DEFINABILITY, AUTOMORPHISMS, AND INFINITARY LANGUAGES

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1. Introduction.

Let τ be a (finitary) similarity type. We consider the languages $L_{\kappa\lambda}(\tau)$ (defined elsewhere in this volume) whose only non-logical symbols are those occurring in τ. Models of type τ (and thus models for the languages $L_{\kappa\lambda}(\tau)$) will be denoted by the symbols $\mathfrak{A}$ and $\mathfrak{B}$. We will follow the convention that the universe of $\mathfrak{A}$ is A and that of $\mathfrak{B}$ is B.

If $\underline{P}$ is a k-place predicate symbol not occurring in τ $(k \in \omega)$, then we shall also consider models of similarity type $\tau \cup \{\underline{P}\}$. Such models will be written as $(\mathfrak{A}, P)$, where $\mathfrak{A}$ is a model of type τ and P is a k-place relation on A. Also, if $\mathfrak{A}$ is a model of type τ and $a_0, \ldots, a_{n-1} \in A$ then $(\mathfrak{A}, a_0, \ldots, a_{n-1})$, or $(\mathfrak{A}, a_i)_{i<n}$, is a model of τ together with n additional individual constant symbols.

We shall supress all reference to the similarity type τ. Thus, we assume throughout that we have a fixed type τ, and write simply $L_{\kappa\lambda}$ instead of $L_{\kappa\lambda}(\tau)$. All models $\mathfrak{A}$ and $\mathfrak{B}$ are assumed to be of this type τ. We further assume throughout the following

Convention. $\underline{P}$ is a unary predicate symbol which does not occur in τ.
Thus, no formula of any language $L_{\kappa\lambda}$ contains the symbol $\underline{P}$.

We assume familiarity with the basic concepts of the infinitary languages $L_{\kappa\lambda}$ and of model theory. Unless otherwise noted, we employ standard terminology and notation. In particular, $\cong$

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denotes the relation of isomorphism between models; the cardinality of a set X is denoted by $|X|$; by the cardinality of a model $\mathfrak{A}$ we mean $|A|$; and $\equiv_{\kappa\lambda}$ denotes the relation of elementary equivalence with respect to sentences of $L_{\kappa\lambda}(\tau')$ (for the appropriate type τ').

In this paper we present several results concerning infinitary definability, particularly in the language $L_{\omega_1\omega}$. Our results all have the form of an equivalence between certain model-theortic condition on a model and some infinitary (syntactical) definability condition on the model. The proofs are all model-theoretic rather than syntactical. The best known result of this sort is the following Countable Definability Theorem of Scott [12].

THEOREM 1.1. Let $\mathfrak{A}$ be a countable model and let $P \subset A$. Then the following are equivalent:

(i) For any $Q \subset A$, if $(\mathfrak{A},P) \cong (\mathfrak{A},Q)$ then $P = Q$.

(ii) There is some formula $\varphi(x)$ of $L_{\omega_1\omega}$ such that

$$(\mathfrak{A},P) \models \forall x\, [\underline{P}(x) \longleftrightarrow \varphi(x)].$$

This theorem can be considered as an $L_{\omega_1\omega}$ analogue for countable models of the well-known definability results of Beth [1] and Svenonuis [13]. As Lopez-Escobar showed in [5], Beth's Theorem holds for theories given by a single sentence of $L_{\omega_1\omega}$. Lopez-Escobar's result, however, appears to be neither weaker nor stronger than Theorem 1.1. On the one hand, there is no way known of obtaining Lopez-Escobar's theorem from Scott's result, or indeed, in any model-theoretic fashion. And on the other hand, Theorem 1.1 does not follow from Lopez-Escobar's result even if we grant the Isomorphism Theorem (see below) because the complete $L_{\omega_1\omega}$ theory of a countable model for an uncountable type τ is not given by a single sentence of $L_{\omega_1\omega}$.

We remark that in Theorem 1.1 we could allow P to be any (finitary) relation or function on A. This is also true for the other results we will present in this paper. We have stated them just for unary relations solely for ease of exposition.

Theorem 1.1 is an easy consequence of the Countable Isomorphism Theorem of Scott [12], which is also the main tool which we shall require. In its simplest form it says that two countable models are isomorphic if and only if they are elementarily equivalent in $L_{\omega_1\omega}$. What we will actually use is the following theorem from which the usual formulation of the Isomorphism Theorem easily follows. (For a detailed account of Isomorphism theorems, see Chang's paper [4] in the present volume.)

THEOREM 1.2. (1) If $\mathfrak{A}$ and $\mathfrak{B}$ are countable models and $\mathfrak{A} \equiv_{\omega_1\omega} \mathfrak{B}$, then for any $a \in A$ there is some $b \in B$ such that

$$(\mathfrak{A},a) \equiv_{\omega_1\omega} (\mathfrak{B},b).$$

(2) If $\mathfrak{A}$ is countable and $a_0,\dots,a_n \in A$ then there is some formula $\varphi(v_0,\dots,v_n)$ of $L_{\omega_1\omega}$ such that for any $b_0,\dots,b_n \in A$

$$\mathfrak{A} \models \varphi(b_0,\dots,b_n) \quad \text{iff}$$

$$(\mathfrak{A},a_0,\dots,a_n) \equiv_{\omega_1\omega} (\mathfrak{A},b_0,\dots,b_n).$$

In the following section we present our main results, Theorems 2.1 and 2.2, Theorem 2.1 is a generalization of Theorem 1.1, and is an $L_{\omega_1\omega}$ analogue of the definability theorem of Chang [3] and Makkai [7]. Theorem 2.1 was obtained independently by G. E. Reyes and the author.

Theorem 2.2 gives an $L_{\omega_1\omega}$ definability characterization of those countable models with fewer than 2^ω automorphisms. From this we show that if a countable model $\mathfrak{A}$ is elementarily equivalent in $L_{\omega_1\omega}$ to an uncountable model, then $\mathfrak{A}$ has 2^ω automorphisms.

In §3 we consider briefly the existence and properties of uncountable models which are $L_{\omega_1\omega}$ elementarily equivalent to countable models. This topic is suggested by the corollary to Theorem 2.2.

Finally, in §4, we consider analogues of the results in §2 for uncountable models. Our results there are mainly negative, except for models of cardinality λ cofinal with ω.

2. Definability in countable models.

For ease in stating our main results, we introduce the following notation.

(1) If $\mathfrak{A}$ is any model and $P \subset A$, then

$$M(\mathfrak{A},P) = \{P' \subset A : (\mathfrak{A},P) \cong (\mathfrak{A},P')\}.$$

(2) For any model $\mathfrak{A}$, $F(\mathfrak{A})$ is the set of all automorphisms of $\mathfrak{A}$.

In this notation, condition (i) of Theorem 1.1 says that $M(\mathfrak{A},P)$ has exactly one element.

The main results of this paper are the following.

THEOREM 2.1. Let $\mathfrak{A}$ be a countable model and let $P \subset A$. Then the following are equivalent:

(i) $|M(\mathfrak{A},P)| \leq \omega$.

(ii) $|M(\mathfrak{A},P)| < 2^{\omega}$.

(iii) There is a formula $\varphi(x,v_0,\ldots,v_k)$ of $L_{\omega_1\omega}$ such that

$$(\mathfrak{A},P) \models \exists v_0,\ldots,v_k\ \forall x[\underline{P}(x) \leftrightarrow \varphi(x,v_0,\ldots,v_k)].$$

THEOREM 2.2. Let $\mathfrak{A}$ be a countable model. Then the following are equivalent:

(i) $|F(\mathfrak{A})| \leq \omega$.

(ii) $|F(\mathfrak{A})| < 2^{\omega}$.

(iii) For every $P \subset A$, $|M(\mathfrak{A},P)| < 2^{\omega}$.

(iv) There are formulas $\varphi_n(x,v_0,\ldots,v_k)$ of $L_{\omega_1\omega}$, for $n \in \omega$, such that

$$\mathfrak{A} \models \exists v_0,\ldots,v_k\ \forall x \bigvee_{n\in\omega} [\varphi_n(x,v_0,\ldots,v_k) \wedge \exists !\ z\ \varphi_n(z,v_0,\ldots,v_k)].$$

COROLLARY 2.3. Let $\mathfrak{A}$ be countable. If $\mathfrak{A}$ is $L_{\omega_1\omega}$ elementarily equivalent to any uncountable model, then $\mathfrak{A}$ has 2^{ω} automorphisms.

REMARKS. (1) Theorem 2.1 was obtained independently by G. E. Reyes. His proof is very different from the author's, and may be found in [11].

(2) Condition (iv) of Theorem 2.2 says that there are a finite number of elements $a_0,\ldots,a_k \in A$ in terms of which every element of A is definable (in $L_{\omega_1\omega}$). It follows that every subset of A is also definable in terms of $a_0,\ldots,a_k$. By Theorem 2.1, condition (iii) of Theorem 2.2 implies that every subset of A is definable in terms of some finite number of points of A, with different subsets perhaps requiring different individuals from A. Thus condition (iii) is apparently weaker than condition (iv).

(3) The equivalence, in Theorem 2.2, of (i), (ii), and (iii) is a purely model-theoretic fact of some interest in itself, and appears to be new.

Corollary 2.3 is an almost immediate consequence of Theorem 2.2, since if $\mathfrak{A}$ is countable and has fewer than 2^{ω} automorphisms, then $\mathfrak{A}$ satisfies condition (iv) of Theorem 2.2, and so any model which is $L_{\omega_1\omega}$ elementarily equivalent to $\mathfrak{A}$ also satisfies (iv). But any model satisfying (iv) must be countable, so $\mathfrak{A}$ cannot be $L_{\omega_1\omega}$ elementarily equivalent to any uncountable model.

It is easy to find examples of countable models with 2^{ω} automorphisms which are not $L_{\omega_1\omega}$ elementarily equivalent to any uncountable model, so the converse to the corollary fails.

The proof of Theorem 2.1 is similar to Chang's proof in [3] of the Chang-Makkai Theorem, by which it was suggested. In addition, we use the same general method in the proof of Theorem 2.2. Because of this, we give here only the proof of Theorem 2.2, which involves several points which do not arise in the other proof.

Proof of Theorem 2.2. The implications from (i) to (ii) and from (ii) to (iii) are immediate. To see that (iv) implies (i) simply note that (iv) says that there are elements $a_0,\ldots,a_k \in A$ in terms of which every element of A is definable by one of the φ_n. Thus every automorphism of $\mathfrak{A}$ is uniquely determined by its action on $a_0,\ldots,a_k$, and so $F(\mathfrak{A})$ is countable.

The remaining direction, from (iii) to (iv), will take more time. We will show that if (iv) fails then (iii) must also fail.

Let S be the set of all finite sequences of 0's and 1's, that is, all functions on some $n \in \omega$ into 2. If $s \in S$ has domain n, then we define

$$s^0 = s \cup \{\langle n,0\rangle\},$$

$$s^1 = s \cup \{\langle n,1\rangle\}.$$

Thus s^0 and s^1 are the immediate successors of s in the natural ordering of inclusion on S.

Let the sequence $\{a_k\}_{k\in\omega}$ enumerate the elements of A.

Assuming that (iv) fails, we construct functions G and H such that the following hold:

(1) domain of G = domain of $H = S$.

(2) for each $s \in S$, $G(s)$ and $H(s)$ are functions on the domain of s into A.

(3) if $s,t \in S$ and $s \subset t$ then $G(s) \subset G(t)$ and $H(s) \subset H(t)$.

(4) if $s \in S$ has fomain n, then

$$(\mathfrak{A}, G(s)(i))_{i<n} \equiv_{\omega_1\omega} (\mathfrak{A}, H(s)(i))_{i<n} .$$

(5) if $s \in S$ has domain $n = 3k$, then

$$G(s^0) = G(s^1) = G(s) \cup \{\langle n, a_k \rangle\}.$$

(6) if $s \in S$ has domain $n = 3k + 1$, then

$$H(s^0) = H(s^1) = H(s) \cup \{\langle n, a_k \rangle\}.$$

(7) if $s \in S$ has domain $n = 3k + 2$, then

(a) $H(s^0) = H(s^1)$

(b) $G(s^0)(n) \neq G(s^1)(n)$

(c) for any $t \in S$ of domain n,

$$G(s^0)(n) = G(t^0)(n), \text{ and}$$

$$G(s^1)(n) = G(t^1)(n)$$

(d) for any $t \in S$ of domain $\leq n$ and for any $j <$ domain of t,

$$G(s^0)(n) \neq G(t)(j) \text{ and}$$

$$G(s^1)(n) \neq G(t)(j).$$

$G(s)$ and $H(s)$ are defined by induction on the domain of s. So, let $n \in \omega$ and assume that G and H have been defined for all $s \in S$ with domain $\leq n$. We define G and H on those sequences in S with domain $n + 1$ by cases on n.

First, if $n = 3k$ or $n = 3k + 1$ then it is easy, using Theorem 1.2 (1), to define G and H on the sequences with domain $n + 1$. So we assume that $n = 3k + 2$.

Let $b_1, \ldots, b_m$ enumerate all the points of A of the form $G(t)(j)$ for $t \in S$ of domain $\leq n$ and $j <$ domain of t.

We first show

(A) There are points $c,c' \in A$ such that $c \neq c'$ and

$$(\mathfrak{A},b_1,\ldots,b_m,c) \equiv_{\omega_1\omega} (\mathfrak{A},b_1,\ldots,b_m,c').$$

Assume that no such points c, c' existed. For each $i \in \omega$ let $\varphi_i(x,v_1,\ldots,v_m)$ be a formula of $L_{\omega_1\omega}$ which determines the complete $L_{\omega_1\omega}$ type of the sequence a_i, $b_1,\ldots,b_m$ in $\mathfrak{A}$. Such formulas φ_i exist by Theorem 1.2 (2). In particular, then, for any $a' \in A$ we have

$$\mathfrak{A} \models \varphi_i(a',b_1,\ldots,b_m) \quad \text{iff}$$

$$(\mathfrak{A},a_i,b_1,\ldots,b_m) \equiv_{\omega_1\omega} (\mathfrak{A},a',b_1,\ldots,b_m);$$

and by our assumption that (A) fails, this happens if and only if $a' = a_i$. Hence

$$\mathfrak{A} \models \forall x \bigvee_{i\in\omega} [\varphi_i(x,b_1,\ldots,b_m) \wedge \exists!\, z\, \varphi_i(z,b_1,\ldots,b_m)].$$

But this implies that (iv) holds, contradicting our hypothesis that it fails.

Choosing points c, c' such that (A) holds, we define

$$G(s^0) = G(s) \cup \{\langle n,c\rangle\} \quad \text{and}$$

$$G(s^1) = G(s) \cup \{\langle n,c'\rangle\},$$

for all $s \in S$ with domain n.

Then (7)(b) and (7)(c) hold by definition. Also, $c \neq b_i$ for each $i = 1,\ldots,m$ since if $c = b_i$ then, by (A), we would also have $c' = b_i$ and hence $c = c'$. Similarly $c' \neq b_i$ for each i. Thus (7)(d) holds.

Now, if $s \in S$ has domain n, then $G(s)$ is a sequence $(c_0,\ldots,c_{n-1})$ and $H(s)$ is a sequence $(d_0,\ldots,d_{n-1})$ of points of A such that

$$(\mathfrak{A},c_0,\ldots,c_{n-1}) \equiv_{\omega_1\omega} (\mathfrak{A},d_0,\ldots,d_{n-1}).$$

Notice that $c_0,\ldots,c_{n-1}$ is a subsequence of $b_1,\ldots,b_m$ and hence (A) implies that

$$(\mathfrak{A},c_0,\ldots,c_{n-1},c) \equiv_{\omega_1\omega} (\mathfrak{A},c_0,\ldots,c_{n-1},c').$$

By Theorem 1.2 (1) choose $d \in A$ such that

$$(\mathfrak{A},d_0,\ldots,d_{n-1},d) \equiv_{\omega_1\omega} (\mathfrak{A},c_0,\ldots,c_{n-1},c),$$

and define

$$H(s^0) = H(s^1) = H(s) \cup \{\langle n,d\rangle\}.$$

Then for this definition of $H(s^0)$ and $H(s^1)$ we have shown that (7)(a) and (4) hold. Since all the other conditions obviously hold, we have defined G and H on all sequences of domain $n + 1$. This completes our induction.

We now define

$$P = \{G(s^0)(n) : s \in S \text{ of domain } n = 3k + 2\}.$$

It is for this set P for which we will eventually show that $|M(\mathfrak{A},P)| = 2^\omega$. But first we show

(B) For every $s \in S$ of domain $n = 3k + 2$,

$$G(s^1)(n) \notin P.$$

If, on the contrary, $G(s^1)(n) \in P$, then $G(s^1)(n) = G(t^0)(n')$ for some $t \in S$ of domain $n' = 3k'+ 2$. But if $k = k'$ this contradicts (7)(b)(c), and if $k \neq k'$ then this contradicts (7)(d).

We now extend G and H to be defined on any function s on ω into 2 by defining

$$G(s) = \bigcup_{n\in\omega} G(s|n) \quad \text{and}$$

$$H(s) = \bigcup_{n\in\omega} H(s|n).$$

Recall that $s|n$ is the restriction of s to domain n. By (3) $G(s)$ and $H(s)$ are well-defined ω-termed sequences of elements of A, and they each enumerate A by (5) and (6).

Finally, if s is a function on ω into 2, we define f_s on A by

$$f_s(G(s)(n)) = H(s)(n), \quad \text{for all } n \in \omega.$$

Then f_s maps A onto A, and by (4) f_s is an isomorphism. Hence for any such s, f_s is an automorphism of $\mathfrak{A}$.

Let S' be the set of all functions s on ω into 2 such that $s(n) = 0$ if n is not of the form $3k + 2$. Notice that $|S'| = 2^{\omega}$.

For any $s \in S'$, let $P_s = f_s(P)$. Then $(\mathfrak{A},P) \cong (\mathfrak{A},P_s)$ under the isomorphism f_s, so $P_s \in M(\mathfrak{A},P)$. Thus, we will have shown that $|M(\mathfrak{A},P)| = 2^{\omega}$, and hence that (iii) fails, as soon as we have shown

(C) If $s,t \in S'$ and $s \neq t$, then $P_s \neq P_t$.

If $s,t \in S'$ and $s \neq t$, then there is some $n = 3k + 2$ such that $s(m) = t(m)$ for $m < n$ but $s(n) \neq t(n)$. Say that $s(n) = 0$, $t(n) = 1$. Then $s|n = t|n$ is some $r \in S$ with domain n, and $s|(n + 1) = r^0$, $t|(n + 1) = r^1$. Therefore $G(s)(n) \in P$ by the definition of P, and $G(t)(n) \notin P$ by (B). Hence

$$H(s)(n) = f_s(G(s)(n)) \in P_s \quad \text{and}$$

$$H(t)(n) = f_t(G(t)(n)) \notin P_t.$$

But $H(s)(n) = H(t)(n)$ by (7)(a), hence we have shown that $P_s \neq P_t$, which completes the proof.

3. Complete $L_{\omega_1\omega}$ sentences.

In this section we depart briefly from definability in order to consider some questions raised by Corollary 2.3.

The first question is whether or not the hypothesis of the corollary is satisfied by a significant class of models; in other words, are there many non-trivial examples of countable models which are

$L_{\omega_1\omega}$ elementarily equivalent to uncountable models? The answer to this is yes; in fact one can show the following.

THEOREM 3.1. Let T be any $L_{\omega\omega}$ theory for a countable type τ which has infinite models. Then T has a countable model $\mathfrak{A}$ which is $L_{\omega_1\omega}$ elementarily equivalent to models of arbitrarily large cardinality.

Theorem 3.1 was obtained independently by Chang, Makkai, and the author. A proof of some more general results, using the method of indiscernables, appears in Chang [2].

The second question is less precise. It asks what properties are possessed by uncountable models which are $L_{\omega_1\omega}$ elementarily equivalent to countable models. To make the discussion clearer we introduce the following definition.

DEFINITION. A sentence σ of $L_{\omega_1\omega}$ is complete for $L_{\omega_1\omega}$ (or simply complete) if σ has a model, and any two models of σ are $L_{\omega_1\omega}$ elementarily equivalent.

By the Isomorphism theorem and the downward Löwenheim-Skolem Theorem, a complete $L_{\omega_1\omega}$ sentence is simply the complete $L_{\omega_1\omega}$ theory of a countable model (for a countable type τ). So Corollary 2.3 implies that if σ is a complete $L_{\omega_1\omega}$ sentence with uncountable models, then the countable models of σ have 2^{ω} automorphisms.

Our second question, then, is what can one say about the uncountable models of a complete $L_{\omega_1\omega}$ sentence, particularly with regard to the existence of many automorphisms. The main positive result here is the following theorem, due to Malitz (unpublished).

THEOREM 3.2. Let σ be a sentence of $L_{\omega_1\omega}$ which has models of arbitrarily large cardinality. Then in every infinite power α, σ has a model with 2^{α} automorphisms.

Theorem 3.2 may be derived from the following two facts:

(1) As Lopez-Escobar [6] has shown, the class of models of a sentence of $L_{\omega_1\omega}$ is precisely the class of reducts of ω-models of a theory in ω-logic with at most a countable number of new predicates.

(2) Morley's proof in [9] that the Hanf number of ω-logic is $\beth_{\omega_1}$ actually shows that the Ehrenfeucht-Mostowski results on indiscernables hold for theories of ω-logic with ω-models of

arbitrarily large cardinality; in particular, such theories have ω-models with 2^{α} automorphisms in each infinite power α.

As Malitz shows in his paper [8] in this volume, there are complete $L_{\omega_1\omega}$ sentences which have uncountable models but not models of arbitrarily large cardinality. Thus, Theorem 3.2 does not apply to every complete $L_{\omega_1\omega}$ sentence with uncountable models (in particular, it does not imply Corollary 2.3, even for models $\mathfrak{A}$ for a countable type τ). In fact, using Malitz's examples one can show the following (assuming the G.C.H.):

(A) For every ordinal $\xi < \omega_1$ there is a complete $L_{\omega_1\omega}$ sentence σ which has models in all infinite powers $\leq \beth_{\xi}$, but such that none of its models has more than 2^{ω} automorphisms.

Let σ be the complete sentence that Malitz defines which has models in all infinite powers $\leq \beth_{\xi}$, but in no larger powers. Note that if $\mathfrak{A}$ is a model of σ, then any automorphism of $\mathfrak{A}$ is uniquely determined by what it does to the tree part of $\mathfrak{A}$. Since the tree is countable, $\mathfrak{A}$ can then have at most 2^{ω} automorphisms.

Finally, we have the following example which indicates that the conclusion of Theorem 3.2 cannot be strengthened even if we assume that the sentence σ is complete.

(B) There is a complete $L_{\omega_1\omega}$ sentence σ which has models in arbitrarily large powers which admit only the trivial automorphism.

Let σ be the sentence of $L_{\omega_1\omega}$ which characterizes dense linear orderings without end points (in fact, σ is a sentence of $L_{\omega\omega}$). Then σ is complete for $L_{\omega_1\omega}$ since any two countable models of σ are isomorphic. But it is well-known (cf. [10]) that there are dense linear orders of arbitrarily large powers which admit only the trivial automorphism.

4. Generalizations to higher powers.

In this section we consider the question of generalizing Theorem 1.1 and the results of §2 to uncountable powers. Any such generalization must depend on first finding a suitable generalization of the Isomorphism Theorem, Theorem 1.2. The result one would want would be the following:

(A) If $\mathfrak{A}$ and $\mathfrak{B}$ have power λ and $\mathfrak{A} \equiv_{\infty\lambda} \mathfrak{B}$ then $\mathfrak{A} \cong \mathfrak{B}$.
($L_{\infty\lambda}$ is the union of the languages $L_{\kappa\lambda}$ over all κ).

As Chang shows in his paper [4] in this volume, (A) is true if λ is cofinal with ω. But Morley has found an example (unpublished) which shows that (A) is false if λ is regular and uncountable.

One can also use Morley's example to show that the natural generalizations of Theorems 1.1 and 2.1 fail for uncountable regular cardinals. In fact, if λ is regular and uncountable, then there is a model $\mathfrak{A}$ of power λ and a set $P \subset A$ such that $M(\mathfrak{A},P)$ has exactly one element, but P is not definable by any formula of $L_{\infty\lambda}$ in terms of fewer than λ elements of A.

In the remainder of this section we give the positive results which may be obtained when λ is cofinal with ω. First define $\lambda^* = [\lambda^{\underline{\lambda}}]^+$, where $\lambda^{\underline{\lambda}}$ is the sum of all λ^μ for $\mu < \lambda$. Then Chang actually shows that (A) is true, for λ cofinal with ω, assuming only elementary equivalence in $L_{\lambda^*\lambda}$.

The first result is a straightforward generalization of Theorem 1.1.

THEOREM 4.1. <u>Let</u> $\mathfrak{A}$ <u>be a model of cardinality</u> λ, <u>where</u> λ <u>is cofinal with</u> ω. <u>Let</u> $P \subset A$. <u>Then the following are equivalent</u>:

(i) $M(\mathfrak{A},P)$ <u>has exactly one element.</u>

(ii) <u>There is some formula</u> $\varphi(x)$ <u>of</u> $L_{\lambda^*\lambda}$ <u>such that</u>

$$(\mathfrak{A},P) \models \forall x\,[\underline{P}(x) \longleftrightarrow \varphi(x)].$$

<u>Proof</u>. It is sufficient to show that if (ii) fails then (i) fails. If (ii) fails then one can show (using the fact that the complete $L_{\lambda^*\lambda}$ type of any $a \in A$ is determined by a single formula of $L_{\lambda^*\lambda}$) that there are elements $a,b \in A$ such that

$$a \in P,\quad b \notin P,\quad \text{but}\quad (\mathfrak{A},a) \equiv_{\lambda^*\lambda} (\mathfrak{B},b).$$

Then $(\mathfrak{A},a) \cong (\mathfrak{A},b)$ under some isomorphism f. Let $P' = f(P)$. Then $P' \in M(\mathfrak{A},P)$ and $b \in P'$ (since $a \in P$ and $f(a) = b$). Therefore $P \neq P'$, and so (i) fails.

The situation as regards generalizations of Theorems 2.1 and 2.2 is not as pleasant. We can establish the following "one-directional" generalization of Theorem 2.1.

THEOREM 4.2. <u>Let</u> $\mathfrak{A}$ <u>be a model of cardinality</u> λ, <u>where</u> λ <u>is cofinal with</u> ω. <u>Let</u> $P \subset A$. <u>Assume</u> (i) $|M(\mathfrak{A},P)| < \lambda^\omega$.

Then (ii) There is a formula $\varphi(x,v)$ of $L_{\lambda^*\lambda}$, where v is a sequence of fewer than λ variables, such that

$$(\mathfrak{A},P) \models \exists v\, \forall x\, [\underline{P}(x) \longleftrightarrow \varphi(x,v)].$$

The "easy" direction, from (ii) to (i), no longer holds for $\lambda > \omega$. In fact it is easy to find examples of $\mathfrak{A}$ and P such that (ii) holds for a countable sequence v of variables, but $|M(\mathfrak{A},P)| = \lambda^\omega$. And we certainly cannot allow $|M(\mathfrak{A},P)| = \lambda^\omega$ in (i), since it is consistent to have to have $\lambda^\omega = 2^\lambda$, but not every subset of A need be definable.

On the other hand, one cannot get an equivalence by restricting v in (ii) to be a finite sequence of variables, since one can find $\mathfrak{A}$ and P such that $|M(\mathfrak{A},P)| < \lambda$ but (ii) does not hold for a finite sequence v. These remarks seem to indicate that the definability of a subset P in terms of individual parameters is no longer solely dependent on the cardinality of $M(\mathfrak{A},P)$.

If we add to (ii) the condition that

$$|\{P' \subset A : (\mathfrak{A},P') \models \exists v\, \forall x\, [\underline{P}(x) \longleftrightarrow \varphi(x,v)]\}| < \lambda^\omega,$$

then we do obtain an equivalence, in a trivial fashion. But this added condition seems rather artificial.

A similar partial generalization of Theorem 2.2 can also be obtained. From it we derive the following generalization of Corollary 2.3.

THEOREM 4.3. Let $\mathfrak{A}$ have cardinality λ, where λ is cofinal with ω. If $\mathfrak{A}$ is $L_{\lambda^*\lambda}$ elementary equivalent to a model of cardinality $> \lambda$, then $\mathfrak{A}$ has (at least) λ^ω automorphisms.

We omit the proofs of these last results, since they are similar in outline to the proofs of the corresponding results of §2. We indicate here the main differences between them. Let $\lambda = \Sigma \lambda_n$ $(n \in \omega)$ where $\lambda_n < \lambda_{n+1}$ for all $n \in \omega$. Then instead of using the tree S in which every branch has length ω and which branches twice at every node (as in the proof of Theorem 2.2), we use the tree in which every branch has length ω and which branches λ_n times at every node on the nth level of the tree. An additional variation is needed to ensure that the resulting sequences each enumerate A. And finally, at the end of the proof, we use the fact that $\lambda^\omega = \prod \lambda_n$ $(n \in \omega)$.

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