \quad (\text{B23})$$

in agreement with (68b) where the leading-order term comes from the expansion of $(k_m\rho)^{-1}H_1^{(1)}(k_m\rho)$ and $(k_m\rho)^{-1}Y_1(k_m\rho)$ for $|k_m\rho| \ll 1$ under $|\omega\mu_0\sigma| \ll |k|$. The remaining component of the electric field is evaluated on the basis of (26c), viz.,

$$E_{2z} \sim -\frac{i\omega\mu_0}{2\pi} \cos\phi \frac{1}{i\omega\mu_0\sigma} \int_0^\infty d\lambda \lambda J_1(\lambda\rho) = -\frac{\omega\mu_0}{2\pi} \cos\phi \frac{1}{\omega\mu_0\sigma} \frac{1}{\rho^2}. \quad (\text{B24})$$

This result smoothly connects with (65b) where the leading-order contribution comes from the $(k_m\rho)^{-2}$ term of the second line.

We proceed to the computation of the magnetic field. By (27a), we write

$$\begin{aligned} B_{2\rho}^{(\text{hd})} &\sim \frac{\mu_0}{4\pi} \sin\phi \int_0^\infty d\lambda \lambda \left\{ [J_0(\lambda\rho) + J_2(\lambda\rho)] \frac{k^2}{i\lambda\omega\mu_0\sigma} + [J_0(\lambda\rho) - J_2(\lambda\rho)] \frac{1}{2} \right\} \\ &\sim \frac{\mu_0}{4\pi} \sin\phi \int_0^\infty d\lambda \frac{\partial}{\partial\rho} J_1(\lambda\rho) = -\frac{\mu_0}{4\pi} \sin\phi \frac{1}{\rho^2}. \end{aligned} \quad (\text{B25})$$

This formula also results from (69b); the leading-order term comes from the expansion of $(k_m\rho)^{-1}H_1^{(1)}(k_m\rho)$ and $(k_m\rho)^{-1}Y_1(k_m\rho)$. By (27b), we obtain

$$\begin{aligned} B_{2\phi}^{(\text{hd})} &\sim \frac{\mu_0}{4\pi} \cos\phi \int_0^\infty d\lambda \lambda \left\{ [J_0(\lambda\rho) - J_2(\lambda\rho)] \frac{k^2}{i\lambda\omega\mu_0\sigma} + [J_0(\lambda\rho) + J_2(\lambda\rho)] \frac{1}{2} \right\} \\ &\sim \frac{\mu_0}{4\pi} \cos\phi \frac{1}{\rho} \int_0^\infty d\lambda J_1(\lambda\rho) = \frac{\mu_0}{4\pi} \frac{\cos\phi}{\rho^2}. \end{aligned} \quad (\text{B26})$$

Now compare this approximation to (70b); the latter reduces to the above expression by expansion in powers of $k\rho$ and $k_m\rho$. The remaining component of the magnetic field is

$$B_{2z}^{(\text{hd})} \sim \frac{i\mu_0}{2\pi} \sin\phi \int_0^\infty d\lambda \lambda^2 J_1(\lambda\rho) \frac{1}{2i\lambda} = \frac{\mu_0}{4\pi} \frac{\sin\phi}{\rho^2}. \quad (\text{B27})$$

This formula manifestly agrees with (66b) if $|k\rho| \ll 1$ and $|\omega\mu_0\sigma| \ll |k|$.

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