# Problem Set

| Category | $z = f(x,y)$ | Cylinder (polar) | Cone (spherical) |
|---|---|---|---|
| **Parametrization** | $\mathbf{r}(x,y) = \begin{pmatrix} x \\ y \\ f(x,y) \end{pmatrix}$ | $\mathbf{r}(\theta,z) = \begin{pmatrix} R\cos\theta \\ R\sin\theta \\ z \end{pmatrix}$ | $\mathbf{r}(\rho,\theta) = \begin{pmatrix} \rho\sin\alpha\cos\theta \\ \rho\sin\alpha\sin\theta \\ \rho\cos\alpha \end{pmatrix}$ |
| $\mathbf{r}_x \times \mathbf{r}_y$ | $\begin{pmatrix} -f_x \\ -f_y \\ 1 \end{pmatrix}$ | $\begin{pmatrix} -R\sin\theta \\ R\cos\theta \\ 0 \end{pmatrix}$ | $\begin{pmatrix} -\rho\sin\alpha\cos\alpha\cos\theta \\ -\rho\sin\alpha\cos\alpha\sin\theta \\ \rho\sin^2\alpha \end{pmatrix}$ |
| $\lVert\mathbf{r}_x \times \mathbf{r}_y\rVert$ | $\sqrt{1+f_x^2+f_y^2}$ | $R$ | $\rho\sin\alpha$ |

| Category | Cone (polar) | Sphere (spherical) | Surface of Revolution |
|---|---|---|---|
| **Parametrization** | $\mathbf{r}(r,\theta) = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ r \end{pmatrix}$ | $\mathbf{r}(\theta,\phi) = \begin{pmatrix} R\sin\phi\cos\theta \\ R\sin\phi\sin\theta \\ R\cos\phi \end{pmatrix}$ | $\mathbf{r}(x,\theta) = \begin{pmatrix} x \\ f(x)\cos\theta \\ f(x)\sin\theta \end{pmatrix}$ |
| $\mathbf{r}_x \times \mathbf{r}_y$ | $\begin{pmatrix} -r\cos\theta \\ -r\sin\theta \\ r \end{pmatrix}$ | $\begin{pmatrix} R^2\sin^2\phi\cos\theta \\ R^2\sin^2\phi\sin\theta \\ R^2\sin\phi\cos\phi \end{pmatrix}$ | $\begin{pmatrix} 0 \\ -f'(x)\sin\theta \\ f'(x)\cos\theta \end{pmatrix}$ |
| $\lVert\mathbf{r}_x \times \mathbf{r}_y\rVert$ | $r\sqrt{2}$ | $R^2\sin\phi$ | $\sqrt{(f'(x))^2+(f(x))^2}$ |

## Exercises

1. Let $R$ be the region enclosed by the ellipse $x^2 + 4y^2 = 9$.

   Use an appropriate change of variables to evaluate the integral $\iint_R x^2\,dA$.

2. Let $R$ be the region in the first quadrant between the lines $y = x$ and $y = \sqrt{3}x$ and between the curves $xy = 1$ and $xy = 2$.

   Use an appropriate change of variables to evaluate the integral $\iint_R y^2\,dA$. Sketch the new region obtained as a result of the change of variables.

3. Let $S$ be the surface given by $x^2 + z^2 = 4$ that lies between $y = 1$ and $y = 4$. Find an equation of the plane tangent to the surface at $(2, 2, 0)$.

4. Let $S$ be the surface of revolution of the graph of $y = x^2$ around the $x$-axis.

   Find an equation of the tangent plane to $S$ at point $\left(1, \frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

5. Let $\Sigma$ be the portion of the cylinder $x^2 + y^2 = 4$ bounded below by the $xy-$plane, above by $z = y + 1$. Find the area of $\Sigma$

6. Let $\Sigma$ is the portion of the sphere

   $$\mathbf{r} = 2\sin\phi\cos\theta\,\mathbf{i} + 2\sin\phi\sin\theta\,\mathbf{j} + 2\cos\phi\,\mathbf{k}$$

   that lies above the plane $z = 1$. Find the area of $\Sigma$

7. Find the area $S$ of the part of the sphere $x^2 + y^2 + z^2 = 1$ that lies above the $xy$-plane and inside the (vertical) cylinder $r = \sin\theta$ (where $r$ and $\theta$ represent polar coordinates in the $xy$-plane).