DEGENERACY LOCUS FORMULAS FOR AMENABLE WEYL GROUP ELEMENTS

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ABSTRACT. We define a class of amenable Weyl group elements in the Lie types B, C, and D, which we propose as the analogues of vexillary permutations in these Lie types. Our amenable signed permutations index flagged theta and eta polynomials, which generalize the double theta and eta polynomials of Wilson and the author. In geometry, we obtain corresponding formulas for the cohomology classes of symplectic and orthogonal degeneracy loci.

1. Introduction

A fundamental problem in Schubert calculus is that of finding polynomial representatives for the cohomology classes of the Schubert varieties. In the mid 1990s, Fulton and Pragacz [FP] asked a relative version of the same question, seeking explicit formulas which represent the classes of degeneracy loci for the classical groups, in the sense of [F1, F2, PR]. These loci pull back from the universal Schubert varieties in a G/P-bundle, where G is a classical Lie group, and P a parabolic subgroup of G. As such, they are indexed by elements w in the Weyl group of G, which describe the relative position of two flags of (isotropic) subspaces of a fixed (symplectic or orthogonal) vector space.

The above *Giambelli* and *degeneracy locus* problems were solved in full generality in [T1]. The answer given there is a positive Chern class formula which respects the symmetries of the Weyl group element w and its inverse. The paper [T1] introduced a new, intrinsic point of view in Schubert calculus, showing that formulas native to the homogeneous space G/P are possible, for any parabolic subgroup P, and in all classical Lie types. In special cases, there are alternatives to the general formulas of [T1], but they must all be equivalent to the formulas found there, modulo an explicit ideal of relations among the variables involved.

The seminal work of Lascoux and Schützenberger [LS1, LS2] on Schubert polynomials exposed intrinsic formulas for an important class of permutations, which they called *vexillary*. They defined the *shape* of a general permutation to be the partition obtained by arranging the entries of its code in decreasing order. The key defining property of a vexillary permutation was that its Schubert polynomial can be expressed as a flagged Schur polynomial indexed by its shape. In particular, the prototype for vexillary permutations were the *Grassmannian permutations*, whose Schubert polynomials are the classical Schur polynomials. Our aim in the present paper is to define a family of *amenable signed permutations*, which serve as the analogues of vexillary permutations in the other classical Lie types.

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There have been two attempts in the past to define a notion of vexillary signed permutation in the Lie types B, C, and D, by Billey and Lam [BL] and Anderson and Fulton [AF1]. These definitions miss the mark because according to either of them, the Grassmannian signed permutations are not all vexillary. The revision [AF2] of [AF1] sought to generalize the latter paper by incorporating the theta and eta polynomials of Buch, Kresch, and the author [BKT2, BKT3] and Wilson [W, TW], which are the analogues of the Schur polynomials in the aforementioned Lie types. Unfortunately, although [AF2] is in the right direction, the proofs given there contain serious errors in all Lie types except type A, and the main theorem is false, at least in type D. Moreover, Anderson and Fulton have not acknowledged that intrinsic Chern class formulas for the cohomology classes of all the degeneracy loci are known in any of their writings to date.

Our approach to amenable elements is based on a careful study of how the corresponding raising operator formulas transform under divided differences. The outline of the argument is similar to the one found in Macdonald's notes [M], but there are important differences in the details. The proof is new even in type A, where we obtain a new characterization of vexillary permutations (see below). It is critical to work with double polynomials throughout and use both left and right divided differences to maximum effect, starting from the known formula for the top polynomial, which is indexed by the longest length element. In types B, C, and D, we employ the Schubert polynomials of Ikeda, Mihalcea, and Naruse [IMN], which extend the work of Billey and Haiman [BH] to a theory suitable for applications to equivariant cohomology and degeneracy loci. The paper [T6] provides another key ingredient: the definition of the *shape* of a signed permutation, which plays the role of Lascoux and Schützenberger's shape in the latter Lie types.

The difficulty when working with sequences of divided differences applied to polynomials lies in choosing which path to follow in the weak Bruhat order, as the Leibnitz rule tends to destroy any nice formulas. The papers [TW, T4, T5] showed how divided differences can be used to obtain combinatorial proofs of the raising operator formulas for double theta and double eta polynomials, exploiting the fact that these polynomials behave well under the action of *left* divided differences. Therefore, as long as one remains among the Grassmannian elements, the choice of path through the left weak Bruhat order is immaterial. However this surprising property, first observed in the symplectic case by Ikeda and Matsumura [IM], completely fails once one leaves the Grassmannian regime.

To solve this problem, we introduce the notion of *leading elements* of the Weyl group, which generalize the Grassmannian elements. In the Lie types A, B, and C, a (signed) permutation $w = (w_1, \ldots, w_n)$ is leading if the A-code of the extended sequence $(0, w_1, \ldots, w_n)$ is unimodal. The analogous treatment of type D elements involves some subtleties, which we discuss later. The leading signed permutations are partitioned into equivalence classes defined by their *truncated A-code*. Each of them is in bijection with the class of Grassmannian elements, where the truncated A-code vanishes. The longest length elements within each class give rise to Pfaffian formulas which are proved using divided differences, starting from the formula for the longest element in the Weyl group. Following this, any sequence of left divided differences used to establish the double theta/eta polynomial formula in the Grassmannian case works – in the same way! – to prove a corresponding 'factorial' formula for the elements of the other equivalence classes.

Once the formulas for leading elements are obtained, one can continue to apply type A divided differences, in a manner that preserves the shape of these formulas, and proceed a bit further down the left weak order. We thus arrive at our definition of *amenable elements* of the Weyl group: they are modifications of leading elements, obtained by multiplying them on the left by suitable permutations. In the symmetric group, this reflects the (apparently new) fact that the vexillary permutations are exactly those which can be written as products $\omega\varpi$, with $\ell(\omega\varpi) = \ell(\varpi) - \ell(\omega)$, where ω and ϖ are 312-avoiding and 132-avoiding permutations, respectively.

Finally, one has to deal with the problem that the above formulas do not respect the symmetries (that is, the descent sets) of the amenable Weyl group element involved. This issue was dealt with in [M] by exploiting the alternating properties of determinants, and a similar argument works for the Pfaffian examples of [Ka, AF1]. In the situation at hand, we require variants of the key technical lemmas obtained in [BKT2], which exposed the more subtle alternating properties of the raising operator expressions that define theta polynomials.

We now describe our main result in the symplectic case. Fix an amenable signed permutation w in the hyperoctahedral group W_n. Let $k \geq 0$ be the first right descent of w, list the entries $w_{k+1}, \ldots, w_n$ in increasing order:

$$u_1 < \cdots < u_m < 0 < u_{m+1} < \cdots < u_{n-k}$$

and define

$$\beta := (u_1 + 1, \ldots, u_m + 1, u_{m+1}, \ldots, u_{n-k}),$$

$$D := \{(i,j) \mid 1 \leq i < j \leq n-k \ \text{ and } \ u_i + u_j < 0\},$$

and the raising operator expression

$$R^D := \prod_{i<j}(1 - R_{ij}) \prod_{i<j\,:\,(i,j)\in D}(1 + R_{ij})^{-1}.$$

The A-code of w is the sequence γ with $\gamma_i := \#\{j > i \mid w_j < w_i\}$. Define two partitions ν and ξ by setting $\nu_j := \#\{i \mid \gamma_i \geq j\}$ and $\xi_j := \#\{i \mid \gamma_{k+i} \geq j\}$ for each $j \geq 1$. Following [T6], the shape of w is the partition $\lambda = \mu + \nu$, where $\mu := (-u_1, \ldots, -u_m)$. If ℓ denotes the length of λ, we say that $\mathfrak{q} \in [1, \ell]$ is a *critical index* if $\beta_{\mathfrak{q}+1} > \beta_{\mathfrak{q}} + 1$, or if $\lambda_{\mathfrak{q}} > \lambda_{\mathfrak{q}+1} + 1$ (respectively, $\lambda_{\mathfrak{q}} > \lambda_{\mathfrak{q}+1}$) and $\mathfrak{q} < m$ (respectively, $\mathfrak{q} > m$). Define two sequences $\mathfrak{f}$ and $\mathfrak{g}$ of length ℓ by setting

$$\mathfrak{f}_j := k + \max(i \mid \gamma_{k+i} \geq j)$$

for each j, and $\mathfrak{g}_j := \mathfrak{f}_{\mathfrak{q}} + \beta_{\mathfrak{q}} - \xi_{\mathfrak{q}} - k$, where $\mathfrak{q}$ is the least critical index such that $\mathfrak{q} \geq j$. The sequences $\mathfrak{f}$ and $\mathfrak{g}$ are the *right* and *left flags* of w, and $\mathfrak{f}$ (respectively $|\,\mathfrak{g}\,|$) consists of right (respectively left) descents of w.

Let $E \to \mathfrak{X}$ be a symplectic vector bundle of rank $2n$ on a smooth complex algebraic variety $\mathfrak{X}$. We are given two complete flags of subbundles of E

$$0 \subset E_1 \subset \cdots \subset E_{2n} = E \quad \text{and} \quad 0 \subset F_1 \subset \cdots \subset F_{2n} = E$$

with $\operatorname{rank} E_r = \operatorname{rank} F_r = r$ for each r, while $E_{n+s} = E_{n-s}^{\perp}$ and $F_{n+s} = F_{n-s}^{\perp}$ for $0 \leq s < n$. Consider the *degeneracy locus* $\mathfrak{X}_w \subset \mathfrak{X}$, which we assume has pure codimension $\ell(w)$ in $\mathfrak{X}$ (the precise definition of $\mathfrak{X}_w$ is given in Section 5.3). Then the *flagged theta polynomial formula*

$$[\mathfrak{X}_w] = \Theta_w(E - E_{n-\mathfrak{f}} - F_{n+\mathfrak{g}}) = R^D\, c_\lambda(E - E_{n-\mathfrak{f}} - F_{n+\mathfrak{g}}) \tag{1}$$

holds in the cohomology ring $\mathrm{H}^*(\mathfrak{X})$. Following [TW], the Chern polynomial in (1) is interpreted as the image of $R^D \mathfrak{c}_\lambda$ under the $\mathbb{Z}$-linear map which sends the noncommutative monomial $\mathfrak{c}_\alpha = \mathfrak{c}_{\alpha_1}\mathfrak{c}_{\alpha_2}\cdots$ to $\prod_j c_{\alpha_j}(E - E_{n-\mathfrak{f}_j} - F_{n+\mathfrak{g}_j})$, for every integer sequence α.

To understand some of the additional challenges one faces in the even orthogonal type D, consider first the question of how to define the shape of an element w in the associated Weyl group $\widetilde{W}_n$. There seems to be no consistent way to do this, since e.g. the element $(\overline{3}, \overline{1}, 2)$ has shape $\lambda = 2$ when considered as a $\square$-Grassmannian element, but shape $\lambda = (1,1)$ when considered as a 1-Grassmannian element. The definition given in [T6, Def. 5] prefers the latter shape over the former, but the more difficult question before us here requires a further refinement.

Our solution is to define the shape of w to be a *typed* partition, where the type is an integer in $\{0,1,2\}$, extending the corresponding notion for Grassmannian elements from [BKT1]. The $\square$-Grassmannian elements and their Pfaffian formulas are abandoned entirely; instead, we view them all as 1-Grassmannian elements! This fits in well with our previous papers [BKT1, BKT3, T2, T4] on the orthogonal Grassmannians $\mathrm{OG}(n-k, 2n)$ and (double) eta polynomials, where we assumed $k \geq 1$ from the beginning – but for a different reason.

Another obstacle appears when one tries to define the leading elements of $\widetilde{W}_n$. It was observed in [T4, Sec. 3.3] that the compatibility of double eta polynomials with left divided differences is more delicate than the corresponding fact in types B and C. In order to preserve this crucial property for the polynomials indexed by leading elements, we must demand that they are all *proper* elements of $\widetilde{W}_n$ (Definition 12). There is no analogue of this subtle condition in the other classical Lie types. Once all the definitions which are special to the type D theory are found, the proof of the main result proceeds in a manner parallel to the other three types.

This paper is organized as follows. Section 2 contains background material on divided differences and Schubert polynomials, and defines the shape of a (signed) permutation in all the classical types. Section 3 deals with raising operators and provides variants of the lemmas from [BKT2] that we require here. Sections 4, 5, and 6 define and study amenable elements and their applications in types A, C, and B/D, respectively. In particular, we give our notion of *flagged theta* and *flagged eta polynomials*; these are indexed by amenable Weyl group elements. Finally, Appendix A contains counterexamples to several statements in [AF2].

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2. Preliminaries

This section gathers together background material on the divided differences and Schubert polynomials used in this work. We also discuss the notion of the shape of a (signed) permutation. Our notation is compatible with that found in [T6].

2.1. **Lie type A.** Throughout this paper we will employ integer sequences $\alpha = (\alpha_1, \alpha_2, \ldots)$, which are assumed to have finite support, and we identify with integer vectors. The integer sequence α is a *composition* if $\alpha_j \geq 0$ for all j. A weakly decreasing composition is called a *partition.* If λ is a partition, the *length* of λ is the integer $\ell(\lambda) := \#\{i \mid \lambda_i \neq 0\}$, and the *conjugate* of λ is the partition λ' with $\lambda'_j := \#\{i \mid \lambda_i \geq j\}$ for all $j \geq 1$. As is customary, we identify partitions with their Young diagrams of boxes, arranged in left justified rows. An inclusion $\mu \subset \lambda$ of partitions corresponds to the containment of their respective diagrams; in this case, the skew diagram λ/μ is the set-theoretic difference $\lambda \smallsetminus \mu$. For each integer $r \geq 1$, let $\delta_r := (r, r-1, \ldots, 1)$, $\delta_r^\vee := (1, 2, \ldots, r)$, and set $\delta_0 := 0$. Denote by ϵ_r the sequence whose rth term is 1 and all other terms are zero.

The symmetric group S_n is generated by the simple transpositions $s_i = (i, i+1)$ for $1 \leq i \leq n-1$. There is a natural embedding of S_n in S_{n+1} by adjoining $n+1$ as a fixed point, and we let $S_\infty := \cup_n S_n$. We will write a permutation $\varpi \in S_n$ using one line notation, as the word $(\varpi_1, \ldots, \varpi_n)$ where $\varpi_i = \varpi(i)$.

The *length* of a permutation ϖ, denoted $\ell(\varpi)$, is the least integer r such that we have an expression $\varpi = s_{i_1} \cdots s_{i_r}$. The word $s_{i_1} \cdots s_{i_r}$ is called a *reduced decomposition* for ϖ. An element $\varpi \in S_\infty$ has a *left descent* (respectively, a *right descent*) at position $i \geq 1$ if $\ell(s_i \varpi) < \ell(\varpi)$ (respectively, if $\ell(\varpi s_i) < \ell(\varpi)$). The permutation $\varpi = (\varpi_1, \varpi_2, \ldots)$ has a right descent at i if and only if $\varpi_i > \varpi_{i+1}$, and a left descent at i if and only if $\varpi^{-1}(i) > \varpi^{-1}(i+1)$.

The *code* $\gamma = \gamma(\varpi)$ of a permutation $\varpi \in S_n$ is the sequence $\{\gamma_i\}$ with $\gamma_i := \#\{j > i \mid \varpi_j < \varpi_i\}$. The code γ determines ϖ, as follows. We have $\varpi_1 = \gamma_1 + 1$, and for $i > 1$, ϖ_i is the $(\gamma_i + 1)$st element in the complement of $\{\varpi_1, \ldots, \varpi_{i-1}\}$ in the sequence $(1, \ldots, n)$. Following [LS1, M], the *shape* $\lambda = \lambda(\varpi)$ of ϖ is the partition whose parts are the non-zero entries γ_i of the code $\gamma(\varpi)$, arranged in weakly decreasing order. We have $|\lambda| := \sum_i \lambda_i = \sum_i \gamma_i = \ell(\varpi)$.

For any integer $p \geq 0$ and sequence of variables $Z := (z_1, z_2, \ldots)$, the elementary and complete symmetric functions $e_p(Z)$ and $h_p(Z)$ are defined by the generating series

$$\prod_{i=1}^{\infty}(1 + z_i t) = \sum_{p=0}^{\infty} e_p(Z) t^p \quad \text{and} \quad \prod_{i=1}^{\infty}(1 - z_i t)^{-1} = \sum_{p=0}^{\infty} h_p(Z) t^p,$$

respectively. If $r \geq 1$ then we let $e_p^r(Z) := e_p(z_1, \ldots, z_r)$ and $h_p^r(Z) := h_p(z_1, \ldots, z_r)$ denote the polynomials obtained from $e_p(Z)$ and $h_p(Z)$ by setting $z_i = 0$ for all $i > r$. Let $e_p^0(Z) = h_p^0(Z) := \delta_{0p}$, where δ_{0p} denotes the Kronecker delta, and for $r < 0$, define $h_p^r(Z) := e_p^{-r}(Z)$ and $e_p^r(Z) := h_p^{-r}(Z)$.

Let $X := (x_1, x_2, \ldots)$ and $Y := (y_1, y_2, \ldots)$ be two sequences of independent variables. There is an action of S_∞ on $\mathbb{Z}[X, Y]$ by ring automorphisms, defined by letting the simple reflections s_i act by interchanging x_i and x_{i+1} while leaving all the remaining variables fixed. Define the *divided difference operator* ∂_i^x on $\mathbb{Z}[X, Y]$ by

$$\partial_i^x f := \frac{f - s_i f}{x_i - x_{i+1}} \quad \text{for } i \geq 1.$$

Consider the ring involution $\pi : \mathbb{Z}[X, Y] \to \mathbb{Z}[X, Y]$ determined by $\pi(x_i) = -y_i$ and $\pi(y_i) = -x_i$ for each i, and set $\partial_i^y := \pi \partial_i^x \pi$.

For any $p, r, s \in \mathbb{Z}$, define the polynomial ${}^r h_p^s$ by

$${}^r h_p^s := \sum_{i=0}^{p} h_i^r(X) e_{p-i}^s(-Y).$$

We have the following basic lemma.

Lemma 1. *Suppose that $p, r, s \in \mathbb{Z}$. For all $i \geq 1$, we have*

$$\partial_i^x({}^r h^s_p) = \begin{cases} {}^{r+1}h^s_{p-1} & \text{if } r = \pm i, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \partial_i^y({}^r h^s_p) = \begin{cases} {}^{r}h^{s-1}_{p-1} & \text{if } s = \pm i, \\ 0 & \text{otherwise.} \end{cases}$$

The double Schubert polynomials $\mathfrak{S}_\varpi$ for $\varpi \in S_\infty$ of Lascoux and Schützenberger [Las, LS1] are the unique family of polynomials in $\mathbb{Z}[X,Y]$ such that

$$\partial_i^x \mathfrak{S}_\varpi = \begin{cases} \mathfrak{S}_{\varpi s_i} & \text{if } \ell(\varpi s_i) < \ell(\varpi), \\ 0 & \text{otherwise,} \end{cases} \qquad \partial_i^y \mathfrak{S}_\varpi = \begin{cases} \mathfrak{S}_{s_i \varpi} & \text{if } \ell(s_i \varpi) < \ell(\varpi), \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

for all $i \geq 1$, together with the condition that the constant term of $\mathfrak{S}_\varpi$ is 1 if $\varpi = 1$, and 0 otherwise.

2.2. **Lie type C.** The Weyl group for the root system of type C_n is the group of signed permutations on the set $\{1, \ldots, n\}$, denoted W_n. The group W_n is generated by the simple transpositions $s_i = (i, i+1)$ for $1 \leq i \leq n-1$ together with the sign change s_0, which fixes all $j \in [2, n]$ and sends 1 to $\overline{1}$ (a bar over an integer here means a negative sign). We write the elements of W_n as n-tuples $(w_1, \ldots, w_n)$, where $w_i := w(i)$ for each $i \in [1, n]$. There is a natural embedding of W_n in W_{n+1} by adjoining $n+1$ as a fixed point, and we let $W_\infty := \cup_n W_n$. The symmetric groups S_n and S_∞ are the subgroups of W_n and W_∞, respectively, generated by the reflections s_i for i positive. The *length* $\ell(w)$ and the reduced decompositions of an element $w \in W_\infty$ is defined as in type A. We have

$$\ell(w) = \#\{i < j \mid w_i > w_j\} + \sum_{i\,:\,w_i < 0} |w_i|$$

for every $w \in W_\infty$.

An element $w \in W_\infty$ has a *right descent* (respectively, a *left descent*) at position $i \geq 0$ if $\ell(ws_i) < \ell(w)$ (respectively, if $\ell(s_i w) < \ell(w)$). The signed permutation $w = (w_1, w_2, \ldots)$ has a right descent at 0 if and only if $w_1 < 0$, and a right descent at $i \geq 1$ if and only if $w_i > w_{i+1}$. The element w has a left descent at 0 if and only if $w^{-1}(1) < 0$, that is, $w = (\cdots \overline{1} \cdots)$. The element w has a left descent at $i \geq 1$ if and only if $w^{-1}(i) > w^{-1}(i+1)$, that is, w has one of the following four forms:

$$(\cdots i+1 \cdots i \cdots), \quad (\cdots i \cdots \overline{i+1} \cdots), \quad (\cdots \overline{i+1} \cdots i \cdots), \quad (\cdots \overline{i} \cdots \overline{i+1} \cdots).$$

Let $w \in W_\infty$ be a signed permutation. Following [T6, Def. 2], the strict partition $\mu = \mu(w)$ is the one whose parts are the absolute values of the negative entries of w, arranged in decreasing order. The *A-code* of w is the sequence $\gamma = \gamma(w)$ with $\gamma_i := \#\{j > i \mid w_j < w_i\}$. We define a partition $\nu = \nu(w)$ by

$$\nu_j = \#\{i \mid \gamma_i \geq j\}, \quad \text{for all } j \geq 1.$$

Finally, the *shape* of w is the partition $\lambda(w) := \mu(w) + \nu(w)$. The element w is uniquely determined by $\mu(w)$ and $\gamma(w)$, and we have $|\lambda(w)| = \ell(w)$.

Example 1. (a) For the signed permutation $w := (\overline{5}, 3, \overline{4}, 7, \overline{1}, \overline{6}, 2)$ in W_7, we obtain $\mu = (6,5,4,1)$, $\gamma = (1,4,1,3,1,0,0)$, $\nu = (5,2,2,1)$, and $\lambda = (11,7,6,2)$.

(b) Let $k \geq 0$. An element $w \in W_\infty$ is k-Grassmannian if $\ell(ws_i) > \ell(w)$ for all $i \neq k$. This is equivalent to the conditions

$$0 < w_1 < \cdots < w_k \quad \text{and} \quad w_{k+1} < w_{k+2} < \cdots .$$

If w is a k-Grassmannian element of W_∞, then $\lambda(w)$ is the k-strict partition associated to w in [BKT2, Sec. 6.1].

(c) Suppose that the first right descent of $w \in W_n$ is $k \geq 0$, and let $m = \ell(\mu)$ and $\ell = \ell(\lambda)$. Then μ is a strict partition and $\nu \subset k^{n-k} + \delta_{n-k-1}$, with $\nu_j \geq k$ for all $j \in [1, m]$. It follows that

$$\lambda_1 > \lambda_2 > \cdots > \lambda_m > \max(\lambda_{m+1}, k) \geq \lambda_{m+1} \geq \lambda_{m+2} \geq \cdots \geq \lambda_\ell.$$

Lemma 2 ([M, T6]). *If $i \geq 1$, $w \in W_\infty$, and $\gamma = \gamma(w)$, then*

$$\gamma_i > \gamma_{i+1} \Leftrightarrow w_i > w_{i+1} \Leftrightarrow \ell(ws_i) = \ell(w) - 1.$$

If any of the above conditions hold, then

$$\gamma(ws_i) = (\gamma_1, \ldots, \gamma_{i-1}, \gamma_{i+1}, \gamma_i - 1, \gamma_{i+2}, \gamma_{i+3}, \ldots).$$

Let $c := (c_1, c_2, \ldots)$ be a sequence of commuting variables, and set $c_0 := 1$ and $c_p := 0$ for $p < 0$. Consider the graded ring Γ which is the quotient of the polynomial ring $\mathbb{Z}[c]$ modulo the ideal generated by the relations

$$c_p c_p + 2\sum_{i=1}^{p}(-1)^i c_{p+i}c_{p-i} = 0, \quad \text{for all } p \geq 1. \tag{3}$$

Let $X := (x_1, x_2, \ldots)$ and $Y := (y_1, y_2, \ldots)$ be two sequences of variables. Following [BH, IMN], there is an action of W_∞ on $\Gamma[X, Y]$ by ring automorphisms, defined as follows. The simple reflections s_i for $i \geq 1$ act by interchanging x_i and x_{i+1} while leaving all the remaining variables fixed. The reflection s_0 maps x_1 to $-x_1$, fixes the x_j for $j \geq 2$ and all the y_j, and satisfies

$$s_0(c_p) := c_p + 2\sum_{j=1}^{p} x_1^j c_{p-j} \ \text{ for all } p \geq 1.$$

For each $i \geq 0$, define the *divided difference operator* ∂_i^x on $\Gamma[X, Y]$ by

$$\partial_0^x f := \frac{f - s_0 f}{-2x_1}, \qquad \partial_i^x f := \frac{f - s_i f}{x_i - x_{i+1}} \quad \text{for } i \geq 1.$$

Consider the ring involution $\varphi : \Gamma[X, Y] \to \Gamma[X, Y]$ determined by

$$\varphi(x_j) = -y_j, \qquad \varphi(y_j) = -x_j, \qquad \varphi(c_p) = c_p$$

and set $\partial_i^y := \varphi\partial_i^x\varphi$ for each $i \geq 0$. The right and left divided difference operators ∂_i^x and ∂_i^y on $\Gamma[X, Y]$ satisfy the right and left Leibnitz rules

$$\partial_i^x(fg) = (\partial_i^x f)g + (s_i f)\partial_i^x g \quad \text{and} \quad \partial_i^y(fg) = (\partial_i^y f)g + (s_i^y f)\partial_i^y g, \tag{4}$$

where $s_i^y := \varphi s_i \varphi$, for every $i \geq 0$.

For any $p, r, s \in \mathbb{Z}$, define the polynomial ${}^r c_p^s$ by

$${}^r c_p^s := \sum_{i=0}^{p}\sum_{j=0}^{p} c_{p-i-j} e_i^r(X) h_j^s(-Y).$$

We have the following basic lemma, which stems from [IM, Sec. 5.1].

Lemma 3. (a) *Suppose that $p, r, s \in \mathbb{Z}$. For all $i \geq 0$, we have*

$$\partial_i^x({}^r c_p^s) = \begin{cases} {}^{r-1}c_{p-1}^s & \textit{if } r = \pm i, \\ 0 & \textit{otherwise.} \end{cases} \quad \text{and} \quad \partial_i^y({}^r c_p^s) = \begin{cases} {}^r c_{p-1}^{s+1} & \textit{if } s = \pm i, \\ 0 & \textit{otherwise.} \end{cases}$$

(b) *For all $i \geq 1$, $r, s \geq 0$, and indices p and q, we have*

$$\partial_i^y({}^r c_p^{-i}\, {}^s c_q^i) = {}^r c_{p-1}^{-i+1}\, {}^s c_q^{i+1} + {}^r c_p^{-i+1}\, {}^s c_{q-1}^{i+1}.$$

Suppose $r, s \geq 0$, and let $\mathfrak{c}_p := {}^r c_p^{-s}$ for each $p \in \mathbb{Z}$. We then have the relations

$$\mathfrak{c}_p \mathfrak{c}_p + 2\sum_{i=1}^{p} (-1)^i \mathfrak{c}_{p+i}\mathfrak{c}_{p-i} = 0 \quad \text{for all} \quad p > r+s \tag{5}$$

in $\Gamma[X, Y]$. Indeed, if $\mathcal{C}(t) := \sum_{p=0}^{\infty} \mathfrak{c}_p t^p$ is the generating function for the $\mathfrak{c}_p$, we have

$$\mathcal{C}(t) = \prod_{i=1}^{r}(1+x_i t)\prod_{j=1}^{s}(1-y_j t)\left(\sum_{p=0}^{\infty} c_p t^p\right)$$

and hence

$$\mathcal{C}(t)\mathcal{C}(-t) = \prod_{i=1}^{r}(1-x_i^2 t^2)\prod_{j=1}^{s}(1-y_j^2 t^2),$$

which is a polynomial in t of degree $2(r+s)$.

The type C double Schubert polynomials $\mathfrak{C}_w$ for $w \in W_\infty$ of Ikeda, Mihalcea, and Naruse [IMN] are the unique family of elements of $\Gamma[X, Y]$ such that

$$\partial_i^x \mathfrak{C}_w = \begin{cases} \mathfrak{C}_{ws_i} & \text{if } \ell(ws_i) < \ell(w), \\ 0 & \text{otherwise,} \end{cases} \qquad \partial_i^y \mathfrak{C}_w = \begin{cases} \mathfrak{C}_{s_i w} & \text{if } \ell(s_i w) < \ell(w), \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

for all $i \geq 0$, together with the condition that the constant term of $\mathfrak{C}_w$ is 1 if $w = 1$, and 0 otherwise.

2.3. **Lie types B and D.** When working with the orthogonal Lie types, we use coefficients in the ring $\mathbb{Z}[\frac{1}{2}]$. For any $w \in W_\infty$, the type B double Schubert polynomial $\mathfrak{B}_w$ of [IMN] satisfies $\mathfrak{B}_w = 2^{-s(w)}\mathfrak{C}_w$, where $s(w)$ is the number of indices j such that $w_j < 0$. The odd orthogonal case is therefore entirely similar to the symplectic case. In the rest of this section we provide the corresponding preliminaries for the even orthogonal group, that is, in Lie type D, and assume that $n \geq 2$.

The Weyl group $\widetilde{W}_n$ for the root system D_n is the subgroup of W_n consisting of all signed permutations with an even number of sign changes. The group $\widetilde{W}_n$ is an extension of S_n by the element $s_\square = s_0 s_1 s_0$, which acts on the right by

$$(w_1, w_2, \ldots, w_n)s_\square = (\overline{w}_2, \overline{w}_1, w_3, \ldots, w_n).$$

There is a natural embedding $\widetilde{W}_n \hookrightarrow \widetilde{W}_{n+1}$ of Weyl groups, induced by the embedding $W_n \hookrightarrow W_{n+1}$, and we let $\widetilde{W}_\infty := \cup_n \widetilde{W}_n$. The elements of the set $\mathbb{N}_\square := \{\square, 1, \ldots\}$ index the simple reflections in $\widetilde{W}_\infty$. The *length* $\ell(w)$ and reduced decompositions of an element $w \in \widetilde{W}_\infty$ are defined as before. We have

$$\ell(w) = \#\{i < j \mid w_i > w_j\} + \sum_{i\,:\,w_i<0}(|w_i| - 1)$$

for every $w \in \widetilde{W}_\infty$.

An element $w \in \widetilde{W}_\infty$ has a *right descent* (respectively, a *left descent*) at position $i \in \mathbb{N}_\square$ if $\ell(ws_i) < \ell(w)$ (respectively, if $\ell(s_i w) < \ell(w)$). The element $w = (w_1, w_2, \ldots)$ has a right descent at $\square$ if and only if $w_1 < -w_2$, and a right descent at $i \geq 1$ if and only if $w_i > w_{i+1}$. We use the notation $\widehat{1}$ to denote 1 or

$\overline{1}$, determined by the parity of the number of negative entries of w. The following result corrects [T4, Lemma 4]:

Lemma 4. *Suppose that w is an element of $\widetilde{W}_\infty$.*

(a) *We have $\ell(s_\square w) < \ell(w)$ if and only if w has one of the following four forms:*

$$(\cdots \widehat{1} \cdots \overline{2} \cdots), \quad (\cdots \overline{2} \cdots \overline{1} \cdots), \quad (\cdots 2 \cdots \overline{1} \cdots).$$

(b) *Assume that $i \geq 1$. We have $\ell(s_i w) < \ell(w)$ if and only if w has one of the following four forms:*

$$(\cdots i+1 \cdots i \cdots), \quad (\cdots i \cdots \overline{i+1} \cdots), \quad (\cdots \overline{i+1} \cdots i \cdots), \quad (\cdots \overline{i} \cdots \overline{i+1} \cdots).$$

Definition 1. We say that w has *type 0* if $|w_1| = 1$, *type 1* if $w_1 > 1$, and *type 2* if $w_1 < -1$.

There is an involution $\iota : \widetilde{W}_\infty \to \widetilde{W}_\infty$ which interchanges $s_\square$ and s_1; we have $\iota(w) = s_0 w s_0$ in the hyperoctahedral group W_∞. We deduce that $\iota(w) = w$ if and only if w has type 0, while if w has positive type and $|w_r| = 1$ for some $r > 1$, then

$$\iota(w) = (-w_1, w_2, \ldots, w_{r-1}, -w_r, w_{r+1}, \ldots).$$

It follows that ι interchanges type 1 and type 2 elements. The next definition refines the notion of the shape of an element of $\widetilde{W}_\infty$ introduced in [T6, Def. 5].

Definition 2. Let $w \in \widetilde{W}_\infty$ have type 0 or type 1. The strict partition $\mu(w)$ is the one whose parts are the absolute values of the negative entries of w minus one, arranged in decreasing order. Let $\gamma = \gamma(w)$ be the A-code of w, and define the parts of the partition $\nu = \nu(w)$ by $\nu_j := \#\{i \mid \gamma_i \geq j\}$. If w has type 2, then set $\mu(w) := \mu(\iota(w))$, $\gamma(w) := \gamma(\iota(w))$, and $\nu(w) := \nu(\iota(w))$.

A *typed partition* is a pair consisting of a partition λ together with an integer $\mathrm{type}(\lambda) \in \{0, 1, 2\}$. The *shape* of w is the typed partition $\lambda = \lambda(w)$ defined by $\lambda(w) := \mu(w) + \nu(w)$, with $\mathrm{type}(\lambda) := \mathrm{type}(w)$.

Observe that the element w is uniquely determined by $\mu(w)$, $\gamma(w)$, and $\mathrm{type}(w)$. Moreover, we have $|\lambda(w)| = \ell(w)$.

Definition 3. Let $w \in \widetilde{W}_\infty \smallsetminus \{1\}$, let d denote the first right descent of w, and set $k := d$, if $d \neq \square$, and $k := 1$, if $d = \square$. We call k the *primary index* of w.

Example 2. (a) For the signed permutation $w := (3, 2, \overline{7}, 1, 5, 4, \overline{6})$ in $\widetilde{W}_7$, we obtain $\mu = (6, 5)$, $\gamma = (4, 3, 0, 1, 2, 1, 0)$, $\nu = (5, 3, 2, 1)$, and $\lambda = (11, 8, 2, 1)$ with $\mathrm{type}(\lambda) = 1$. The element $\iota(w) = (\overline{3}, 2, \overline{7}, \overline{1}, 5, 4, \overline{6})$ has shape $\overline{\lambda} = (11, 8, 2, 1)$ with $\mathrm{type}(\overline{\lambda}) = 2$. Both w and $\iota(w)$ have primary index $k = 1$.

(b) Let $k \geq 1$. An element w of $\widetilde{W}_\infty$ is k-Grassmannian if $\ell(ws_i) > \ell(w)$ for all $i \neq k$, if $k > 1$, and for all $i \notin \{\square, 1\}$, if $k = 1$. This is equivalent to the conditions

$$|w_1| < w_2 < \cdots < w_k \quad \text{and} \quad w_{k+1} < w_{k+2} < \cdots,$$

the first condition being vacuous if $k = 1$. If w is a k-Grassmannian element of $\widetilde{W}_\infty$, then $\lambda(w)$ is the typed k-strict partition associated to w in [BKT3, Sec. 6.1].

(c) Suppose that the primary index of $w \in \widetilde{W}_n$ is $k \geq 1$, and let $m = \ell(\mu)$ and $\ell = \ell(\lambda)$. Then μ is a strict partition and $\nu \subset k^{n-k} + \delta_{n-k-1}$, with $\nu_j \geq k$ for all $j \in [1, m]$. We therefore have

$$\lambda_1 > \lambda_2 > \cdots > \lambda_m > \max(\lambda_{m+1}, k) \geq \lambda_{m+1} \geq \lambda_{m+2} \geq \cdots \geq \lambda_\ell.$$

Let $b := (b_1, b_2, \ldots)$ be a sequence of commuting variables, and set $b_0 := 1$ and $b_p := 0$ for $p < 0$. Consider the graded ring Γ' which is the quotient of the polynomial ring $\mathbb{Z}[b]$ modulo the ideal generated by the relations

$$b_p b_p + 2\sum_{i=1}^{p-1}(-1)^i b_{p+i}b_{p-i} + (-1)^p b_{2p} = 0, \quad \text{for all } p \geq 1.$$

We regard Γ as a subring of Γ' via the injective ring homomorphism which sends c_p to $2b_p$ for every $p \geq 1$.

Following [BH, IMN], we define an action of $\widetilde{W}_\infty$ on $\Gamma'[X, Y]$ by ring automorphisms as follows. The simple reflections s_i for $i \geq 1$ act by interchanging x_i and x_{i+1} and leaving all the remaining variables fixed. The reflection $s_\Box$ maps (x_1, x_2) to $(-x_2, -x_1)$, fixes the x_j for $j \geq 3$ and all the y_j, and satisfies, for any $p \geq 1$,

$$s_\Box(b_p) := b_p + (x_1 + x_2)\sum_{j=0}^{p-1}\left(\sum_{a+b=j} x_1^a x_2^b\right) c_{p-1-j}.$$

For each $i \in \mathbb{N}_\Box$, define the divided difference operator ∂_i^x on $\Gamma'[X, Y]$ by

$$\partial_\Box^x f := \frac{f - s_\Box f}{-x_1 - x_2}, \qquad \partial_i^x f := \frac{f - s_i f}{x_i - x_{i+1}} \quad \text{for } i \geq 1.$$

Consider the ring involution $\varphi' : \Gamma'[X, Y] \to \Gamma'[X, Y]$ determined by

$$\varphi'(x_j) = -y_j, \qquad \varphi'(y_j) = -x_j, \qquad \varphi'(b_p) = b_p$$

and set $\partial_i^y := \varphi' \partial_i^x \varphi'$ for each $i \in \mathbb{N}_\Box$. The right and left divided difference operators ∂_i^x and ∂_i^y on $\Gamma'[X, Y]$ satisfy the right and left Leibnitz rules

$$\partial_i^x(fg) = (\partial_i^x f)g + (s_i f)\partial_i^x g \quad \text{and} \quad \partial_i^y(fg) = (\partial_i^y f)g + (s_i^y f)\partial_i^y g, \tag{7}$$

where $s_i^y := \varphi' s_i \varphi'$, for every $i \in \mathbb{N}_\Box$.

Let $r \geq 0$ and set ${}^r c_p := \sum_{i=0}^p c_{p-i} h_i^{-r}(X)$. Define ${}^r b_p := {}^r c_p$ for $p < r$, ${}^r b_p := \frac{1}{2}{}^r c_p$ for $p > r$, and set

$${}^r b_r := \frac{1}{2}{}^r c_r + \frac{1}{2}e_r^r(X) \quad \text{and} \quad {}^r\widetilde{b}_r := \frac{1}{2}{}^r c_r - \frac{1}{2}e_r^r(X).$$

For $s \in \{0, 1\}$, let ${}^r a_p^s := \frac{1}{2}{}^r c_p + \sum_{i=1}^p {}^r c_{p-i} h_i^s(-Y)$, and define

$${}^r b_r^s := {}^r b_r + \sum_{i=1}^r {}^r c_{r-i} h_i^s(-Y), \quad \text{and} \quad {}^r\widetilde{b}_r^s := {}^r\widetilde{b}_r + \sum_{i=1}^r {}^r c_{r-i} h_i^s(-Y).$$

We have the following propositions, which are proved as in [T4, Sec. 2].

Proposition 1. *Suppose that $p, q \in \mathbb{Z}$ and $r, s \geq 1$.*

(a) *For all $i \geq 1$, we have*

$$\partial_i^x({}^r c_p^q) = \begin{cases} {}^{r-1}c_{p-1}^q & \text{if } r = \pm i, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \partial_i^y({}^r c_p^q) = \begin{cases} {}^r c_{p-1}^{q+1} & \text{if } q = \pm i, \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$\partial^y_\square\left({}^r c^q_p\right) = \begin{cases} {}^r c^2_{p-1} & \text{if } q = 1, \\ 2\left({}^r c^2_{p-1}\right) & \text{if } q = 0, \\ 2\left({}^r c^1_{p-1}\right) - {}^r c_{p-1} & \text{if } q = -1, \\ 0 & \text{if } |q| \geq 2. \end{cases}$$

(b) *For all $i \geq 1$, we have*

$$\partial^y_i({}^r c^{-i}_p\, {}^s c^i_q) = {}^r c^{-i+1}_{p-1}\, {}^s c^{i+1}_q + {}^r c^{-i+1}_p\, {}^s c^{i+1}_{q-1}.$$

We also require certain variations of the above identities. Let f_r be an indeterminate of degree r, which will equal ${}^r b_r$, ${}^r\widetilde{b}_r$, or $\frac{1}{2}\,{}^r c_r$ in the sequel. We also let $f_0 \in \{0, 1\}$. For any $p, s \in \mathbb{Z}$, define ${}^r\widehat{c}^s_p$ by

$${}^r\widehat{c}^s_p := {}^r c^s_p + \begin{cases} (2f_r - {}^r c_r) e^{p-r}_{p-r}(-Y) & \text{if } s = r - p < 0, \\ 0 & \text{otherwise.} \end{cases}$$

For $s \in \{0, 1\}$, define

$$f^s_r := f_r + \sum_{j=1}^{r} {}^r c_{r-j} h^s_j(-Y),$$

set $\widetilde{f}_r := {}^r c_r - f_r$ and $\widetilde{f}^s_r := {}^r c_r - 2f_r + f^s_r$.

Proposition 2. *Suppose that $p \in \mathbb{Z}$ and $p > r$.*

(a) *For all $i \geq 1$, we have*

$$\partial^x_i({}^r\widehat{c}^{r-p}_p) = \begin{cases} {}^{r-1}\widehat{c}^{r-p}_{p-1} & \text{if } i = p - r \geq 2, \\ 2\varphi'(f_r) & \text{if } i = p - r = 1, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\partial^y_i({}^r\widehat{c}^{r-p}_p) = \begin{cases} {}^r\widehat{c}^{r-p+1}_{p-1} & \text{if } i = p - r \geq 2, \\ 2f_r & \text{if } i = p - r = 1, \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$\partial^y_\square\left({}^r\widehat{c}^{r-p}_p\right) = \begin{cases} 2\widetilde{f}^1_r & \text{if } r - p = -1, \\ 0 & \text{if } r - p < -1. \end{cases}$$

(b) *For all $i \geq 2$, we have*

$$\partial^y_i({}^r\widehat{c}^{-i}_p\, {}^s c^i_q) = {}^r\widehat{c}^{-i+1}_{p-1}\, {}^s c^{i+1}_q + {}^r\widehat{c}^{-i+1}_p\, {}^s c^{i+1}_{q-1}.$$

Fix $r, s \geq 0$, and define $\mathfrak{c}_p := {}^r c^{-s}_p$ for each $p \in \mathbb{Z}$. For $p = r + s$, set $\mathfrak{d}_p := e^r_r(X) e^s_s(-Y)$. Then, in addition to the relations (5), we have the relations

$$(\mathfrak{c}_p + \mathfrak{d}_p)(\mathfrak{c}_p - \mathfrak{d}_p) + 2\sum_{i=1}^{p}(-1)^i \mathfrak{c}_{p+i}\mathfrak{c}_{p-i} = 0 \quad \text{for } p = r + s \tag{8}$$

in $\Gamma'[X, Y]$.

Following [IMN], the type D double Schubert polynomials $\mathfrak{D}_w$ for $w \in \widetilde{W}_\infty$ are the unique family of elements of $\Gamma'[X,Y]$ satisfying the equations

$$\partial_i^x \mathfrak{D}_w = \begin{cases} \mathfrak{D}_{ws_i} & \text{if } \ell(ws_i) < \ell(w), \\ 0 & \text{otherwise,} \end{cases} \qquad \partial_i^y \mathfrak{D}_w = \begin{cases} \mathfrak{D}_{s_iw} & \text{if } \ell(s_iw) < \ell(w), \\ 0 & \text{otherwise,} \end{cases} \tag{9}$$

for all $i \in \mathbb{N}_\Box$, together with the condition that the constant term of $\mathfrak{D}_w$ is 1 if $w = 1$, and 0 otherwise.

3. Raising operators

For each pair $i < j$ of distinct positive integers, the operator R_{ij} acts on integer sequences $\alpha = (\alpha_1, \alpha_2, \ldots)$ by

$$R_{ij}(\alpha) := (\alpha_1, \ldots, \alpha_i + 1, \ldots, \alpha_j - 1, \ldots).$$

A *raising operator* R is any monomial in these R_{ij}'s.

Following [BKT2, Sec. 1.2], let $\Delta^\circ := \{(i,j) \in \mathbb{Z} \times \mathbb{Z} \mid 1 \leq i < j\}$ and define a partial order on Δ° by agreeing that $(i',j') \leq (i,j)$ if $i' \leq i$ and $j' \leq j$. We call a finite subset D of Δ° a *valid set of pairs* if it is an order ideal in Δ°. Any valid set of pairs D defines the raising operator expression

$$R^D := \prod_{i<j} (1 - R_{ij}) \prod_{i<j\,:\,(i,j)\in D} (1 + R_{ij})^{-1}.$$

We also use the raising operator expressions

$$R^\emptyset := \prod_{i<j} (1 - R_{ij}) \quad \text{and} \quad R^\infty := \prod_{i<j} \frac{1 - R_{ij}}{1 + R_{ij}}.$$

3.1. **Alternating properties in types A, B, and C.** For each $r \geq 1$, let $\sigma^r = (\sigma_i^r)_{i\in\mathbb{Z}}$ be a sequence of variables, with $\sigma_0^r = 1$ and $\sigma_i^r = 0$ for each $i < 0$, and let $\mathbb{Z}[\sigma]$ denote the polynomial ring in the variables σ_i^r for $i, r \geq 1$. For any integer sequence α, let $\sigma_\alpha := \sigma^1_{\alpha_1} \sigma^2_{\alpha_2} \cdots$, and for any raising operator R, set $R\,\sigma_\alpha := \sigma_{R\alpha}$.

Fix $j \geq 1$, let z be a variable, set $\tau^r := \sigma^r$ for each $r \neq j$ and $\tau^j_p = \sigma^j_p + z\,\sigma^j_{p-1}$ for each $p \in \mathbb{Z}$. If $\alpha := (\alpha_1, \ldots, \alpha_\ell)$ and $\alpha' := (\alpha'_1, \ldots, \alpha'_{\ell'})$ are two integer vectors and $r, s \in \mathbb{Z}$, we let (α, r, s, α') denote the integer vector $(\alpha_1, \ldots, \alpha_\ell, r, s, \alpha'_1, \ldots, \alpha'_{\ell'})$. The following two lemmas are generalizations of [BKT2, Lemmas 1.2 and 1.3].

Lemma 5. *Let $\lambda = (\lambda_1, \ldots, \lambda_{j-1})$ and $\mu = (\mu_{j+2}, \ldots, \mu_\ell)$ be integer vectors, and D be a valid set of pairs. Assume that $\sigma^j = \sigma^{j+1}$, $(j, j+1) \notin D$, and that for each $h < j$, $(h,j) \in D$ if and only if $(h, j+1) \in D$.*

(a) *For any integers r and s, we have*

$$R^D\,\sigma_{\lambda,r,s,\mu} = -R^D\,\sigma_{\lambda,s-1,r+1,\mu}$$

in $\mathbb{Z}[\sigma]$.

(b) *For any integer r, we have*

$$R^D\,\tau_{\lambda,r,r,\mu} = R^D\,\sigma_{\lambda,r,r,\mu}$$

in $\mathbb{Z}[\sigma, z]$.

Proof. The proof of (a) is identical to that of [BKT2, Lemma 1.2]. For part (b), we use linearity in the j-th position to obtain $R\,\tau_{\lambda,r,r,\mu} = R\,\sigma_{\lambda,r,r,\mu} + z\,R\,\sigma_{\lambda,r-1,r,\mu}$, for any raising operator R that appears in the expansion of the power series R^D. Adding these equations gives

$$R^D\,\tau_{\lambda,r,r,\mu} = R^D\,\sigma_{\lambda,r,r,\mu} + z\,R^D\,\sigma_{\lambda,r-1,r,\mu}.$$

Now part (a) implies that $R^D\,\sigma_{\lambda,r-1,r,\mu} = 0$. □

Fix $k \geq 0$, let $\mathfrak{c} = (\mathfrak{c}_i)_{i\in\mathbb{Z}}$ be another sequence of variables, and consider the relations

$$\mathfrak{c}_p\mathfrak{c}_p + 2\sum_{i=1}^{p}(-1)^i\mathfrak{c}_{p+i}\mathfrak{c}_{p-i} = 0 \quad \text{for all } \ p > k. \tag{10}$$

Lemma 6. *Let $\lambda = (\lambda_1,\ldots,\lambda_{j-1})$ and $\mu = (\mu_{j+2},\ldots,\mu_\ell)$ be integer vectors, and D be a valid set of pairs. Assume that $\sigma^j = \sigma^{j+1} = \mathfrak{c}$, $(j,j+1) \in D$, and that for each $h > j+1$, $(j,h) \in D$ if and only if $(j+1,h) \in D$.*

(a) *If $r,s \in \mathbb{Z}$ are such that $r+s > 2k$, then we have*

$$R^D\,\sigma_{\lambda,r,s,\mu} = -R^D\,\sigma_{\lambda,s,r,\mu}$$

in the ring $\mathbb{Z}[\sigma]$ modulo the relations coming from (10).

(b) *For any integer $r > k$, we have*

$$R^D\,\tau_{\lambda,r+1,r,\mu} = R^D\,\sigma_{\lambda,r+1,r,\mu}$$

in the ring $\mathbb{Z}[\sigma,z]$ modulo the relations coming from (10).

Proof. The proof of (a) is identical to that of [BKT2, Lemma 1.3]. For part (b), we expand $R^D\,\tau_{\lambda,r+1,r,\mu}$ and use linearity in the j-th position to obtain

$$R^D\,\tau_{\lambda,r+1,r,\mu} = R^D\,\sigma_{\lambda,r+1,r,\mu} + z\,R^D\,\sigma_{\lambda,r,r,\mu}.$$

Now part (a) implies that $R^D\,\sigma_{\lambda,r,r,\mu}$ vanishes modulo the relations (10). □

3.2. **Alternating properties in type D.** In type D we will require certain variations of Lemma 5 and Lemma 6. For each $r \geq 1$, we introduce a new sequence of variables $\upsilon^r = (\upsilon^r_i)_{i\in\mathbb{Z}}$ such that $\upsilon^r_i = 0$ for each $i \leq 0$. Let $\mathbb{Z}[\sigma,\upsilon]$ denote the polynomial ring in the variables σ^r_i, υ^r_i for $i,r \geq 1$. For each $r \geq 1$, define the sequence $\widehat{\sigma}^r$ by $\widehat{\sigma}^r_i := \sigma^r_i + (-1)^r\upsilon^r_i$ for each i, and for any integer sequence α, let $\widehat{\sigma}_\alpha := \widehat{\sigma}^1_{\alpha_1}\widehat{\sigma}^2_{\alpha_2}\cdots$.

Fix an integer $d \geq 0$ such that $\upsilon^r_i = 0$ for all i whenever $r > d$. If $R := \prod_{i<j} R_{ij}^{n_{ij}}$ is a raising operator, denote by $\mathrm{supp}_d(R)$ the set of all indices i and j such that $n_{ij} > 0$ and $j \leq d$. Let D be a valid set of pairs and R be any raising operator appearing in the expansion of the power series R^D. Let $\lambda = (\lambda_1,\ldots,\lambda_\ell)$ be any integer vector and set $\rho := R\lambda$. Define

$$R \star \widehat{\sigma}_\lambda = \overline{\sigma}_\rho := \overline{\sigma}^1_{\rho_1}\cdots\overline{\sigma}^\ell_{\rho_\ell}$$

where, for each $i \geq 1$ and $p \in \mathbb{Z}$,

$$\overline{\sigma}^i_p := \begin{cases} \sigma^i_p & \text{if } i \in \mathrm{supp}_d(R), \\ \widehat{\sigma}^i_p & \text{otherwise.} \end{cases}$$

Fix $j \geq 1$, set $\widehat{\tau}^i := \widehat{\sigma}^i$ for each $i \neq j$ and $\widehat{\tau}^j_p = \widehat{\sigma}^j_p + z\,\widehat{\sigma}^j_{p-1}$ for each $p \in \mathbb{Z}$.

Lemma 7. *Let $\lambda = (\lambda_1, \ldots, \lambda_{j-1})$ and $\mu = (\mu_{j+2}, \ldots, \mu_\ell)$ be integer vectors, and D be a valid set of pairs. Assume that $j > d$, $\sigma^j = \sigma^{j+1}$, $(j, j+1) \notin D$, and that for each $h < j$, $(h, j) \in D$ if and only if $(h, j+1) \in D$.*

(a) *For any integers r and s, we have*

$$R^D \star \widehat{\sigma}_{\lambda,r,s,\mu} = -R^D \star \widehat{\sigma}_{\lambda,s-1,r+1,\mu}$$

in $\mathbb{Z}[\sigma, \upsilon]$.

(b) *For any integer r, we have*

$$R^D \star \widehat{\tau}_{\lambda,r,r,\mu} = R^D \star \widehat{\sigma}_{\lambda,r,r,\mu}$$

in $\mathbb{Z}[\sigma, \upsilon, z]$.

Proof. Since $j > d$, the argument used in the proof of [BKT2, Lemma 1.2] works here as well to establish part (a). Part (b) is an easy consequence of (a). □

Fix $k \geq 0$, let $\mathfrak{c} = (\mathfrak{c}_i)_{i \in \mathbb{Z}}$ and $\mathfrak{d} = (\mathfrak{d}_i)_{i \in \mathbb{Z}}$ be two other sequences of variables such that $\mathfrak{d}_p = 0$ for all $p > k+1$, and consider the relations

$$(\mathfrak{c}_p + \mathfrak{d}_p)(\mathfrak{c}_p - \mathfrak{d}_p) + 2\sum_{i=1}^{p}(-1)^i \mathfrak{c}_{p+i}\mathfrak{c}_{p-i} = 0 \quad \text{for all } p > k. \tag{11}$$

Lemma 8. *Let $\lambda = (\lambda_1, \ldots, \lambda_{j-1})$ and $\mu = (\mu_{j+2}, \ldots, \mu_\ell)$ be integer vectors, and D be a valid set of pairs. Assume that $j < d$, $\sigma^j = \sigma^{j+1} = \mathfrak{c}$, $\upsilon^j = \upsilon^{j+1} = \mathfrak{d}$, $(j, j+1) \in D$, and that for each $h > j+1$, $(j, h) \in D$ if and only if $(j+1, h) \in D$.*

(a) *If $r, s \in \mathbb{Z}$ are such that $r + s > 2k + 2$, then we have*

$$R^D \star \widehat{\sigma}_{\lambda,r,s,\mu} = -R^D \star \widehat{\sigma}_{\lambda,s,r,\mu} \tag{12}$$

and

$$R^D \star \widehat{\sigma}_{\lambda,k+1,k+1,\mu} = 0 \tag{13}$$

in the ring $\mathbb{Z}[\sigma, \upsilon]$ modulo the relations coming from (11).

(b) *For any integer $r > k$, we have*

$$R^D \star \widehat{\tau}_{\lambda,r+1,r,\mu} = R^D \star \widehat{\sigma}_{\lambda,r+1,r,\mu}$$

in the ring $\mathbb{Z}[\sigma, \upsilon, z]$ modulo the relations coming from (11).

Proof. The proof of (12) is identical to that of [BKT2, Lemma 1.3]. The proof of (13) follows the same argument, using (12) and induction to reduce to the case when μ is empty. For any integer vector ρ with at most d components, define $T_\rho := R^D \star \widehat{\sigma}_\rho$. If g is the least integer such that $2g \geq \ell$ and $\rho := (\lambda, r, s)$, then we have the relation

$$T_\rho = \sum_{j=2}^{2g} (-1)^j T_{\rho_1,\rho_j} T_{\rho_2,\ldots,\widehat{\rho}_j,\ldots,\rho_{2g}}.$$

The proof is now completed by induction, as in loc. cit. For part (b), we expand $R^D \star \widehat{\tau}_{\lambda,r+1,r,\mu}$ and use linearity in the j-th position to obtain

$$R^D \star \widehat{\tau}_{\lambda,r+1,r,\mu} = R^D \star \widehat{\sigma}_{\lambda,r+1,r,\mu} + z\, R^D\, \widehat{\sigma}_{\lambda,r,r,\mu}.$$

Now part (a) implies that $R^D \star \widehat{\sigma}_{\lambda,r,r,\mu}$ vanishes modulo the relations (11). □

4. Amenable elements: Type A theory

4.1. **Definitions and main theorem.** As Lie theorists know well, type A is very special when compared to the other Lie types. In the theory of amenable elements, this manifests itself in the fact that we can work with dominant elements instead of leading elements. The result is the simplified treatment given here, which does not have a direct analogue in types B, C, and D. Another difference in type A is that the order of application of the divided difference operators is switched: we first use the left divided differences, then the right ones. But by far the main distinction between type A and the other classical types is that one can use Jacobi-Trudi determinants, represented here by $R^{\emptyset}$, instead of the more general raising operator expressions R^D that define theta and eta polynomials, which are essential ingredients of the theory for the symplectic and orthogonal groups.

If $\varpi \in S_n$ and $v \in S_m$, then ϖ is called *v-avoiding* if ϖ does not contain a subword $(\varpi_{i_1},\ldots,\varpi_{i_m})$ having the same relative order as $(v_1,\ldots,v_m)$. The notion of v-avoidance also makes sense when ϖ is any integer vector $(\varpi_1,\ldots,\varpi_n)$ with distinct components ϖ_i. We say that ϖ is *dominant* if its code $\gamma(\varpi)$ is a partition, or equivalently, if ϖ is 132-avoiding (see [M, (1.30)] and [R, Thm. 2.2]).

For the next result, we refer to [Kn, Exercise 2.2.1.5] and [Stu, §2.2].

Lemma 9. *The following conditions on a permutation $\omega \in S_n$ are equivalent:*

(a) ω *is 312-avoiding;* (b) ω^{-1} *is 231-avoiding;*

(c) ω *has a reduced decomposition of the form $R_1\cdots R_{n-1}$ where each R_j is a (possibly empty) subword of $s_1\cdots s_{n-1}$ and furthermore all simple reflections in R_p are also contained in R_{p+1}, for each $p < n-1$.*

Definition 4. A (right) *modification* of $\varpi \in S_n$ is a permutation $\varpi\omega$, where $\omega \in S_n$ is such that $\ell(\varpi\omega) = \ell(\varpi) - \ell(\omega)$, and ω is 231-avoiding. A permutation is *amenable* if it is a modification of a dominant permutation.

For any three integer vectors $\alpha, \beta, \rho \in \mathbb{Z}^{\ell}$, which we view as integer sequences with finite support, define ${}^{\rho}h_{\alpha}^{\beta} := {}^{\rho_1}h_{\alpha_1}^{\beta_1}\,{}^{\rho_2}h_{\alpha_2}^{\beta_2}\cdots$. Given any raising operator $R = \prod_{i<j} R_{ij}^{n_{ij}}$, let $R\,{}^{\rho}h_{\alpha}^{\beta} := {}^{\rho}h_{R\alpha}^{\beta}$.

Proposition 3. [M, (6.14)] *Suppose that $\varpi \in S_n$ is dominant. Then we have*

$$\mathfrak{S}_{\varpi} = R^{\emptyset}\,{}^{\delta_{n-1}^{\vee}}h_{\lambda(\varpi)}^{\lambda(\varpi)}.$$

Proof. We use descending induction on $\ell(\varpi)$. Let $\varpi_0 := (n,\ldots,1)$ denote the longest element in S_n. One knows from [Las] and [M, (3.5)] that the equation

$$\mathfrak{S}_{\varpi_0} = R^{\emptyset}\,{}^{\delta_{n-1}^{\vee}}h_{\delta_{n-1}}^{\delta_{n-1}}$$

holds in $\mathbb{Z}[X,Y]$, so the result is true when $\varpi = \varpi_0$.

Suppose that $\varpi \neq \varpi_0$ and ϖ is dominant of shape λ. Then $\lambda \subset \delta_{n-1}$ and $\lambda \neq \delta_{n-1}$. Let $r \geq 1$ be the largest integer such that $\lambda_i = n-i$ for $i \in [1,r]$, and let $j := \lambda_{r+1}+1 = \varpi_{r+1} \leq n-r-1$. Then $s_j\varpi$ is dominant of length $\ell(\varpi)+1$ and $\lambda(s_j\varpi) = \lambda(\varpi) + \epsilon_{r+1}$. Using Lemma 1 and the left Leibnitz rule, we deduce that for any integer sequence $\alpha = (\alpha_1,\ldots,\alpha_n)$, we have

$$\partial_j^y\big({}^{\delta_{n-1}^{\vee}}h_{\alpha}^{\lambda(s_j\varpi)}\big) = {}^{\delta_{n-1}^{\vee}}h_{\alpha-\epsilon_{r+1}}^{\lambda(\varpi)}.$$

We conclude that

$$\mathfrak{S}_\varpi = \partial_j^y(\mathfrak{S}_{s_j\varpi}) = \partial_j^y(R^{\emptyset\;\delta_{n-1}^\vee}h_{\lambda(s_j\varpi)}^{\lambda(s_j\varpi)}) = R^{\emptyset\;\delta_{n-1}^\vee}h_{\lambda(\varpi)}^{\lambda(\varpi)}.$$

□

Definition 5. Let ϖ be an amenable permutation with code γ and shape λ, with $\ell = \ell(\lambda)$. Define two sequences $\mathfrak{f} = \mathfrak{f}(\varpi)$ and $\mathfrak{g} = \mathfrak{g}(\varpi)$ of length ℓ as follows. For $1 \leq j \leq \ell$, set

$$\mathfrak{f}_j := \max(i \mid \gamma_i \geq \lambda_j)$$

and let

$$\mathfrak{g}_j := \mathfrak{f}_\mathfrak{q} + \lambda_\mathfrak{q} - \mathfrak{q},$$

where $\mathfrak{q}$ is the least integer such that $\mathfrak{q} \geq j$ and $\lambda_\mathfrak{q} > \lambda_{\mathfrak{q}+1}$. We call $\mathfrak{f}$ the *right flag* of ϖ, and $\mathfrak{g}$ the *left flag* of ϖ.

It is clear from Lemma 2 that the right flag $\mathfrak{f}$ of an amenable permutation is a weakly increasing sequence consisting of right descents of ϖ. We will show that the left flag $\mathfrak{g}$ is a weakly decreasing sequence consisting of left descents of ϖ.

Proposition 4. *Suppose that $\widehat{\varpi} \in S_n$ is dominant with $\widehat{\lambda} := \lambda(\widehat{\varpi})$. Let ω be a 231-avoiding permutation such that $\ell(\widehat{\varpi}\omega) = \ell(\widehat{\varpi}) - \ell(\omega)$, and set $\varpi := \widehat{\varpi}\omega$, $\gamma := \gamma(\varpi)$, and $\lambda := \lambda(\varpi)$. Then the sequence $\delta_{n-1}^\vee + \widehat{\lambda} - \lambda$ is weakly increasing, and*

$$\mathfrak{S}_\varpi = R^{\emptyset\;\delta_{n-1}^\vee + \widehat{\lambda} - \lambda}h_\lambda^{\widehat{\lambda}}.$$

Moreover, if $\lambda_\mathfrak{q} > \lambda_{\mathfrak{q}+1}$, then $\widehat{\lambda}_\mathfrak{q}$ is a left descent of ϖ, $\mathfrak{q} + \widehat{\lambda}_\mathfrak{q} - \lambda_\mathfrak{q}$ is a right descent of ϖ, and we have $\mathfrak{q} + \widehat{\lambda}_\mathfrak{q} - \lambda_\mathfrak{q} = \max(i \mid \gamma_i \geq \lambda_\mathfrak{q})$.

Proof. Suppose that $\widehat{\varpi}$ is of shape $\widehat{\lambda} = \widehat{\gamma} = (p_1^{n_1}, p_2^{n_2} \ldots, p_t^{n_t})$, where $p_1 > \cdots > p_t$. Then the right descents of $\widehat{\varpi}$ are at positions $d_1 := n_1, d_2 := n_1 + n_2, \ldots, d_t := n_1 + \cdots + n_t$. Since we have $\widehat{\varpi}_j < \widehat{\varpi}_{j+1}$ for all $j \neq d_r$ for $r \in [1,t]$, we deduce that

$$\widehat{\varpi}_1 = p_1 + 1, \widehat{\varpi}_{d_1+1} = p_2 + 1, \ldots, \widehat{\varpi}_{d_{t-1}+1} = p_t + 1, \widehat{\varpi}_{d_t+1} = 1.$$

Moreover, since $\widehat{\varpi}$ is 132-avoiding, it follows that the left descents of $\widehat{\varpi}$ are $p_1, \ldots, p_t$. Finally, Proposition 3 gives

$$\mathfrak{S}_{\widehat{\varpi}} = R^{\emptyset\;\delta_{n-1}^\vee}h_{\widehat{\lambda}}^{\widehat{\lambda}} \tag{14}$$

so the result holds when $\omega = 1$ and $\varpi = \widehat{\varpi}$ is dominant.

Suppose next that $\varpi := \widehat{\varpi}\omega$ for some 231-avoiding permutation ω such that $\ell(\widehat{\varpi}\omega) = \ell(\widehat{\varpi}) - \ell(\omega)$. Lemma 9 implies that ω has a reduced decomposition of the form $R_1 \cdots R_{n-1}$ where each R_j is a (possibly empty) subword of $s_{n-1} \cdots s_1$ and all simple reflections in R_{p+1} are also contained in R_p, for every $p \geq 1$. Now repeated application of (2), Lemma 1, and the right Leibnitz rule (4) in equation (14) give

$$\mathfrak{S}_\varpi = \partial_{\omega^{-1}}^x(\mathfrak{S}_{\widehat{\varpi}}) = R^{\emptyset\;\delta_{n-1}^\vee + \widehat{\lambda} - \lambda}h_\lambda^{\widehat{\lambda}}.$$

We will show that the sequence $\delta_{n-1}^\vee + \widehat{\lambda} - \lambda$ is weakly increasing and verify the last assertion, about the left and right descents of ϖ.

Using Lemma 2, we study the right action of the successive simple transpositions in the reduced decomposition $R_1 \cdots R_{n-1}$ for ω on the code $\widehat{\gamma}$ of $\widehat{\varpi}$. The action of these on $\widehat{\gamma}$ is by a finite sequence of *moves* $\alpha \mapsto \alpha'$, where $\alpha := \gamma(v)$ and $\alpha' := \gamma(v')$ for some $v, v' \in S_n$. Here $v' = vs_{j-1} \cdots s_i$ for some $i < j$ such that

$\ell(v') = \ell(v) - j + i$, and $s_{j-1} \cdots s_i$ is a subword of some R_p with $j - i$ maximal. Since the initial code $\widehat{\gamma}$ is weakly decreasing, we have $\alpha_i \geq \cdots \geq \alpha_{j-1} > \alpha_j$, and

$$\alpha' = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_j, \alpha_i - 1, \ldots, \alpha_{j-1} - 1, \alpha_{j+1}, \alpha_{j+2}, \ldots).$$

We say that the move is performed on the interval $[i, j]$, or is an $[i, j]$-move. The *procedure* is defined as the performance of finitely many moves to $\widehat{\gamma}$, ending in the code γ. This describes the effect of multiplying $\widehat{\varpi}$ on the right by ω.

Example 3. Suppose that $\widehat{\varpi} = (5, 6, 7, 4, 3, 8, 2, 1)$ in S_8 with code

$$\widehat{\gamma} = (4, 4, 4, 3, 2, 2, 1, 0).$$

If $\omega = s_7s_6s_5s_4s_3s_2s_7s_6s_5s_4s_6s_5$, then $\widehat{\varpi}\omega = (5, 1, 6, 2, 3, 7, 4, 8)$ and $\gamma = \gamma(\varpi\omega) = (4, 0, 3, 0, 0, 1, 0, 0)$. The procedure from $\widehat{\gamma}$ to γ consists of a $[2, 8]$-move, followed by a $[4, 8]$-move, followed by a $[5, 7]$-move:

$$(4, 4, 4, 3, 2, 2, 1, 0) \mapsto (4, 0, 3, 3, 2, 1, 1, 0) \mapsto (4, 0, 3, 0, 2, 1, 0, 0) \mapsto (4, 0, 3, 0, 0, 1, 0, 0).$$

Notice that after an $[i, j]$-move $\alpha \mapsto \alpha'$, we have

$$\alpha'_i = \gamma_i = \min(\alpha'_r \mid r \in [i, j]) \geq \max(\alpha'_r \mid r > j). \tag{15}$$

Let μ and μ' be the shapes of v and v', respectively, and set $f := \delta^\vee_{n-1} + \widehat{\lambda} - \mu$ (respectively, $f' := \delta^\vee_{n-1} + \widehat{\lambda} - \mu'$). We then have

$$\mu' = (\mu_1, \ldots, \mu_{r-1}, \mu_r - 1, \ldots, \mu_{s-1} - 1, \mu_s, \mu_{s+1}, \ldots)$$

for some $r < s$ with $s - r = j - i$, and

$$f' = (f_1, \ldots, f_{r-1}, f_r + 1, \ldots, f_{s-1} + 1, f_s, f_{s+1}, \ldots).$$

Since $\mu_s = \alpha_j \leq \alpha_{j-1} - 1 = \mu_{s-1} - 1$, we deduce that $f'_s - f'_{s-1} = f_s - f_{s-1} - 1 = \mu_s - \mu_{s-1} - 1 \geq 0$. It follows by induction on the number of moves that the sequence f is weakly increasing.

Suppose that $\mu'_d > \mu'_{d+1}$ for some d. Using (15) and induction on the number of moves, we deduce that $f'_d = \max(i \mid \alpha'_i \geq \mu'_d)$. This implies that for any $\mathfrak{q}$ such that $\lambda_\mathfrak{q} > \lambda_{\mathfrak{q}+1}$, we have $\mathfrak{q} + \widehat{\lambda}_\mathfrak{q} - \lambda_\mathfrak{q} = \max(i \mid \gamma_i \geq \lambda_\mathfrak{q})$, and hence that $\mathfrak{q} + \widehat{\lambda}_\mathfrak{q} - \lambda_\mathfrak{q}$ is a right descent of ϖ, in view of Lemma 2.

We claim that $\widehat{\lambda}_d$ is a left descent of v'. Clearly the left descents of v and v' are subsets of $\{p_1, \ldots, p_t\}$. There is at most one left descent p_e of v that is not a left descent of v', and this occurs if and only if $v_j = p_e$ and $v_h = p_e + 1$ for some $h \in [i, j-1]$. Since $\alpha_h \geq \cdots \geq \alpha_{j-1} > \alpha_j$, we deduce that $\alpha_h = \cdots = \alpha_{j-1} = \alpha_j + 1$, and hence $\mu'_{s-(j-h)+1} = \cdots = \mu'_s = \mu_s$. We conclude that $\widehat{\lambda}_d \neq p_e$, completing the proof of the claim, and the proposition. $\square$

Example 4. Let $\widehat{\varpi} := (4, 5, 6, 2, 1, 3)$, a dominant permutation in S_6 with shape $\widehat{\lambda} = \gamma(\widehat{\varpi}) = (3, 3, 3, 1)$. Take $\omega := s_4s_3s_2s_1s_4s_3$ in Proposition 4, so that $\varpi = \widehat{\varpi}\omega = (1, 4, 2, 5, 6, 3)$, with $\gamma(\widehat{\varpi}\omega) = (0, 2, 0, 1, 1, 0)$ and $\lambda = (2, 1, 1)$. We have $\delta^\vee_5 + \widehat{\lambda} - \lambda = (2, 4, 5, 5, 5)$, and deduce that

$$\mathfrak{S}_\varpi = R^{\emptyset\ (2,4,5,5,5)} h^{(3,3,3,1)}_{(2,1,1,0)} = R^{\emptyset\ (2,4,5)} h^{(3,3,3)}_{(2,1,1)}.$$

Theorem 1. *For any amenable permutation ϖ, we have*

$$\mathfrak{S}_\varpi = R^{\emptyset\ \mathfrak{f}(\varpi)} h^{\mathfrak{g}(\varpi)}_{\lambda(\varpi)}.$$

Proof. We may assume we are in the situation of Proposition 4, so that $\varpi = \widehat{\varpi}\omega$, with $\widehat{\lambda} = \lambda(\widehat{\varpi})$ and $\lambda = \lambda(\varpi)$. Choose $j \in [1, \ell]$ and let $\mathfrak{q}$ be the least integer such that $\mathfrak{q} \geq j$ and $\lambda_{\mathfrak{q}} > \lambda_{\mathfrak{q}+1}$. Then we have $\lambda_j = \lambda_{j+1} = \cdots = \lambda_{\mathfrak{q}}$. As the sequence $f := \delta^\vee_{n-1} + \widehat{\lambda} - \lambda$ is weakly increasing, we deduce that if $\lambda_r = \lambda_{r+1} > 0$, then either (i) $\widehat{\lambda}_r = \widehat{\lambda}_{r+1}$ and $f_r = f_{r+1} - 1$, or (ii) $\widehat{\lambda}_r = \widehat{\lambda}_{r+1} + 1$ and $f_r = f_{r+1}$. Theorem 1 follows from this and induction on $\mathfrak{q} - j$, using Lemma 5(b) in Proposition 4. □

Example 5. Consider the amenable permutation $\varpi = (3, 4, 6, 1, 5, 2)$ in S_6. We then have $\gamma(\varpi) = (2, 2, 3, 0, 1, 0)$, $\lambda(\varpi) = (3, 2, 2, 1)$, $\mathfrak{f}(\varpi) = (3, 3, 3, 5)$, and $\mathfrak{g}(\varpi) = (5, 2, 2, 2)$. Theorem 1 gives

$$\mathfrak{S}_{346152} = R^{\emptyset\ (3,3,3,5)} h^{(5,2,2,2)}_{(3,2,2,1)}.$$

Recall from [LS1, LS2] that a permutation ϖ is *vexillary* if and only if it is 2143-avoiding. Equivalently, ϖ is vexillary if and only if $\lambda(\varpi^{-1}) = \lambda(\varpi)'$.

Theorem 2. *The permutation ϖ is amenable if and only if ϖ is vexillary.*

Proof. According to [M, (1.32)], a permutation is vexillary if and only if its code γ satisfies the following two conditions, for any $i < j$: (i) If $\gamma_i \leq \gamma_j$, then $\gamma_i \leq \gamma_k$ for any k with $i < k < j$; (ii) If $\gamma_i > \gamma_j$, then the number of k with $i < k < j$ and $\gamma_k < \gamma_j$ is at most $\gamma_i - \gamma_j$.

Assume first that ϖ is amenable, so that $\varpi = \widehat{\varpi}\omega$ for some dominant permutation $\widehat{\varpi}$ and 231-avoiding permutation ω. Using Lemma 2, we see that the code $\widehat{\gamma}$ of $\widehat{\varpi}$ is transformed into the code γ of ϖ by the moves of the procedure described in the proof of Proposition 4.

We claim that the sequence γ is a vexillary code. It follows from the inequalities (15) that for any $[i, j]$-move of the procedure, we have $\gamma_s \leq \gamma_i \leq \gamma_r$ for every $r \in [i, j]$ and $s > j$. Moreover, if $r \neq i$ for all $[i, j]$-moves of the procedure, then $\gamma_s \leq \gamma_r$ for all $s > r$. It is easy to see from this that γ satisfies the vexillary conditions (i) and (ii). Indeed, choose $r < s$ such that $\gamma_r \leq \gamma_s$, and some $t \in [r, s]$. If $r = i$ for some $[i, j]$-move, then we have $\gamma_r \leq \gamma_k$ for all $k \in [i, j]$, while if $k > j$, then we must have $\gamma_r = \gamma_s = \gamma_k$. Therefore, $\gamma_r \leq \gamma_t$. If $r \neq i$ for all $[i, j]$-moves, then $\gamma_r = \gamma_s$ and hence $\gamma_r = \gamma_t$. To prove (ii), suppose that $r < k < s$ and $\gamma_r > \gamma_s > \gamma_k$. Then we must have $k = i$ for some $[i, j]$-move of the procedure, where $s \leq j$. We conclude that the number of such k is at most $\gamma_r - \gamma_s$.

Conversely, suppose that $\varpi \in S_n$ is a vexillary permutation with code γ. We call an integer $i \geq 1$ an *initial index* if there exists an $s > i$ with $\gamma_i < \gamma_s$. We claim that there is a canonical 312-avoiding permutation ω such that $\ell(\varpi\omega) = \ell(\varpi) + \ell(\omega)$, $\varpi\omega$ is dominant, and $\omega_a = a$ if $a < i$ for every initial index i. This will complete the proof of the theorem, by applying Lemma 9.

To establish the claim, we argue by descending induction on the length of ϖ. Observe that ϖ has no initial index if and only if ϖ is a dominant permutation. Hence, if ϖ is already dominant, then we must take ω to be the identity.

Assume that ϖ is not dominant. We say that the index j is *associated* to the initial index i of ϖ if j is the maximum s such that $\gamma_i < \gamma_s$. Let i be the smallest initial index, let j be associated to i, and set $\varpi' := \varpi s_i \cdots s_{j-1}$. The vexillary condition (i) and Lemma 2 imply that

$$\gamma(\varpi') = (\gamma_i, \ldots, \gamma_{i-1}, \gamma_{i+1} + 1, \ldots, \gamma_j + 1, \gamma_i, \gamma_{j+1}, \ldots)$$

and $\ell(\varpi') = \ell(\varpi) + j - i > \ell(\varpi)$. It follows by checking conditions (i) and (ii) that ϖ' is vexillary and that every initial index i' of ϖ' satisfies $i' \geq i$.

By the inductive hypothesis, there exists a canonical 312-avoiding permutation ω' with $\ell(\varpi'\omega') = \ell(\varpi') + \ell(\omega')$, $\varpi'\omega'$ dominant, and $\omega'_a = a$ if $a < i'$ for every initial index i' of ϖ'. The claim is proved with $\omega := s_i \cdots s_{j-1}\omega'$, once we check that ω is 312-avoiding. Indeed, since $\omega'_a = a$ for all $a < i$ and $\ell(s_i \cdots s_{j-1}\omega') = j - i + \ell(\omega')$, we must have

$$\omega' = (1, 2, \ldots, i-1, \ldots, a_1, \ldots, a_2, \ldots, a_{j-i}, \ldots, j, \ldots)$$

and

$$\omega = (1, 2, \ldots, i-1, \ldots, a_1 + 1, \ldots, a_2 + 1, \ldots, a_{j-i} + 1, \ldots, i, \ldots)$$

where the set $\{a_1, \ldots, a_{j-i}\}$ is equal to $\{i, \ldots, j-1\}$. As ω' is 312-avoiding, there are no integers $a < b < c$ such that $\omega'_a > \omega'_c > \omega'_b$. It is easy to see from this and the above relation between ω' and ω that the latter permutation has the same property, and therefore is also 312-avoiding. □

Remark 1. (a) Define a *left modification* of $\varpi \in S_n$ to be a permutation $\omega\varpi$, where $\omega \in S_n$ is 312-avoiding and such that $\ell(\omega\varpi) = \ell(\varpi) - \ell(\omega)$. Then a permutation is amenable if and only if it is a left modification of a dominant permutation. This follows from Lemma 9, Theorem 2, and the fact that ϖ is dominant (resp. vexillary) if and only if ϖ^{-1} is dominant (resp. vexillary).

(b) It is not hard to show that a definition of amenable permutations as left modifications of leading permutations, in the same manner as Definition 8 in type C, results in the same class of permutations as that given in Definition 4.

Let ϖ be a vexillary permutation with code γ and shape λ, and let ω be the canonical 312-avoiding permutation associated to ϖ in the proof of Theorem 2. Define a new sequence $\widehat{\gamma}$ by the prescription

$$\widehat{\gamma}_\alpha := \gamma_\alpha + \#\{i \mid i \text{ is an initial index with associated index } j \text{ and } i < \alpha \leq j\}$$

for each $\alpha \geq 1$. Let $\widehat{\lambda}$ be the partition obtained by listing the entries of $\widehat{\gamma}$ in weakly decreasing order. Then $\widehat{\lambda}$ is the shape of $\varpi\omega$.

Consider the skew Young diagram $\tau(\varpi) := \widehat{\lambda}/\lambda$. For each $i \geq 1$, fill the boxes in row i of $\tau(\varpi)$ with a strictly decreasing sequence of consecutive positive integers ending in i. In this way, we obtain a tableaux $T = T(\varpi)$ of shape $\tau(\varpi)$ with strictly decreasing rows. Define the *depth* of a box B of T to be the distance from B to the end of the row it occupies. Form a reduced decomposition for a permutation ω_T by listing the entries in the boxes of T in decreasing order of depth, with the entries of a fixed depth listed in *increasing* order. It then follows from the definition of ω that $\omega_T = \omega$.

Example 6. Let $\varpi = (1, 3, 6, 7, 9, 4, 8, 2, 5)$ be the vexillary permutation in S_9 with code $\gamma = (0, 1, 3, 3, 4, 1, 2, 0, 0)$. The initial indices are 1, 2, 3, 4, and 6 with associated indices 7, 7, 5, 5, and 7, respectively. The reduced decomposition for the canonical permutation ω is

$$s_4 s_3 s_4 s_6 s_2 s_3 s_4 s_5 s_6 s_1 s_2 s_3 s_4 s_5 s_6$$

and we have $\varpi\omega = (9, 7, 6, 8, 4, 3, 1, 2, 5)$, with code $(8, 6, 5, 5, 3, 2, 0, 0, 0)$. We also have $\lambda = (4, 3, 3, 2, 1, 1)$, $\widehat{\gamma} = (0, 2, 5, 6, 8, 3, 5, 0)$, and $\widehat{\lambda} = (8, 6, 5, 5, 3, 2)$. The tableau $T(\varpi)$ on the skew diagram $\tau(\varpi)$ is displayed in Figure 1.

				4	**3**	**2**	**1**
			4	**3**	**2**		
			4	**3**			
		6	**5**	**4**			
	6	**5**					
	6						

FIGURE 1. The tableau T on the skew diagram $\widehat{\lambda}/\lambda$.

It would be interesting to find analogues of the canonical permutation ω and the tableaux $T(\varpi)$ for the amenable elements in the other classical Lie types.

4.2. **Type A degeneracy loci.** Let $E \to \mathfrak{X}$ be a vector bundle of rank n on a complex algebraic variety $\mathfrak{X}$, assumed to be smooth for simplicity. Let $\varpi \in S_n$ be amenable of shape λ, and let $\mathfrak{f}$ and $\mathfrak{g}$ be the left and right flags of ϖ, respectively. Consider two complete flags of subbundles of E

$$0 \subset E_1 \subset \cdots \subset E_n = E \quad \text{and} \quad 0 \subset F_1 \subset \cdots \subset F_n = E$$

with $\operatorname{rank} E_r = \operatorname{rank} F_r = r$ for each r. Define the *degeneracy locus* $\mathfrak{X}_\varpi \subset \mathfrak{X}$ as the locus of $x \in \mathfrak{X}$ such that

$$\dim(E_r(x) \cap F_s(x)) \geq \#\{\, i \leq r \mid \varpi_i > n - s \,\} \ \ \forall\, r, s.$$

Assume further that $\mathfrak{X}_\varpi$ has pure codimension $\ell(\varpi)$ in $\mathfrak{X}$. The next result, which follows from Theorem 1 and Fulton's work [F1], will be a formula for the cohomology class $[\mathfrak{X}_\varpi]$ in $\mathrm{H}^{2\ell(\varpi)}(\mathfrak{X})$ in terms of the Chern classes of the bundles E_r and F_s. Recall that for any integer p, the class $c_p(E - E_r - F_s)$ is defined by the equation

$$c(E - E_r - F_s) := c(E)c(E_r)^{-1}c(F_s)^{-1}$$

of total Chern classes.

Theorem 3 ([F1])**.** *For any amenable permutation $\varpi \in S_n$, we have*

$$[\mathfrak{X}_\varpi] = s_{\lambda'}(E - E_{\mathfrak{f}} - F_{n-\mathfrak{g}}) = R^{\emptyset}\, c_\lambda(E - E_{\mathfrak{f}} - F_{n-\mathfrak{g}}) \tag{16}$$

in the cohomology ring $\mathrm{H}^*(\mathfrak{X})$.

The Chern polynomial in (16) is interpreted as the image of the Schur polynomial $s_{\lambda'}(\mathfrak{c}) := R^{\emptyset}\mathfrak{c}_\lambda$ under the $\mathbb{Z}$-linear map which sends the noncommutative monomial $\mathfrak{c}_\alpha$ to $\prod_j c_{\alpha_j}(E - E_{\mathfrak{f}_j} - F_{n-\mathfrak{g}_j})$, for every integer sequence α.

Remark 2. Theorem 3 and its companion Theorems 5, 7, and 8 in the other classical Lie types are results about cohomology groups, taken with rational coefficients in types B and D. However, from these, one may obtain corresponding results for cohomology with integer coefficients, and for the Chow groups of algebraic cycles modulo rational equivalence. For the latter transition, see [F2, G].

5. Amenable elements: Type C theory

5.1. **Definitions and main theorem.** Let w be a signed permutation with A-code γ and shape $\lambda = \mu + \nu$, with $\ell = \ell(\lambda)$ and $m = \ell(\mu)$. Choose $k \geq 0$, and assume that w is increasing up to k. If $k = 0$, this condition is vacuous, while if $k \geq 1$ it means that $0 < w_1 < \cdots < w_k$. Eventually, k will be the first right descent of w, but the increased flexibility is useful.

List the entries $w_{k+1}, \ldots, w_n$ in increasing order:

$$u_1 < \cdots < u_m < 0 < u_{m+1} < \cdots < u_{n-k}.$$

Define a sequence $\beta(w)$ by

$$\beta(w) := (u_1 + 1, \ldots, u_m + 1, u_{m+1}, \ldots, u_{n-k}).$$

and the *denominator set* $D(w)$ by

$$D(w) := \{(i,j) \mid 1 \leq i < j \leq n-k \text{ and } u_i + u_j < 0\}. \tag{17}$$

This notation suppresses the dependence of $\beta(w)$ and $D(w)$ on k. Observe that the inequality $u_i + u_j < 0$ in (17) is equivalent to $\beta_i(w) + \beta_j(w) \leq 0$.

Definition 6. Suppose that $w \in W_n$ has code $\gamma = \gamma(w)$ and $k \geq 0$. The *k-truncated A-code* ${}^k\gamma = {}^k\gamma(w)$ is defined by

$${}^k\gamma(w) := (\gamma_{k+1}, \gamma_{k+2}, \ldots, \gamma_n).$$

If k is the first right descent of w, then we call ${}^k\gamma(w)$ the *truncated A-code* of w. We let $\xi = \xi(w)$ be the conjugate of the partition whose parts are the non-zero entries of ${}^k\gamma(w)$ arranged in weakly decreasing order.

Clearly an element $w \in W_n$ increasing up to k with a given k-truncated A-code C is uniquely determined by the *set* of elements $\{w_{k+1}, \ldots, w_n\}$, or equivalently, by the sequence $\beta(w)$.

Let $v := v(w)$ be the unique k-Grassmannian element obtained by reordering the parts $w_{k+1}, \ldots, w_n$ to be strictly increasing. For example, if $w = (2, 4, 7, 5, 8, \overline{3}, 1, \overline{6})$ and $k = 3$, then $v = (2, 4, 7, \overline{6}, \overline{3}, 1, 5, 8)$. Note that the map $w \mapsto v(w)$ is a bijection from the set of elements in W_n increasing up to k with k-truncated A-code C onto the set of k-Grassmannian elements in W_n, such that $\beta(v(w)) = \beta(w)$ and

$$\ell(v(w)) = \ell(w) - \sum_{i=k+1}^{n} C_i = \ell(w) - \sum_{j=1}^{n-k} \gamma_{k+j}.$$

In particular, if $w, \overline{w}$ are two such elements, then $\ell(w) > \ell(\overline{w})$ if and only if $\ell(v(w)) > \ell(v(\overline{w}))$.

Lemma 10. *Let w and $\overline{w}$ be elements in W_n increasing up to k and with the same k-truncated A-code C, such that $\ell(w) = \ell(\overline{w}) + 1$. Suppose that $v(\overline{w}) = s_i v(w)$ for some simple reflection s_i. Then $\overline{w} = s_i w$.*

Proof. There are 4 possible cases for i and $v(w)$: (a) $i = 0$ and $v(w) = (\cdots \overline{1} \cdots)$; (b) $i \geq 1$ and $v(w) = (\cdots i \cdots \overline{i+1} \cdots)$; (c) $i \geq 1$ and $v(w) = (\cdots \overline{i+1} \cdots i \cdots)$; (d) $i \geq 1$ and $v(w) = (\cdots i+1 \cdots i \cdots)$. In the first three cases, the result is clear. In case (d), the $i+1$ must be among the first k entries of $v(w)$, which coincide with the first k entries of w, while i lies among the last $n-k$ entries of w. Hence the the result follows. □

For any three integer vectors $\alpha, \beta, \rho \in \mathbb{Z}^\ell$, define ${}^\rho c^\beta_\alpha := {}^{\rho_1}c^{\beta_1}_{\alpha_1}\,{}^{\rho_2}c^{\beta_2}_{\alpha_2}\cdots$. Given any raising operator $R = \prod_{i<j} R_{ij}^{n_{ij}}$, let $R\,{}^\rho c^\beta_\alpha := {}^\rho c^\beta_{R\alpha}$.

Proposition 5. *Fix an integer $k \geq 0$. Suppose that w and $\overline{w}$ are elements in W_n increasing up to k with the same k-truncated A-code C, such that $\ell(w) = \ell(\overline{w}) + 1$ and $s_i v(w) = v(\overline{w})$ for some simple reflection s_i. Assume that we have*

$$\mathfrak{C}_w = R^{D(w)}\,{}^\kappa c^{\beta(w)}_{\lambda(w)}$$

for some integer sequence κ. Then we have

$$\mathfrak{C}_{\overline{w}} = R^{D(\overline{w})}\,{}^\kappa c^{\beta(\overline{w})}_{\lambda(\overline{w})}.$$

Proof. Set $F_w := R^{D(w)}\,{}^\kappa c^{\beta(w)}_{\lambda(w)}$, so we know that $\mathfrak{C}_w = F_w$. As equation (6) gives $\partial_i^y \mathfrak{C}_w = \mathfrak{C}_{\overline{w}}$, it will suffice to show that $\partial_i^y F_w = F_{\overline{w}}$. The proof of this will follow the argument of [TW, Prop. 5].

Let $\mu := \mu(w)$, $\nu := \nu(w)$, $\lambda := \lambda(w) = \mu+\nu$, $\overline{\mu} := \mu(\overline{w})$, $\overline{\nu} := \nu(\overline{w})$, $\overline{\lambda} := \lambda(\overline{w}) = \overline{\mu} + \overline{\nu}$, $\beta = \beta(w)$, and $\overline{\beta} = \beta(\overline{w})$. There are 4 possible cases for w, discussed below. In each case, we have $\overline{\lambda} \subset \lambda$, so that $\overline{\lambda}_p = \lambda_p - 1$ for some $p \geq 1$ and $\overline{\lambda}_j = \lambda_j$ for all $j \neq p$.

(a) $v(w) = (\cdots \overline{1} \cdots)$ with $i = 0$. In this case we have $D(w) = D(\overline{w})$. Since clearly $\nu = \overline{\nu}$ and $\overline{\mu}_p = \mu_p - 1$ for $p = \ell(\mu)$, while $\overline{\mu}_j = \mu_j$ for all $j \neq p$, it follows that $\beta_p = i$, $\overline{\beta}_p = i + 1$, while $\beta_j = \overline{\beta}_j$ for all $j \neq p$.

(b) $v(w) = (\cdots i \cdots \overline{i+1} \cdots)$. In this case $D(w) = D(\overline{w})$, and we have $\nu = \overline{\nu}$ and $\overline{\mu}_p = \mu_p - 1$, while $\overline{\mu}_j = \mu_j$ for all $j \neq p$. It follows that $\beta_p = -i$, $\overline{\beta}_p = -i + 1$, and $\beta_j = \overline{\beta}_j$ for all $j \neq p$.

(c) $v(w) = (\cdots \overline{i+1} \cdots i \cdots)$. In this case $D(w) = D(\overline{w}) \cup \{(p, q)\}$, where $\beta_p = -i$ and $\beta_q = i$. We see similarly that $\overline{\beta}_p = -i + 1$ and $\overline{\beta}_q = i + 1$, while $\beta_j = \overline{\beta}_j$ for all $j \notin \{p, q\}$.

(d) $v(w) = (\cdots i + 1 \cdots i \cdots)$. In this case we have $D(w) = D(\overline{w})$ while clearly $\mu = \overline{\mu}$. We deduce that $\overline{\nu}_p = \nu_p - 1$, while $\overline{\nu}_j = \nu_j$ for all $j \neq p$. We must show that $\beta_p = i$, and hence $\overline{\beta}_p = i + 1$, and $\beta_j = \overline{\beta}_j$ for all $j \neq p$.

Note that if $w_r = i + 1$, then $r \in [1, k]$. Since $w_j \geq w_r$ for all $j \in [r, k]$, and the sequence $\beta(w)$ is strictly increasing, we deduce that $\beta_g = i$ exactly when $g = \gamma_r(w)$. We have $\gamma_r(\overline{w}) = \gamma_r(w) - 1 = g - 1$, while $\gamma_j(w) = \gamma_j(\overline{w})$ for $j \neq r$. It follows that $\overline{\nu}_g = \nu_g - 1$, while $\overline{\nu}_j = \nu_j$ for all $j \neq g$. In other words, $g = p$, as desired.

To simplify the notation, set $c^\rho_\alpha := {}^\kappa c^\rho_\alpha$, for any integer sequences α and ρ. In cases (a), (b), or (d), it follows using the left Leibnitz rule and Lemma 3(a) that for any integer sequence $\alpha = (\alpha_1, \ldots, \alpha_\ell)$, we have

$$\begin{aligned}
\partial_i^y c^\beta_\alpha &= c^{(\beta_1,\ldots,\beta_{p-1})}_{(\alpha_1,\ldots,\alpha_{p-1})}\left(\partial_i^y(c^{\beta_p}_{\alpha_p})c^{(\beta_{p+1},\ldots,\beta_\ell)}_{(\alpha_{p+1},\ldots,\alpha_\ell)} + s_i^y(c^{\beta_p}_{\alpha_p})\partial_i^y(c^{(\beta_{p+1},\ldots,\beta_\ell)}_{(\alpha_{p+1},\ldots,\alpha_\ell)})\right) \\
&= c^{(\beta_1,\ldots,\beta_{p-1})}_{(\alpha_1,\ldots,\alpha_{p-1})}\left(c^{\beta_p+1}_{\alpha_p-1}c^{(\beta_{p+1},\ldots,\beta_\ell)}_{(\alpha_{p+1},\ldots,\alpha_\ell)} + s_i^y(c^{\beta_p}_{\alpha_p})\cdot 0\right) = c^{(\beta_1,\ldots,\beta_p+1,\ldots,\beta_\ell)}_{(\alpha_1,\ldots,\alpha_p-1,\ldots,\alpha_\ell)} = c^{\overline{\beta}}_{\alpha-\epsilon_p}.
\end{aligned}$$

Since $\lambda - \epsilon_p = \overline{\lambda}$, we deduce that if R is any raising operator, then

$$\partial_i^y(R\,c^\beta_\lambda) = \partial_i^y(c^\beta_{R\lambda}) = c^{\overline{\beta}}_{R\lambda-\epsilon_p} = R\,c^{\overline{\beta}}_{\overline{\lambda}}.$$

As $R^{D(w)} = R^{D(\overline{w})}$, we conclude that

$$\partial_i^y(F_w) = \partial_i^y(R^{D(w)} c^{\beta}_{\lambda}) = R^{D(\overline{w})} c^{\overline{\beta}}_{\overline{\lambda}} = F_{\overline{w}}.$$

In case (c), it follows from the left Leibnitz rule as in the proof of Lemma 3(b) that for any integer sequence $\alpha = (\alpha_1, \ldots, \alpha_\ell)$, we have

$$\begin{aligned}\partial_i^y c^{\beta}_{\alpha} &= \partial_i^y c^{(\beta_1,\ldots,-i,\ldots,i,\ldots,\beta_\ell)}_{(\alpha_1,\ldots,\alpha_p,\ldots,\alpha_q,\ldots,\alpha_\ell)} \\ &= c^{(\beta_1,\ldots,-i+1,\ldots,i+1,\ldots,\beta_\ell)}_{(\alpha_1,\ldots,\alpha_p-1,\ldots,\alpha_q,\ldots,\alpha_\ell)} + c^{(\beta_1,\ldots,-i+1,\ldots,i+1,\ldots,\beta_\ell)}_{(\alpha_1,\ldots,\alpha_p,\ldots,\alpha_q-1,\ldots,\alpha_\ell)} = c^{\overline{\beta}}_{\alpha-\epsilon_p} + c^{\overline{\beta}}_{\alpha-\epsilon_q}.\end{aligned}$$

Since $\lambda - \epsilon_p = \overline{\lambda}$, we deduce that if R is any raising operator, then

$$\partial_i^y(R\, c^{\beta}_{\lambda}) = \partial_i^y(c^{\beta}_{R\lambda}) = c^{\overline{\beta}}_{R\lambda-\epsilon_p} + c^{\overline{\beta}}_{R\lambda-\epsilon_q} = R\, c^{\overline{\beta}}_{\overline{\lambda}} + RR_{pq}\, c^{\overline{\beta}}_{\overline{\lambda}}.$$

As $R^{D(w)} + R^{D(w)}R_{pq} = R^{D(\overline{w})}$, we conclude that

$$\partial_i^y(F_w) = \partial_i^y(R^{D(w)} c^{\beta}_{\lambda}) = R^{D(w)} c^{\overline{\beta}}_{\overline{\lambda}} + R^{D(w)} R_{pq} c^{\overline{\beta}}_{\overline{\lambda}} = R^{D(\overline{w})} c^{\overline{\beta}}_{\overline{\lambda}} = F_{\overline{w}}.$$

□

Proposition 6. *Suppose that $w \in W_n$ is such that $\gamma(w)$ is a partition. Then we have*

$$\mathfrak{C}_w = R^{D(w)}\, {}^{\nu(w)}c^{\beta(w)}_{\lambda(w)}. \tag{18}$$

Proof. Assume first that $w_i < 0$ for each i, and $\gamma(w)$ is a partition. We claim that

$$\mathfrak{C}_w = R^{\infty}\, {}^{\nu(w)}c^{-\delta_{n-1}}_{\lambda(w)}. \tag{19}$$

The proof of (19) is by descending induction on $\ell(w)$. One knows from [IMN, Thm. 1.2] and [T5, Prop. 3.2] that (19) is true for the longest element w_0 in W_n, since $\nu(w_0) = \delta_{n-1}$ and $\lambda(w_0) = \delta_n + \delta_{n-1}$.

Suppose that $w \neq w_0$ is such that $\gamma(w)$ is a partition, and the shape of w equals $\delta_n + \nu$. Then $\nu \subset \delta_{n-1}$ and $\nu \neq \delta_{n-1}$. Let $r \geq 1$ be the largest integer such that $\nu_i = n-i$ for $i \in [1, r]$, and let $j := \nu_{r+1}+1 \leq n-r-1$. Then ws_j is of length $\ell(w)+1$ and satisfies the same conditions, $\nu(ws_j) = \nu(w)+\epsilon_{r+1}$, and $\lambda(ws_j) = \lambda(w)+\epsilon_{r+1}$. Using Lemma 3(a), for any integer sequence $\alpha = (\alpha_1, \ldots, \alpha_n)$, we have

$$\partial_j^x({}^{\nu(ws_j)}c^{-\delta_{n-1}}_{\alpha}) = {}^{\nu(w)}c^{-\delta_{n-1}}_{\alpha-\epsilon_{r+1}}.$$

We deduce that

$$\mathfrak{C}_w = \partial_j^x(\mathfrak{C}_{ws_j}) = \partial_j^x(R^{\infty}\, {}^{\nu(ws_j)}c^{-\delta_{n-1}}_{\lambda(ws_j)}) = R^{\infty}\, {}^{\nu(w)}c^{-\delta_{n-1}}_{\lambda(w)},$$

proving the claim. Equation (18) now follows, by combining (19) with the $k = 0$ case of Proposition 5. □

Corollary 1. *Suppose that $w \in W_n$ is increasing up to k and the k-truncated A-code ${}^k\gamma$ is a partition. Let $k^{n-k} + \xi(w) = (k+\xi_1, \ldots, k+\xi_{n-k})$. Then we have*

$$\mathfrak{C}_w = R^{D(w)}\, {}^{k^{n-k}+\xi(w)}c^{\beta(w)}_{\lambda(w)}. \tag{20}$$

Proof. If $w = (1, \ldots, k, w_{k+1}, \ldots, w_n)$ with $w_{k+j} < 0$ for all $j \in [1, n-k]$, then $\gamma(w) = k^{n-k} + \xi(w)$ is a partition, and (20) is a direct application of Proposition 6. In this case, $v(w) = (1, \ldots, k, -n, \ldots, -k-1)$ is the longest k-Grassmannian element in W_n. The general result now follows from Proposition 5, as the k-Grassmannian elements of W_n form an ideal for the left weak Bruhat order (see for example [Ste, Prop. 2.5]). □

Definition 7. Let $k \geq 0$ denote the first right descent of $w \in W_n$. List the entries $w_{k+1}, \ldots, w_n$ in increasing order:

$$u_1 < \cdots < u_m < 0 < u_{m+1} < \cdots < u_{n-k}.$$

We say that a simple transposition s_i for $i \geq 1$ is *w-negative* (respectively, *w-positive*) if $\{i, i{+}1\}$ is a subset of $\{-u_1, \ldots, -u_m\}$ (respectively, of $\{u_{m+1} \ldots, u_{n-k}\}$). Let σ^- (respectively, σ^+) be the longest subword of $s_{n-1} \cdots s_1$ (respectively, of $s_1 \cdots s_{n-1}$) consisting of w-negative (respectively, w-positive) simple transpositions. A *modification* of $w \in W_n$ is an element ωw, where $\omega \in S_n$ is such that $\ell(\omega w) = \ell(w) - \ell(\omega)$, and ω has a reduced decomposition of the form $R_1 \cdots R_{n-1}$ where each R_j is a (possibly empty) subword of $\sigma^-\sigma^+$ and all simple reflections in R_p are also contained in R_{p+1}, for each $p < n-1$.

Definition 8. Suppose that $w \in W_n$ has first right descent at $k \geq 0$ and A-code γ. We say that w is *leading* if $(\gamma_{k+1}, \gamma_{k+2}, \ldots, \gamma_n)$ is a partition. We say that w is *amenable* if w is a modification of a leading element.

Remark 3. (a) The integer vector $\alpha = (\alpha_1, \ldots, \alpha_p)$ is called *unimodal* if for some $j \in [1, p]$, we have

$$\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_j \geq \alpha_{j+1} \geq \cdots \geq \alpha_p.$$

The element $w \in W_n$ is leading if and only if the A-code of the extended sequence $(0, w_1, w_2, \ldots, w_n)$, where we have set $w(0) := 0$, is unimodal.

(b) Given an element $w \in W_n$, there is an easy algorithm to decide whether or not w is amenable. One simply applies all possible *inverse modifications* to w and checks if any of these result in a leading element.

Example 7. Consider the leading element $w = (2, 4, 6, 5, \overline{1}, \overline{3})$ in W_6, with $k = 3$, $\gamma(w) = (2, 2, 3, 2, 1, 0)$, $\mu(w) = (3, 1)$, $\nu(w) = (5, 4, 1)$, $\xi(w) = (2, 1)$, and $\lambda(w) = (8, 5, 1)$. We have $\beta(w) = (-2, 0, 5)$ and $D(w) = \{(1, 2)\}$, so Corollary 1 gives

$$\mathfrak{C}_w = R^{\{12\}}\,{}^{(5,4,3)}c^{(-2,0,5)}_{(8,5,1)} = \frac{1 - R_{12}}{1 + R_{12}}(1 - R_{13})(1 - R_{23})\,{}^{(5,4,3)}c^{(-2,0,5)}_{(8,5,1)}.$$

In the following we will assume that w has first right descent at $k \geq 0$ and ξ is as in Definition 6. Let $\psi := (\gamma_k, \ldots, \gamma_1)$, $\phi := \psi'$, $\ell := \ell(\lambda)$ and $m := \ell(\mu)$. We then have

$$\lambda = \phi + \xi + \mu \tag{21}$$

and $\lambda_1 > \cdots > \lambda_m > \lambda_{m+1} \geq \cdots \geq \lambda_\ell$.

Definition 9. Say that $\mathfrak{q} \in [1, \ell]$ is a *critical index* if $\beta_{\mathfrak{q}+1} > \beta_{\mathfrak{q}} + 1$, or if $\lambda_{\mathfrak{q}} > \lambda_{\mathfrak{q}+1} + 1$ (respectively, $\lambda_{\mathfrak{q}} > \lambda_{\mathfrak{q}+1}$) and $\mathfrak{q} < m$ (respectively, $\mathfrak{q} > m$). Define two sequences $\mathfrak{f} = \mathfrak{f}(w)$ and $\mathfrak{g} = \mathfrak{g}(w)$ of length ℓ as follows. For $1 \leq j \leq \ell$, set

$$\mathfrak{f}_j := k + \max(i \mid \gamma_{k+i} \geq j)$$

and let

$$\mathfrak{g}_j := \mathfrak{f}_{\mathfrak{q}} + \beta_{\mathfrak{q}} - \xi_{\mathfrak{q}} - k,$$

where $\mathfrak{q}$ is the least critical index such that $\mathfrak{q} \geq j$. We call $\mathfrak{f}$ the *right flag* of w, and $\mathfrak{g}$ the *left flag* of w.

If $m \geq 1$, then m is a critical index, since u_m is the largest negative entry of w. We will show that for any amenable element w, $\mathfrak{f}$ is a weakly decreasing sequence consisting of right descents of w, and $\mathfrak{g}$ is a weakly increasing sequence whose absolute values consist of left descents of w.

Lemma 11. (a) *If $\beta_{s+1} > \beta_s + 1$, then $|\beta_s|$ is a left descent of w.*

(b) *If $s \leq m$, then $\phi_s = k$, while if $s > m$ and $\beta_{s+1} = \beta_s + 1$, then $\phi_s = \phi_{s+1}$.*

Proof. Let $i := |\beta_s|$, and suppose that $1 \leq s \leq m$. If $i = 0$ then $u_s = -1$, so clearly i is a left descent of w. If $i \geq 1$ and $\beta_{s+1} > \beta_s + 1$, then i is a left descent of w, since $w^{-1}(i) > 0$ and $w^{-1}(i+1) < 0$. As $w_j > 0$ for all $j \in [1, k]$, we have $\psi_j \geq m$ for all $j \in [1, k]$, and hence $\phi_s = k$.

Next suppose that $s > m$. If $\beta_{s+1} > \beta_s + 1 = i + 1$, then we have $w^{-1}(i+1) < 0$ or $w_j = i + 1$ for some $j \in [1, k]$. In either case, it is clear that i is a left descent of w. Finally, assume that $\beta_{s+1} = \beta_s + 1$. If $\phi_s > \phi_{s+1}$, there must exist $j \in [1, k]$ such that $\gamma_j = s$, that is, $\#\{r > k \mid w_r < w_j\} = s$. We deduce that

$$\{w_r \mid r > k \text{ and } w_r < w_j\} = \{u_1, \ldots, u_s\},$$

which is a contradiction, since $u_s < w_j \Rightarrow u_{s+1} = u_s + 1 < w_j$, for any $j \in [1, k]$. This completes the proof of (a) and (b). □

Proposition 7. *Suppose that $\widehat{w} \in W_n$ is leading with first right descent at $k \geq 0$, let $\widehat{\lambda} := \lambda(\widehat{w})$, and $\widehat{\xi} := \xi(\widehat{w})$. Let $w = \omega\widehat{w}$ be a modification of $\widehat{w}$, and set $\gamma := \gamma(w)$, $\lambda := \lambda(w)$, $\beta := \beta(w)$, and $\xi := \xi(w)$. Then the sequence $\beta + \widehat{\lambda} - \lambda$ is weakly increasing, and*

$$\mathfrak{C}_w = R^{D(w)\ k^{n-k}+\widehat{\xi}} c_{\lambda(w)}^{\beta(w)+\widehat{\xi}-\xi} = R^{D(w)\ k^{n-k}+\widehat{\xi}} c_{\lambda}^{\beta+\widehat{\lambda}-\lambda}.$$

Moreover, if $\mathfrak{q} \in [1, \ell]$ is a critical index of w, then $k + \widehat{\xi}_{\mathfrak{q}}$ is a right descent of w, the absolute value of $\beta_{\mathfrak{q}} + \widehat{\xi}_{\mathfrak{q}} - \xi_{\mathfrak{q}}$ is a left descent of w, and $\widehat{\xi}_{\mathfrak{q}} = \max(i \mid \gamma_{k+i} \geq \mathfrak{q})$.

Proof. Suppose that the truncated A-code of $\widehat{w}$ is

$${}^k\widehat{\gamma} = (p_1^{n_1}, \ldots, p_t^{n_t})$$

for some parts $p_1 > p_2 > \cdots > p_t > 0$, and we let $d_j := n_1 + \cdots + n_j$ for $j \in [1, t]$. Then we have

$$\widehat{\xi} = (d_t^{p_t}, d_{t-1}^{p_{t-1}-p_t}, \ldots, d_1^{p_1-p_2})$$

and it follows that

$$\widehat{w}_{k+1} = u_{p_1+1},\ \widehat{w}_{k+d_1+1} = u_{p_2+1},\ \ldots,\ \widehat{w}_{k+d_{t-1}+1} = u_{p_t+1}$$

and $\widehat{w}_j < \widehat{w}_{j+1}$ for all $j \notin \{k, k + d_1, \ldots, k + d_t\}$. Hence the set of components of $k^{n-k} + \widehat{\xi}$ coincides with the set of all right descents of $\widehat{w}$.

If $\mathfrak{q} \in [1, \ell]$ is a critical index, we have shown that $f_{\mathfrak{q}}$ is a right descent of $\widehat{w}$. We claim that $i := |g_{\mathfrak{q}}| = |\beta_{\mathfrak{q}}|$ is a left descent of $\widehat{w}$. By Lemma 11(a), we may assume that $\beta_{\mathfrak{q}+1} = \beta_{\mathfrak{q}} + 1$, which implies that $\mathfrak{q} \neq m$.

Suppose that $\mathfrak{q} < m$. Then we have $\widehat{\lambda}_{\mathfrak{q}} > \widehat{\lambda}_{\mathfrak{q}+1} + 1$ and $\widehat{\mu}_{\mathfrak{q}} = \widehat{\mu}_{\mathfrak{q}+1} + 1$, so (21) gives $\widehat{\xi}_{\mathfrak{q}} > \widehat{\xi}_{\mathfrak{q}+1}$. We therefore have $\mathfrak{q} = p_j$ for some $\mathfrak{q} \in [1, t]$, and hence $i = \widehat{\mu}_{p_j} - 1 = \widehat{\mu}_{p_j+1} = -u_{p_j+1} = -\widehat{w}_{k+d_{j-1}+1}$. Since we have

$$\widehat{w}_{k+1} > \widehat{w}_{k+d_1+1} > \cdots > \widehat{w}_{k+d_{j-1}+1} = -i,$$

and the sequence $(\widehat{w}_{k+1},\ldots,\widehat{w}_n)$ is 132-avoiding, we conclude that $\widehat{w}^{-1}(-i) = k+d_{j-1}+1 < \widehat{w}^{-1}(-i-1)$, as desired.

Suppose next that $\mathfrak{q} > m$. Then we have $\widehat{\lambda}_{\mathfrak{q}} > \widehat{\lambda}_{\mathfrak{q}+1}$, so Lemma 11(b) and equation (21) imply that $\widehat{\xi}_{\mathfrak{q}} > \widehat{\xi}_{\mathfrak{q}+1}$. We deduce that $\mathfrak{q} = p_j$ for some j, hence $i+1 = u_{p_j+1}$ and the result follows.

According to Corollary 1, we have

$$\mathfrak{C}_{\widehat{w}} = R^{D(\widehat{w})\ k^{n-k}+\widehat{\xi}} c_{\lambda(\widehat{w})}^{\beta(\widehat{w})}, \tag{22}$$

so the proposition holds for leading elements. Suppose next that $w := \omega\widehat{w}$ is a modification of $\widehat{w}$. Then repeated application of (6), Lemma 3(a), and the left Leibnitz rule (4) in equation (22) give

$$\mathfrak{C}_w = R^{D(w)\ k^{n-k}+\widehat{\xi}} c_{\lambda(w)}^{\beta(w)+\widehat{\xi}-\xi}.$$

It remains to check the last assertion, about the left and right descents of w.

Let $R_1\cdots R_{n-1}$ be the reduced decomposition for ω from Definition 7. We will study the left action of the successive simple transpositions in $R_1\cdots R_{n-1}$ on $\widehat{w}$. Observe that σ^- and σ^+ are disjoint and $\sigma^-\sigma^+ = \sigma^+\sigma^-$. Moreover, the actions of the $\widehat{w}$-positive and $\widehat{w}$-negative simple transpositions on $\widehat{w}$ are similar, and we can consider them separately. Let $A := \{k+1, k+d_1+1, \ldots, k+d_t+1\}$.

We begin with the $\widehat{w}$-positive simple transpositions. The action of these on $\widehat{w}$ is by a finite sequence of *moves* $v \mapsto v'$, where $v' = s_i\cdots s_{j-1}v$ for some i,j with $1\leq i<j$, $\ell(v') = \ell(v)-j+i$, and $s_i\cdots s_{j-1}$ is a subword of some R_p with $j-i$ maximal. We call such a move an $[i,j]$-move at position r if $v_r = j$ and $v'_r = i$, so that v' is obtained from v by cyclically permuting the values $i, i+1, \ldots, j$. Observe that we must have $r\in A$, and subsequent $[i',j']$-moves for $[i',j']\subset[i,j]$ are at positions $r'\in A$ with $r<r'$. This follows from the fact that the sequence $(\widehat{w}_{k+1},\ldots,\widehat{w}_n)$ is 132-avoiding, and by induction on the number of moves.

Let α denote the truncated A-code of v, and ξ_v, $g_v := \beta+\widehat{\xi}-\xi_v$ the associated statistics, with $\alpha', \xi' := \xi_{v'}$, $g' = g_{v'} := \beta+\widehat{\xi}-\xi'$ the corresponding ones for v'. If $\alpha_r = e$, then we have $\alpha'_r = d$ for $d = e+i-j$, and $\alpha'_s = \alpha_s$ for all $s\neq r$. If $\xi_v = (\xi_1,\xi_2,\ldots)$ and $g_v = (g_1, g_2, \ldots)$, then $g_{d+1} = i, \ldots, g_e = j-1$, while

$$\xi' = (\xi_1,\ldots,\xi_d,\xi_{d+1}-1,\ldots,\xi_e-1,\xi_{e+1},\ldots)$$

and

$$\begin{aligned} g' &= (g_1,\ldots,g_d,g_{d+1}+1,\ldots,g_e+1,g_{e+1},\ldots)\\ &= (g_1,\ldots,g_d,i+1,\ldots,j,g_{e+1},\ldots). \end{aligned}$$

Lemma 11 implies that the critical indices of v and v' can only differ in positions d and e. Since the simple transpositions $s_i,\ldots,s_{j-1}$ are all $\widehat{w}$-positive, we have $\beta_s+1 = \beta_{s+1}$ for all $s\in[d+1,e]$. If $\beta_d+1<\beta_{d+1}$, then $|g_d| = |\beta_d|$ is a left descent of both $\widehat{w}$ and v', by Lemma 11. We may therefore assume that $\beta_s+1 = \beta_{s+1}$ for all $s\in[d,e]$, and only need to study the $s\in[d,e]$ where $\xi'_s > \xi'_{s+1}$. If $s\in[d+1,e]$, since the values $i,\ldots,j$ of v are cyclically permuted in v', it follows by induction on the number of moves that g'_s is a left descent of v'. Notice that we must have $i\geq 2$ in this situation. It remains to prove that $g'_d = g_d = i-1$ is a left descent of v'. But since $\widehat{w}^{-1}(i-1) > k$ and the sequence $(\widehat{w}_{k+1},\ldots,\widehat{w}_n)$ is 132-avoiding, we deduce that $(v')^{-1}(i) = r < (v')^{-1}(i-1)$, as desired.

The above procedure shows that for any critical index h of v', we must have

$$\widehat{\xi}_h = \max(i \mid \widehat{\gamma}_{k+i} \geq h) = \max(i \mid \alpha'_i \geq h),$$

while $k + \max(i \mid \alpha'_i \geq h)$ is a right descent of v', by Lemma 2. Finally, the action of the $\widehat{w}$-negative simple transpositions on $\widehat{w}$ is studied in the same way. □

Theorem 4. *For any amenable element $w \in W_\infty$, we have*

$$\mathfrak{C}_w = R^{D(w)\ \mathfrak{f}(w)} c_{\lambda(w)}^{\mathfrak{g}(w)} \tag{23}$$

in $\Gamma[X, Y]$.

Proof. We may assume that we are in the situation of Proposition 7, so that $w = \omega\widehat{w}$, with $\widehat{\lambda} = \lambda(\widehat{w})$ and $\lambda = \lambda(w)$. Suppose that $j \in [1, \ell]$ and let $\mathfrak{q}$ be the least critical index of w such that $\mathfrak{q} \geq j$. Then we have $\lambda_j = \lambda_{j+1} = \cdots = \lambda_{\mathfrak{q}}$, if $\mathfrak{q} > m$, and $\lambda_j = \lambda_{j+1} + 1 = \cdots = \lambda_{\mathfrak{q}} + (\mathfrak{q} - j)$, if $\mathfrak{q} \leq m$. Moreover, in either case, we have $\xi_j = \cdots = \xi_{\mathfrak{q}}$, and the values $\beta_j, \ldots, \beta_{\mathfrak{q}}$ are consecutive integers. As the sequence $g := \beta + \widehat{\xi} - \xi$ is weakly increasing, we deduce that for any $r \in [j, \mathfrak{q} - 1]$, either (i) $\widehat{\xi}_r = \widehat{\xi}_{r+1}$ and $g_r = g_{r+1} - 1$, or (ii) $\widehat{\xi}_r = \widehat{\xi}_{r+1} + 1$ and $g_r = g_{r+1}$. Theorem 4 follows from this and induction on $\mathfrak{q} - j$, by employing Lemmas 5(b) and 6(b) in Proposition 7. The required conditions on $D(w)$ in these two lemmas and the corresponding relations (5) are both easily checked. □

Remark 4. The equalities such as (23) in this section occur in $\Gamma[X, Y]$, which is a ring with relations coming from Γ. Therefore they are not equalities of polynomials in independent variables, in contrast to the situation in type A. The same remark applies to the corresponding equalities in Section 6.1.

Anderson and Fulton [AF2] have introduced a family of signed permutations, each determined by an algorithm starting from an equivalence class of 'triples'. These were named 'theta-vexillary' and studied further by Lambert [Lam]. It seems plausible that the theta-vexillary signed permutations coincide with our amenable elements in types B and C, but we do not examine this question here.

5.2. **Flagged theta polynomials.** In this section, we define a family of polynomials Θ_w indexed by amenable elements $w \in W_\infty$ that generalize Wilson's double theta polynomials [W, TW]. For each $k \geq 0$, let ${}^k\mathfrak{c} := ({}^k\mathfrak{c}_p)_{p\in\mathbb{Z}}$ be a family of variables, such that ${}^k\mathfrak{c}_0 = 1$ and ${}^k\mathfrak{c}_p = 0$ for $p < 0$, and let $t := (t_1, t_2, \ldots)$. The polynomial Θ_w represents an equivariant Schubert class in the T-equivariant cohomology ring of the symplectic partial flag variety associated to the right flag $\mathfrak{f}(w)$, which is defined in [T1, Sec. 4.1]. The t variables come from the characters of the maximal torus T, as explained in [TW].

For any integers p and r, define

$${}^k\mathfrak{c}_p^r := \sum_{j=0}^{p} {}^k\mathfrak{c}_{p-j} h_j^r(-t).$$

Given integer sequences κ, α, and ρ, let ${}^\kappa\mathfrak{c}_\alpha^\rho := {}^{\kappa_1}\mathfrak{c}_{\alpha_1}^{\rho_1}\, {}^{\kappa_2}\mathfrak{c}_{\alpha_2}^{\rho_2} \cdots$, and let any raising operator R act in the usual way, by $R\, {}^\kappa\mathfrak{c}_\alpha^\rho := {}^\kappa\mathfrak{c}_{R\alpha}^\rho$.

If $w \in W_n$ is amenable with left flag $\mathfrak{f}(w)$ and right flag $\mathfrak{g}(w)$, then the *flagged double theta polynomial* $\Theta_w(\mathfrak{c}\,|\,t)$ is defined by

$$\Theta_w(\mathfrak{c}\,|\,t) := R^{D(w)\ \mathfrak{f}(w)}\mathfrak{c}_{\lambda(w)}^{\mathfrak{g}(w)}. \tag{24}$$

The *flagged single theta polynomial* is given by $\Theta_w(\mathfrak{c}) := \Theta_w(\mathfrak{c}\,|\,0)$. If w is a leading element, then (24) can be written in the 'factorial' form

$$\Theta_w(\mathfrak{c}\,|\,t) = R^{D(w)\ \mathfrak{f}(w)}\mathfrak{c}_{\lambda(w)}^{\beta(w)}.$$

When w is a k-Grassmannian element, the above formulas specialize to the double theta polynomial $\Theta_\lambda(\mathfrak{c}\,|\,t)$ found in [TW]; here λ is the k-strict partition corresponding to w. Moreover, the single theta polynomial $\Theta_\lambda(\mathfrak{c})$ agrees with that of [BKT2].

5.3. **Symplectic degeneracy loci.** Let $E \to \mathfrak{X}$ be a vector bundle of rank $2n$ on a smooth complex algebraic variety $\mathfrak{X}$. Assume that E is a *symplectic* bundle, so that E is equipped with an everywhere nondegenerate skew-symmetric form $E \otimes E \to \mathbb{C}$. Let $w \in W_n$ be amenable of shape λ, and let $\mathfrak{f}$ and $\mathfrak{g}$ be the left and right flags of w, respectively. Consider two complete flags of subbundles of E

$$0 \subset E_1 \subset \cdots \subset E_{2n} = E \quad \text{and} \quad 0 \subset F_1 \subset \cdots \subset F_{2n} = E$$

with $\operatorname{rank} E_r = \operatorname{rank} F_r = r$ for each r, while $E_{n+s} = E_{n-s}^{\perp}$ and $F_{n+s} = F_{n-s}^{\perp}$ for $0 \leq s < n$.

There is a group monomorphism $\zeta : W_n \hookrightarrow S_{2n}$ with image

$$\zeta(W_n) = \{\, \varpi \in S_{2n} \mid \varpi_i + \varpi_{2n+1-i} = 2n+1, \quad \text{for all } i \,\}.$$

The map ζ is determined by setting, for each $w = (w_1, \ldots, w_n) \in W_n$ and $1 \leq i \leq n$,

$$\zeta(w)_i := \begin{cases} n+1-w_{n+1-i} & \text{if } w_{n+1-i} \text{ is unbarred,} \\ n+\overline{w}_{n+1-i} & \text{otherwise.} \end{cases}$$

Define the *degeneracy locus* $\mathfrak{X}_w \subset \mathfrak{X}$ as the locus of $x \in \mathfrak{X}$ such that

$$\dim(E_r(x) \cap F_s(x)) \geq \#\,\{\, i \leq r \mid \zeta(w)_i > 2n-s \,\} \ \text{ for } 1 \leq r \leq n,\ 1 \leq s \leq 2n.$$

We assume that $\mathfrak{X}_w$ has pure codimension $\ell(w)$ in $\mathfrak{X}$, and give a formula for the class $[\mathfrak{X}_w]$ in $\mathrm{H}^{2\ell(w)}(\mathfrak{X})$.

Theorem 5. *For any amenable element $w \in W_n$, we have*

$$[\mathfrak{X}_w] = \Theta_w(E - E_{n-\mathfrak{f}} - F_{n+\mathfrak{g}}) = R^{D(w)}\, c_\lambda(E - E_{n-\mathfrak{f}} - F_{n+\mathfrak{g}}) \tag{25}$$

in the cohomology ring $\mathrm{H}^*(\mathfrak{X})$.

As in [TW, Eqn. (7)], the Chern polynomial in (25) is interpreted as the image of the polynomial $R^{D(w)}\mathfrak{c}_\lambda$ under the $\mathbb{Z}$-linear map which sends the noncommutative monomial $\mathfrak{c}_\alpha = \mathfrak{c}_{\alpha_1}\mathfrak{c}_{\alpha_2}\cdots$ to $\prod_j c_{\alpha_j}(E - E_{n-\mathfrak{f}_j} - F_{n+\mathfrak{g}_j})$, for every integer sequence α. Theorem 5 is proved by applying the type C geometrization map of [IMN, Sec. 10] to both sides of (23), following [T1, Sec. 4.2].

6. Amenable elements: Type D theory

6.1. **Definitions and main theorem.** Let w be an element in $\widetilde{W}_\infty$ with A-code γ and shape $\lambda = \mu + \nu$, with $\ell = \ell(\lambda)$ and $m = \ell(\mu)$. Choose $k \geq 1$, and assume that w is increasing up to k. If $k = 1$, this condition is vacuous, while if $k > 1$ it means that $|w_1| < w_2 < \cdots < w_k$. Eventually, k will be set equal to the primary index of w.

List the entries $w_{k+1}, \ldots, w_n$ in increasing order:

$$u_1 < \cdots < u_{m'} < 0 < u_{m'+1} < \cdots < u_{n-k},$$

where $m' \in \{m, m+1\}$. Define a sequence $\beta(w)$ by

$$\beta(w) := (u_1+1, \ldots, u_{m'}+1, u_{m'+1}, \ldots, u_{n-k}).$$

and the *denominator set* $D(w)$ by

$$D(w) := \{(i,j) \mid 1 \leq i < j \leq n-k \text{ and } u_i + u_j < 0\}.$$

As in Section 5.1, the notation suppresses the dependence of $\beta(w)$ and $D(w)$ on k.

Definition 10. Suppose that $w \in \widetilde{W}_n$ has code $\gamma = \gamma(w)$ and $k \geq 1$. The *k-truncated A-code* ${}^k\gamma = {}^k\gamma(w)$ is defined by ${}^k\gamma(w) := (\gamma_{k+1}, \gamma_{k+2}, \ldots, \gamma_n)$. If k is the primary index of w, then we call ${}^k\gamma(w)$ the *truncated A-code* of w. Let $\xi = \xi(w)$ be the partition whose parts satisfy $\xi_j := \#\{i \mid \gamma_{k+i} \geq j\}$ for each $j \geq 1$.

We let $v(w)$ be the unique k-Grassmannian element obtained by reordering the parts $w_{k+1}, \ldots, w_n$ to be strictly increasing. For example, if $w = (\overline{2}, 4, 7, 5, \overline{8}, \overline{3}, 1, \overline{6})$ and $k = 3$, then $v(w) = (\overline{2}, 4, 7, \overline{8}, \overline{6}, \overline{3}, 1, 5)$. The map $w \mapsto v(w)$ is a type-preserving bijection from the set of elements in $\widetilde{W}_n$ increasing up to k with a given k-truncated A-code C onto the set of k-Grassmannian elements in $\widetilde{W}_n$, such that $\beta(v(w)) = \beta(w)$ and

$$\ell(v(w)) = \ell(w) - \sum_{i=k+1}^{n} C_i = \ell(w) - \sum_{j=1}^{n-k} \gamma_{k+j}.$$

Lemma 12. *Let w and $\overline{w}$ be elements in $\widetilde{W}_n$ increasing up to $k \geq 1$ and with the same k-truncated A-code C, such that $\ell(w) = \ell(\overline{w})+1$. Suppose that $v(\overline{w}) = s_i v(w)$ for some simple reflection s_i. Then $\overline{w} = s_i w$.*

Proof. We have seven possible cases for i and $v(w)$: (a) $i = \Box$ and $v(w) = (\widehat{1} \cdots \overline{2} \cdots)$; (b) $i = \Box$ and $v(w) = (\cdots \overline{2} \cdots \overline{1} \cdots)$; (c) $i = \Box$ and $v(w) = (2 \cdots \overline{1} \cdots)$; (d) $i \geq 1$ and $v(w) = (\cdots i \cdots \overline{i+1} \cdots)$; (e) $i \geq 1$ and $v(w) = (\cdots \overline{i+1} \cdots i \cdots)$; (f) $i \geq 1$ and $v(w) = (\overline{i} \cdots \overline{i+1} \cdots)$; (g) $i \geq 1$ and $v(w) = (\cdots i+1 \cdots i \cdots)$. In the first five cases, it is clear that $\overline{w} = s_i w$. In case (f) (resp. (g)), the $\overline{i}$ (resp. $i+1$) must be among the first k entries of $v(w)$, which coincide with the first k entries of w, while $\overline{i+1}$ (resp. i) lies among the last $n-k$ entries of w. Hence we again deduce that $\overline{w} = s_i w$. $\Box$

If $R := \prod_{i<j} R_{ij}^{n_{ij}}$ is any raising operator and $d \geq 0$, denote by $\mathrm{supp}_d(R)$ the set of all indices i and j such that $n_{ij} > 0$ and $j \leq d$.

Definition 11. Let $w \in \widetilde{W}_n$ be of shape $\lambda = \mu + \nu$, with $\ell = \ell(\lambda)$ and $m = \ell(\mu)$. Let $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ be a composition such that $\alpha_{m+1} = \lambda_{m+1}$, if $\mathrm{type}(w) > 0$, and $\upsilon = (\upsilon_1, \ldots, \upsilon_\ell)$ be an integer vector such that $\upsilon_{m+1} \in \{0, 1\}$. For any integer vector ρ, define

$${}^{\alpha}\widehat{c}_{\rho}^{\,\upsilon} := {}^{\alpha_1}\widehat{c}_{\rho_1}^{\,\upsilon_1}\, {}^{\rho_2}\widehat{c}_{\rho_2}^{\,\upsilon_2} \cdots$$

where, for each $i \geq 1$,

$$(26) \qquad {}^{\alpha_i}\widehat{c}_{\rho_i}^{\,\upsilon_i} := {}^{\alpha_i}c_{\rho_i}^{\upsilon_i} + \begin{cases} (-1)^i e_{\alpha_i}^{\alpha_i}(X) e_{\rho_i - \alpha_i}^{\rho_i - \alpha_i}(-Y) & \text{if } \upsilon_i = \alpha_i - \rho_i < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let R be any raising operator appearing in the expansion of the power series $R^{D(w)}$ and set $\rho := R\lambda$. If $\mathrm{type}(w) = 0$, then define

$$R \star {}^{\alpha}\widehat{c}_{\lambda}^{\,\upsilon} = {}^{\alpha}\overline{c}_{\rho}^{\,\upsilon} := {}^{\alpha_1}\overline{c}_{\rho_1}^{\,\upsilon_1} \cdots {}^{\alpha_\ell}\overline{c}_{\rho_\ell}^{\,\upsilon_\ell}$$

where, for each $i \geq 1$,

$$ {}^{\alpha_i}\overline{c}^{\,\upsilon_i}_{\rho_i} := \begin{cases} {}^{\alpha_i}c^{\upsilon_i}_{\rho_i} & \text{if } i \in \mathrm{supp}_m(R), \\ {}^{\alpha_i}\widehat{c}^{\,\upsilon_i}_{\rho_i} & \text{otherwise.} \end{cases} $$

If $\mathrm{type}(w) > 0$ and R involves any factors R_{ij} with $i = m+1$ or $j = m+1$, then define

$$ R \star {}^{\alpha}\widehat{c}^{\,\upsilon}_{\lambda} := {}^{\alpha_1}\overline{c}^{\,\upsilon_1}_{\rho_1} \cdots {}^{\alpha_m}\overline{c}^{\,\upsilon_m}_{\rho_m}\, {}^{\alpha_{m+1}}a^{\upsilon_{m+1}}_{\rho_{m+1}}\, {}^{\alpha_{m+2}}c^{\upsilon_{m+2}}_{\rho_{m+2}} \cdots {}^{\alpha_\ell}c^{\upsilon_\ell}_{\rho_\ell}. $$

If R has no such factors, then define

$$ R \star {}^{\alpha}\widehat{c}^{\,\upsilon}_{\lambda} := \begin{cases} {}^{\alpha_1}\overline{c}^{\,\upsilon_1}_{\rho_1} \cdots {}^{\alpha_m}\overline{c}^{\,\upsilon_m}_{\rho_m}\, {}^{\alpha_{m+1}}b^{\upsilon_{m+1}}_{\lambda_{m+1}}\, {}^{\alpha_{m+2}}c^{\upsilon_{m+2}}_{\rho_{m+2}} \cdots {}^{\alpha_\ell}c^{\upsilon_\ell}_{\rho_\ell} & \text{if } \mathrm{type}(w) = 1, \\ {}^{\alpha_1}\overline{c}^{\,\upsilon_1}_{\rho_1} \cdots {}^{\alpha_m}\overline{c}^{\,\upsilon_m}_{\rho_m}\, {}^{\alpha_{m+1}}\widetilde{b}^{\upsilon_{m+1}}_{\lambda_{m+1}}\, {}^{\alpha_{m+2}}c^{\upsilon_{m+2}}_{\rho_{m+2}} \cdots {}^{\alpha_\ell}c^{\upsilon_\ell}_{\rho_\ell} & \text{if } \mathrm{type}(w) = 2. \end{cases} $$

Let $w \in \widetilde{W}_\infty$ have shape $\lambda = \mu + \nu$ and κ be any integer sequence. We say that κ is *compatible with* w if $\kappa_p = \nu_p$ for $p \in [1, m]$, and $\kappa_{m+1} = \nu_{m+1}$ whenever $\mathrm{type}(w) > 0$.

Proposition 8. *Fix an integer $k \geq 1$. Suppose that w and $\overline{w}$ are elements in $\widetilde{W}_n$ increasing up to k with the same k-truncated A-code C, such that $\ell(w) = \ell(\overline{w}) + 1$ and $s_i v(w) = v(\overline{w})$ for some simple reflection s_i. Assume that we have*

$$ \mathfrak{D}_w = 2^{-\ell(\mu(w))}\, R^{D(w)} \star {}^{\kappa}\widehat{c}^{\,\beta(w)}_{\lambda(w)} $$

in $\Gamma'[X, Y]$, for some integer sequence κ compatible with w. Moreover, if $i \in \{\Box, 1\}$ and $|w_1| > 2$, assume that $\kappa_m = \kappa_{m+1}$. Then κ is compatible with $\overline{w}$, and we have

$$ \mathfrak{D}_{\overline{w}} = 2^{-\ell(\mu(\overline{w}))}\, R^{D(\overline{w})} \star {}^{\kappa}\widehat{c}^{\,\beta(\overline{w})}_{\lambda(\overline{w})} $$

in $\Gamma'[X, Y]$.

Proof. Set

$$ F_w := 2^{-\ell(\mu(w))}\, R^{D(w)} \star {}^{\kappa}\widehat{c}^{\,\beta(w)}_{\lambda(w)}, \tag{27} $$

so we know that $\mathfrak{D}_w = F_w$. As equation (9) gives $\partial^y_i \mathfrak{D}_w = \mathfrak{D}_{\overline{w}}$, it will suffice to show that $\partial^y_i F_w = F_{\overline{w}}$. The proof of this will follow the argument of [T4, Prop. 5], and correct it by including the case (h) below, which was missing there.

Let $\mu := \mu(w)$, $\nu := \nu(w)$, $\lambda := \lambda(w) = \mu + \nu$, $\overline{\mu} := \mu(\overline{w})$, $\overline{\nu} := \nu(\overline{w})$, $\overline{\lambda} := \lambda(\overline{w}) = \overline{\mu} + \overline{\nu}$, $\beta = \beta(w)$, and $\overline{\beta} = \beta(\overline{w})$. Using Lemma 4, we distinguish eight possible cases for w. In each case, we have $\overline{\lambda} \subset \lambda$, so that $\overline{\lambda}_p = \lambda_p - 1$ for some $p \geq 1$ and $\overline{\lambda}_j = \lambda_j$ for all $j \neq p$. Moreover, we must have $\mathrm{type}(w) + \mathrm{type}(\overline{w}) \neq 3$.

First, we consider the four cases with $i \geq 1$:

(a) $v(w) = (\cdots i+1 \cdots i \cdots)$. In this case $D(w) = D(\overline{w})$ while clearly $\mu = \overline{\mu}$. We deduce that $\overline{\nu}_p = \nu_p - 1$, while $\overline{\nu}_j = \nu_j$ for all $j \neq p$. We must show that $\beta_p = i$, and hence $\overline{\beta}_p = i + 1$, while $\beta_j = \overline{\beta}_j$ for all $j \neq p$.

Note that if $w_r = i + 1$, then $r \in [1, k]$. Since $w_j \geq w_r$ for all $j \in [r, k]$, and the sequence $\beta(w)$ is strictly increasing, we deduce that $\beta_g = i$ exactly when $g = \gamma_r(w)$. We have $\gamma_r(\overline{w}) = \gamma_r(w) - 1 = g - 1$, while $\gamma_j(w) = \gamma_j(\overline{w})$ for $j \neq r$. It follows that $\overline{\nu}_g = \nu_g - 1$, while $\overline{\nu}_j = \nu_j$ for all $j \neq g$. In other words, $g = p$, as desired.

Finally, observe that (i) if $i = 1$, then $p = m + 1$, $\mathrm{type}(w) > 0$, and $\mathrm{type}(\overline{w}) = 0$; (ii) if $i \geq 2$, then $p > m$ and $\mathrm{type}(w) = \mathrm{type}(\overline{w})$, while $p > m + 1$ if $\mathrm{type}(w) > 0$. It follows that κ is compatible with $\overline{w}$.

(b) $v(w) = (\cdots i \cdots \overline{i+1} \cdots)$. In this case $w^{-1}(i) \in [1,k]$, $D(w) = D(\overline{w})$, $\nu = \overline{\nu}$, $\beta_p = -i$, $\overline{\beta}_p = -i+1$, and $\beta_j = \overline{\beta}_j$ for all $j \neq p$.

(c) $v(w) = (\overline{i} \cdots \overline{i+1} \cdots)$. In this case $w_1 = \overline{i}$, $\mathrm{type}(w) = 2$ if $i \geq 2$, $D(w) = D(\overline{w})$, $\nu = \overline{\nu}$, $\beta_p = -i$, $\overline{\beta}_p = -i+1$, and $\beta_j = \overline{\beta}_j$ for all $j \neq p$.

(d) $v(w) = (\cdots \overline{i+1} \cdots i \cdots)$. We distinguish two subcases here:

Case (d1): $w_1 \neq \overline{i+1}$. Then $\nu = \overline{\nu}$, $\beta_p = -i$, $\overline{\beta}_p = -i+1 = \beta_p + 1$, and $D(w) = D(\overline{w}) \cup \{(p,q)\}$, where $v(w)_{k+p} = \overline{i+1}$ and $v(w)_{k+q} = i$. It follows that $\beta_q = i$ and $\overline{\beta}_q = i+1 = \beta_q + 1$, while $\beta_j = \overline{\beta}_j$ for all $j \notin \{p,q\}$.

Case (d2): $w_1 = \overline{i+1}$ and we have $w^{-1}(i) > k$. In this case $\mathrm{type}(w) = 2$, $D(w) = D(\overline{w})$, while clearly $\mu = \overline{\mu}$. We deduce that $\overline{\nu}_p = \nu_p - 1$, while $\overline{\nu}_j = \nu_j$ for all $j \neq p$. We must show that $\beta_p = i$, and hence $\overline{\beta}_p = i+1$, while $\beta_j = \overline{\beta}_j$ for all $j \neq p$. Indeed, observe that $\nu(w) = \nu(\iota(w))$, and $\iota(w)_1 = i+1$, the argument used in case (a) applies; this is true even when $i = 1$.

Next, we consider the four cases where $i = \Box$.

(e) $v(w) = (\widehat{1} \cdots \overline{2} \cdots)$. In this case $\mathrm{type}(w) = 0$, $D(w) = D(\overline{w})$, $\nu = \overline{\nu}$, $\beta_p = -1$, and $\overline{\beta}_p = 1$. We also have $\beta_j = \overline{\beta}_j$ for all $j \neq p$.

(f) $v(w) = (\overline{2} \cdots \overline{1} \cdots)$. In this case $\mathrm{type}(w) = 2$, $D(w) = D(\overline{w})$, and $\mu = \overline{\mu}$. We deduce that $\overline{\nu}_p = \nu_p - 1$, while $\overline{\nu}_j = \nu_j$ for all $j \neq p$. We must show that $\beta_p = 0$, and hence $\overline{\beta}_p = 2$, while $\overline{\beta}_j = \beta_j$ for all $j \neq p$. Indeed, we have $\iota(v(w)) = (2 \cdots 1 \cdots)$, so the analysis in case (a) applies.

(g) $v(w) = (\cdots \overline{2}\overline{1} \cdots)$, with $|w_1| > 2$. In this case $\mathrm{type}(w)$ and $\mathrm{type}(\overline{w})$ are both positive, $\nu = \overline{\nu}$, and $D(w) = D(\overline{w}) \cup \{(p,p+1)\}$, where $v(w)_{k+p} = \overline{2}$ and $v(w)_{k+p+1} = \overline{1}$, and thus $p = \ell(\mu) = m$. It follows that $\beta_p = -1$, $\beta_{p+1} = 0$, $\overline{\beta}_p = 1$, $\overline{\beta}_{p+1} = 2$, and $\beta_j = \overline{\beta}_j$ for all $j \notin \{p, p+1\}$. We also have $\lambda_m = k + \xi_m + 1$, $\lambda_{m+1} = k + \xi_{m+1} = \lambda_m - 1$.

(h) $v(w) = (2 \cdots \overline{1} \cdots)$. In this case $\mathrm{type}(w) = 1$, $D(w) = D(\overline{w})$, and $\mu = \overline{\mu}$. We deduce that $\overline{\nu}_p = \nu_p - 1$, while $\overline{\nu}_j = \nu_j$ for all $j \neq p$. We must show that $\beta_p = 0$, so that $\overline{\beta}_p = 2$, and $\beta_j = \overline{\beta}_j$ for all $j \neq p$. This is proved as in case (f).

To simplify the notation, set $c_\alpha^\upsilon := {}^\kappa c_\alpha^\upsilon$ and $\widehat{c}_\alpha^\upsilon := {}^\kappa \widehat{c}_\alpha^\upsilon$, for any integer sequences α and υ. We now distinguish the following cases.

Case 1. $\mathrm{type}(w) = \mathrm{type}(\overline{w}) = 0$.

Note that we have $|w_1| = |\overline{w}_1| = 1$, and hence $i \geq 2$ and $\ell(\mu) = \ell(\overline{\mu})$. We must be in one among cases (a), (b), or (d1) above. In cases (a) or (b), it follows from Propositions 1 and 2 and the left Leibnitz rule that for any integer sequence $\alpha = (\alpha_1, \ldots, \alpha_\ell)$, we have

$$\begin{aligned}
\partial_i^y \, \widehat{c}_\alpha^{\beta(w)} &= \widehat{c}_{(\alpha_1,\ldots,\alpha_{p-1})}^{(\beta_1,\ldots,\beta_{p-1})} \left(\partial_i(\widehat{c}_{\alpha_p}^{\beta_p}) \widehat{c}_{(\alpha_{p+1},\ldots,\alpha_\ell)}^{(\beta_{p+1},\ldots,\beta_\ell)} + s_i^y(\widehat{c}_{\alpha_p}^{\beta_p}) \partial_i(\widehat{c}_{(\alpha_{p+1},\ldots,\alpha_\ell)}^{(\beta_{p+1},\ldots,\beta_\ell)}) \right) \\
&= \widehat{c}_{(\alpha_1,\ldots,\alpha_{p-1})}^{(\beta_1,\ldots,\beta_{p-1})} \left(\widehat{c}_{\alpha_p-1}^{\beta_p+1} \widehat{c}_{(\alpha_{p+1},\ldots,\alpha_\ell)}^{(\beta_{p+1},\ldots,\beta_\ell)} + 0 \right) = \widehat{c}_{(\alpha_1,\ldots,\alpha_p-1,\ldots,\alpha_\ell)}^{(\beta_1,\ldots,\beta_p+1,\ldots,\beta_\ell)} = \widehat{c}_{\alpha-\epsilon_p}^{\beta(\overline{w})}.
\end{aligned}$$

Since $\lambda - \epsilon_p = \overline{\lambda}$, it follows that if R is any raising operator, then

$$\partial_i^y(R \star \widehat{c}_\lambda^{\beta(w)}) = \partial_i^y(\overline{c}_{R\lambda}^{\beta(w)}) = \overline{c}_{R\lambda-\epsilon_p}^{\beta(\overline{w})} = R \star \widehat{c}_{\overline{\lambda}}^{\beta(\overline{w})}.$$

As $R^{D(w)} = R^{D(\overline{w})}$, we deduce that

$$\partial_i^y F_w = 2^{-\ell(\mu)}\partial_i^y(R^{D(w)}\star\widehat{c}_\lambda^{\,\beta(w)}) = 2^{-\ell(\overline{\mu})}R^{D(\overline{w})}\star\widehat{c}_{\overline{\lambda}}^{\,\beta(\overline{w})} = F_{\overline{w}}.$$

In case (d1), for any integer sequence $\alpha = (\alpha_1,\ldots,\alpha_\ell)$, we compute that

$$\begin{aligned}\partial_i^y\,\widehat{c}_\alpha^{\,\beta(w)} &= \partial_i^y\,\widehat{c}^{\,(\beta_1,\ldots,-i,\ldots,i,\ldots,\beta_\ell)}_{(\alpha_1,\ldots,\alpha_p,\ldots,\alpha_q,\ldots,\alpha_\ell)}\\ &= \widehat{c}^{\,(\beta_1,\ldots,-i+1,\ldots,i+1,\ldots,\beta_\ell)}_{(\alpha_1,\ldots,\alpha_p-1,\ldots,\alpha_q,\ldots,\alpha_\ell)} + \widehat{c}^{\,(\beta_1,\ldots,-i+1,\ldots,i+1,\ldots,\beta_\ell)}_{(\alpha_1,\ldots,\alpha_p,\ldots,\alpha_q-1,\ldots,\alpha_\ell)} = \widehat{c}^{\,\beta(\overline{w})}_{\alpha-\epsilon_p} + \widehat{c}^{\,\beta(\overline{w})}_{\alpha-\epsilon_q}.\end{aligned}$$

This follows from the left Leibnitz rule, as in the proof of Proposition 1(b). Since $i \geq 2$, we must have $q > \ell(\mu)$. Hence if R is any raising operator, then $q \notin \mathrm{supp}_m(RR_{pq})$, where $m = \ell(\mu)$. As $\lambda - \epsilon_p = \overline{\lambda}$, we deduce that

$$\partial_i^y(R\star\widehat{c}_\lambda^{\,\beta(w)}) = \partial_i^y(\overline{c}^{\,\beta(w)}_{R\lambda}) = \overline{c}^{\,\beta(\overline{w})}_{R\lambda-\epsilon_p} + \overline{c}^{\,\beta(\overline{w})}_{R\lambda-\epsilon_q} = R\star\widehat{c}_{\overline{\lambda}}^{\,\beta(\overline{w})} + RR_{pq}\star\widehat{c}_{\overline{\lambda}}^{\,\beta(\overline{w})}.$$

Since $R^{D(w)} + R^{D(w)}R_{pq} = R^{D(\overline{w})}$, it follows that $\partial_i^y F_w = F_{\overline{w}}$.

Case 2. $\mathrm{type}(w) = 0$ and $\mathrm{type}(\overline{w}) > 0$.

In this case, we have $|w_1| = 1$ and $|\overline{w}_1| > 1$, so $i \in \{\Box, 1\}$. We must be in one of cases (b), (c), or (e) above, hence $D(w) = D(\overline{w})$. We also have $(p,p+1)\notin D(w)$, $\ell(\mu) = \ell(\overline{\mu})+1$, $\beta_p = -1$, $\overline{\beta}_p = 0$ if $i = 1$, and $\overline{\beta}_p = 1$ if $i = \Box$.

Observe that for any integer sequence $\alpha = (\alpha_1,\ldots,\alpha_\ell)$, we have

$$\begin{aligned}\partial_i^y\widehat{c}_\alpha^{\,\beta(w)} &= \widehat{c}^{\,(\beta_1,\ldots,\beta_{p-1})}_{(\alpha_1,\ldots,\alpha_{p-1})}\left(\partial_i^y(\widehat{c}^{\,-1}_{\alpha_p})\widehat{c}^{\,(\beta_{p+1},\ldots,\beta_\ell)}_{(\alpha_{p+1},\ldots,\alpha_\ell)} + s_i^y(\widehat{c}^{\,-1}_{\alpha_p})\partial_i^y(\widehat{c}^{\,(\beta_{p+1},\ldots,\beta_\ell)}_{(\alpha_{p+1},\ldots,\alpha_\ell)})\right)\\ &= \widehat{c}^{\,(\beta_1,\ldots,\beta_{p-1})}_{(\alpha_1,\ldots,\alpha_{p-1})}\partial_i^y(\widehat{c}^{\,-1}_{\alpha_p})\widehat{c}^{\,(\beta_{p+1},\ldots,\beta_\ell)}_{(\alpha_{p+1},\ldots,\alpha_\ell)}.\end{aligned}$$

We now compute using Propositions 1 and 2(a) that

$$\partial_1^y\left({}^r\widehat{c}_q^{\,-1}\right) = \begin{cases}2\left({}^ra^0_{q-1}\right) & \text{if } q \neq r+1\\ 2f_r & \text{if } q = r+1.\end{cases}$$

Propositions 1(a) and 2 give

$$\partial_\Box^y\left({}^r\widehat{c}_q^{\,-1}\right) = \begin{cases}2\left({}^ra^1_{q-1}\right) & \text{if } q \neq r+1\\ 2\widetilde{f}_r^1 & \text{if } q = r+1.\end{cases}$$

Note that the choice of f_r in these equations is specified by formula (26). The rest is straightforward from the definitions, arguing as in Case 1.

Case 3. $\mathrm{type}(w) > 0$ and $\mathrm{type}(\overline{w}) = 0$.

We have $|w_1| > 1$ and $|\overline{w}_1| = 1$, so $i \in \{\Box, 1\}$, and we are in one of cases (a), (d2), (f), or (h) above, hence $D(w) = D(\overline{w})$.

We also have $\beta_p \in \{0,1\}$, $\overline{\beta}_p = 2$, and $\ell(\mu) = \ell(\overline{\mu})$. Recall that ${}^r\widehat{c}^{\,s}_q = {}^rc^s_q$ whenever $q \leq r$, ${}^rb^1_r = {}^rc^1_r - {}^r\widetilde{b}_r$, ${}^r\widetilde{b}^1_r = {}^rc^1_r - {}^rb_r$, and ${}^ra^s_q = {}^rc^s_q - \frac{1}{2}{}^rc_q$. We deduce the calculations

$$\begin{gathered}\partial_\Box^y\left({}^rb_r\right) = \partial_\Box^y\left({}^r\widetilde{b}_r\right) = \partial_1^y\left({}^rb^1_r\right) = \partial_1^y\left({}^r\widetilde{b}^1_r\right) = {}^rc^2_{r-1}\\ \partial_\Box^y\left({}^ra^0_q\right) = \partial_1^y\left({}^ra^1_q\right) = {}^rc^2_{q-1}.\end{gathered}$$

As in the previous cases, it follows that $\partial_i^y F_w = F_{\overline{w}}$.

Case 4. $\mathrm{type}(w) = \mathrm{type}(\overline{w}) > 0$.

We have $|w_1| > 1$ and $|\overline{w}_1| > 1$. If $i \geq 2$, then we must be in one of cases (a), (b), (c), or (d1) above, and the result is proved by arguing as in Case 1. It remains to study (i) case (d1) with $v(w) = (\cdots \overline{2}1 \cdots)$ and $i = 1$, or (ii) case (g) with $v(w) = (\cdots \overline{21} \cdots)$ and $i = \Box$. In both of these subcases, we have $p = m$, $\ell(\mu) = \ell(\overline{\mu})+1$, $D(w) = D(\overline{w}) \cup \{(m, m+1)\}$, $\beta_m(w) = -1$, $\overline{\beta}_{m+1} = 2$, and $\beta_j = \overline{\beta}_j$ for all $j \notin \{m, m+1\}$. In subcase (i), we have $\beta_{m+1} = 1$ and $\overline{\beta}_m = 0$, while in subcase (ii), we have $\beta_{m+1} = 0$ and $\overline{\beta}_m = 1$. Finally, we have $\lambda_m = k + 1 + \xi_m$ and $\lambda_{m+1} = k + \xi_{m+1} = \lambda_m - 1$, since the assumption $\nu_m = \nu_{m+1}$ implies that $\xi_m = \xi_{m+1}$.

The rest of the argument now follows the proof of [T4, Prop. 5]. We first assume that λ has length $m + 1$, let $r := \lambda_{m+1} = \lambda_m - 1$, and use [T4, Prop. 3] and the key relations

$$f_r\widetilde{f}_r + 2\sum_{j=1}^{r}(-1)^j({}^ra^0_{r+j}\,{}^ra^0_{r-j}) = \widetilde{f}^1_r f^1_r + 2\sum_{j=1}^{r}(-1)^j({}^ra^1_{r+j}\,{}^ra^1_{r-j}) = 0$$

in $\Gamma'[X, Y]$, which are easily deduced from the relations (8), as in op. cit. Finally, if $\ell(\lambda) > m+1$, induction as in the proof of [BKT2, Lemma 1.3] and similar arguments show that the contribution of all the residual terms in that appear with a negative sign in [T4, Prop. 3] vanishes. $\Box$

Definition 12. An element $w \in \widetilde{W}_\infty$ is called *proper* if (i) $|w_1| \leq 2$, or (ii) $|w_1| > 2$ and $w_j = 2$ implies $j > 2 > w_{j-1}$.

Example 8. Let $n = 3$ and $w = (3, 2, 1)$. Then $\text{type}(w) = 1$, $\lambda = \nu = (2, 1)$, $\ell = 2$, $\mu = 0$, $m = 0$, $k = 1$, $\beta(w) = (1, 2)$, $D(w) = \emptyset$, and w is not proper. We have ${}^1c^2_0 = 1$, while

$$\begin{gathered}{}^2b^1_2 = {}^2b_2 + {}^2c_1\, h^1_1(-Y) + h^1_2(-Y)\\ = \frac{1}{2}(c_2 + c_1e^2_1(X)) + e^2_2(X) + (c_1 + e^2_1(X))h^1_1(-Y) + h^1_2(-Y),\\ {}^1c^2_1 = c_1 + e^1_1(X) + h^2_1(-Y),\\ {}^2c^1_3 = c_3 + c_2(e^2_1(X) + h^1_1(-Y)) + c_1(e^2_2(X) + e^2_1(X)h^1_1(-Y) + h^1_2(-Y))\\ +(e^2_2(X)h^1_1(-Y) + e^2_1(X)h^1_2(-Y) + h^1_3(-Y)).\end{gathered}$$

One checks using the table of [IMN, Sec. 13] that

$$(1 - R_{12}) \star {}^{(2,1)}\widehat{c}^{(1,2)}_{(2,1)} = {}^2b^1_2\,{}^1c^2_1 - \frac{1}{2}{}^2c^1_3\,{}^1c^2_0 \neq \mathfrak{D}_{321}.$$

Now consider $w' := \iota(w) = (\overline{3}, 2, \overline{1})$. Then $\text{type}(w') = 2$, $\lambda = \nu = (2, 1)$, $\ell = 2$, $\mu = 0$, $m = 0$, $k = 1$, $\beta(w') = (0, 2)$, $D(w') = \emptyset$, and w' is not proper. We have

$$\begin{gathered}{}^2b^0_2 = {}^2b_2 = \frac{1}{2}(c_2 + c_1e^2_1(X)) + e^2_2(X),\\ {}^2c^0_3 = c_3 + c_2e^2_1(X) + c_1e^2_2(X).\end{gathered}$$

Using the table of [IMN, Sec. 13], we observe that

$$(1 - R_{12}) \star {}^{(2,1)}\widehat{c}^{(0,2)}_{(2,1)} = {}^2b^0_2\,{}^1c^2_1 - \frac{1}{2}{}^2c^0_3\,{}^1c^2_0 \neq \mathfrak{D}_{\overline{3}2\overline{1}}.$$

Notice that if w (respectively $\overline{w}$) is increasing up to $k \geq 1$ and proper (respectively not proper) with truncated A-code equal to a fixed partition C, such that

$\ell(w) = \ell(\overline{w}) + 1$ and $s_i v(w) = v(\overline{w})$, then $i = \Box$ or $i = 1$. Moreover, referring to equation (27), we can have $\partial_i^y F_w \neq F_{\overline{w}}$, as Example 8 shows, with $k = 1$, $w = (3, \overline{1}, \overline{2})$, and $\overline{w} = (3, 2, 1)$.

Lemma 13. *Fix an integer $k \geq 1$. We say that an element of $\widetilde{W}_n$ is* valid *if w is increasing up to k, has k-truncated A-code a partition C, and is proper. Let $w(C)$ be the longest valid element in $\widetilde{W}_n$, which has type 0. If $\overline{w}$ is valid and $\overline{w} \neq w(C)$, then there exists a valid $w \in \widetilde{W}_n$ such that $\ell(w) = \ell(\overline{w}) + 1$ and $v(\overline{w}) = s_i v(w)$ for some $i \in \mathbb{N}_\Box$.*

Proof. We distinguish the following cases for $\overline{w}$:

Case 1: $|\overline{w}_1| = 1$. Let w be any element with the same truncated A-code C such that $\ell(w) = \ell(\overline{w}) + 1$ and $v(\overline{w}) = s_i v(w)$ for some $i \in \mathbb{N}_\Box$.

Case 2: $|\overline{w}_1| = 2$. If $\overline{w} = (\widehat{2} \cdots 1 \cdots)$, then let $w := s_\Box \overline{w}$, while if $\overline{w} = (\widehat{2} \cdots \overline{1} \cdots)$, then let $w := s_1 \overline{w}$.

Case 3: $|\overline{w}_1| > 2$. If $\overline{w}_j = 2$ then $j > 2$ and $\overline{w}_{j-1} < 2$. Since the A-code C is a partition, the sequence $(\overline{w}_{k+1}, \ldots, \overline{w}_n)$ is 132-avoiding. We deduce that if $\overline{w}_i = \widehat{1}$, then $i < j$. If $\overline{w} = (\cdots \overline{1} \cdots 2 \cdots)$, then let $w := s_1 \overline{w}$, while if $\overline{w} = (\cdots 1 \cdots 2 \cdots)$, then let $w := s_\Box \overline{w}$.

In all three cases, the element w satisfies the required conditions. ∎

Proposition 9. *Suppose that $w \in \widetilde{W}_n$ is an element with primary index k such that $(w_1, \ldots, w_k) = (\widehat{1}, 2, \ldots, k)$, $w_{k+j} < 0$ for $1 \leq j \leq n-k$, and ${}^k\gamma(w)$ is a partition. Then we have*

$$\mathfrak{D}_w = 2^{k-n}\, R^{D(w)} \star {}^{\nu(w)}\widehat{c}_{\lambda(w)}^{(1-n,2-n,\ldots,-k)}. \tag{28}$$

Proof. The proof of (28) is by descending induction on $\ell(w)$. One knows from [T5, §4.4] that (28) is true for the longest element $w_0^{(k,n)} := (\widehat{1}, 2, \ldots, k, -k-1, \ldots, -n)$, which has shape $(n+k-1, \ldots, 2k) + \delta_{n-k-1}$ of type 0.

Suppose that $w \neq w_0^{(k,n)}$ satisfies the conditions of the proposition, and the shape of w equals $(n+k-1, \ldots, 2k) + \rho(w)$ (of type 0). Then $\rho \subset \delta_{n-k-1}$ and $\rho \neq \delta_{n-k-1}$. Let $r \geq 1$ be the largest integer such that $\rho_i = n-k-i$ for $i \in [1, r]$, and let $j := \rho_{r+1} + 1 \leq n-k-r-1$. Then ws_{k+j} is of length $\ell(w)+1$ and satisfies the same conditions, $\nu(ws_{k+j}) = \nu(w) + \epsilon_{r+1}$, and $\lambda(ws_{k+j}) = \lambda(w) + \epsilon_{r+1}$. Using Proposition 1(a), for any integer sequence $\alpha = (\alpha_1, \ldots, \alpha_n)$, we have

$$\partial_{k+j}^x({}^{\nu(ws_{k+j})}\widehat{c}_\alpha^{(1-n,\ldots,-k)}) = {}^{\nu(w)}\widehat{c}_{\alpha-\epsilon_{r+1}}^{(1-n,\ldots,-k)}.$$

By induction, we deduce that

$$\begin{aligned}\mathfrak{D}_w = \partial_{k+j}^x(\mathfrak{D}_{ws_{k+j}}) &= 2^{k-n}\, \partial_{k+j}^x(R^\infty \star {}^{\nu(ws_{k+j})}\widehat{c}_{\lambda(ws_{k+j})}^{(1-n,\ldots,-k)}) \\ &= 2^{k-n}\, R^\infty \star {}^{\nu(w)}\widehat{c}_{\lambda(w)}^{(1-n,\ldots,-k)},\end{aligned}$$

proving the proposition. ∎

Corollary 2. *Suppose that $w \in \widetilde{W}_n$ is increasing up to $k \geq 1$, proper, and the k-truncated A-code ${}^k\gamma$ is a partition. Let $k^{n-k} + \xi(w) = (k+\xi_1, \ldots, k+\xi_{n-k})$. Then we have*

$$\mathfrak{D}_w = 2^{-\ell(\mu(w))}\, R^{D(w)} \star {}^{k^{n-k}+\xi(w)}\widehat{c}_{\lambda(w)}^{\beta(w)}. \tag{29}$$

Proof. If $w = (\widehat{1}, 2, \ldots, k, w_{k+1}, \ldots, w_n)$ with $w_{k+j} < 0$ for all $j \in [1, n-k]$, then (29) follows from Proposition 9. In this case, $v(w) = (\widehat{1}, 2, \ldots, k, -n, \ldots, -k-1)$ is the longest k-Grassmannian element in $\widetilde{W}_n$. We deduce the result in the general case from Proposition 8 and Lemma 13, using the fact that the k-Grassmannian elements of $\widetilde{W}_n$ form an ideal for the left weak Bruhat order. Indeed, the hypotheses required in Proposition 8 are satisfied, as long as w and $\overline{w}$ are proper. The key point is to show that if $i \in \{\Box, 1\}$ and $|w_1| > 2$, then $\xi_m = \xi_{m+1}$, which implies that $\nu_m = \nu_{m+1}$, and hence $\kappa_m = \kappa_{m+1}$. For if not, then $\xi_m > \xi_{m+1}$, so there exists a $j > k$ such that $\gamma_j = m$. As ${}^k\gamma$ is a partition, $(w_{k+1}, \ldots, w_n)$ is a 132-avoiding sequence. It follows that $w_j = \widehat{1}$, and furthermore $j = 2$, or $j > 2$ and $w_{j-1} > w_j$. We conclude that $\overline{w}_j = 2$ and $\overline{w}$ is not proper, completing the proof. $\Box$

Definition 13. Let $k \geq 1$ be the primary index of $w \in \widetilde{W}_n$, and list the entries $w_{k+1}, \ldots, w_n$ in increasing order:

$$u_1 < \cdots < u_{m'} < 0 < u_{m'+1} < \cdots < u_{n-k},$$

where $m' \in \{m, m+1\}$. We say that a simple transposition s_i for $i \geq 2$ is *w-negative* (respectively, *w-positive*) if $\{i, i+1\}$ is a subset of $\{-u_1, \ldots, -u_m\}$ (respectively, of $\{u_{m'+1}, \ldots, u_{n-k}\}$). Let σ^- (respectively, σ^+) be the longest subword of $s_{n-1} \cdots s_2$ (respectively, of $s_2 \cdots s_{n-1}$) consisting of w-negative (respectively, w-positive) simple transpositions. A *modification* of $w \in \widetilde{W}_n$ is an element ωw, where $\omega \in S_n$ is such that $\ell(\omega w) = \ell(w) - \ell(\omega)$, and ω has a reduced decomposition of the form $R_1 \cdots R_{n-2}$ where each R_j is a (possibly empty) subword of $\sigma^- \sigma^+$ and all simple reflections in R_p are also contained in R_{p+1}, for each $p < n-2$.

Definition 14. Suppose that $w \in \widetilde{W}_n$ has primary index $k \geq 1$ and A-code γ. We say that w is *leading* if w is proper and $(\gamma_{k+1}, \gamma_{k+2}, \ldots, \gamma_n)$ is a partition. We say that w is *amenable* if w is a modification of a leading element.

Remark 5. The proper element $w \in \widetilde{W}_n$ of type 0 or 1 is leading if and only if the A-code of the extended sequence $(0, w_1, w_2, \ldots, w_n)$ is unimodal. Indeed, if $\text{type}(w) = 0$ and $w_1 = \overline{1}$, then this is ensured since there is more than one negative entry in w. If $\text{type}(w) = 2$, then w is leading if and only if it is proper and the A-code of the extended sequence $(0, w'_1, w'_2, \ldots, w'_n)$ is unimodal, where $w' := \iota(w)$.

In the following we will assume that w has primary index $k \geq 1$ and the partition ξ is specified as in Definition 10. Let $\psi := (\gamma_k, \ldots, \gamma_1)$, $\phi := \psi'$, $\ell := \ell(\lambda)$ and $m := \ell(\mu)$. We then have

$$\lambda = \phi + \xi + \mu \tag{30}$$

and $\lambda_1 > \cdots > \lambda_m > \lambda_{m+1} \geq \cdots \geq \lambda_\ell$.

Definition 15. Say that $\mathfrak{q} \in [1, \ell]$ is a *critical index* if $\beta_{\mathfrak{q}+1} > \beta_{\mathfrak{q}}+1$, or $(\beta_{\mathfrak{q}}, \beta_{\mathfrak{q}+1}) = (1, 2)$, or if $\lambda_{\mathfrak{q}} > \lambda_{\mathfrak{q}+1}+1$ (respectively, $\lambda_{\mathfrak{q}} > \lambda_{\mathfrak{q}+1}$) and $\mathfrak{q} \leq m$ (respectively, $\mathfrak{q} > m$). Define two sequences $\mathfrak{f} = \mathfrak{f}(w)$ and $\mathfrak{g} = \mathfrak{g}(w)$ of length ℓ as follows. For $1 \leq j \leq \ell$, set

$$\mathfrak{f}_j := k + \max(i \mid \gamma_{k+i} \geq j)$$

and let

$$\mathfrak{g}_j := \mathfrak{f}_{\mathfrak{q}} + \beta_{\mathfrak{q}} - \xi_{\mathfrak{q}} - k,$$

where $\mathfrak{q}$ is the least critical index such that $\mathfrak{q} \geq j$. We call $\mathfrak{f}$ the *right flag* of w, and $\mathfrak{g}$ the *left flag* of w.

We will show that for any amenable element w, $\mathfrak{f}$ is a weakly decreasing sequence, which consists of right descents of w, unless $\mathfrak{f}_j = 1$ and $w_1 < -|w_2|$, when $\mathfrak{f}_j$ is a right descent of $\iota(w)$. Moreover, $\mathfrak{g}$ is a weakly increasing sequence, whose absolute values consist of left descents of w, unless $\mathfrak{g}_j = 0$ and $w = (\cdots \overline{1} \cdots 2 \cdots)$, or $\mathfrak{g}_j = 1$ and $w = (\cdots 1 \cdots 2 \cdots)$.

Lemma 14. (a) *If $\beta_{s+1} > \beta_s + 1$ and $(\beta_s, \beta_{s+1}) \neq (0,2)$, then $|\beta_s|$ is a left descent of w.*

(b) *If $s \leq m$ then $\phi_s = k$. If $s > m$ and $\beta_{s+1} = \beta_s + 1$, then $\phi_s = \phi_{s+1}$.*

(c) *If $\beta_s = 0$ or $\beta_s = 1$, then $s = m+1$, $\mathrm{type}(w) > 0$, and $\phi_{m+1} = k$. If $\beta_s = 0$ then $\Box$ is a left descent of w, unless $w = (\cdots \overline{1} \cdots 2 \cdots)$. If $\beta_s = 1$ then 1 is a left descent of w, unless $w = (\cdots 1 \cdots 2 \cdots)$.*

Proof. Let $i := |\beta_s|$, and suppose that $1 \leq s \leq m$. If $\beta_{s+1} > \beta_s + 1 \neq 1$ then $i \geq 1$ is a left descent of w. Indeed, if $w_1 = -i$ then i is a left descent of w, while if $w_1 \neq -i$, then $w^{-1}(i) > 0$ and $w^{-1}(i+1) < 0$, so this is clear. If $\mathrm{type}(w) \neq 2$, then $w_j \geq -1$ for all $j \in [1,k]$, and hence $\psi_j \geq m$ for all $j \in [1,k]$, and so $\phi_s = k$.

Next suppose that $s > m$. If $\beta_{s+1} > \beta_s + 1 \neq 1$, then we have $w^{-1}(i+1) < 0$ or $w_j = i+1$ for some $j \in [1,k]$. In either case, it is clear that i is a left descent of w. Assume that $\beta_{s+1} = \beta_s + 1$, so that $\beta_s = u_s \geq 1$. If $\phi_s > \phi_{s+1}$, there must exist $j \in [1,k]$ such that $\gamma_j = s$, that is, $\#\{r > k \mid w_r < |w_j|\} = s$. We deduce that

$$\{w_r \mid r > k \ \text{ and } \ w_r < |w_j|\} = \{u_1, \ldots, u_s\},$$

which is a contradiction, since $u_s < |w_j| \Rightarrow u_{s+1} = u_s + 1 < |w_j|$, for any $j \in [1,k]$. This completes the proof of (a) and (b) except in the case $\beta_s = 0$, which is dealt with below.

If $\beta_s = 0$ or $\beta_s = 1$, then clearly $s = m+1$ and $\mathrm{type}(w) > 0$. We have $\psi_j \geq m+1$ for all $j \in [1,k]$, and hence $\phi_{m+1} = k$. If $\beta_s = 0$, then since $w_{m+1} = \overline{1}$ we see that $\Box$ is a left descent of w, unless $w = (\cdots \overline{1} \cdots 2 \cdots)$. Finally, if $\beta_s = 1$, then since $w_{m+1} = 1$, it is clear that 1 is a left descent of w, unless $w = (\cdots 1 \cdots 2 \cdots)$. □

Proposition 10. *Suppose that $\widehat{w} \in \widetilde{W}_n$ is leading with primary index $k \geq 1$, let $\widehat{\lambda} := \lambda(\widehat{w})$, and $\widehat{\xi} := \xi(\widehat{w})$. Let $w = \omega\widehat{w}$ be a modification of $\widehat{w}$, and set $\gamma := \gamma(w)$, $\lambda := \lambda(w)$, $\beta := \beta(w)$, and $\xi := \xi(w)$. Then the sequence $\beta + \widehat{\lambda} - \lambda$ is weakly increasing, and*

$$\mathfrak{D}_w = 2^{-\ell(\mu(w))}\, R^{D(w)} \star {}^{k^{n-k}+\widehat{\xi}}\widehat{c}^{\,\beta(w)+\widehat{\xi}-\xi}_{\lambda(w)} = 2^{-\ell(\mu(w))}\, R^{D(w)} \star {}^{k^{n-k}+\widehat{\xi}}\widehat{c}^{\,\beta+\widehat{\lambda}-\lambda}_{\lambda}.$$

If $\mathfrak{q} \in [1,\ell]$ is a critical index of w, then $k + \widehat{\xi}_{\mathfrak{q}}$ is a right descent of w, unless $k+\widehat{\xi}_{\mathfrak{q}} = 1$ and $w_1 < -|w_2|$, when $k+\widehat{\xi}_{\mathfrak{q}}$ is a right descent of $\iota(w)$. The absolute value of $g_{\mathfrak{q}} := \beta_{\mathfrak{q}} + \widehat{\xi}_{\mathfrak{q}} - \xi_{\mathfrak{q}}$ is a left descent of w, unless $g_{\mathfrak{q}} = 0$ and $w = (\cdots \overline{1} \cdots 2 \cdots)$, or $g_{\mathfrak{q}} = 1$ and $w = (\cdots 1 \cdots 2 \cdots)$. Moreover, we have $\widehat{\xi}_{\mathfrak{q}} = \max(i \mid \gamma_{k+i} \geq \mathfrak{q})$.

Proof. Suppose that the truncated A-code of $\widehat{w}$ is

$${}^k\widehat{\gamma} = (p_1^{n_1}, \ldots, p_t^{n_t})$$

for some parts $p_1 > p_2 > \cdots > p_t > 0$, and we let $d_j := n_1 + \cdots + n_j$ for $j \in [1,t]$. Then we have

$$\widehat{\xi} = (d_t^{p_t}, d_{t-1}^{p_{t-1}-p_t}, \ldots, d_1^{p_1-p_2})$$

and it follows that

$$\widehat{w}_{k+1} = u_{p_1+1},\ \widehat{w}_{k+d_1+1} = u_{p_2+1},\ \ldots,\ \widehat{w}_{k+d_{t-1}+1} = u_{p_t+1}$$

and $\widehat{w}_j < \widehat{w}_{j+1}$ for all $j \notin \{k, k+d_1, \ldots, k+d_t\}$. Recall that 1 is a right descent of w if and only if $w_1 > w_2$, and $\Box$ is a right descent of w if and only if $w_1 < -w_2$. Hence, if the primary index k equals 1, then k is not a right descent of w if and only if and $w_1 < -|w_2|$, in which case $\mathrm{type}(w) = 2$ and k is a right descent of $\iota(w)$. We deduce that the set of components of $k^{n-k} + \widehat{\xi}$ coincides with the set of all positive right descents of $\widehat{w}$, or of $\iota(\widehat{w})$ if $\widehat{w}_1 < -|\widehat{w}_2|$.

If $\mathfrak{q} \in [1, \ell]$ is a critical index, we have shown that $k + \widehat{\xi}_{\mathfrak{q}}$ is a right descent of $\widehat{w}$, except in the case when $k + \widehat{\xi}_{\mathfrak{q}} = 1$ and $\widehat{w}_1 < -|\widehat{w}_2|$, when $k + \widehat{\xi}_{\mathfrak{q}}$ is a right descent of $\iota(\widehat{w})$. We claim that $i := |g_{\mathfrak{q}}| = |\beta_{\mathfrak{q}}|$ is a left descent of $\widehat{w}$, unless $i = 0$ and $\widehat{w} = (\cdots \overline{1} \cdots 2 \cdots)$, or $i = 1$ and $\widehat{w} = (\cdots 1 \cdots 2 \cdots)$. By Lemma 14, we may assume that $\beta_{\mathfrak{q}} \neq 0$ and $\beta_{\mathfrak{q}+1} = \beta_{\mathfrak{q}} + 1$.

We first prove that $\mathfrak{q} \neq m$. Indeed, $\beta_{m+1} = \beta_m + 1$ implies that $\beta_m = -1$ and $\beta_{m+1} = 0$, so in particular $|\widehat{w}_1| > 2$. Since $\widehat{w}$ is proper and ${}^k\widehat{\gamma}$ is a partition, it follows that there is no $j \geq 1$ such that $\widehat{\gamma}_{k+j} = m$. This implies that $\xi_m = \xi_{m+1}$, and since $\phi_m = \phi_{m+1} = k$ by Lemma 14(b), we deduce that $\lambda_m = \lambda_{m+1} + 1$, which contradicts the fact that $\mathfrak{q}$ is a critical index.

Suppose that $\mathfrak{q} < m$ and let $\widehat{\mu} := \mu(\widehat{w})$. Then we have $\widehat{\lambda}_{\mathfrak{q}} > \widehat{\lambda}_{\mathfrak{q}+1} + 1$ and $\widehat{\mu}_{\mathfrak{q}} = \widehat{\mu}_{\mathfrak{q}+1} + 1$, so (30) gives $\widehat{\xi}_{\mathfrak{q}} > \widehat{\xi}_{\mathfrak{q}+1}$. We therefore have $\mathfrak{q} = p_j$ for some $\mathfrak{q} \in [1, t]$, and hence $i = \widehat{\mu}_{p_j} - 1 = \widehat{\mu}_{p_j+1} = -u_{p_j+1} = -\widehat{w}_{k+d_{j-1}+1}$. Since we have

$$\widehat{w}_{k+1} > \widehat{w}_{k+d_1+1} > \cdots > \widehat{w}_{k+d_{j-1}+1} = -i,$$

and the sequence $(\widehat{w}_{k+1}, \ldots, \widehat{w}_n)$ is 132-avoiding, we conclude that $\widehat{w}^{-1}(-i) = k + d_{j-1} + 1 < \widehat{w}^{-1}(-i-1)$, as desired.

Suppose next that $\mathfrak{q} > m$. Then we have $\widehat{\lambda}_{\mathfrak{q}} > \widehat{\lambda}_{\mathfrak{q}+1}$, so Lemma 14(b) and equation (30) imply that $\widehat{\xi}_{\mathfrak{q}} > \widehat{\xi}_{\mathfrak{q}+1}$. We deduce that $\mathfrak{q} = p_j$ for some j, hence $i + 1 = u_{p_j+1}$ and the claim follows.

According to Corollary 2, we have

$$\mathfrak{D}_{\widehat{w}} = 2^{-\ell(\mu(\widehat{w}))}\, R^{D(\widehat{w})} \star {}^{k^{n-k}+\widehat{\xi}}\widehat{c}_{\lambda(\widehat{w})}^{\beta(\widehat{w})}, \tag{31}$$

so the proposition holds for leading elements. Suppose next that $w := \omega\widehat{w}$ is a modification of $\widehat{w}$. Then repeated application of (9), Propositions 1(a), 2(a), and the left Leibnitz rule in equation (31) give

$$\mathfrak{D}_w = 2^{-\ell(\mu(w))}\, R^{D(w)} \star {}^{k^{n-k}+\widehat{\xi}}\widehat{c}_{\lambda(w)}^{\beta(w)+\widehat{\xi}-\xi}.$$

It remains to check the last assertion, about the left and right descents of w. This is done exactly as in the proof of Proposition 7. $\Box$

Theorem 6. *For any amenable element $w \in \widetilde{W}_\infty$, we have*

$$\mathfrak{D}_w = 2^{-\ell(\mu(w))}\, R^{D(w)} \star {}^{\mathfrak{f}(w)}\widehat{c}_{\lambda(w)}^{\mathfrak{g}(w)} \tag{32}$$

in $\Gamma'[X, Y]$.

Proof. We may assume that we are in the situation of Proposition 10, so that $w = \omega\widehat{w}$, with $\widehat{\lambda} = \lambda(\widehat{w})$ and $\lambda = \lambda(w)$. Suppose that $j \in [1, \ell]$ and let $\mathfrak{q}$ be the least critical index of w such that $\mathfrak{q} \geq j$. Then we have $\lambda_j = \lambda_{j+1} = \cdots = \lambda_{\mathfrak{q}}$, if $\mathfrak{q} > m$, and $\lambda_j = \lambda_{j+1} + 1 = \cdots = \lambda_{\mathfrak{q}} + (\mathfrak{q} - j)$, if $\mathfrak{q} \leq m$. Moreover, in either case, we have $\xi_j = \cdots = \xi_{\mathfrak{q}}$, and the values $\beta_j, \ldots, \beta_{\mathfrak{q}}$ are consecutive integers. As the sequence $g := \beta + \widehat{\xi} - \xi$ is weakly increasing, we deduce that for any $r \in [j, \mathfrak{q}-1]$, either (i) $\widehat{\xi}_r = \widehat{\xi}_{r+1}$ and $g_r = g_{r+1} - 1$, or (ii) $\widehat{\xi}_r = \widehat{\xi}_{r+1} + 1$ and $g_r = g_{r+1}$. Equation (32) follows from this and induction on $\mathfrak{q} - j$, by using Lemmas 7(b) and 8(b) in Proposition 10. The required conditions on $D(w)$ in these two lemmas and the corresponding relations (5) and (8) are all easily checked. □

6.2. **Flagged eta polynomials.** In this section, we define a family of polynomials H_w indexed by amenable elements $w \in \widetilde{W}_\infty$ that generalize the double eta polynomials of [T4]. As in Section 5.2, the polynomial H_w represents an equivariant Schubert class in the T-equivariant cohomology ring of the even orthogonal partial flag variety associated to the right flag $\mathfrak{f}(w)$.

For every $k \geq 1$, let ${}^k\mathfrak{b} = ({}^k\widetilde{\mathfrak{b}}_k, {}^k\mathfrak{b}_1, {}^k\mathfrak{b}_2, \ldots)$ and ${}^k\mathfrak{c} = ({}^k\mathfrak{c}_1, {}^k\mathfrak{c}_2, \ldots)$ be families of commuting variables, set ${}^k\mathfrak{b}_0 = {}^k\mathfrak{c}_0 = 1$ and ${}^k\mathfrak{b}_p = {}^k\mathfrak{c}_p = 0$ for each $p < 0$, and let $t = (t_1, t_2, \ldots)$. These variables are related by the equations

$${}^k\mathfrak{c}_p = \begin{cases} {}^k\mathfrak{b}_p & \text{if } p < k, \\ {}^k\mathfrak{b}_k + {}^k\widetilde{\mathfrak{b}}_k & \text{if } p = k, \\ 2\left({}^k\mathfrak{b}_p\right) & \text{if } p > k. \end{cases}$$

For any $p, r \in \mathbb{Z}$ and for $s \in \{0,1\}$, define the polynomials ${}^k\mathfrak{c}_p^r$ and ${}^k\mathfrak{a}_p^s$ by

$${}^k\mathfrak{c}_p^r := \sum_{j=0}^{p} {}^k\mathfrak{c}_{p-j}\, h_j^r(-t) \quad \text{and} \quad {}^k\mathfrak{a}_p^s := \frac{1}{2}\left({}^k\mathfrak{c}_p\right) + \sum_{j=1}^{p} {}^k\mathfrak{c}_{p-j}\, h_j^s(-t).$$

Moreover, define

$${}^k\mathfrak{b}_k^s := {}^k\mathfrak{b}_k + \sum_{j=1}^{k} {}^k\mathfrak{c}_{k-j}\, h_j^s(-t) \quad \text{and} \quad {}^k\widetilde{\mathfrak{b}}_k^s := {}^k\widetilde{\mathfrak{b}}_k + \sum_{j=1}^{k} {}^k\mathfrak{c}_{k-j}\, h_j^s(-t).$$

For any integer sequences α, ρ, κ with $\kappa_i \geq 1$ for each i, let

$${}^\kappa\widehat{\mathfrak{c}}_\alpha^\rho := {}^{\kappa_1}\widehat{\mathfrak{c}}_{\alpha_1}^{\rho_1}\, {}^{\kappa_2}\widehat{\mathfrak{c}}_{\alpha_2}^{\rho_2} \cdots$$

where, for each $i \geq 1$,

$${}^{\kappa_i}\widehat{\mathfrak{c}}_{\alpha_i}^{\rho_i} := {}^{\kappa_i}\mathfrak{c}_{\alpha_i}^{\rho_i} + \begin{cases} (2({}^{\kappa_i}\widetilde{\mathfrak{b}}_{\kappa_i}) - {}^{\kappa_i}\mathfrak{c}_{\kappa_i}) e_{\alpha_i-\kappa_i}^{\alpha_i-\kappa_i}(-t) & \text{if } \rho_i = \kappa_i - \alpha_i < 0 \text{ and } i \text{ is odd}, \\ (2({}^{\kappa_i}\mathfrak{b}_{\kappa_i}) - {}^{\kappa_i}\mathfrak{c}_{\kappa_i}) e_{\alpha_i-\kappa_i}^{\alpha_i-\kappa_i}(-t) & \text{if } \rho_i = \kappa_i - \alpha_i < 0 \text{ and } i \text{ is even}, \\ 0 & \text{otherwise}. \end{cases}$$

If $w \in \widetilde{W}_n$ is amenable with left flag $\mathfrak{f}(w)$ and right flag $\mathfrak{g}(w)$, the *flagged double eta polynomial* $H_w(\mathfrak{c}\,|\,t)$ is defined by

$$H_w(\mathfrak{c}\,|\,t) := 2^{-\ell(\mu(w))}\, R^{D(w)} \star {}^{\mathfrak{f}(w)}\widehat{\mathfrak{c}}_{\lambda(w)}^{\mathfrak{g}(w)}, \tag{33}$$

where the action $\star$ of the raising operator expression $R^{D(w)}$ is as in Definition 11. The *flagged single eta polynomial* is given by $H_w(\mathfrak{c}) := H_w(\mathfrak{c}\,|\,0)$. If w is a leading element, then (33) can be written in the 'factorial' form

$$H_w(\mathfrak{c}\,|\,t) = 2^{-\ell(\mu(w))}\, R^{D(w)} \star {}^{\mathfrak{f}(w)}\widehat{\mathfrak{c}}_{\lambda(w)}^{\beta(w)}.$$

When w is a k-Grassmannian element, the above formulas specialize to the double eta polynomial $H_\lambda(\mathfrak{c}\,|\,t)$ found in [T4]; here λ is the typed k-strict partition corresponding to w. Moreover, the single eta polynomial $H_\lambda(\mathfrak{c})$ agrees with that introduced in [BKT3]; see also [T2, T3].

6.3. **Orthogonal degeneracy loci.**

6.3.1. *Odd orthogonal loci.* Let $E \to \mathfrak{X}$ be a vector bundle of rank $2n+1$ on a smooth complex algebraic variety $\mathfrak{X}$. Assume that E is an *orthogonal* bundle, i.e. E is equipped with an everywhere nondegenerate symmetric form $E \otimes E \to \mathbb{C}$. Let $w \in W_n$ be amenable of shape λ, and let $\mathfrak{f}$ and $\mathfrak{g}$ be the left and right flags of w, respectively. Consider two complete flags of subbundles of E

$$0 \subset E_1 \subset \cdots \subset E_{2n+1} = E \quad \text{and} \quad 0 \subset F_1 \subset \cdots \subset F_{2n+1} = E$$

with $\operatorname{rank} E_r = \operatorname{rank} F_r = r$ for each r, while $E_{n+s} = E_{n+1-s}^{\perp}$ and $F_{n+s} = F_{n+1-s}^{\perp}$ for $1 \leq s \leq n$.

There is a group monomorphism $\zeta' : W_n \hookrightarrow S_{2n+1}$ with image

$$\zeta'(W_n) = \{\, \varpi \in S_{2n+1} \mid \varpi_i + \varpi_{2n+2-i} = 2n+2, \quad \text{for all } i \,\}.$$

The map ζ is determined by setting, for each $w = (w_1, \ldots, w_n) \in W_n$ and $1 \leq i \leq n$,

$$\zeta'(w)_i := \begin{cases} n+1-w_{n+1-i} & \text{if } w_{n+1-i} \text{ is unbarred,} \\ n+1+\overline{w}_{n+1-i} & \text{otherwise.} \end{cases}$$

Define the *degeneracy locus* $\mathfrak{X}_w \subset \mathfrak{X}$ as the locus of $x \in \mathfrak{X}$ such that

$$\dim(E_r(x) \cap F_s(x)) \geq \#\{\, i \leq r \mid \zeta'(w)_i > 2n+1-s \,\} \text{ for } 1 \leq r \leq n,\ 1 \leq s \leq 2n.$$

As in the symplectic case, we assume that $\mathfrak{X}_w$ has pure codimension $\ell(w)$ in $\mathfrak{X}$, and give a formula for the class $[\mathfrak{X}_w]$ in $\mathrm{H}^{2\ell(w)}(\mathfrak{X})$.

Theorem 7. *For any amenable element $w \in W_n$, we have*

$$\begin{aligned} [\mathfrak{X}_w] &= 2^{-\ell(\mu(w))}\, \Theta_w(E - E_{n-\mathfrak{f}} - F_{n+1+\mathfrak{g}}) \\ &= 2^{-\ell(\mu(w))}\, R^{D(w)}\, c_\lambda(E - E_{n-\mathfrak{f}} - F_{n+1+\mathfrak{g}}) \end{aligned}$$

in the cohomology ring $\mathrm{H}^*(\mathfrak{X})$.

Theorem 7 is derived from equation (23) in the same way as Theorem 5, using the type B geometrization map of [IMN, Sec. 10]; compare with [T3, Sec. 6.3.1].

6.3.2. *Even orthogonal loci.* Let $E \to \mathfrak{X}$ be an orthogonal vector bundle of rank $2n$ on a smooth complex algebraic variety $\mathfrak{X}$. Let $w \in \widetilde{W}_n$ be an amenable element of shape λ, and let $\mathfrak{f}$ and $\mathfrak{g}$ be the left and right flags of w, respectively. Two maximal isotropic subbundles L and L' of E are said to be in the same family if $\operatorname{rank}(L \cap L') \equiv n \pmod 2$. Consider two complete flags of subbundles of E

$$0 \subset E_1 \subset \cdots \subset E_{2n} = E \quad \text{and} \quad 0 \subset F_1 \subset \cdots \subset F_{2n} = E$$

with $\operatorname{rank} E_r = \operatorname{rank} F_r = r$ for each r, while $E_{n+s} = E_{n-s}^{\perp}$ and $F_{n+s} = F_{n-s}^{\perp}$ for $0 \leq s < n$. We assume that E_n is in the same family as F_n, if n is even, and in the opposite family, if n is odd.

We have a group monomorphism $\zeta : \widetilde{W}_n \hookrightarrow S_{2n}$, defined by restricting the map ζ of Section 5.3 to $\widetilde{W}_n$. Let $\widetilde{w}_0$ denote the longest element of $\widetilde{W}_n$, and define the *degeneracy locus* $\mathfrak{X}_w \subset \mathfrak{X}$ as the closure of the locus of $x \in \mathfrak{X}$ such that

$$\dim(E_r(x) \cap F_s(x)) = \#\{\, i \leq r \mid \zeta(\widetilde{w}_0 w \widetilde{w}_0)_i > 2n-s \,\} \text{ for } 1 \leq r \leq n-1,\ 1 \leq s \leq 2n$$

with the reduced scheme structure. Assume further that $\mathfrak{X}_w$ has pure codimension $\ell(w)$ in $\mathfrak{X}$, and consider its cohomology class $[\mathfrak{X}_w]$ in $\mathrm{H}^{2\ell(w)}(\mathfrak{X})$.

Theorem 8. *For any amenable element $w \in \widetilde{W}_n$, we have*

$$[\mathfrak{X}_w] = H_w(E - E_{n-\mathfrak{f}} - F_{n+\mathfrak{g}}) = 2^{-\ell(\mu(w))}\, R^{D(w)} \star \widehat{c}_\lambda(E - E_{n-\mathfrak{f}} - F_{n+\mathfrak{g}}) \tag{34}$$

in the cohomology ring $\mathrm{H}^*(\mathfrak{X})$.

The Chern polynomial in (34) is defined by employing the substitutions

$$\begin{aligned} {}^r\mathfrak{b}_p &\longmapsto \begin{cases} c_p(E - E_{n-r} - F_n) & \text{if } p < r, \\ \frac{1}{2} c_p(E - E_{n-r} - F_n) & \text{if } p > r, \end{cases} \\ {}^r\mathfrak{b}_r &\longmapsto \frac{1}{2}(c_r(E - E_{n-r} - F_n) + c_r(E_n - E_{n-r})), \\ {}^r\widetilde{\mathfrak{b}}_r &\longmapsto \frac{1}{2}(c_r(E - E_{n-r} - F_n) - c_r(E_n - E_{n-r})) \end{aligned}$$

in equation (33), for any integer p and $r \geq 1$. The proof of Theorem 8 is obtained by applying the type D geometrization map of [IMN, Sec. 10] to equation (32), and using the computations in [T3, Sec. 7.4].

Appendix A. Counterexamples to statements in [AF2]

The following two examples exhibit errors in the proofs – in all types except type A – and in the main type D result of [AF2]. We use the notation in op. cit.

Example 9. We show that Lemma A.1(i) of [AF2] is incorrect. Set $\rho = (0,1,0)$, $\lambda = (2,1,1)$, $k = m = 2$, and $n = \ell = 3$. The assumptions are that $c(2) = c(3)$, $c'(2) = c(2)(1+b_1)$, so that $c'_j(2) = c_j(2) + b_1 c_{j-1}(2)$ for each j, and $c'(i) = c(i)$ for $i = 1,3$.

We compute that $R^{(\rho,\ell)} = (1+R_{12})^{-1}(1-R_{12})(1-R_{13})(1-R_{23})$ and hence

$$\Theta_\lambda^{(\rho)}(c) = c_2(1)c_1(2)c_1(2) - 2c_3(1)c_1(2) + c_3(1)c_1(2) - c_2(1)c_2(2)$$

while

$$\Theta_\lambda^{(\rho)}(c') = c_2(1)(c_1(2)+b_1)c_1(2) - 2c_3(1)c_1(2) + c_3(1)(c_1(2)+b_1) - c_2(1)(c_2(2)+b_1c_1(2)).$$

It follows that

$$\Theta_\lambda^{(\rho)}(c') - \Theta_\lambda^{(\rho)}(c) = b_1 c_3(1) \neq 0.$$

One can show similarly that Lemma A.1(ii) and Lemma A.2 of op. cit. are also wrong.

Example 10. We show that Theorem 4 of [AF2] is false. Consider the type D triple $\tau = (\mathbf{k},\mathbf{p},\mathbf{q}) := ((1,2),(2,1),(0,-2))$, which corresponds to the Weyl group element $\overline{3}2\overline{1} \in \widetilde{W}_3$ (or to the element 321, depending on the type convention). We have $\rho = (0,0)$ and $\lambda = (2,1)$, while $\ell = 2$ and $r = 1$, so $\widetilde{R}^{(\rho,r,\ell)} = 1 - R_{12}$ and

$$H_{\lambda(\tau)}^{\rho(\tau)}(c(1),c(2)) = (1-R_{12})(c(1)_2 c(2)_1) = c_2(1)c_1(2) - c_3(1)c_0(2).$$

The computations of Example 8 show that $2\,[\Omega_\tau] \neq H_{\lambda(\tau)}^{\rho(\tau)}(c(1),c(2))$.

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