Bose–Einstein condensation in an external potential at zero temperature: Solitary-wave theory

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(Received 28 June 1999; accepted for publication 26 July 1999)

For a trapped, dilute atomic gas of short-range, repulsive interactions at extremely low temperatures, when Bose–Einstein condensation is nearly complete, some special forms of the time-dependent condensate wave function and the pair-excitation function, the latter being responsible for phonon creation, are investigated. Specifically, (i) a class of external potentials $V_e(\mathbf{r},t)$ that allow for localized, shape-preserving solutions to the nonlinear Schrödinger equation for the condensate wave function, each recognized as a solitary wave moving along an arbitrary trajectory, is derived and analyzed in any number of space dimensions; and (ii) for any such external potential and condensate wave function, the nonlinear integro-differential equation for the pair-excitation function is shown to admit solutions of the same nature. Approximate analytical results are presented for a sufficiently slowly varying trapping potential. Numerical results are obtained for the condensate wave function when V_e is a time-independent, spherically symmetric harmonic potential. © *1999 American Institute of Physics.* [S0022-2488(99)03211-9]

I. INTRODUCTION

The first successful experiments on Bose–Einstein condensation in dilute atomic gases were reported recently by the groups at JILA,[1] Rice University,[2] and MIT.[3] In their respective experiments, vapors of ^{87}Rb, ^{7}Li, and ^{23}Na atoms were confined by traps of inhomogeneous magnetic fields acting on the spin of the unpaired electron of each atom. A combination of laser and evaporative cooling techniques were employed to cool each gas below the phase transition point. Many similar experiments followed soon after these pioneering works. These experimental observations have, in turn, stimulated theoretical interest, with emphasis on the study of the effect on condensation of parameters that can be controlled externally, aiming at new predictions or designs of future experiments. Major problems related to Bose–Einstein condensation in a trap include equilibrium and nonequilibrium properties of the boson gas, such as collective excitations and vortices, and description of time evolution under the influence of time-dependent trapping potentials.

An entirely quantum mechanical treatment of Bose–Einstein condensation in dilute systems of hard spheres lacking translational symmetry at extremely low temperatures, when condensation into a single-particle state is nearly complete, was given in 1961 by Wu.[4] In his approach, the two crucial quantities for the minimal description of the Bose system are: (i) the condensate wave function $\Phi(\mathbf{r},t)$, which, to the lowest approximation in the particle density, satisfies a Schrödinger equation with a self-coupling term of third order, also derived by Gross[5] and Pitaevskii[6] by other methods, and (ii) the pair-excitation function $K_0(\mathbf{r},\mathbf{r}';t)$, which describes the scattering of two atoms from the condensate to other states at positions $\mathbf{r}$ and $\mathbf{r}'$, offers a systematic treatment of physical effects such as sound vibrations, and provides corrections to higher orders in the particle density; $K_0(\mathbf{r},\mathbf{r}';t)$ was shown to satisfy a nonlinear integro-differential equation. To the lowest approximation, this analysis has recently been extended both for zero and finite temperatures to incorporate the effect of a sufficiently smooth external potential that increases rapidly at large

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distances.[7–9] Of particular significance is the underlying ansatz for the many-body Schrödinger state vector at zero temperature:[4,7,8]

$$\mathbf{\Psi}(t)=\mathcal{N}(t)e^{\mathcal{P}(t)}(N!)^{-1/2}a_0^*(t)^N|\mathrm{vac}\rangle, \tag{1.1}$$

where $\mathcal{P}(t)$ describes the creation of pairs from the condensate

$$\mathcal{P}(t)=(2N)^{-1}\int d\mathbf{r}\,d\mathbf{r}'\,\psi_1^*(\mathbf{r},t)\psi_1^*(\mathbf{r}',t)K_0(\mathbf{r},\mathbf{r}';t)a_0(t)^2, \tag{1.2}$$

$\mathcal{N}(t)$ is the normalization constant, which is immaterial for present purposes, N is the total number of atoms, $a_0^*(t)$ and $a_0(t)$ are the creation and annihilation operators for the condensate, respectively, and $\psi_1^*(\mathbf{r},t)$ is the boson creation field operator corresponding to the space orthogonal to the condensate wave function. Formula (1.1), being combined with a consistent approximation for the N-body Hamiltonian, is a nontrivial generalization of the many-body wave function of Lee, Huang, and Yang[10] for the case with translational invariance and periodic boundary conditions, where the main effect of particle interactions is the creation and annihilation of pairs of opposite momenta. The inclusion of pair excitation according to Eq. (1.1) necessarily modifies the equation of motion for $\Phi(\mathbf{r},t)$. Some physically interesting implications of this second-order approximation without any external potential, such as the difference between a compressional wave and a phonon, are discussed in Ref. 4.

Recent numerical or analytical studies of properties of nonuniform atomic gases undergoing Bose–Einstein condensation at extremely low temperatures have focused on the nonlinear Schrödinger equation for the condensate wave function either in its time-independent[8,11–14] or its time-dependent form.[15] A different approach by Benjamin, Quiroga, and Johnson[16] deals with the relative motion of the atoms in a hyperspherical coordinate system, with application to two-dimensional harmonic traps. In other contexts, several types of nonlinear Schrödinger equations are examined in the light of soliton theory,[17] often with emphasis on the description and conditions of existence of a pulselike solution—from now on referred to as a solitary wave—whose main feature is the preservation of its shape during propagation. A summary and discussion of some of these approaches can be found in the very recent comprehensive paper by Morgan *et al.*,[18] whose terminology is mainly adopted here.

Soliton theory usually describes nonlinear waves that interact like classical elastic particles, in the sense that the initial shape and velocity of the waves are regained asymptotically, yet possibly with a phase shift. Studies of such a behavior are believed to have been motivated from some unusual findings in a computation by Fermi, Pasta, and Ulam in 1955.[19,20] Significant advances toward the understanding of solutions to the underlying Korteweg–deVries (or KdV) equation were made ten years later by Zabusky and Kruskal,[21] followed by systematic investigations of Gardner *et al.*[22] A good list of references and exposition of methods or concepts germane to widely known types of evolution equations are given in Ref. 23. It has been realized that a central role in soliton theory is played by the ''Bäcklund transformations,'' which have provided a test for solitonic behavior and led to higher soliton solutions to some equations. (For a review of the mathematically advanced theory, see Ref. 19 and the references therein.)

It is well-known that the Schrödinger equation with a self-coupling term of third order and zero external potential admits soliton solutions in the sense of Ref. 24. In general, the inclusion of a term accounting for an external potential modifies the nature of the associated solutions, as is pointed out in Ref. 18. Specifically, Morgan *et al.*[18] examine conditions on nonlinear terms and accompanying external potentials that allow for localized solitary-wave solutions, and provide a physical interpretation of their results. They justifiably conclude that (i) such nonlinearities should not explicitly depend on the space variable x in $(1+1)$ dimensions, and (ii) the change in the potential experienced by the wave must be linear in x. They subsequently attempt to extend their results to higher dimensions, with restriction to motion along fixed axes in space. This in turn imposes conditions on the external potential, which they briefly describe. Notably, one-dimensional motion of shape-preserving pulses of the condensate wave function is also studied in

Refs. 25 and 26 for positive and negative scattering lengths, respectively, with restriction to time-independent parabolic potentials of weak confinement along one specified axis (cigar-shaped traps).

It should be emphasized, however, that, although it simplifies the treatment, the assumption of rectilinear motion in a space of dimensions higher than one is not necessary for the existence of solitary-wave solutions: motion of the solitary wave along an arbitrary trajectory in any number of space dimensions is possible, provided the external potential is consistently chosen. Furthermore, in Refs. 18, 25, and 26 the effects of scattering processes due to atomic interactions are ignored. Such a simplified approach, though adequate for some cases of experimental relevance, is certainly physically incomplete and needs improvement. It has been argued by others, for instance, that predictions based on the usual nonlinear Schrödinger equation become, in general, questionable for time-dependent systems, when the number of noncondensed particles may grow in time.[27] In the present paper, scattering processes are minimally taken into account through the *joint* consideration of the condensate wave function and the pair-excitation function.[4,7–9] The purpose of this work is to study solitary-wave motion by addressing the aforementioned issues in some detail, complementing, therefore, the analysis in Ref. 18, as a step toward an understanding of more complicated nonequilibrium properties of the trapped Bose gas. An outline of the paper is provided below.

In Sec. II, external potentials $V_e(\mathbf{r},t)$ in $(d+1)$ dimensions $(d\geqslant 1)$ are analyzed under the assumption that they sustain a condensate wave function identified with a single pulse that preserves its shape while moving along an arbitrarily prescribed trajectory in the d-dimensional Euclidean space. Focus is on the Schrödinger equation containing a cubic self-coupling term and positive scattering length a. The analysis starts with $d=1$, but with a perspective different from Ref. 18, and proceeds to generalizing to $d\geqslant 2$. Given a consistent V_e, the initial condition for the condensate wave function, when the nonlinearity plays an important role, is discussed. An argument is sketched to verify that, as a consequence of the requisite decomposition for the potential, the harmonic potentials constitute the sole class of admissible time-independent potentials that allow for solitary-wave solutions.[28] Furthermore, the assumption of nonuniqueness of the derived decomposition for the potential furnishes a class of time-dependent harmonic potentials. In Sec. III, it is demonstrated that the corresponding lowest-order nonlinear integro-differential equation for the pair-excitation function admits solitary waves in $(2d+1)$ dimensions. Section IV proceeds to determine approximately the initial amplitudes for the condensate wave function and the pair-excitation function corresponding to the lowest state of the condensate in a case of experimental interest, namely, when the trapping potential is slowly varying in space. In Sec. V, both analytical and numerical results are obtained for the lowest-energy condensate wave function under a three-dimensional, spherically symmetric harmonic potential.

II. THE CONDENSATE WAVE FUNCTION

The time-dependent nonlinear Schrödinger equation for the condensate wave function $\Phi(\mathbf{r},t)$ in an external potential $V_e(\mathbf{r},t)$ is $(\hbar=2m=1)$[7,8]

$$i(\partial/\partial t)\Phi(\mathbf{r},t)=[-\nabla^2+V_e(\mathbf{r},t)+8\pi aN\Omega^{-1}|\Phi(\mathbf{r},t)|^2-4\pi aN\Omega^{-1}\zeta(t)]\Phi(\mathbf{r},t), \tag{2.1}$$

where

$$\Omega^{-1}\int d\mathbf{r}|\Phi(\mathbf{r},t)|^2=1, \tag{2.2}$$

$$\zeta(t)=\Omega^{-1}\int d\mathbf{r}|\Phi(\mathbf{r},t)|^4, \tag{2.3}$$

a is the scattering length, assumed to be positive, N is the number of particles, and Ω is the volume of the system.

For mathematical convenience, Eq. (2.1) is cast in the form

$$i(\partial/\partial t)F(\mathbf{r},t)=[-\nabla^2+V_e(\mathbf{r},t)+|F(\mathbf{r},t)|^2]F(\mathbf{r},t), \tag{2.4}$$

where

$$F(\mathbf{r},t)=(8\pi a\rho_0)^{1/2}e^{-i4\pi\alpha\rho_0\sigma(t)}\Phi(\mathbf{r},t), \quad \rho_0=N/\Omega, \tag{2.5}$$

provided that

$$\sigma(t)=\int^t dt\,\zeta(t)+\text{const}, \tag{2.6}$$

where $\int^t$ denotes an indefinite integral. The normalization condition (2.2) now reads

$$\int d\mathbf{r}|F(\mathbf{r},t)|^2=8\pi aN. \tag{2.7}$$

A. The one-dimensional nonlinear Schrödinger equation

In the one-dimensional case, both the external potential and the condensate wave function depend on one space variable, say x. Equation (2.4) then becomes

$$i\frac{\partial F(x,t)}{\partial t}=\left[-\frac{\partial^2}{\partial x^2}+V_e(x,t)+|F(x,t)|^2\right]F(x,t). \tag{2.8}$$

For $V_e=0$, this reduces to the more or less standard form of the nonlinear Schrödinger equation.[24] Solitary-wave solutions of this equation are assumed to be of the form (see the Appendix):

$$F(x,t)=f(x-\alpha(t))e^{-i\theta(x,t)}, \tag{2.9}$$

where $f(x)$ and $\theta(x,t)$ are real functions, sufficiently smooth in x and t, and $\alpha(t)$ is a continuously differentiable function of time. Under the assumption of a potential $V_e(x,t)$ increasing sufficiently rapidly for $x\to\pm\infty$, it is necessary to require that

$$f(x)\to 0 \quad \text{rapidly} \quad \text{as } |x|\to\infty. \tag{2.10}$$

The example of the one-dimensional harmonic oscillator (briefly reviewed in the Appendix) suggests that f should decrease faster than exponentially in $|x|$ for large values of $|x|$. The same conclusion can be reached by employing the Wentzel–Kramers–Brillouin method.

The substitution of Eq. (2.9) into Eq. (2.8), and separation of real and imaginary parts, yield a system of coupled differential equations for f and θ:

$$f(x-\alpha(t))\frac{\partial^2\theta}{\partial x^2}+2f'(x-\alpha(t))\frac{\partial\theta}{\partial x}=-\alpha'(t)f'(x-\alpha(t)), \tag{2.11}$$

$$-f''(x-\alpha(t))+\left(\frac{\partial\theta}{\partial x}\right)^2 f(x-\alpha(t))+[V_e(x,t)+f(x-\alpha(t))^2]f(x-\alpha(t))=\frac{\partial\theta}{\partial t}f(x-\alpha(t)), \tag{2.12}$$

where the prime denotes differentiation with respect to argument. Equation (2.11) can be rewritten as

$$\frac{\partial}{\partial x}\left[f(x-\alpha(t))^2\frac{\partial\theta}{\partial x}\right]=-\alpha'(t)f'(x-\alpha(t))f(x-\alpha(t)). \tag{2.13}$$

This is explicitly integrated to give

$$\frac{\partial\theta}{\partial x}=-\frac{1}{2}\alpha'(t)+\frac{A_1(t)}{f(x-\alpha(t))^2}, \tag{2.14}$$

except at points $x=x(t)$ where $f(x-\alpha(t))$ vanishes. It immediately follows that

$$\theta(x,t)=\int^x dx\,\frac{\partial\theta}{\partial x}+A(t)=-\frac{1}{2}\alpha'(t)x+\int^x dx\,\frac{A_1(t)}{f(x-\alpha(t))^2}+A(t), \tag{2.15}$$

where x lies between consecutive zeros of $f(x-\alpha(t))$, calling for the possible use of different corresponding A_1's and A's. Consider the simplest case where f has no zeros. According to the preceding formula, for nonzero $A_1(t)$, the limiting behavior of f at large distances x and fixed time t gives rise to increasingly rapid oscillations in x of the real and imaginary parts of the condensate wave function. This in turn implies an infinite expectation value of the kinetic energy term $-\partial^2/\partial x^2$ in the Hamiltonian of the system. To eliminate this unphysical possibility, it is necessary to set $A_1(t)$ equal to zero:

$$A_1(t)\equiv 0. \tag{2.16}$$

To put this argument on a firm foundation, it is expedient to invoke the following conditions.

(i) Normalizability of $F(x,t)$ from Eq. (2.7), viz.

$$\int dx|F(x,t)|^2=\int dx\, f(x-\alpha(t))^2<\infty. \tag{2.17}$$

(ii) Finite kinetic energy of the condensate, viz.

$$\int dx\, F^*(x,t)\left(-\frac{\partial^2}{\partial x^2}\right)F(x,t)=\int dx\left|\frac{\partial F}{\partial x}\right|^2<\infty. \tag{2.18}$$

The last condition entails

$$\int dx\, f'(x-\alpha(t))^2<\infty, \tag{2.19a}$$

$$\int dx\left(\frac{\partial\theta}{\partial x}\right)^2 f(x-\alpha(t))^2<\infty. \tag{2.19b}$$

The use of Eqs. (2.14) and (2.17) in Eq. (2.19b) gives

$$\int dx\, A_1(t)\left[-\alpha'(t)+\frac{A_1(t)}{f(x-\alpha(t))^2}\right]<\infty, \tag{2.20}$$

which is impossible unless identity (2.16) holds. A similar argument can be applied to the case where f has any number of zeros.

For smooth real f, the resulting phase $\theta(x,t)$ is

$$\theta(x,t)=-\tfrac{1}{2}\alpha'(t)x+A(t), \tag{2.21}$$

in agreement with Eq. (9) of Ref. 18. The substitution of Eq. (2.21) into Eq. (2.12) yields a consistency equation for $V_e(x,t)$:

$$f''(x-\alpha(t))=[V_e(x,t)+f(x-\alpha(t))^2+\tfrac{1}{2}\alpha''(t)x+\tfrac{1}{4}\alpha'(t)^2-A'(t)]f(x-\alpha(t)). \tag{2.22}$$

It is inferred that $V_e(x,t)$ must be expressed as

$$V_e(x,t)=\mathcal{V}_1(x-\alpha(t))+x\mathcal{V}_2(t)+\mathcal{V}_3(t), \tag{2.23}$$

where

$$\mathcal{V}_1(x)=\frac{f''(x)}{f(x)}-f(x)^2, \tag{2.24}$$

$$\mathcal{V}_2(t)=-\tfrac{1}{2}\alpha''(t), \tag{2.25}$$

$$\mathcal{V}_3(t)=-\tfrac{1}{4}\alpha'(t)^2+A'(t). \tag{2.26}$$

Equation (2.23) gives the requisite form of potentials for given $f(x)$, $\alpha(t)$, and $A(t)$. Note that some of the inflection points of $f(x)$ need to coincide with its zeros. By close examination of Eqs. (2.23)–(2.26), the following should be pointed out.

(1) Given a $\mathcal{V}_1(x)$, the differential equation (2.24) suggests, in some sense, an eigenvalue problem. More particularly, when $|x|$ is sufficiently large, condition (2.10) becomes effective, indicating that $f^2 \ll |f''/f|$. Under this approximation, Eq. (2.24) becomes

$$f''(x)\sim\mathcal{V}_1(x)f(x), \tag{2.27a}$$

which is a linear equation. Hence, only discrete shifts ϵ_m of $\mathcal{V}_1(x)=\mathcal{V}_{1m}(x)$ are permissible, corresponding to ''eigenfunctions'' $f=f_m$ (m=non-negative integer). These shifts in turn induce discrete amounts of shift in $A'(t)$ through Eqs. (2.23) and (2.26). Accordingly, $F(x,t)$ exhibits a behavior of the form $e^{-i\epsilon_m t}f_m(x-\alpha(t))$ in the fixed trapping potential

$$\mathcal{V}_e(x)=\mathcal{V}_{1m}(x)+\sum_{l\leq m-1}\epsilon_l+C_0 \tag{2.27b}$$

experienced by the pulse, where C_0 is a constant.[29]

(2) For $\alpha(t)$ different from a constant, the only class of time-independent potentials $V_e(x,t)=V_e(x)$ of the form (2.23) consists of the harmonic potentials. Indeed, differentiation in x of both sides of Eq. (2.23) twice yields

$$V_e''(x)=\mathcal{V}_l''(x-\alpha(t))=K=\text{const}>0. \tag{2.28}$$

Hence,

$$V_e(x)=\tfrac{1}{2}Kx^2+\overline{K}x+\mathcal{C}. \tag{2.29}$$

(3) If $V_e(x,t)$ admits a second decomposition

$$V_e(x,t)=\mathcal{U}_1(x-\beta(t))+x\mathcal{U}_2(t)+\mathcal{U}_3(t), \tag{2.30}$$

where

$$\mathcal{U}_1(x)=\frac{\breve{f}''(x)}{\breve{f}(x)}-\breve{f}(x)^2, \tag{2.31}$$

$$\mathcal{U}_2(t)=-\tfrac{1}{2}\beta''(t), \tag{2.32}$$

$$\mathcal{U}_3(t)=-\tfrac{1}{4}\beta'(t)^2+B'(t), \tag{2.33}$$

and $\mathcal{U}_1(x)\neq\mathcal{V}_1(x)$, $\mathcal{U}_3(t)\neq\mathcal{V}_3(t)$, two cases for $\alpha(t)$ and $\beta(t)$ need to be distinguished.

(i) $\alpha(t)-\beta(t)\neq$const. Differentiation of Eqs. (2.23) and (2.30) with respect to x twice yields

$$\mathcal{V}_1''(x-\alpha(t))=\mathcal{U}_1''(x-\beta(t))=K. \tag{2.34}$$

Therefore,

$$V_e(x,t)=\tfrac{1}{2}K[x-\alpha(t)]^2+K_1[x-\alpha(t)]+K_2+x\mathcal{V}_2(t)+\mathcal{V}_3(t) \tag{2.35a}$$

$$=\tfrac{1}{2}K[x-\beta(t)]^2+M_1[x-\beta(t)]+M_2+x\mathcal{U}_2(t)+\mathcal{U}_3(t), \tag{2.35b}$$

i.e., $V_e(x,t)$ is the *time-dependent* harmonic potential

$$V_e(x,t)=\tfrac{1}{2}Kx^2+\overline{K}(t)x+\mathcal{C}(t) \quad (K>0). \tag{2.36}$$

A comparison of Eqs. (2.35a) and (2.35b) furnishes the consistency equations

$$-K\alpha(t)+K_1+\mathcal{V}_2(t)=-K\beta(t)+M_1+\mathcal{U}_2(t), \tag{2.37a}$$

$$\tfrac{1}{2}K\alpha(t)^2-K_1\alpha(t)+K_2+\mathcal{V}_3(t)=\tfrac{1}{2}K\beta(t)^2-M_1\beta(t)+M_2+\mathcal{U}_3(t). \tag{2.37b}$$

(ii) $\alpha(t)-\beta(t)=\mathcal{C}_1=$const. From Eqs. (2.25) and (2.32),

$$\mathcal{V}_2(t)=\mathcal{U}_2(t). \tag{2.38}$$

Equations (2.23) and (2.30) combined give

$$\mathcal{V}_3(t)-\mathcal{U}_3(t)=\mathcal{U}_1(x-\mathcal{C}_1)-\mathcal{V}_1(x)=\epsilon=\text{const.} \tag{2.39}$$

In view of (2.26) and (2.33),

$$A(t)=B(t)+\epsilon t+\text{const.} \tag{2.40}$$

The meaning of this ϵ becomes apparent from Eqs. (2.27): it is the discrete amount of shift in $\mathcal{V}_1(x)$ corresponding to a shift from the ''eigenfunction'' $f(x)$ to another ''eigenfunction'' $\breve{f}(x)$ under the same trapping potential $\mathcal{V}_e$ experienced by the solitary wave.

B. The nonlinear Schrödinger equation in *d* space dimensions, $d\geqslant 2$

The foregoing analysis in one dimension can be extended to higher dimensions. For definiteness, consider $d=3$. In accord with the conditions in the recent experiments,[1–3] it is assumed that

$$V_e(\mathbf{r},t)\to+\infty, \quad \text{uniformly in } \hat{\mathbf{r}}=\mathbf{r}/|\mathbf{r}| \quad \text{as } r=|\mathbf{r}|\to\infty. \tag{2.41}$$

Instead of assuming motion of the solitary wave along a fixed axis, as is the case in Ref. 18, let

$$F(\mathbf{r},t)=f(\mathbf{r}-\boldsymbol{\alpha}(t))e^{-i\theta(\mathbf{r},t)}, \tag{2.42}$$

where $\boldsymbol{\alpha}(t)$ is a twice differentiable vector function of time, $f(\mathbf{r})$ and $\theta(\mathbf{r},t)$ are real and sufficiently smooth, and from Eq. (2.7),

$$\int d\mathbf{r}\, f(r-\boldsymbol{\alpha}(t))^2=8\pi aN. \tag{2.43}$$

In view of condition (2.41), it is reasonable to assume that

$$f\to 0 \quad \text{rapidly}, \quad \text{uniformly in } \hat{\mathbf{r}}=\mathbf{r}/|\mathbf{r}| \quad \text{as } r\to\infty, \tag{2.44}$$

ensuring that the condensate is localized and has a finite kinetic energy, as indicated in the Appendix. The substitution of Eq. (2.42) in Eq. (2.4) gives

$$-i\boldsymbol{\alpha}'(t)\cdot\nabla f(\mathbf{r}-\boldsymbol{\alpha}(t))+f(\mathbf{r}-\boldsymbol{\alpha}(t))\frac{\partial\theta(\mathbf{r},t)}{\partial t}=-\nabla^2 f(\mathbf{r}-\boldsymbol{\alpha}(t))+2i\nabla f(\mathbf{r}-\boldsymbol{\alpha}(t))\cdot\nabla\theta(\mathbf{r},t)$$
$$+if(\mathbf{r}-\boldsymbol{\alpha}(t))\nabla^2\theta(\mathbf{r},t)+f(\mathbf{r}-\boldsymbol{\alpha}(t))|\nabla\theta(\mathbf{r},t)|^2$$
$$+[V_e(\mathbf{r},t)+f(\mathbf{r}-\boldsymbol{\alpha}(t))^2]f(\mathbf{r}-\boldsymbol{\alpha}(t)). \tag{2.45}$$

Upon separation of real and imaginary parts, the preceding equation decomposes into

$$-\boldsymbol{\alpha}'(t)\cdot\nabla f(\mathbf{r}-\boldsymbol{\alpha}(t))=2\nabla f(\mathbf{r}-\boldsymbol{\alpha}(t))\cdot\nabla\theta(\mathbf{r},t)+f(\mathbf{r}-\boldsymbol{\alpha}(t))\nabla^2\theta(\mathbf{r},t), \tag{2.46}$$

$$f(\mathbf{r}-\boldsymbol{\alpha}(t))\frac{\partial\theta(\mathbf{r},t)}{\partial t}=-\nabla^2 f(\mathbf{r}-\boldsymbol{\alpha}(t))+|\nabla\theta(\mathbf{r},t)|^2 f(\mathbf{r}-\boldsymbol{\alpha}(t))$$
$$+[V_e(\mathbf{r},t)+f(\mathbf{r}-\boldsymbol{\alpha}(t))^2]f(\mathbf{r}-\boldsymbol{\alpha}(t)). \tag{2.47}$$

Equation (2.46) is recast in the form

$$\nabla\cdot(f^2\nabla\theta)=-f\boldsymbol{\alpha}'(t)\cdot\nabla f, \quad f=f(\mathbf{r}-\boldsymbol{\alpha}(t)), \tag{2.48}$$

which holds regardless of the specific form for the shape $f=f(\mathbf{r},t)$ of $F(\mathbf{r},t)$. A particular solution to this equation is

$$\theta_p(\mathbf{r},t)=-\tfrac{1}{2}\boldsymbol{\alpha}'(t)\cdot\mathbf{r}+A(t). \tag{2.49}$$

With $\theta=\theta_p+\theta_1$, $\theta_1(\mathbf{r},t)$ satisfies the homogeneous equation

$$\nabla\cdot(f^2\nabla\theta_1)=0. \tag{2.50}$$

Integration by parts over a finite region $\mathcal{R}$ bounded by a surface $\mathcal{S}$ yields

$$0=\int d\mathbf{r}\,\theta_1\nabla\cdot(f^2\nabla\theta_1)=\oint_{\mathcal{S}}d\mathcal{S}\,f^2\theta_1\hat{\mathbf{n}}\cdot\nabla\theta_1-\int d\mathbf{r}\,f^2|\nabla\theta_1|^2, \tag{2.51}$$

where $\hat{\mathbf{n}}$ is the unit vector normal to $\mathcal{S}$ pointing outward. When $\mathcal{R}$ extends to infinity, the surface integral becomes arbitrarily small because of the condition (2.44), in analogy with the one-dimensional case. Consequently,

$$f^2|\nabla\theta_1|^2=0 \quad \text{almost everywhere}, \tag{2.52}$$

i.e., except for a set of points of measure zero. When $f\neq 0$, this in turn entails

$$\theta_1(\mathbf{r},t)=C_1(t) \quad \text{almost everywhere}. \tag{2.53}$$

At the zeros of f, $|\nabla\theta_1|$ seems to be indeterminate, calling for the use of different C_1's in Eq. (2.53). However, for a sufficiently smooth $\theta(\mathbf{r},t)$, $C_1(t)$ can be taken to be zero everywhere without loss of generality. Accordingly, $\theta(\mathbf{r},t)$ reads

$$\theta(\mathbf{r},t)=\theta_p(\mathbf{r},t)=-\tfrac{1}{2}\boldsymbol{\alpha}'(t)\cdot\mathbf{r}+A(t), \tag{2.54}$$

which is a generalization of Eq. (2.21).

The external potential consistent with Eqs. (2.47) and (2.54) is

$$V_e(\mathbf{r},t)=\mathcal{V}_1(\mathbf{r}-\boldsymbol{\alpha}(t))+\boldsymbol{\mathcal{V}}_2(t)\cdot\mathbf{r}+\mathcal{V}_3(t), \tag{2.55}$$

where

$$\mathcal{V}_1(\mathbf{r}) = \frac{\nabla^2 f(\mathbf{r})}{f(\mathbf{r})} - f(\mathbf{r})^2, \tag{2.56}$$

$$\boldsymbol{\mathcal{V}}_2(t) = -\tfrac{1}{2}\boldsymbol{\alpha}''(t), \tag{2.57}$$

$$\mathcal{V}_3(t) = -\tfrac{1}{4}|\boldsymbol{\alpha}'(t)|^2 + A'(t). \tag{2.58}$$

Notably, $\nabla^2 f(\mathbf{r})$ needs to vanish at any surface where $f(\mathbf{r})$ vanishes.

A few important remarks are in order.

(1) For an external potential increasing in $|\mathbf{r}|$, Eq. (2.56) bears the features of an eigenvalue problem. Specifically, for $|\mathbf{r}|\to\infty$, a linear equation is recovered approximately:

$$\nabla^2 f(\mathbf{r}) \sim \mathcal{V}_1(\mathbf{r}) f(\mathbf{r}). \tag{2.59}$$

Analogies with the one-dimensional case are easily drawn from this equation.

(2) When $\boldsymbol{\alpha}(t)$ is not a constant, the only time-independent potential of the form (2.55) that satisfies condition (2.41) is the d-dimensional harmonic potential. The justification for this is somewhat more demanding than for the one-dimensional case. With $V_e(\mathbf{r},t) = V_e(\mathbf{r})$, the application of the Laplacian to both sides of Eq. (2.55) gives

$$\nabla^2 V_e(\mathbf{r}) = \nabla^2 \mathcal{V}_1(\mathbf{r} - \boldsymbol{\alpha}(t)) = K = \text{const} > 0. \tag{2.60}$$

In three dimensions, a solution to Eq. (2.60) for $V_e(\mathbf{r})$ is:

$$V_p(\mathbf{r}) = \frac{1}{2}\sum_{i,j=1,2,3} K_{ij} x_i x_j + \sum_{j=1,2,3} \overline{K}_j x_j + \mathcal{C}, \tag{2.61}$$

where $(x_1, x_2, x_3) = \mathbf{r} = (x,y,z)$,

$$\mathrm{Tr}[K_{ij}] = K, \tag{2.62}$$

and the matrix $[K_{ij}]$ is symmetric and positive definite. Every admissible solution to Eq. (2.60) can be written as

$$V_e(\mathbf{r}) = V_p(\mathbf{r}) + V_1(\mathbf{r}), \tag{2.63}$$

where $V_1(\mathbf{r})$ is a smooth function satisfying Laplace's equation:

$$\nabla^2 V_1(\mathbf{r}) = 0 \quad \text{everywhere}. \tag{2.64}$$

If $\mathcal{S}$ is now a spherical surface with center $\mathbf{r}$ and radius R, then according to Gauss' mean value theorem[30]

$$V_1(\mathbf{r}) = \frac{1}{4\pi R^2} \oint_{\mathcal{S}} dS' \, V_1(\mathbf{r}'). \tag{2.65}$$

Since R can be taken to be arbitrarily large, it follows that V_1 cannot be forced to comply with condition (2.41). Consequently, $V_1(\mathbf{r})$ is equal to a constant. Without loss of generality,

$$V_1(\mathbf{r}) \equiv 0. \tag{2.66}$$

(3) Let $V_e(\mathbf{r},t)$ admit an alternative decomposition,

$$V_e(\mathbf{r},t) = \mathcal{U}_1(\mathbf{r} - \boldsymbol{\beta}(t)) + \mathbf{r}\cdot\boldsymbol{\mathcal{U}}_2(t) + \mathcal{U}_3(t), \tag{2.67}$$

where

$$\mathcal{U}_1(\mathbf{r})=\frac{\nabla^2\breve{f}(\mathbf{r})}{\breve{f}(\mathbf{r})}-\breve{f}(\mathbf{r})^2, \tag{2.68}$$

$$\boldsymbol{\mathcal{U}}_2(t)=-\tfrac{1}{2}\boldsymbol{\beta}''(t), \tag{2.69}$$

$$\mathcal{U}_3(t)=-\tfrac{1}{4}|\boldsymbol{\beta}'(t)|^2+B'(t), \tag{2.70}$$

and $\mathcal{U}_1(\mathbf{r})\neq\mathcal{V}_1(\mathbf{r})$, $\mathcal{U}_3(t)\neq\mathcal{V}_3(t)$. In analogy with the one-dimensional case, there are two distinct possibilities.

(i) $\boldsymbol{\alpha}(t)-\boldsymbol{\beta}(t)\neq$ const. Then,

$$\nabla^2\mathcal{V}_1(\mathbf{r}-\boldsymbol{\alpha}(t))=\nabla^2\mathcal{U}_1(\mathbf{r}-\boldsymbol{\beta}(t))=K, \tag{2.71}$$

which in turn implies that

$$\begin{aligned}V_e(\mathbf{r},t)&=\frac{1}{2}\sum_{i,j=1,2,3}K_{ij}[x_i-\alpha_i(t)][x_j-\alpha_j(t)]+[\mathbf{r}-\boldsymbol{\alpha}(t)]\cdot\mathbf{K}_1+K_2+\mathbf{r}\cdot\boldsymbol{\mathcal{V}}_2(t)+\mathcal{V}_3(t)\\&=\frac{1}{2}\sum_{i,j=1,2,3}M_{ij}[x_i-\beta_i(t)][x_j-\beta_j(t)]+[\mathbf{r}-\boldsymbol{\beta}(t)]\cdot\mathbf{M}_1+M_2+\mathbf{r}\cdot\boldsymbol{\mathcal{U}}_2(t)+\mathcal{U}_3(t),\end{aligned} \tag{2.72}$$

where

$$\mathrm{Tr}[K_{ij}]=\mathrm{Tr}[M_{ij}]=K, \tag{2.73}$$

and K_2, M_2 are immaterial constants. Therefore, $V_e(\mathbf{r},t)$ is the time-dependent harmonic potential

$$V_e(\mathbf{r},t)=\frac{1}{2}\sum_{i,j=1,2,3}K_{ij}x_ix_j+\mathbf{r}\cdot\overline{\mathbf{K}}(t)+\mathcal{C}(t). \tag{2.74}$$

(ii) $\boldsymbol{\alpha}(t)-\boldsymbol{\beta}(t)=\mathbf{C}_1=$const. Without loss of generality, $\mathbf{C}_1=0$. It is easily found that

$$\boldsymbol{\mathcal{V}}_2(t)=\boldsymbol{\mathcal{U}}_2(t), \tag{2.75}$$

$$\mathcal{V}_3(t)-\mathcal{U}_3(t)=\mathcal{U}_1(\mathbf{r})-\mathcal{V}_1(\mathbf{r})=\epsilon=\text{const.} \tag{2.76}$$

Equation (2.76) implies that

$$A(t)=B(t)+\epsilon t+\text{const.} \tag{2.77}$$

Therefore, $\breve{f}(\mathbf{r})$ is just another ''eigenfunction'' of Eq. (2.56) under the same trapping potential $\mathcal{V}_e$ seen by the pulse.

III. THE PAIR-EXCITATION FUNCTION

The pair-excitation function $K_0(\mathbf{r},\mathbf{r}';t)$ satisfies the integro-differential equation[8]

$$\left[i\frac{\partial}{\partial t} - 2E(t) \right] K_0(\mathbf{r},\mathbf{r}';t) = -\nabla^2 K_0(\mathbf{r},\mathbf{r}';t) - \nabla'^2 K_0(\mathbf{r},\mathbf{r}';t) + 8\pi a\rho_0 \Phi(\mathbf{r},t)^2 \delta(\mathbf{r}-\mathbf{r}')$$
$$+ \{ -2\overline{\zeta}(t) - 16\pi a\rho_0 \zeta(t) - 2\zeta_e(t) + V_e(\mathbf{r},t) + V_e(\mathbf{r}',t)$$
$$+ 16\pi a\rho_0 [|\Phi(\mathbf{r},t)|^2 + |\Phi(\mathbf{r}',t)|^2] \} K_0(\mathbf{r},\mathbf{r}';t)$$
$$+ 8\pi a\rho_0 \int d\mathbf{r}'' \, \Phi^*(\mathbf{r}'',t)^2 K_0(\mathbf{r},\mathbf{r}'';t) K_0(\mathbf{r}',\mathbf{r}'';t)$$
$$- 8\pi a\rho_0 \Omega^{-1} \Big\{ \Phi(\mathbf{r},t)\Phi(\mathbf{r}',t)[|\Phi(\mathbf{r},t)|^2 + |\Phi(\mathbf{r}',t)|^2 - \zeta(t)]$$
$$+ \Phi(\mathbf{r},t) \int d\mathbf{r}'' \, K_0(\mathbf{r}',\mathbf{r}'';t) |\Phi(\mathbf{r}'',t)|^2 \Phi^*(\mathbf{r}'',t)$$
$$+ \Phi(\mathbf{r}',t) \int d\mathbf{r}'' \, K_0(\mathbf{r},\mathbf{r}'';t) |\Phi(\mathbf{r}'',t)|^2 \Phi^*(\mathbf{r}'',t) \Big\}, \tag{3.1}$$

where

$$E(t) = i\Omega^{-1} \int d\mathbf{r} \frac{\partial \Phi(\mathbf{r},t)}{\partial t} \Phi^*(\mathbf{r},t), \tag{3.2}$$

$$\overline{\zeta}(t) = \Omega^{-1} \int d\mathbf{r} |\nabla \Phi(\mathbf{r},t)|^2, \quad \zeta_e(t) = \Omega^{-1} \int d\mathbf{r} \, V_e(\mathbf{r},t) |\Phi(\mathbf{r},t)|^2, \tag{3.3}$$

and $\nabla \equiv \nabla_{\mathbf{r}}$, $\nabla' \equiv \nabla_{\mathbf{r}'}$. Without loss of generality, $K_0(\mathbf{r},\mathbf{r}';t)$ has been chosen to satisfy

$$K_0(\mathbf{r},\mathbf{r}';t) = K_0(\mathbf{r}',\mathbf{r};t), \tag{3.4}$$

$$\int d\mathbf{r} \, \Phi^*(\mathbf{r},t) K_0(\mathbf{r},\mathbf{r}';t) = 0. \tag{3.5}$$

In order to investigate the possibility for solitary-wave solutions to Eq. (3.1), the following preliminary steps are taken:

(i) By virtue of Eq. (2.5), $\Phi(\mathbf{r},t)$ is replaced by $(8\pi a\rho_0)^{-1/2} e^{i4\pi a\rho_0 \sigma(t)} F(\mathbf{r},t)$.
(ii) To balance out the exponential factor introduced above, $K_0(\mathbf{r},\mathbf{r}';t)$ is written as

$$K_0(\mathbf{r},\mathbf{r}';t) = e^{i8\pi a\rho_0 \sigma(t)} \mathcal{K}_0(\mathbf{r},\mathbf{r}';t). \tag{3.6}$$

The resulting equation for this $\mathcal{K}_0(\mathbf{r},\mathbf{r}';t)$ is

$$i\frac{\partial \mathcal{K}_0(\mathbf{r},\mathbf{r}';t)}{\partial t} = -\nabla^2 \mathcal{K}_0(\mathbf{r},\mathbf{r}';t) - \nabla'^2 \mathcal{K}_0(\mathbf{r},\mathbf{r}';t) + F(\mathbf{r},t)^2 \delta(\mathbf{r}-\mathbf{r}') + \{ V_e(\mathbf{r},t) + V_e(\mathbf{r}',t)$$
$$+ 2[|F(\mathbf{r},t)|^2 + |F(\mathbf{r}',t)|^2] \} \mathcal{K}_0(\mathbf{r},\mathbf{r}';t) + \int d\mathbf{r}'' \, F^*(\mathbf{r}'',t)^2 \mathcal{K}_0(\mathbf{r},\mathbf{r}'';t) \mathcal{K}_0(\mathbf{r}',\mathbf{r}'';t)$$
$$- (8\pi a N)^{-1} \Big\{ F(\mathbf{r},t) F(\mathbf{r}',t) [|F(\mathbf{r},t)|^2 + |F(\mathbf{r}',t)|^2 - \hat{\zeta}(t)]$$
$$+ F(\mathbf{r},t) \int d\mathbf{r}'' \, \mathcal{K}_0(\mathbf{r}',\mathbf{r}'';t) |F(\mathbf{r}'',t)|^2 F^*(\mathbf{r}'',t)$$
$$+ F(\mathbf{r}',t) \int d\mathbf{r}'' \, \mathcal{K}_0(\mathbf{r},\mathbf{r}'';t) |F(\mathbf{r}'',t)|^2 F^*(\mathbf{r}'',t) \Big\}, \tag{3.7}$$

where

$$\hat{\zeta}(t)=8\pi a\rho_0\zeta(t), \tag{3.8}$$

and $E(t)$ was replaced by

$$E(t)=\overline{\zeta(t)}+\zeta_e(t)+4\pi a\rho_0\zeta(t), \tag{3.9}$$

by employing Eq. (2.1).

Given Eqs. (2.42) and (3.4), solitary-wave solutions are sought in the form

$$\mathcal{K}_0(\mathbf{r},\mathbf{r}';t)=\kappa_0(\mathbf{r}-\boldsymbol{\gamma}(t),\mathbf{r}'-\boldsymbol{\gamma}(t))e^{-i\chi(\mathbf{r},\mathbf{r}';t)}, \tag{3.10}$$

where $\kappa_0(\mathbf{r},\mathbf{r}')$, $\chi(\mathbf{r},\mathbf{r}';t)$, and $\boldsymbol{\gamma}(t)$ are sufficiently smooth real functions satisfying

$$\kappa_0(\mathbf{r},\mathbf{r}')=\kappa_0(\mathbf{r}',\mathbf{r}), \qquad \chi(\mathbf{r},\mathbf{r}';t)=\chi(\mathbf{r}',\mathbf{r};t), \tag{3.11}$$

$$\int d\mathbf{r}\, f(\mathbf{r}-\boldsymbol{\alpha}(t))\kappa_0(\mathbf{r}-\boldsymbol{\gamma}(t),\mathbf{r}'-\boldsymbol{\gamma}(t))e^{i[\theta(\mathbf{r},t)-\chi(\mathbf{r},\mathbf{r}';t)]}=0. \tag{3.12}$$

The substitution of Eq. (3.10) into Eq. (3.7) by virtue of Eqs. (2.42) and (2.55) yields

$$\begin{aligned}
&-i\boldsymbol{\gamma}'(t)\cdot(\nabla\kappa_0+\nabla'\kappa_0)+\kappa_0(\mathbf{r}-\boldsymbol{\gamma}(t),\mathbf{r}'-\boldsymbol{\gamma}(t))\frac{\partial\chi(\mathbf{r},\mathbf{r}';t)}{\partial t}\\
&=-\nabla^2\kappa_0-\nabla'^2\kappa_0+2i(\nabla\kappa_0\cdot\nabla\chi+\nabla'\kappa_0\cdot\nabla'\chi)+i\kappa_0(\mathbf{r}-\boldsymbol{\gamma}(t),\mathbf{r}'-\boldsymbol{\gamma}(t))(\nabla^2\chi+\nabla'^2\chi)\\
&\quad+\kappa_0(\mathbf{r}-\boldsymbol{\gamma}(t),\mathbf{r}'-\boldsymbol{\gamma}(t))(|\nabla\chi|^2+|\nabla'\chi|^2)+f(\mathbf{r}-\boldsymbol{\alpha}(t))^2\delta(\mathbf{r}-\mathbf{r}')e^{i\chi(\mathbf{r},\mathbf{r}';t)-i2\theta(\mathbf{r},t)}\\
&\quad+\{\mathcal{V}_1(\mathbf{r}-\boldsymbol{\alpha}(t))+\mathcal{V}_1(\mathbf{r}'-\boldsymbol{\alpha}(t))+(\mathbf{r}+\mathbf{r}')\cdot\boldsymbol{\mathcal{V}}_2(t)+2\mathcal{V}_3(t)\\
&\quad+2[f(\mathbf{r}-\boldsymbol{\alpha}(t))^2+f(\mathbf{r}'-\boldsymbol{\alpha}(t))^2]\}\kappa_0(\mathbf{r}-\boldsymbol{\gamma}(t),\mathbf{r}'-\boldsymbol{\gamma}(t))\\
&\quad+\int d\mathbf{r}''\, f(\mathbf{r}''-\boldsymbol{\alpha}(t))^2\kappa_0(\mathbf{r}-\boldsymbol{\gamma}(t),\mathbf{r}''-\boldsymbol{\gamma}(t))\kappa_0(\mathbf{r}'-\boldsymbol{\gamma}(t),\mathbf{r}''-\boldsymbol{\gamma}(t))\\
&\quad\times\exp\{2i\theta(\mathbf{r}'',t)-i[\chi(\mathbf{r},\mathbf{r}'';t)+\chi(\mathbf{r}',\mathbf{r}'';t)-\chi(\mathbf{r},\mathbf{r}';t)]\}\\
&\quad-(8\pi aN)^{-1}\Big\{f(\mathbf{r}-\boldsymbol{\alpha}(t))f(\mathbf{r}'-\boldsymbol{\alpha}(t))[f(\mathbf{r}-\boldsymbol{\alpha}(t))^2+f(\mathbf{r}'-\boldsymbol{\alpha}(t))^2-\hat{\zeta}]\\
&\quad\times\exp\{i\chi(\mathbf{r},\mathbf{r}';t)-i[\theta(\mathbf{r},t)+\theta(\mathbf{r}',t)]\}\\
&\quad+f(\mathbf{r}-\boldsymbol{\alpha}(t))\int d\mathbf{r}''\,\kappa_0(\mathbf{r}'-\boldsymbol{\gamma}(t),\mathbf{r}''-\boldsymbol{\gamma}(t))f(\mathbf{r}''-\boldsymbol{\alpha}(t))^3\\
&\quad\times\exp\{i[\theta(\mathbf{r}'',t)-\theta(\mathbf{r},t)]+i[\chi(\mathbf{r},\mathbf{r}';t)-\chi(\mathbf{r}',\mathbf{r}'';t)]\}\\
&\quad+f(\mathbf{r}'-\boldsymbol{\alpha}(t))\int d\mathbf{r}''\,\kappa_0(\mathbf{r}-\boldsymbol{\gamma}(t)),\mathbf{r}''-\boldsymbol{\gamma}(t))f(\mathbf{r}''-\boldsymbol{\alpha}(t))^3\\
&\quad\times\exp\{i[\theta(\mathbf{r}'',t)-\theta(\mathbf{r}',t)]+i[\chi(\mathbf{r},\mathbf{r}';t)-\chi(\mathbf{r},\mathbf{r}'';t)]\}\Big\},
\end{aligned} \tag{3.13}$$

where it is understood that $\kappa_0=\kappa_0(\mathbf{r}-\boldsymbol{\gamma}(t),\mathbf{r}'-\boldsymbol{\gamma}(t))$ and $\chi=\chi(\mathbf{r},\mathbf{r}';t)$, and $\hat{\zeta}$ is now time independent. Elimination of the above phase factors succeeds if χ is taken equal to

$$\chi(\mathbf{r},\mathbf{r}';t)=\theta(\mathbf{r},t)+\theta(\mathbf{r}',t)=-\tfrac{1}{2}\boldsymbol{\alpha}'(t)\cdot(\mathbf{r}+\mathbf{r}')+2A(t). \tag{3.14}$$

In view of Eq. (3.14), separation of the real and imaginary parts in Eq. (3.13) leads to

$$-\boldsymbol{\gamma}'(t)\cdot(\nabla\kappa_0+\nabla'\kappa_0)=2(\nabla\kappa_0\cdot\nabla\chi+\nabla'\kappa_0\cdot\nabla'\chi)+\kappa_0(\mathbf{r}-\boldsymbol{\gamma}(t),\mathbf{r}'-\boldsymbol{\gamma}(t))(\nabla^2\chi+\nabla'^2\chi), \tag{3.15}$$

$$\begin{aligned}
&-\tfrac{1}{2}\kappa_0(\mathbf{r}-\boldsymbol{\gamma}(t),\mathbf{r}'-\boldsymbol{\gamma}(t))\boldsymbol{\alpha}''(t)\cdot(\mathbf{r}+\mathbf{r}')+2\kappa_0(\mathbf{r}-\boldsymbol{\gamma}(t),\mathbf{r}'-\boldsymbol{\gamma}(t))A'(t)\\
&\quad=-\nabla^2\kappa_0-\nabla'^2\kappa_0+\tfrac{1}{2}\kappa_0(\mathbf{r}-\boldsymbol{\gamma}(t),\mathbf{r}'-\boldsymbol{\gamma}(t))|\boldsymbol{\alpha}'(t)|^2+f(\mathbf{r}-\boldsymbol{\alpha}(t))^2\delta(\mathbf{r}-\mathbf{r}')\\
&\qquad+\{\mathcal{V}_1(\mathbf{r}-\boldsymbol{\alpha}(t))+\mathcal{V}_1(\mathbf{r}'-\boldsymbol{\alpha}(t))+(\mathbf{r}+\mathbf{r}')\cdot\boldsymbol{\mathcal{V}}_2(t)+2\mathcal{V}_3(t)\\
&\qquad+2[f(\mathbf{r}-\boldsymbol{\alpha}(t))^2+f(\mathbf{r}'-\boldsymbol{\alpha}(t))^2]\}\kappa_0(\mathbf{r}-\boldsymbol{\gamma}(t),\mathbf{r}'-\boldsymbol{\gamma}(t))\\
&\qquad+\int d\mathbf{r}''\,f(\mathbf{r}''-\boldsymbol{\alpha}(t))^2\kappa_0(\mathbf{r}-\boldsymbol{\gamma}(t)),\mathbf{r}''-\boldsymbol{\gamma}(t))\kappa_0(\mathbf{r}'-\boldsymbol{\gamma}(t),\mathbf{r}''-\boldsymbol{\gamma}(t))\\
&\qquad-(8\pi aN)^{-1}\Big\{f(\mathbf{r}-\boldsymbol{\alpha}(t))f(\mathbf{r}'-\boldsymbol{\alpha}(t))[f(\mathbf{r}-\boldsymbol{\alpha}(t))^2+f(\mathbf{r}'-\boldsymbol{\alpha}(t))^2-\hat{\zeta}]\\
&\qquad+f(\mathbf{r}-\boldsymbol{\alpha}(t))\int d\mathbf{r}''\,\kappa_0(\mathbf{r}'-\boldsymbol{\gamma}(t),\mathbf{r}''-\boldsymbol{\gamma}(t))f(\mathbf{r}''-\boldsymbol{\alpha}(t))^3\\
&\qquad+f(\mathbf{r}'-\boldsymbol{\alpha}(t))\int d\mathbf{r}''\,\kappa_0(\mathbf{r}-\boldsymbol{\gamma}(t),\mathbf{r}''-\boldsymbol{\gamma}(t))f(\mathbf{r}''-\boldsymbol{\alpha}(t))^3\Big\},
\end{aligned} \tag{3.16}$$

of which the first one is satisfied if

$$\boldsymbol{\gamma}(t)=\boldsymbol{\alpha}(t)+\boldsymbol{\alpha}_0, \tag{3.17}$$

where $\boldsymbol{\alpha}_0$ is a vector constant. Without loss of generality, this $\boldsymbol{\alpha}_0$ is set equal to zero.

In Eq. (3.16), $\boldsymbol{\mathcal{V}}_2(t)$ and $\mathcal{V}_3(t)$ are replaced by $-\frac{1}{2}\boldsymbol{\alpha}''(t)$ and $-\frac{1}{4}|\boldsymbol{\alpha}'(t)|^2+A'(t)$ from Eqs. (2.57) and (2.58), respectively. With a subsequent shift both of $\mathbf{r}$ and $\mathbf{r}'$ by $\boldsymbol{\alpha}(t)$, all time dependencies are eliminated and an equation for $\kappa_0(\mathbf{r},\mathbf{r}')$ is obtained:

$$\begin{aligned}
&-\nabla^2\kappa_0(\mathbf{r},\mathbf{r}')-\nabla'^2\kappa_0(\mathbf{r},\mathbf{r}')+f(\mathbf{r})^2\delta(\mathbf{r}-\mathbf{r}')+\{\mathcal{V}_1(\mathbf{r})+\mathcal{V}_1(\mathbf{r}')+2[f(\mathbf{r})^2+f(\mathbf{r}')^2]\}\kappa_0(\mathbf{r},\mathbf{r}')\\
&\quad+\int d\mathbf{r}''\,f(\mathbf{r}'')^2\kappa_0(\mathbf{r},\mathbf{r}'')\kappa_0(\mathbf{r}',\mathbf{r}'')-(8\pi aN)^{-1}\Big\{f(\mathbf{r})f(\mathbf{r}')[f(\mathbf{r})^2+f(\mathbf{r}')^2-\hat{\zeta}]\\
&\quad+f(\mathbf{r})\int d\mathbf{r}''\,\kappa_0(\mathbf{r}',\mathbf{r}'')f(\mathbf{r}'')^3+f(\mathbf{r}')\int d\mathbf{r}''\,\kappa_0(\mathbf{r},\mathbf{r}'')f(\mathbf{r}'')^3\Big\}=0,
\end{aligned} \tag{3.18}$$

where

$$\int d\mathbf{r}\,f(\mathbf{r})\kappa_0(\mathbf{r},\mathbf{r}')=0. \tag{3.19}$$

When the number of particles, N, is sufficiently large, Eq. (3.18) is approximated by

$$\begin{aligned}
&-\nabla^2\kappa_0(\mathbf{r},\mathbf{r}')-\nabla'^2\kappa_0(\mathbf{r},\mathbf{r}')+f(\mathbf{r})^2\delta(\mathbf{r}-\mathbf{r}')+\{\mathcal{V}_1(\mathbf{r})+\mathcal{V}_1(\mathbf{r}')+2[f(\mathbf{r})^2+f(\mathbf{r}')^2]\}\kappa_0(\mathbf{r},\mathbf{r}')\\
&\quad+\int d\mathbf{r}''\,f(\mathbf{r}'')^2\kappa_0(\mathbf{r},\mathbf{r}'')\kappa_0(\mathbf{r}',\mathbf{r}'')=0.
\end{aligned} \tag{3.20}$$

IV. SLOWLY VARYING TRAPPING POTENTIAL

In order to elucidate the dependence on the physical parameters of the problem, let

$$\tilde{\Phi}(\mathbf{r},t)=\sqrt{\rho_0}\Phi(\mathbf{r},t), \quad \rho_0=N/\Omega. \tag{4.1}$$

$\tilde{\Phi}(\mathbf{r},t)$ satisfies

$$i(\partial/\partial t)\tilde{\Phi}(\mathbf{r},t)=[-\nabla^2+V_e(\mathbf{r},t)+8\pi a|\tilde{\Phi}(\mathbf{r},t)|^2-4\pi a\tilde{\zeta}(t)]\tilde{\Phi}(\mathbf{r},t), \tag{4.2}$$

and the normalization condition

$$N^{-1}\int d\mathbf{r}|\tilde{\Phi}(\mathbf{r},t)|^2=1. \tag{4.3}$$

In the above,

$$\tilde{\zeta}(t)=N^{-1}\int d\mathbf{r}|\tilde{\Phi}(\mathbf{r},t)|^4. \tag{4.4}$$

Equation (2.42) reads

$$\tilde{\Phi}(\mathbf{r},t)=\tilde{f}(\mathbf{r}-\boldsymbol{\alpha}(t))\exp\{i\tfrac{1}{2}\boldsymbol{\alpha}'(t)\cdot\mathbf{r}-iA(t)\}, \tag{4.5}$$

where

$$N^{-1}\int d\mathbf{r}\,\tilde{f}(\mathbf{r})^2=1. \tag{4.6}$$

The external potential is

$$V_e(\mathbf{r},t)=\tilde{\mathcal{V}}_1(\mathbf{r}-\boldsymbol{\alpha}(t))+\mathbf{r}\cdot\tilde{\boldsymbol{\mathcal{V}}}_2(t)+\tilde{\mathcal{V}}_3(t), \tag{4.7}$$

where

$$\tilde{\mathcal{V}}_1(\mathbf{r})=\frac{\nabla^2\tilde{f}(\mathbf{r})}{\tilde{f}(\mathbf{r})}-8\pi a\tilde{f}(\mathbf{r})^2+4\pi a\tilde{\zeta}, \quad \tilde{\zeta}=N^{-1}\int d\mathbf{r}\,\tilde{f}(\mathbf{r})^4, \tag{4.8a}$$

and $\tilde{\boldsymbol{\mathcal{V}}}_2(t)$, $\tilde{\mathcal{V}}_3(t)$ are given by equations similar to Eqs. (2.57) and (2.58). Therefore, $\tilde{f}(\mathbf{r})=\tilde{f}_m(\mathbf{r})(m=0,1,...,)$ correspond to states of the condensate with energies $\mathcal{E}_m$ under the external potential

$$\mathcal{V}_e=\tilde{\mathcal{V}}_{1m}+\mathcal{E}_m \quad (\tilde{\mathcal{V}}_1=\tilde{\mathcal{V}}_{1m}). \tag{4.8b}$$

Given a $\mathcal{V}_e(\mathbf{r})$, Eq. (4.2) can be solved approximately for the lowest state of the condensate when $\mathcal{V}_e(\mathbf{r})$ is sufficiently slowly varying. This is the case in the recent experiments on Bose–Einstein condensation, where the trap is of macroscopic dimensions. By applying the procedure of Refs. 8 and 13, neglect of the Laplacian furnishes

$$[\mathcal{V}_e(\mathbf{r})+8\pi a\tilde{f}(\mathbf{r})^2-4\pi a\tilde{\zeta}-\mathcal{E}]\tilde{f}(\mathbf{r})=0, \tag{4.9}$$

where $\mathcal{E}=\mathcal{E}_0$, or,

$$\tilde{f}(\mathbf{r})\sim\begin{cases}(8\pi a)^{-1/2}[\mathcal{E}+4\pi a\tilde{\zeta}-\mathcal{V}_e(\mathbf{r})]^{1/2}, & \mathbf{r}\ \text{inside}\ \mathcal{R}_0\\ 0, & \mathbf{r}\ \text{outside}\ \mathcal{R}_0\end{cases}, \tag{4.10}$$

since $\tilde{f}(\mathbf{r})$ can be chosen to be non-negative. The region $\mathcal{R}_0$ is determined by

$$\mathcal{V}_e(\mathbf{r})<\mathcal{E}+4\pi a\tilde{\zeta}, \quad \mathbf{r}\in\mathcal{R}_0. \tag{4.11}$$

At the boundary $\partial\mathcal{R}_0$ of $\mathcal{R}_0$,

$$\mathcal{E}+4\pi a\tilde{\zeta}=\mathcal{V}_e(\mathbf{r}), \quad \mathbf{r}\in\partial\mathcal{R}_0. \tag{4.12}$$

Under this approximation, an expression for $\mathcal{E}$ is obtained via multiplication of Eq. (4.9) by $\tilde{f}(\mathbf{r})$ and integration over $\mathbf{r}$:

$$\mathcal{E}\sim 4\pi a\tilde{\zeta}+\tilde{\zeta}_e, \tag{4.13}$$

where

$$\tilde{\zeta}_e=N^{-1}\int d\mathbf{r}\,\mathcal{V}_e(\mathbf{r})|\tilde{f}(\mathbf{r})|^2. \tag{4.14}$$

Formula (4.10) breaks down in the vicinity of $\partial\mathcal{R}_0$. A remedy to this problem is provided in Refs. 8 and 13.

It remains to discuss the pair-excitation function $K_0(\mathbf{r},\mathbf{r}';t)$. With

$$K_0(\mathbf{r},\mathbf{r}';t)=\tilde{\kappa}_0(\mathbf{r}-\boldsymbol{\alpha}(t),\mathbf{r}'-\boldsymbol{\alpha}(t))e^{-i\chi(\mathbf{r},\mathbf{r}';t)}, \tag{4.15}$$

and use of Eq. (3.14), $\tilde{\kappa}_0(\mathbf{r},\mathbf{r}')$ should satisfy

$$-\nabla^2\tilde{\kappa}_0(\mathbf{r},\mathbf{r}')-\nabla'^2\tilde{\kappa}_0(\mathbf{r},\mathbf{r}')+8\pi a\tilde{f}(\mathbf{r})^2\delta(\mathbf{r}-\mathbf{r}')+\{-2\check{\zeta}-16\pi a\tilde{\zeta}-2\tilde{\zeta}_e+\mathcal{V}_e(\mathbf{r})+\mathcal{V}_e(\mathbf{r}')$$
$$+16\pi a[\tilde{f}(\mathbf{r})^2+\tilde{f}(\mathbf{r}')^2]\}\tilde{\kappa}_0(\mathbf{r},\mathbf{r}')+8\pi a\int d\mathbf{r}''\,\tilde{f}(\mathbf{r}'')^2\tilde{\kappa}_0(\mathbf{r},\mathbf{r}'')\tilde{\kappa}_0(\mathbf{r}',\mathbf{r}'')=0, \tag{4.16}$$

where

$$\check{\zeta}=N^{-1}\int d\mathbf{r}|\nabla\tilde{f}(\mathbf{r})|^2. \tag{4.17}$$

Note that shifting $\mathcal{V}_e$ by a constant does not affect the equation of motion.

Following Ref. 8, let

$$p_0(\mathbf{R},\mathbf{r})=\tilde{\kappa}_0(\mathbf{r}_1,\mathbf{r}_2), \tag{4.18}$$

where

$$\mathbf{R}=\tfrac{1}{2}(\mathbf{r}_1+\mathbf{r}_2), \quad \mathbf{r}=\mathbf{r}_1-\mathbf{r}_2. \tag{4.19}$$

Hence,

$$p_0(\mathbf{R},-\mathbf{r})=p_0(\mathbf{R},\mathbf{r}). \tag{4.20}$$

The integro-differential equation for $p_0(\mathbf{R},\mathbf{r})$ reads

$$-\tfrac{1}{2}\nabla_{\mathbf{R}}^2 p_0(\mathbf{R},\mathbf{r})-2\nabla_{\mathbf{r}}^2 p_0(\mathbf{R},\mathbf{r})+8\pi a\tilde{f}(\mathbf{R})^2\delta(\mathbf{r})+\{-2\check{\zeta}-16\pi a\tilde{\zeta}-2\tilde{\zeta}_e+\mathcal{V}_e(\mathbf{R}+\tfrac{1}{2}\mathbf{r})$$
$$+\mathcal{V}_e(\mathbf{R}-\tfrac{1}{2}\mathbf{r})+16\pi a[\tilde{f}(\mathbf{R}+\tfrac{1}{2}\mathbf{r})^2+\tilde{f}(\mathbf{R}-\tfrac{1}{2}\mathbf{r})^2]\}p_0(\mathbf{R},\mathbf{r})$$
$$+8\pi a\int d\mathbf{r}'\,\tilde{f}(\mathbf{R}+\mathbf{r}')^2 p_0(\mathbf{R}+\tfrac{1}{4}\mathbf{r}+\tfrac{1}{2}\mathbf{r}',\tfrac{1}{2}\mathbf{r}-\mathbf{r}')p_0(\mathbf{R}-\tfrac{1}{4}\mathbf{r}+\tfrac{1}{2}\mathbf{r}',-\tfrac{1}{2}\mathbf{r}-\mathbf{r}')=0. \tag{4.21}$$

In the spirit of Eq. (4.9), $\nabla_{\mathbf{R}}^2$ is neglected, while

$$\mathcal{V}_e(\mathbf{R}+\tfrac{1}{2}\mathbf{r})\sim\mathcal{V}_e(\mathbf{R})\sim\mathcal{V}_e(\mathbf{R}-\tfrac{1}{2}\mathbf{r}), \tag{4.22}$$

$$\tilde{f}(\mathbf{R}+\tfrac{1}{2}\mathbf{r})\sim\tilde{f}(\mathbf{R})\sim\tilde{f}(\mathbf{R}-\tfrac{1}{2}\mathbf{r}), \tag{4.23}$$

$$p_0(\mathbf{R}+\tfrac{1}{4}\mathbf{r}+\tfrac{1}{2}\mathbf{r}',\tfrac{1}{2}\mathbf{r}-\mathbf{r}')\sim p_0(\mathbf{R},\tfrac{1}{2}\mathbf{r}-\mathbf{r}'), \tag{4.24a}$$

$$p_0(\mathbf{R}-\tfrac{1}{4}\mathbf{r}+\tfrac{1}{2}\mathbf{r}',-\tfrac{1}{2}\mathbf{r}-\mathbf{r}')\sim p_0(\mathbf{R},-\tfrac{1}{2}\mathbf{r}-\mathbf{r}'). \tag{4.24b}$$

Equation (4.21) then reduces to

$$\begin{aligned}&-\nabla_{\mathbf{r}}^2 p_0(\mathbf{R},\mathbf{r})+4\pi a\tilde{f}(\mathbf{R})^2\delta(\mathbf{r})+\{-\check{\zeta}-8\pi a\tilde{\zeta}-\tilde{\zeta}_e+\mathcal{V}_e(\mathbf{R})+16\pi a\tilde{f}(\mathbf{R})^2\}p_0(\mathbf{R},\mathbf{r})\\&\quad+4\pi a\tilde{f}(\mathbf{R})^2\int d\mathbf{r}'\, p_0(\mathbf{R},\mathbf{r}')p_0(\mathbf{R},\mathbf{r}-\mathbf{r}')=0.\end{aligned} \tag{4.25}$$

Because the nonlinear term is a convolution integral, the equation of motion can be solved *exactly* with recourse to the Fourier transform in $\mathbf{r}$ of $p_0(\mathbf{R},\mathbf{r})$:

$$\overline{p_0}(\mathbf{R},\mathbf{k})=\int d\mathbf{r}e^{i\mathbf{k}\cdot\mathbf{r}}p_0(\mathbf{R},\mathbf{r}), \tag{4.26}$$

which transforms Eq. (4.25) into

$$4\pi a\tilde{f}(\mathbf{R})^2\overline{p_0}(\mathbf{R},\mathbf{k})^2+[k^2+k_0(\mathbf{R})^2]\overline{p_0}(\mathbf{R},\mathbf{k})+4\pi a\tilde{f}(\mathbf{R})^2=0, \tag{4.27}$$

where

$$k_0(\mathbf{R})^2=-\check{\zeta}-8\pi a\tilde{\zeta}-\tilde{\zeta}_e+\mathcal{V}_e(\mathbf{R})+16\pi a\tilde{f}(\mathbf{R})^2. \tag{4.28}$$

Equation (4.27) is solved explicitly to give

$$\overline{p_0}(\mathbf{R},\mathbf{k})=[8\pi a\tilde{f}(\mathbf{R})^2]^{-1}\{-k^2-k_0(\mathbf{R})^2+\sqrt{[k^2+k_0(\mathbf{R})^2]^2-(8\pi a)^2\tilde{f}(\mathbf{R})^4}\}. \tag{4.29}$$

In view of formula (4.10),

$$\overline{p_0}(\mathbf{R},\mathbf{k})\sim\begin{cases}-k_0(\mathbf{R})^{-2}\{k^2+k_0(\mathbf{R})^2-k[k^2+2k_0(\mathbf{R})^2]^{1/2}\}, & \mathbf{R}\ \text{inside}\ \mathcal{R}_0\\ 0, & \mathbf{R}\ \text{outside}\ \mathcal{R}_0\end{cases}, \tag{4.30}$$

by neglecting $\check{\zeta}$ since $|\nabla\tilde{f}(\mathbf{r})|\simeq 0$ unless $\mathbf{r}$ is sufficiently close to $\partial\mathcal{R}_0$, so that

$$k_0(\mathbf{R})^2=8\pi a\tilde{f}(\mathbf{R})^2. \tag{4.31}$$

Inversion of $\overline{p_0}(\mathbf{R},\mathbf{k})$ is carried out as follows. For $\mathbf{R}$ outside $\mathcal{R}_0$,

$$p_0(\mathbf{R},\mathbf{r})=0. \tag{4.32}$$

If $\mathbf{R}$ lies inside $\mathcal{R}_0$,

$$
\begin{aligned}
p_0(\mathbf{R},\mathbf{r}) &= \frac{1}{(2\pi)^3} \lim_{\delta\to 0^+} \int d\mathbf{k}\, e^{-i\mathbf{k}\cdot\mathbf{r}} e^{-\delta k} \overline{p_0}(\mathbf{R},\mathbf{k}) \\
&= -\frac{1}{k_0^2}\frac{1}{(2\pi)^3} \lim_{\delta\to 0^+} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \int_0^{\infty} dk\, k^2 e^{-ik|\mathbf{r}|\cos\theta} e^{-\delta k} \\
&\quad \times [k^2+k_0^2-k(k^2+2k_0^2)^{1/2}] \\
&= \frac{2}{k_0^2|\mathbf{r}|}\frac{1}{(2\pi)^2} \lim_{\delta\to 0^+} \mathrm{Im} \int_0^{\infty} dk\, e^{ik|\mathbf{r}|} e^{-\delta k} k^2 (k^2+2k_0^2)^{1/2} \quad (k=\sqrt{2}k_0 \sinh t) \\
&= \frac{k_0^2}{2\pi^2|\mathbf{r}|} \lim_{\delta\to 0^+} \mathrm{Im} \int_0^{\infty} dt\, e^{i\sqrt{2}k_0|\mathbf{r}|\sinh t} e^{-\delta\sqrt{2}k_0 \sinh t} (\sinh 2t)^2 \\
&= \pi^{-2}(4\pi a)^{3/2} \tilde{f}(\mathbf{R})^3 \frac{\mathrm{Im}\{S_{0,4}(iw)-S_{0,0}(iw)\}}{w},
\end{aligned} \tag{4.33}
$$

where $k_0=k_0(\mathbf{R})$ and

$$w=(16\pi a)^{1/2}\tilde{f}(\mathbf{R})|\mathbf{r}|, \tag{4.34}$$

and $S_{0,4}$ and $S_{0,0}$ are Lommel's functions.[31]

V. $\tilde{f}(\mathbf{r})$ IN A THREE-DIMENSIONAL SPHERICALLY SYMMETRIC HARMONIC POTENTIAL

In the actual experiments on Bose–Einstein condensation, the trapping potential is of complicated form. This is usually modeled as an anisotropic harmonic potential. In this section, $\tilde{f}(\mathbf{r})$ for the lowest state of the condensate is examined in some detail in the simplifying case of a spherically symmetric harmonic potential. A similar task is undertaken in Ref. 12, where the nonlinear Schrödinger equation is given in terms of the chemical potential.

With an external potential $V_e(\mathbf{r},t)=\frac{1}{4}\omega_0^2 r^2$, $\tilde{\mathcal{V}}_1(\mathbf{r})$ is taken to be

$$\tilde{\mathcal{V}}_1(\mathbf{r})=\tfrac{1}{4}\omega_0^2 r^2-\mathcal{E}, \tag{5.1}$$

as is suggested by the eigenvalue problem associated with Eq. (2.56). Terms linear in x, y, and z are omitted. It follows that

$$\tilde{\boldsymbol{\mathcal{V}}}_2(t)=\tfrac{1}{2}\omega_0^2\boldsymbol{\alpha}(t), \quad \tilde{\mathcal{V}}_3(t)=\mathcal{E}-\tfrac{1}{4}\omega_0^2|\boldsymbol{\alpha}(t)|^2, \tag{5.2}$$

yielding

$$\boldsymbol{\alpha}(t)=\mathbf{r}_0\cos\omega_0 t+\frac{\mathbf{v}_0}{\omega_0}\sin\omega_0 t, \tag{5.3}$$

$$A(t)=\mathcal{E}t+\frac{1}{8}\left(\frac{|\mathbf{v}_0|^2-\omega_0^2|\mathbf{r}_0|^2}{\omega_0}\sin 2\omega_0 t+2\mathbf{v}_0\cdot\mathbf{r}_0\cos 2\omega_0 t\right)+\text{const}, \tag{5.4}$$

where $\mathbf{r}_0$ and $\mathbf{v}_0$ are determined by the initial conditions and the constant is real.

For the state of lowest energy $\mathcal{E}=\mathcal{E}_0$, $\tilde{f}(\mathbf{r})=\tilde{f}_0(r)$ is spherically symmetric.[32] Let

$$q(\xi)=(4\pi)^{1/2}(N^2\omega_0/2)^{-1/4} r\tilde{f}_0(r), \quad \xi=(\omega_0/2)^{1/2} r. \tag{5.5}$$

From Eq. (4.8a), this $q(\xi)$ satisfies

$$-\frac{d^2q(\xi)}{d\xi^2}+\xi^2q(\xi)+\Lambda^2\frac{q(\xi)^3}{\xi^2}=\lambda^2q(\xi)\quad(\xi>0),\tag{5.6}$$

supplemented with the boundary conditions

$$q(0)=0,\tag{5.7}$$

$$\lim_{\xi\to\infty}q(\xi)=0,\tag{5.8}$$

and the normalization condition

$$\int_0^\infty d\xi\, q(\xi)^2=1.\tag{5.9}$$

In the above,

$$\Lambda=(2a^2N^2\omega_0)^{1/4},\tag{5.10}$$

$$\lambda^2=\frac{2\mathcal{E}}{\omega_0}+\frac{1}{2}\Lambda^2\int_0^\infty\frac{d\xi}{\xi^2}q(\xi)^4,\quad \lambda>0.\tag{5.11}$$

Note that, for $\xi\to\infty$, the nonlinear term in Eq. (5.6) can be neglected, and the asymptotic behavior of $q(\xi)$ is found via the direct application of the Wentzel–Kramers–Brillouin method:

$$q(\xi)\sim C(\xi^2-\lambda^2)^{-1/4}\exp\{-(\lambda^2/2)[(\xi/\lambda)\sqrt{(\xi/\lambda)^2-1}-\cosh^{-1}(\xi/\lambda)]\},\tag{5.12}$$

where C is independent of ξ. Compare with Ref. 12. For a discussion on the determination of this C see Ref. 33.

Some insight into the solution to Eqs. (5.6)–(5.9) can be obtained by considering the following cases.

(i) $\Lambda\gg1$. To leading order in Λ, neglect of the second derivative of $q(\xi)$ results in

$$q(\xi)\sim q^{(0)}(\xi)=\begin{cases}(\xi/\Lambda)\sqrt{\lambda^2-\xi^2}, & 0\leqslant\xi<\lambda\\ 0, & \xi>\lambda,\end{cases}\tag{5.13}$$

which trivially satisfies Eqs. (5.7) and (5.8). $q^{(0)}(\xi)$ satisfies Eq. (5.9) provided that λ is

$$\lambda\sim\lambda^{(0)}=(\tfrac{15}{2}\Lambda^2)^{1/5}.\tag{5.14}$$

A similar calculation for an anisotropic potential can be found in Ref. 14, where the chemical potential is employed. From Eq. (5.11),

$$\mathcal{E}^{(0)}=\tfrac{5}{14}(\tfrac{15}{2})^{2/5}(2a^2N^2\omega_0)^{1/5}\omega_0=\tfrac{5}{21}(\tfrac{15}{2})^{2/5}\Lambda^{4/5}\epsilon_0^{\text{ho}},\tag{5.15}$$

where $\epsilon_0^{\text{ho}}=\frac{3}{2}\omega_0$ is the ground-state energy of the three-dimensional harmonic oscillator. Approximation (5.13) starts to break down at a distance of the order of $\Lambda^{-2/15}$ from inside the ''boundary'' $\xi=\lambda$, and then needs to be modified according to the procedure in Refs. 8 and 13. This procedure provides a smooth connection to asymptotic formula (5.12) when $0<\xi-\lambda\ll1$ while $\xi-\lambda\gg O(\Lambda^{-2/15})$.[33]

(ii) $\Lambda\ll1$. To zeroth order in Λ, the known solution for the ground-state wave function of the three-dimensional harmonic oscillator is obtained:

$$q^{(0)}(\xi)=2\pi^{-1/4}\xi e^{-\xi^2/2},\tag{5.16}$$

with energy $\mathcal{E}^{(0)}=(\omega_0/2)\lambda^{(0)^2}=\epsilon_0^{\text{ho}}$. The first-order energy correction $\mathcal{E}^{(1)}$ can be obtained

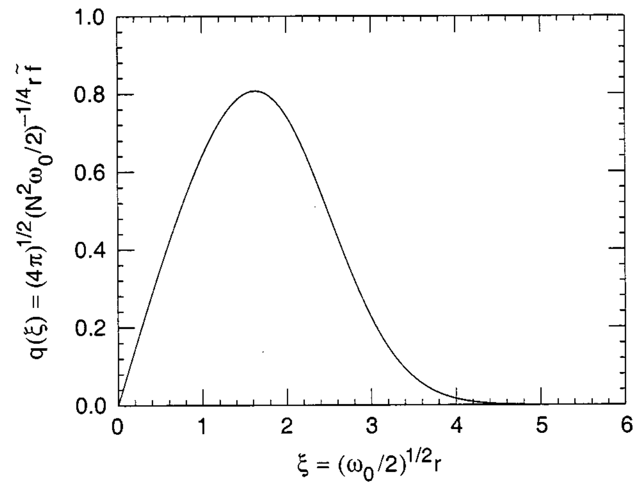

FIG. 1. Solution to Eqs. (5.6)–(5.9) for $\Lambda^2=12.1$ (^{87}Rb atoms, $a\simeq 110a_0$, $N=10^3$, and $\omega_0=(2\pi\times 120)/\sqrt{8}$ rad/s). Numerically computed eigenvalue is $\lambda^2=6.8$.

through the standard perturbation methods, by treating $V^{(0)}(\xi)=\Lambda^2\xi^{-2}q^{(0)}(\xi)^2$ as the perturbing potential. Therefore, $\lambda^{(1)^2}$ equals the matrix element

$$\lambda^{(1)^2}=\int_0^\infty d\xi\, q^{(0)}(\xi)V^{(0)}(\xi)q^{(0)}(\xi). \tag{5.17}$$

By virtue of Eq. (5.11),

$$\mathcal{E}^{(1)}=\frac{\omega_0\Lambda^2}{4}\int_0^\infty \frac{d\xi}{\xi^2}q^{(0)}(\xi)^4=\sqrt{\frac{2}{\pi}}\frac{\omega_0\Lambda^2}{4}, \quad \Lambda\ll 1, \tag{5.18}$$

or

$$\mathcal{E}\sim\mathcal{E}^{(0)}+\mathcal{E}^{(1)}=\frac{3}{2}\omega_0+\sqrt{\frac{2}{\pi}}\frac{\omega_0\Lambda^2}{4}. \tag{5.19}$$

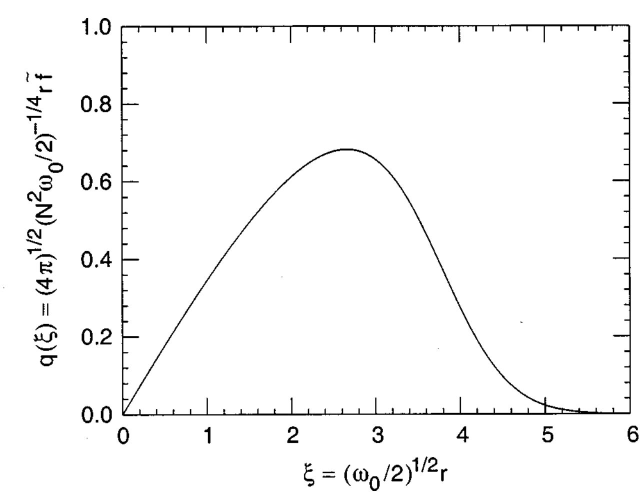

FIG. 2. Solution to Eqs. (5.6)–(5.9) for $\Lambda^2=121$ (^{87}Rb atoms, $N=10^4$, and $\omega_0=(2\pi\times 120)/\sqrt{8}$ rad/s). Eigenvalue is $\lambda^2=15.6$.

Numerical results. In order to make some contact with recent experimental situations, plots of $q(\xi)$ are presented in Figs. 1 and 2 for two different values of Λ, in close relation to the JILA experiments, where ^{87}Rb atoms were used ($a=110a_0$, a_0: the Bohr radius).[1] Specifically, in Fig. 1, $\Lambda^2=12.1$, corresponding, for instance, to $N=10^3$ and $\omega_0=(2\pi\times 120)/\sqrt{8}$ rad/s. The numerically computed eigenvalue there is $\lambda^2=6.8$, giving $\mathcal{E}=1.8\epsilon_0^{\mathrm{ho}}$. Compare with $\mathcal{E}^{(0)}=1.45\epsilon_0^{\mathrm{ho}}$ provided by Eq. (5.15). In Fig. 2, $\Lambda^2=121$. The corresponding eigenvalue is found to be $\lambda^2=15.6$, giving $\mathcal{E}=3.8\epsilon_0^{\mathrm{ho}}$. Compare with $\mathcal{E}^{(0)}=3.63\epsilon_0^{\mathrm{ho}}$ from Eq. (5.15).

VI. CONCLUSIONS AND DISCUSSION

In the theoretical treatment of Bose–Einstein condensation in dilute atomic gases with repulsive interactions, the trap is replaced by a sufficiently smooth external potential $V_e(\mathbf{r},t)$ that acts simultaneously on each atom and increases sufficiently rapidly at large distances. As a consequence, the boson system is no longer translationally invariant. Work carried out 38 years ago[4] turns out to be a suitable starting point. An important element introduced there was the systematic consideration of scattering processes, such as pair creation, with a study of some of their physical consequences. In the presence of a trapping potential, pair creation plays a significant role, being described mathematically by the pair-excitation function $K_0(\mathbf{r},\mathbf{r}';t)$. On the basis of the ansatz (1.1), a nonlinear integro-differential equation is satisfied by $K_0(\mathbf{r},\mathbf{r}';t)$.

Solitary-wave solutions to the nonlinear evolution equations for the condensate wave function $\Phi(\mathbf{r},t)$ and the pair-excitation function are uncovered in any number of space dimensions, if $V_e(\mathbf{r},t)$ can properly be decomposed into (i) a trapping potential $\mathcal{V}_e$ translated by the position vector $\mathbf{r}(t)=\boldsymbol{\alpha}(t)$ of the pulse ''center of mass,'' and (ii) a potential linear in the space coordinates, according to (2.55)–(2.58). It is somewhat tempting to put these statements in the language of classical mechanics, recognizing, for instance, the second term mentioned above as the potential associated with a uniform force. The conclusions here are the natural generalization of results obtained for the one-dimensional case, without any restriction to motion along fixed axes in space.[18] Given an external potential that meets the aforementioned conditions, the initial amplitudes are obtained by solving a nonlinear ''eigenvalue problem'' for $\Phi(\mathbf{r},t=0)$ under $\mathcal{V}_e$, and a nonlinear integro-differential equation for $K_0(\mathbf{r},\mathbf{r}';t=0)$. The motion of the solitary wave in space, i.e., the vector $\boldsymbol{\alpha}(t)$, is determined by the uniform force. In this sense, the solitary wave is expected to behave like a classical particle. Conversely, given an admissible $\Phi(\mathbf{r},t=0)$, i.e., sufficiently smooth and rapidly decreasing to zero as $\mathbf{r}\rightarrow\infty$, it is possible to construct an external potential that permits solitary-wave behavior for both $\Phi(\mathbf{r},t>0)$ and $K_0(\mathbf{r},\mathbf{r}';t>0)$. Of course, in real experimental situations, the form of the external potential may deviate from the one given by Eq. (2.55). The question of the stability of the solitary-wave solutions under variations of $V_e(\mathbf{r},t)$ is not addressed in this paper.

As is also pointed out in Ref. 8, the approximate Hamiltonian that furnishes the equation of motion for K_0 does not include, for instance, the scattering of phonons and the decay of a single phonon into two or three phonons. In other words, under the present approximation, the phonons have infinite lifetimes and remain stable. This in turn implies that the ansatz (1.1) and the existing equations of motion are of rather special forms, being valid only over some moderate time scale. The problem of shorter or longer time scales is not touched upon in this paper; this time limitation may depend on the higher-order terms in the Hamiltonian or the initial condition for the condensate wave function. It is believed that the ansatz for the many-body wave function can be generalized. A challenging open problem is to obtain such generalizations, which must satisfy many consistency conditions.

ACKNOWLEDGMENTS

The author is greatly indebted to Professor Tai Tsun Wu for his unwavering guidance and encouragement. He also wishes to thank Professor Wolfgang Ketterle and Professor Bertrand

Halperin for useful discussions, and Margaret Owens for her valuable assistance throughout the preparation of the manuscript. This work was supported in part by the United States Department of Energy under Grant No. DE-FG02-84ER40158.

APPENDIX

Consider the one-dimensional linear Schrödinger equation in a harmonic-oscillator potential ($\hbar=2m=1,\ \omega_0=2$):

$$i\frac{\partial\varphi(x,t)}{\partial t}=\left(-\frac{\partial^2}{\partial x^2}+x^2\right)\varphi(x,t). \tag{A1}$$

It is well known[34] that an initial displacement of the ground-state wave function $\varphi_0(x)=\pi^{-1/4}e^{-x^2/2}$ at $t=0$ by $x=x_0$ produces the wave packet

$$\varphi(x,t)=\varphi_0(x-x_0e^{-2it})e^{-i\nu(t)},\quad t>0, \tag{A2}$$

where, for definiteness, $\nu(t=0)=0$. Substitution into Eq. (A1) furnishes

$$\nu(t)=t+\frac{ix_0^2}{4}e^{-i4t}-\frac{ix_0^2}{4}. \tag{A3}$$

$\varphi(x,t)$ is subsequently recast in a form where magnitude and phase are separated:

$$\begin{aligned}\varphi(x,t)&=\pi^{-1/4}\exp\left\{-\frac{1}{2}(x-x_0e^{-2it})^2-it+\frac{x_0^2}{4}e^{-i4t}-\frac{x_0^2}{4}\right\}\\&=\pi^{-1/4}\exp\left[-\frac{1}{2}(x-x_0\cos 2t)^2\right]\exp\left[-i\left(t+x_0x\sin 2t-\frac{x_0^2}{4}\sin 4t\right)\right],\end{aligned} \tag{A4}$$

which is a one-dimensional solitary wave. Note that with the units of Eq. (A1) the eigenvalue corresponding to $\varphi_0(x)$ is equal to 1.

The preceding analysis can be extended to the d-dimensional Schrödinger equation

$$i\frac{\partial\varphi(\mathbf{r},t)}{\partial t}=\left(-\nabla^2+\sum_{j=1}^{d}x_j^2\right)\varphi(\mathbf{r},t), \tag{A5}$$

where $\mathbf{r}=(x_1,\ldots,x_d)$, $d\geqslant 2$. With an initial displacement of the ground-state wave function $\varphi_0(\mathbf{r})=\pi^{-d/4}e^{-\mathbf{r}\cdot\mathbf{r}/2}$ by $\mathbf{r}=\mathbf{r}_0$, at later times $\varphi(\mathbf{r},t)$ becomes

$$\varphi(\mathbf{r},t)=\varphi_0(\mathbf{r}-\mathbf{r}_0e^{-2it})e^{-i\nu(t)},\quad t>0. \tag{A6}$$

After some straightforward algebra,

$$\nu(t)=d\cdot t+\frac{i|\mathbf{r}_0|^2}{4}e^{-i4t}-\frac{i|\mathbf{r}_0|^2}{4}, \tag{A7}$$

$$\begin{aligned}\varphi(\mathbf{r},t)&=\pi^{-d/4}\left(\prod_{j=1}^{d}\exp\left[-\frac{1}{2}(x_j-x_{j0}\cos 2t)^2\right]\right)\exp\left[-i\left(d\cdot t+\mathbf{r}_0\cdot\mathbf{r}\sin 2t-\frac{|\mathbf{r}_0|^2}{4}\sin 4t\right)\right]\\&=\varphi_0(\mathbf{r}-\mathbf{r}_0\cos 2t)\exp\left[-i\left(d\cdot t+\mathbf{r}_0\cdot\mathbf{r}\sin 2t-\frac{|\mathbf{r}_0|^2}{4}\sin 4t\right)\right].\end{aligned} \tag{A8}$$

This is a solitary wave in d space dimensions.

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