SCHUBERT POLYNOMIALS AND ARAKELOV THEORY OF ORTHOGONAL FLAG VARIETIES

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ABSTRACT. We propose a theory of combinatorially explicit Schubert polynomials which represent the Schubert classes in the Borel presentation of the cohomology ring of the orthogonal flag variety $\mathfrak{X} = \mathrm{SO}_N/B$. We use these polynomials to describe the arithmetic Schubert calculus on $\mathfrak{X}$. Moreover, we give a method to compute the natural arithmetic Chern numbers on $\mathfrak{X}$, and show that they are all rational numbers.

0. INTRODUCTION

Let V be a complex vector space equipped with a nondegenerate skew-symmetric bilinear form. Let $\mathfrak{X}$ denote the flag variety for the symplectic group, which parametrizes flags of isotropic subspaces in V. In [T5], we defined a family of *symplectic Schubert polynomials* for $\mathfrak{X}$, which represent the classes of the Schubert varieties in the Borel presentation [Bo] of the cohomology ring of $\mathfrak{X}$. These polynomials were applied to understand the structure of the Gillet-Soulé arithmetic Chow ring of $\mathfrak{X}$, thought of as a smooth scheme over the ring of integers. Our aim in this companion paper to [T5] is to explain the analogous theory for the orthogonal group, which arises when the chosen bilinear form on V is symmetric.

The symplectic Schubert polynomials of [T5] are closely related to the type C Schubert polynomials of Billey and Haiman [BH]. As in [BH, Thm. 3], our theory of orthogonal Schubert polynomials for the root system of type B_n is, up to well known scalar factors, the same as that for the root system C_n. Moreover, using these B_n Schubert polynomials, one can describe the arithmetic Chow ring of the flag variety of the odd orthogonal group in a similar fashion to the symplectic group, following [T5, Thm. 3]. Therefore in this paper we will concentrate on the even orthogonal case, and construct Schubert polynomials for the root system of type D_n. For the application to arithmetic intersection theory, we must deal with an extra relation which comes from the vanishing of the top Chern class of the maximal isotropic subbundle of the trivial vector bundle over $\mathfrak{X}$. Fortunately, this relation can be computed using our work [T4] on the Arakelov theory of even orthogonal Grassmannians.

This paper is organized as follows. We begin in §1 with combinatorial preliminaries on $\widetilde{P}$-polynomials and the Lascoux-Schützenberger and Billey-Haiman Schubert polynomials. We introduce our theory of orthogonal Schubert polynomials in §2.2 and list some of their basic properties in §2.3. Section 3 computes the curvature

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of the relevant homogeneous vector bundles over $\mathfrak{X}(\mathbb{C})$, equipped with their natural hermitian metrics. The arithmetic intersection theory of $\mathfrak{X}$ is studied in §4. Our method for computing arithmetic intersections is explained in §4.3, and the arithmetic Schubert calculus is described in §4.4.

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1. Preliminary definitions

1.1. $\widetilde{P}$- and P-functions. We let Π denote the set of all integer partitions. The *length* $\ell(\lambda)$ of a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ is the number of (nonzero) parts λ_i, and the weight $|\lambda|$ is the sum $\sum_i \lambda_i$. We let $\lambda_i = 0$ for any $i > \ell(\lambda)$. A partition is *strict* if no nonzero part is repeated. Let $\mathcal{G}_n = \{\lambda \in \Pi \mid \lambda_1 \leq n\}$ and let $\mathcal{F}_n$ be the set of strict partitions in $\mathcal{G}_n$.

Let $\mathrm{X} = (\mathrm{x}_1, \mathrm{x}_2, \ldots)$ be a sequence of commuting independent variables. Define the elementary symmetric functions $e_k = e_k(\mathrm{X})$ by the generating series

$$\sum_{k=0}^{\infty} e_k(\mathrm{X})t^k = \prod_{i=1}^{\infty}(1 + \mathrm{x}_i t).$$

We will often work with coefficients in the ring $A = \mathbb{Z}[\frac{1}{2}]$; the polynomial ring $\Lambda' = A[e_1, e_2, \ldots]$ is the ring of symmetric functions in the variables X with these coefficients. Next, we define the $\widetilde{P}$-functions of Pragacz and Ratajski [PR]. Set $\widetilde{P}_0 = 1$ and $\widetilde{P}_k = e_k/2$ for $k > 0$. For i, j nonnegative integers, let

$$\widetilde{P}_{i,j} = \widetilde{P}_i \widetilde{P}_j + 2\sum_{r=1}^{j-1}(-1)^r \widetilde{P}_{i+r}\widetilde{P}_{j-r} + (-1)^j \widetilde{P}_{i+j}.$$

If λ is a partition of length greater than two, define

$$\widetilde{P}_\lambda = \mathrm{Pfaffian}(\widetilde{P}_{\lambda_i,\lambda_j})_{1 \leq i < j \leq 2m},$$

where m is the least positive integer with $2m \geq \ell(\lambda)$.

These $\widetilde{P}$-functions have the following properties:

(a) The $\widetilde{P}_\lambda(\mathrm{X})$ for $\lambda \in \Pi$ form an A-basis of Λ'.

(b) $\widetilde{P}_{k,k}(\mathrm{X}) = \frac{1}{4} e_k(\mathrm{X}^2) = \frac{1}{4} e_k(\mathrm{x}_1^2, \mathrm{x}_2^2, \ldots)$ for all $k > 0$.

(c) If $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\lambda^+ = \lambda \cup (k,k) = (\lambda_1, \ldots, k, k, \ldots, \lambda_r)$ then

$$\widetilde{P}_{\lambda^+} = \widetilde{P}_{k,k}\widetilde{P}_\lambda.$$

(d) The coefficients of $\widetilde{P}_\lambda(\mathrm{X})$ are nonnegative rational numbers.

Let $\Lambda'_n = A[\mathrm{x}_1, \ldots, \mathrm{x}_n]^{S_n}$ be the ring of symmetric polynomials in $\mathrm{X}_n = (\mathrm{x}_1, \ldots, \mathrm{x}_n)$. Then we have two additional properties.

(e) If $\lambda_1 > n$, then $\widetilde{P}_\lambda(\mathrm{X}_n) = 0$. The $\widetilde{P}_\lambda(\mathrm{X}_n)$ for $\lambda \in \mathcal{G}_n$ form an A-basis of Λ'_n.

(f) $\widetilde{P}_n(\mathrm{X}_n)\widetilde{P}_\lambda(\mathrm{X}_n) = \widetilde{P}_{(n,\lambda)}(\mathrm{X}_n)$ for all $\lambda \in \mathcal{G}_n$.

Suppose that $Y = (y_1, y_2, \ldots)$ is a second sequence of variables and define symmetric functions $q_k(Y)$ by the equation

$$\sum_{k=0}^{\infty} q_k(Y)t^k = \prod_{i=1}^{\infty}\frac{1 + y_i t}{1 - y_i t}.$$

Let $\Gamma' = A[q_1, q_2 \ldots]$ and define an A-algebra homomorphism $\eta : \Lambda' \to \Gamma'$ by setting $\eta(e_k(\mathrm{X})) = q_k(Y)$ for each $k \geq 1$. For any strict partition λ, the Schur P-function $P_\lambda(Y)$ may be defined as the image of $\widetilde{P}_\lambda(\mathrm{X})$ under η. The P_λ for strict partitions λ have nonnegative *integer* coefficients and form a free A-basis of Γ'.

1.2. **Divided differences and type A Schubert polynomials.** The symmetric group S_n is the Weyl group for the root system A_{n-1}. We write the elements ϖ of S_n using the single-line notation $(\varpi(1), \varpi(2), \ldots, \varpi(n))$. The group S_n is generated by the simple transpositions s_i for $1 \leq i \leq n-1$, where s_i interchanges i and $i+1$ and fixes all other elements of $\{1, \ldots, n\}$.

The elements of the Weyl group $\widetilde{W}_n$ for the root system D_n may be represented by signed permutations; we will adopt the notation where a bar is written over an element with a negative sign. The group $\widetilde{W}_n$ is an extension of S_n by an element s_0 which acts on the right by

$$(u_1, u_2, \ldots, u_n)s_0 = (\overline{u}_2, \overline{u}_1, u_3, \ldots, u_n).$$

A *reduced word* of $w \in \widetilde{W}_n$ is a sequence $a_1 \ldots a_r$ of elements in $\{0, 1, \ldots, n-1\}$ such that $w = s_{a_1} \cdots s_{a_r}$ and r is minimal (so equal to the length $\ell(w)$ of w). If we convert all the 0's which appear in the reduced word $a_1 \ldots a_r$ to 1's, we obtain a *flattened word* of w. For example, 20312 is a reduced word of $\overline{1}4\overline{3}2$, and 21312 is the corresponding flattened word. Note that 21312 is also a word, but not reduced, for 1432. The elements of maximal length in S_n and $\widetilde{W}_n$ are

$$\varpi_0 = (n, n-1, \ldots, 1) \quad \text{and} \quad w_0 = \begin{cases} (\overline{1}, \overline{2}, \ldots, \overline{n}) & \text{if } n \text{ is even,} \\ (1, \overline{2}, \ldots, \overline{n}) & \text{if } n \text{ is odd} \end{cases}$$

respectively.

The group $\widetilde{W}_n$ acts on the ring $A[\mathrm{X}_n]$ of polynomials in X_n: the transposition s_i interchanges x_i and x_{i+1} for $1 \leq i \leq n-1$, while s_0 sends $(\mathrm{x}_1, \mathrm{x}_2)$ to $(-\mathrm{x}_2, -\mathrm{x}_1)$ (all other variables remain fixed). Following [BGG] and [D1, D2], we have divided difference operators $\partial_i : A[\mathrm{X}_n] \to A[\mathrm{X}_n]$. For $1 \leq i \leq n-1$ they are defined by

$$\partial_i(f) = (f - s_i f)/(\mathrm{x}_i - \mathrm{x}_{i+1})$$

while

$$\partial_0(f) = (f - s_0 f)/(\mathrm{x}_1 + \mathrm{x}_2),$$

for any $f \in A[\mathrm{X}_n]$. For each $w \in \widetilde{W}_n$, define an operator ∂_w by setting

$$\partial_w = \partial_{a_1} \circ \cdots \circ \partial_{a_\ell}$$

if $w = a_1 \cdots a_\ell$ is a reduced word for w.

For every permutation $\varpi \in S_n$, Lascoux and Schützenberger [LS] defined a *type A Schubert polynomial* $\mathfrak{S}_\varpi(\mathrm{X}_n) \in \mathbb{Z}[\mathrm{X}_n]$ by

$$\mathfrak{S}_\varpi(\mathrm{X}_n) = \partial_{\varpi^{-1}\varpi_0}\left(\mathrm{x}_1^{n-1}\mathrm{x}_2^{n-2} \cdots \mathrm{x}_{n-1}\right).$$

This definition is stable under the natural inclusion of S_n into S_{n+1}, hence the polynomial $\mathfrak{S}_w$ makes sense for $w \in S_\infty = \cup_{n=1}^\infty S_n$. The $\mathfrak{S}_w$ for $w \in S_\infty$ form a $\mathbb{Z}$-basis of $\mathbb{Z}[\mathrm{X}] = \mathbb{Z}[\mathrm{x}_1, \mathrm{x}_2, \ldots]$. The coefficients of $\mathfrak{S}_w$ are nonnegative integers.

1.3. **Billey-Haiman Schubert polynomials of type D.** We regard $\widetilde{W}_n$ as a subgroup of $\widetilde{W}_{n+1}$ in the obvious way and let $\widetilde{W}_\infty$ denote the union of all the $\widetilde{W}_n$. Let $Z = (z_1, z_2, \ldots)$ be a third sequence of commuting variables. Billey and Haiman [BH] defined a family $\{\mathcal{D}_w\}_{w\in\widetilde{W}_\infty}$ of Schubert polynomials of type D, which form an A-basis of the ring $\Gamma'[Z]$. The expansion coefficients for a product $\mathcal{D}_u\mathcal{D}_v$ in the basis of type D Schubert polynomials agree with the Schubert structure constants on even orthogonal flag varieties for sufficiently large n. For every $w \in \widetilde{W}_n$ there is a unique expression

$$\mathcal{D}_w = \sum_{\substack{\lambda \text{ strict} \\ \varpi\in S_n}} f^w_{\lambda,\varpi} P_\lambda(Y)\mathfrak{S}_\varpi(Z) \tag{1}$$

where the coefficients $f^w_{\lambda,\varpi}$ are nonnegative integers. We proceed to give a combinatorial formula for these numbers.

A sequence $a = (a_1, \ldots, a_m)$ is called *unimodal* if for some $r \leq m$, we have

$$a_1 > a_2 > \cdots > a_{r-1} \geq a_r < a_{r+1} < \cdots < a_m,$$

and if $a_{r-1} = a_r$ then $a_r = 1$.

Let $w \in \widetilde{W}_n$ and λ be a Young diagram with r rows such that $|\lambda| = \ell(w)$. A *Kraśkiewicz-Lam tableau* for w of shape λ is a filling T of the boxes of λ with positive integers in such a way that

a) If t_i is the sequence of entries in the i-th row of T, reading from left to right, then the row word $t_r \ldots t_1$ is a flattened word for w.

b) For each i, t_i is a unimodal subsequence of maximum length in $t_r \ldots t_{i+1}t_i$.

Let T be a Kraśkiewicz-Lam tableau of shape λ with row word $a_1 \ldots a_\ell$. We define $m(T) = \ell(\lambda) + 1 - k$, where k is the number of distinct values of $s_{a_1}\cdots s_{a_j}(1)$ for $0 \leq j \leq \ell$. It follows from [La, Thm. 4.35] that $m(T) \geq 0$.

Example 1. Let $\lambda \in \mathcal{F}_{n-1}$, $\ell = \ell(\lambda)$, $k = n-1-\ell$, and μ be the strict partition whose parts are the numbers from 1 to n which do not lie in the set $\{1, \lambda_\ell + 1, \ldots, \lambda_1 + 1\}$. The barred permutation

$$w_\lambda = (\overline{\lambda_1 + 1}, \ldots, \overline{\lambda_\ell + 1}, \hat{1}, \mu_k, \ldots, \mu_1)$$

where $\hat{1}$ is equal to 1 or $\overline{1}$ according to the parity of ℓ is the *maximal Grassmannian element* of $\widetilde{W}_n$ corresponding to λ. There is a unique Kraśkiewicz-Lam tableau T_λ for w_λ, which has shape λ, and whose i-th row consists of the numbers 1 through λ_i in decreasing order. Moreover, we have $m(T_\lambda) = 0$. For instance, if $\lambda = (6, 4, 3)$ then we obtain

$$T_\lambda \;=\; \begin{matrix} 6\;5\;4\;3\;2\;1 \\ 4\;3\;2\;1 \\ 3\;2\;1. \end{matrix}$$

Proposition 1 (BH, La)**.** *For every $w \in \widetilde{W}_\infty$, we have $f^w_{\lambda,\varpi} = \sum_T 2^{m(T)}$, summed over all Kraśkiewicz-Lam tableaux T for $w\varpi^{-1}$ of shape λ, if $\ell(w\varpi^{-1}) = \ell(w) - \ell(\varpi)$, and $f^w_{\lambda,\varpi} = 0$ otherwise.*

Proof. According to [BH, Thm. 3], the polynomial $\mathcal{D}_w$ satisfies

$$\mathcal{D}_w = \sum_{uv=w} E_u(Y)\mathfrak{S}_v(Z),$$

summed over all factorizations $uv = w$ in $\widetilde{W}_\infty$ such that $\ell(u) + \ell(v) = \ell(w)$ with $v \in S_\infty$. The left factors $E_u(Y)$ are the type D Stanley symmetric functions of [BH, La]. We deduce from [BH, Prop. 3.7] and [La, Thm. 4.35] that for any $u \in \widetilde{W}_\infty$,

$$E_u(Y) = \sum_\lambda d_\lambda^u \, P_\lambda(Y)$$

where $d_\lambda^u = \sum_T 2^{m(T)}$, summed over all Kraśkiewicz-Lam tableaux T for u of shape λ. The result follows by combining these two facts. □

2. Orthogonal Schubert polynomials

2.1. Consider the vector space $\mathbb{C}^{2n}$ with its canonical basis $\{e_i\}_{i=1}^{2n}$ of unit coordinate vectors. We define the *skew diagonal symmetric form* $[\ ,\ ]$ on $\mathbb{C}^{2n}$ by setting $[e_i, e_j] = 0$ for $i + j \neq 2n+1$ and $[e_i, e_{2n+1-i}] = 1$ for $1 \leq i \leq 2n$. The orthogonal group $\mathrm{SO}_{2n}(\mathbb{C})$ is the group of linear automorphisms of $\mathbb{C}^{2n}$ preserving the symmetric form. The upper triangular matrices in SO_{2n} form a Borel subgroup B.

A subspace Σ of $\mathbb{C}^{2n}$ is called isotropic if the restriction of the symmetric form to Σ vanishes. Consider a partial flag of subspaces

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n \subset E_{2n} = \mathbb{C}^{2n}$$

with $\dim E_i = i$ and E_n isotropic. Each such flag can be extended to a complete flag $E_\bullet$ in $\mathbb{C}^{2n}$ by letting $E_{n+i} = E_{n-i}^\perp$ for $1 \leq i \leq n$; we will call such a flag a *complete isotropic flag*. We say that two isotropic subspaces E and F of dimension n are *in the same family* if $\dim(E \cap F) \equiv n \,(\mathrm{mod}\ 2)$; two complete isotropic flags $E_\bullet$ and $F_\bullet$ are in the same family if E_n and F_n are. The variety $\mathfrak{X} = \mathrm{SO}_{2n}/B$ parametrizes complete isotropic flags $E_\bullet$ with E_n in the same family as $\langle e_1, \ldots, e_n \rangle$. We use the same notation to denote the tautological flag $E_\bullet$ of vector bundles over $\mathfrak{X}$.

There is a group monomorphism $\phi : \widetilde{W}_n \hookrightarrow S_{2n}$ whose image consists of those permutations $\varpi \in S_{2n}$ such that $\varpi(i) + \varpi(2n+1-i) = 2n+1$ for all i and the number of $i \leq n$ such that $\varpi(i) > n$ is even. The map ϕ is determined by setting, for each $w = (w_1, \ldots, w_n) \in \widetilde{W}_n$ and $1 \leq i \leq n$,

$$\phi(w)(i) = \begin{cases} n + 1 - w_{n+1-i} & \text{if } w_{n+1-i} \text{ is unbarred,} \\ n + \overline{w}_{n+1-i} & \text{otherwise.} \end{cases}$$

Let $F_\bullet$ be a fixed complete isotropic flag in the same family as the flags in $\mathfrak{X}$. For every $w \in \widetilde{W}_n$ define the *Schubert variety* $\mathfrak{X}_w(F_\bullet) \subset \mathfrak{X}$ as the closure of the locus of $E_\bullet \in \mathfrak{X}$ such that

$$\dim(E_r \cap F_s) = \#\{\, i \leq r \mid \phi(w_0 w w_0)(i) > 2n - s \,\} \quad \text{for} \quad 1 \leq r \leq n-1,\ 1 \leq s \leq 2n.$$

The Schubert class σ_w in $\mathrm{H}^{2\ell(w)}(\mathfrak{X}, \mathbb{Z})$ is the cohomology class which is Poincaré dual to the homology class determined by $\mathfrak{X}_w(F_\bullet)$.

Following Borel [Bo, §29], the cohomology ring $\mathrm{H}^*(\mathfrak{X}, A)$ is presented as a quotient

$$\mathrm{H}^*(\mathfrak{X}, A) \cong A[\mathrm{x}_1, \ldots, \mathrm{x}_n]/J_n \tag{2}$$

where J_n is the ideal generated by the $\widetilde{W}_n$-invariants of positive degree in $A[\mathrm{X}_n]$. The inverse of the isomorphism (2) sends the class of x_i to $-c_1(E_{n+1-i}/E_{n-i})$ for each i with $1 \leq i \leq n$.

2.2. For every $\lambda \in \mathcal{G}_n$ and $\varpi \in S_n$, define the polynomial $\mathfrak{D}_{\lambda,\varpi} = \mathfrak{D}_{\lambda,\varpi}(\mathrm{X}_n)$ by

$$\mathfrak{D}_{\lambda,\varpi} = \widetilde{P}_\lambda(\mathrm{X}_n)\mathfrak{S}_\varpi(-\mathrm{X}_n) = (-1)^{\ell(\varpi)}\widetilde{P}_\lambda(\mathrm{X}_n)\mathfrak{S}_\varpi(\mathrm{X}_n).$$

Lascoux and Pragacz [LP] showed that the products $\widetilde{P}_\lambda(\mathrm{X}_n)\mathfrak{S}_\varpi(\mathrm{X}_n)$ for $\lambda \in \mathcal{F}_{n-1}$ and $\varpi \in S_n$ form a basis for the polynomial ring $A[\mathrm{X}_n]$ as an $A[\mathrm{X}_n]^{\widetilde{W}_n}$-module. Observe that the $\mathfrak{D}_{\lambda,\varpi}(\mathrm{X}_n)$ for $\lambda \in \mathcal{G}_n$ and $\varpi \in S_n$ form a basis of $A[\mathrm{x}_1,\ldots,\mathrm{x}_n]$ as an A-module. The ideal J_n of §2.1 is generated by the polynomials $e_i(\mathrm{X}_n^2) = 4\,\widetilde{P}_{i,i}(\mathrm{X}_n)$ and $e_n(\mathrm{X}_n) = 2\,\widetilde{P}_n(\mathrm{X}_n)$, and the $\widetilde{P}$-polynomials have the factorization properties (c), (f) and the vanishing property (e) of §1.1. We deduce that $\widetilde{P}_\lambda(\mathrm{X}_n) \in J_n$ unless $\lambda \in \mathcal{F}_{n-1}$.

Definition 1. For $w \in \widetilde{W}_n$, define the orthogonal Schubert polynomial $\mathfrak{D}_w = \mathfrak{D}_w(\mathrm{X}_n)$ by

$$\mathfrak{D}_w = \sum_{\substack{\lambda\in\mathcal{F}_{n-1}\\ \varpi\in S_n}} f^w_{\lambda,\varpi}\mathfrak{D}_{\lambda,\varpi}(\mathrm{X}_n)$$

where the coefficients $f^w_{\lambda,\varpi}$ are the same as in (1) and Proposition 1.

Theorem 1. *The orthogonal Schubert polynomial $\mathfrak{D}_w(\mathrm{X}_n)$ is the unique $\mathbb{Z}$-linear combination of the $\mathfrak{D}_{\lambda,\varpi}(\mathrm{X}_n)$ for $\lambda \in \mathcal{F}_{n-1}$ and $\varpi \in S_n$ which represents the Schubert class σ_w in the Borel presentation* (2).

Proof. Recall that a partition is *odd* if all its non-zero parts are odd integers. For each partition μ, let $p_\mu = \prod_i p_{\mu_i}$, where $p_r(\mathrm{X}) = \mathrm{x}_1^k + \mathrm{x}_2^k + \cdots$ denotes the r-th power sum. The $p_\mu(Y)$ for μ odd form a $\mathbb{Q}$-basis of $\Gamma' \otimes_A \mathbb{Q}$. We therefore have a unique expression

$$\mathcal{D}_w = \sum_{\substack{\mu\text{ odd}\\ \varpi\in S_n}} a^w_{\mu,\varpi}\, p_\mu(Y)\mathfrak{S}_\varpi(Z) \tag{3}$$

in the ring $\Gamma'[Z] \otimes_A \mathbb{Q}$.

Józefiak [Jo] showed that the kernel of the homomorphism η from §1.1 is the ideal generated by the symmetric functions of positive degree in $\mathrm{X}^2 = (\mathrm{x}_1^2, \mathrm{x}_2^2, \ldots)$. It follows from this and properties (b), (c) of §1.1 that $\eta(\widetilde{P}_\lambda) = 0$ unless λ is a strict partition. Moreover, we have $\eta(p_k(\mathrm{X})) = 2\,p_k(Y)$, if k is odd, and $\eta(p_k(\mathrm{X})) = 0$, if $k > 0$ is even.

Let $p_{odd} = (p_1, p_3, p_5, \ldots)$. Define a polynomial $\mathcal{D}_w(p_{odd}(\mathrm{X}), \mathrm{X}_{n-1})$ in the variables $p_k := p_k(\mathrm{X})$ for k odd and $\mathrm{x}_1, \ldots, \mathrm{x}_{n-1}$ by substituting $p_k(Y)$ with $p_k(\mathrm{X})/2$ and z_i with $-\mathrm{x}_i$ in (3). We deduce from (1), (3), and the above discussion that $\mathcal{D}_w(p_{odd}(\mathrm{X}), \mathrm{X}_{n-1})$ differs from

$$\sum_{\substack{\lambda\text{ strict}\\ \varpi\in S_n}} f^w_{\lambda,\varpi}\widetilde{P}_\lambda(\mathrm{X})\mathfrak{S}_\varpi(-\mathrm{X}_n)$$

by an element in the ideal of $\Lambda'[\mathrm{X}_{n-1}]$ generated by the $e_i(\mathrm{X}^2)$ for $i > 0$.

According to [BH, §2], for every $w \in \widetilde{W}_n$, the polynomial

$$\mathcal{D}_w(\mathrm{X}_n) := \mathcal{D}_w(p_{odd}(\mathrm{X}_n), \mathrm{X}_{n-1})$$

obtained by setting $\mathrm{x}_i = 0$ for all $i > n$ in $\mathcal{D}_w(p_{odd}(\mathrm{X}), \mathrm{X}_{n-1})$ represents the Schubert class σ_w in the Borel presentation (2). Since $\widetilde{P}_\lambda(\mathrm{X}_n) \in J_n$ unless $\lambda \in \mathcal{F}_{n-1}$, it follows that $\mathfrak{D}_w$ represents the Schubert class σ_w in the presentation (2), as required.

We claim that the $\mathfrak{D}_{\lambda,\varpi}$ for $\lambda \in \mathcal{G}_n \smallsetminus \mathcal{F}_{n-1}$ and $\varpi \in S_n$ form an A-basis of J_n. To see this, note that if h is an element of J_n then $h(\mathrm{X}_n) = \sum_i e_i(\mathrm{X}_n^2) f_i(\mathrm{X}_n) + e_n(\mathrm{X}_n) g(\mathrm{X}_n)$ for some polynomials $f_i, g \in A[\mathrm{X}_n]$. Now the f_i and g are unique A-linear combinations of the $\mathfrak{D}_{\mu,\varpi}$ for $\mu \in \mathcal{G}_n$ and $\varpi \in S_n$, and properties (b), (c), and (f) of §1.1 give

$$e_i(\mathrm{X}_n^2)\mathfrak{D}_{\mu,\varpi}(\mathrm{X}_n) = 4\,\mathfrak{D}_{\mu\cup(i,i),\varpi}(\mathrm{X}_n)$$

and

$$e_n(\mathrm{X}_n)\mathfrak{D}_{\mu,\varpi}(\mathrm{X}_n) = 2\,\mathfrak{D}_{(n,\mu),\varpi}(\mathrm{X}_n),$$

respectively. We deduce that any $h \in J_n$ lies in the A-linear span of the $\mathfrak{D}_{\lambda,\varpi}$ for $\lambda \in \mathcal{G}_n \smallsetminus \mathcal{F}_{n-1}$ and $\varpi \in S_n$. Since the $\mathfrak{D}_{\lambda,\varpi}$ for $\lambda \in \mathcal{G}_n$ and $\varpi \in S_n$ are linearly independent, this proves the claim and the uniqueness assertion in the theorem. □

The statement of Theorem 1 may serve as an alternative definition of the orthogonal Schubert polynomials $\mathfrak{D}_w(\mathrm{X}_n)$.

2.3. We give below some properties of the polynomials $\mathfrak{D}_w(\mathrm{X}_n)$.

(a) The set

$$\{\mathfrak{D}_w \mid w \in \widetilde{W}_n\} \cup \{\mathfrak{D}_{\lambda,\varpi} \mid \lambda \in \mathcal{G}_n \smallsetminus \mathcal{F}_{n-1},\ \varpi \in S_n\}$$

is an A-basis of the polynomial ring $A[\mathrm{x}_1, \ldots, x_n]$. The $\mathfrak{D}_{\lambda,\varpi}$ for $\lambda \in \mathcal{G}_n \smallsetminus \mathcal{F}_{n-1}$ and $\varpi \in S_n$ span the ideal J_n of $A[\mathrm{x}_1, \ldots, \mathrm{x}_n]$ generated by the $e_i(\mathrm{X}_n^2)$ for $1 \leq i \leq n-1$ and $e_n(\mathrm{X}_n) = \mathrm{x}_1 \cdots \mathrm{x}_n$.

(b) For every $u, v \in \widetilde{W}_n$, we have an equation

$$\mathfrak{D}_u \cdot \mathfrak{D}_v = \sum_{w \in \widetilde{W}_n} d_{uv}^w\, \mathfrak{D}_w + \sum_{\substack{\lambda \in \mathcal{G}_n \smallsetminus \mathcal{F}_{n-1} \\ \varpi \in S_n}} d_{uv}^{\lambda\varpi}\, \mathfrak{D}_{\lambda,\varpi} \tag{4}$$

in the ring $A[\mathrm{x}_1, \ldots, \mathrm{x}_n]$. The coefficients d_{uv}^w are nonnegative integers, which vanish unless $\ell(w) = \ell(u) + \ell(v)$, and agree with the structure constants in the equation of Schubert classes

$$\sigma_u \cdot \sigma_v = \sum_{w \in \widetilde{W}_n} d_{uv}^w\, \sigma_w,$$

which holds in $\mathrm{H}^*(\mathfrak{X}, \mathbb{Z})$. The coefficients $d_{uv}^{\lambda\varpi}$ are integers, some of which may be negative. Equation (4) provides a lifting of the Schubert calculus from the cohomology ring $\mathrm{H}^*(\mathfrak{X}, A) \cong A[\mathrm{x}_1, \ldots, \mathrm{x}_n]/J_n$ to the polynomial ring $A[\mathrm{x}_1, \ldots, \mathrm{x}_n]$.

(c) For each $m < n$ let $i = i_{m,n} : \widetilde{W}_m \to \widetilde{W}_n$ be the natural embedding using the first m components. Then for any $w \in \widetilde{W}_m$ we have

$$\mathfrak{D}_{i(w)}(\mathrm{X}_n)\Big|_{x_{m+1}=\cdots=x_n=0} = \mathfrak{D}_w(\mathrm{X}_m).$$

(d) For $\varpi \in S_n$ and $w \in \widetilde{W}_n$, we have

$$\partial_\varpi \mathfrak{D}_w = \begin{cases} (-1)^{\ell(\varpi)}\, \mathfrak{D}_{w\varpi} & \text{if } \ell(w\varpi) = \ell(w) - \ell(\varpi), \\ 0 & \text{otherwise.} \end{cases}$$

The remaining properties listed in [T5, §2.3] also have analogues here, and their proofs are similar.

TABLE 1. Orthogonal Schubert polynomials for $w \in \widetilde{W}_3$

w	$\mathfrak{D}_w(\mathrm{X}_3) = \sum f^w_{\lambda,\varpi}\, \widetilde{P}_\lambda(\mathrm{X}_3)\, \mathfrak{S}_\varpi(-\mathrm{X}_3)$
$123 = 1$	1
$213 = s_1$	$\widetilde{P}_1 - \mathfrak{S}_{213}$
$132 = s_2$	$2\,\widetilde{P}_1 - \mathfrak{S}_{132}$
$231 = s_1 s_2$	$\widetilde{P}_2 - \widetilde{P}_1\, \mathfrak{S}_{132} + \mathfrak{S}_{231}$
$312 = s_2 s_1$	$\widetilde{P}_2 - 2\,\widetilde{P}_1\, \mathfrak{S}_{213} + \mathfrak{S}_{312}$
$321 = s_1 s_2 s_1$	$\widetilde{P}_{21} - \widetilde{P}_2\, \mathfrak{S}_{213} - \widetilde{P}_2\, \mathfrak{S}_{132} + \widetilde{P}_1\, \mathfrak{S}_{312} + 2\,\widetilde{P}_1\, \mathfrak{S}_{231} - \mathfrak{S}_{321}$
$\overline{21}3 = s_0$	$\widetilde{P}_1$
$\overline{12}3 = s_0 s_1$	$\widetilde{P}_2 - \widetilde{P}_1\, \mathfrak{S}_{213}$
$\overline{2}3\overline{1} = s_0 s_2$	$\widetilde{P}_2 - \widetilde{P}_1\, \mathfrak{S}_{132}$
$\overline{1}3\overline{2} = s_0 s_1 s_2$	$-\widetilde{P}_2\, \mathfrak{S}_{132} + \widetilde{P}_1\, \mathfrak{S}_{231}$
$3\overline{21} = s_0 s_2 s_1$	$\widetilde{P}_{21} - \widetilde{P}_2\, \mathfrak{S}_{213} + \widetilde{P}_1\, \mathfrak{S}_{312}$
$3\overline{12} = s_0 s_1 s_2 s_1$	$-\widetilde{P}_{21}\, \mathfrak{S}_{132} + \widetilde{P}_2\, \mathfrak{S}_{312} + \widetilde{P}_2\, \mathfrak{S}_{231} - \widetilde{P}_1\, \mathfrak{S}_{321}$
$\overline{31}2 = s_2 s_0$	$\widetilde{P}_2$
$\overline{13}2 = s_2 s_0 s_1$	$-\widetilde{P}_2\, \mathfrak{S}_{213}$
$\overline{3}2\overline{1} = s_2 s_0 s_2$	$\widetilde{P}_{21} - \widetilde{P}_2\, \mathfrak{S}_{132}$
$\overline{1}2\overline{3} = s_2 s_0 s_1 s_2$	$\widetilde{P}_2\, \mathfrak{S}_{231}$
$2\overline{31} = s_2 s_0 s_2 s_1$	$-\widetilde{P}_{21}\, \mathfrak{S}_{213} + \widetilde{P}_2\, \mathfrak{S}_{312}$
$2\overline{13} = s_2 s_0 s_1 s_2 s_1$	$\widetilde{P}_{21}\, \mathfrak{S}_{231} - \widetilde{P}_2\, \mathfrak{S}_{321}$
$\overline{32}1 = s_1 s_2 s_0$	$\widetilde{P}_{21}$
$\overline{23}1 = s_1 s_2 s_0 s_1$	$-\widetilde{P}_{21}\, \mathfrak{S}_{213}$
$\overline{3}1\overline{2} = s_1 s_2 s_0 s_2$	$-\widetilde{P}_{21}\, \mathfrak{S}_{132}$
$\overline{2}1\overline{3} = s_1 s_2 s_0 s_1 s_2$	$\widetilde{P}_{21}\, \mathfrak{S}_{231}$
$1\overline{32} = s_1 s_2 s_0 s_2 s_1$	$\widetilde{P}_{21}\, \mathfrak{S}_{312}$
$1\overline{23} = s_1 s_2 s_0 s_1 s_2 s_1$	$-\widetilde{P}_{21}\, \mathfrak{S}_{321}$

Example 2. a) We have the equations

$$\mathfrak{D}_{s_0}(\mathrm{X}_n) = \widetilde{P}_1(\mathrm{X}_n) = \frac{1}{2}(x_1 + x_2 + \cdots + x_n)$$

$$\mathfrak{D}_{s_1}(\mathrm{X}_n) = \widetilde{P}_1(\mathrm{X}_n) - \mathfrak{S}_{s_1} = \frac{1}{2}(-x_1 + x_2 + \cdots + x_n)$$

$$\mathfrak{D}_{s_i}(\mathrm{X}_n) = 2\,\widetilde{P}_1(\mathrm{X}_n) - \mathfrak{S}_{s_i} = x_{i+1} + \cdots + x_n \quad \text{for} \quad 2 \leq i \leq n-1.$$

b) For a maximal Grassmannian element $w_\lambda \in \widetilde{W}_n$, we have $\mathfrak{D}_w(\mathrm{X}_n) = \widetilde{P}_\lambda(\mathrm{X}_n)$.

Example 3. The list of all orthogonal Schubert polynomials $\mathfrak{D}_w$ for $w \in \widetilde{W}_3$ is given in Table 1. These polynomials are displayed according to the four orbits of the symmetric group S_3 on $\widetilde{W}_3$. Once the highest degree term in each orbit is known, one can compute the remaining elements easily using type A divided differences, by property (d) above. The reader should compare this table with [BH, Table 3].

3. CURVATURE OF HOMOGENEOUS VECTOR BUNDLES

For any complex manifold X, we denote the space of $\mathbb{C}$-valued smooth differential forms of type (p,q) on X by $\mathcal{A}^{p,q}(X)$. A hermitian vector bundle on X is a pair $\overline{E} = (E,h)$ consisting of a holomorphic vector bundle E over X and a hermitian metric h on E. Let $K(\overline{E}) \in \mathcal{A}^{1,1}(X, \mathrm{End}(E))$ be the curvature of $\overline{E}$ with respect to the hermitian holomorphic connection on $\overline{E}$ and set $K_E = \frac{i}{2\pi} K(\overline{E})$. For any integer k with $1 \leq k \leq \mathrm{rk}(E)$, we have a Chern form $c_k(\overline{E}) := \mathrm{Tr}(\bigwedge^k K_E) \in \mathcal{A}^{k,k}(X)$. The total Chern form of $\overline{E}$ is $c(\overline{E}) = 1 + \sum_{k=1}^n c_k(\overline{E})$. These differential forms are closed and their classes in the de Rham cohomology of X are the Chern classes of E.

To simplify the notation in this section, we will redefine the group $\mathrm{SO}_{2n}(\mathbb{C})$ using the *standard symmetric form* $[\ ,\ ]'$ on $\mathbb{C}^{2n}$ whose matrix $[e_i, e_j]'_{i,j}$ on unit coordinate vectors is $\begin{pmatrix} 0 & \mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix}$, where Id_n denotes the $n \times n$ identity matrix. Let $\mathfrak{X} = \mathrm{SO}_{2n}/B$ be the orthogonal flag variety and $E_\bullet$ its tautological complete isotropic flag of vector bundles. We equip the trivial vector bundle $E_{2n} = \mathbb{C}^{2n}_{\mathfrak{X}}$ with the trivial hermitian metric h compatible with the symmetric form $[\ ,\ ]'$ on $\mathbb{C}^{2n}$.

The metric h on E induces metrics on all the subbundles E_i and the quotient line bundles $Q_i = E_i/E_{i-1}$, for $1 \leq i \leq n$. Our goal here is to compute the $\mathrm{SO}(2n)$-invariant curvature matrices of the homogeneous vector bundles $\overline{E}_i$ and $\overline{Q}_i$ for $1 \leq i \leq n$. As in [T5, §3.2], we do this by pulling back these matrices of $(1,1)$-forms from $\mathfrak{X}$ to the compact Lie group $\mathrm{SO}(2n)$, where their entries may be expressed in terms of the basic invariant forms on $\mathrm{SO}(2n)$.

The Lie algebra of $\mathrm{SO}_{2n}(\mathbb{C})$ is given by

$$\mathfrak{so}(2n, \mathbb{C}) = \{(A, B, C) \mid A, B, C \in M_n(\mathbb{C}),\ B, C \text{ skew symmetric}\},$$

where (A, B, C) denotes the matrix $\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}$. Complex conjugation of the algebra $\mathfrak{so}(2n, \mathbb{C})$ with respect to the Lie algebra of $\mathrm{SO}(2n)$ is given by the map τ with $\tau(A) = -\overline{A}^t$. The Cartan subalgebra $\mathfrak{h}$ consists of all matrices of the form $\{(\mathrm{diag}(t_1, \ldots, t_n), 0, 0) \mid t_i \in \mathbb{C}\}$, where $\mathrm{diag}(t_1, \ldots, t_n)$ denotes a diagonal matrix. Consider the set of roots

$$R = \{\pm t_i \pm t_j \mid i \neq j\} \subset \mathfrak{h}^*$$

and a system of positive roots

$$R^+ = \{t_i - t_j \mid i < j\} \cup \{t_p + t_q \mid p < q\},$$

where the indices run from 1 to n. We use ij to denote a positive root in the first set and pq for a positive root in the second. The corresponding basis vectors are $e_{ij} = (E_{ij}, 0, 0)$ and $e^{pq} = (0, E_{pq} - E_{qp}, 0)$ for $p < q$, where E_{ij} is the matrix with 1 as the ij-th entry and zeroes elsewhere.

Define $\overline{e}_{ij} = \tau(e_{ij})$, $\overline{e}^{pq} = \tau(e^{pq})$, and consider the linearly independent set

$$\mathcal{B}' = \{e_{ij},\, \overline{e}_{ij},\, e^{pq},\, \overline{e}^{pq} \mid i < j,\ p < q\}.$$

The adjoint representation of $\mathfrak{h}$ on $\mathfrak{so}(2n, \mathbb{C})$ gives a root space decomposition

$$\mathfrak{so}(2n, \mathbb{C}) = \mathfrak{h} \oplus \sum_{i<j} (\mathbb{C}\, e_{ij} \oplus \mathbb{C}\, \overline{e}_{ij}) \oplus \sum_{p<q} (\mathbb{C}\, e^{pq} \oplus \mathbb{C}\, \overline{e}^{pq}).$$

Extend $\mathcal{B}'$ to a basis $\mathcal{B}$ of $\mathfrak{so}(2n, \mathbb{C})$ and let $\mathcal{B}^*$ denote the dual basis of $\mathfrak{so}(2n, \mathbb{C})^*$. Let ω^{ij}, $\overline{\omega}^{ij}$, ω_{pq}, $\overline{\omega}_{pq}$ be the vectors in $\mathcal{B}^*$ which are dual to e_{ij}, $\overline{e}_{ij}$, e^{pq}, $\overline{e}^{pq}$, respectively; we regard these elements as left invariant complex one-forms on $\mathrm{SO}(2n)$. If $p > q$ we agree that $\omega_{pq} = -\omega_{qp}$ and $\overline{\omega}_{pq} = -\overline{\omega}_{qp}$. Finally, define $\omega_{ij} = \gamma \omega^{ij}$, $\overline{\omega}_{ij} = \gamma \overline{\omega}^{ij}$, $\omega^{pq} = \gamma \omega_{pq}$, and $\overline{\omega}^{pq} = \gamma \overline{\omega}_{pq}$, where γ is a constant such that $\gamma^2 = \frac{i}{2\pi}$, and set $\Omega_{ij} = \omega_{ij} \wedge \overline{\omega}_{ij}$ and $\Omega^{pq} = \omega^{pq} \wedge \overline{\omega}^{pq}$.

If $\pi : \mathrm{SO}(2n) \to \mathfrak{X}$ denotes the quotient map, the pullbacks of the aforementioned curvature matrices under π can now be written explicitly, following [GrS, $(4.13)_X$] and [T5, §3.2]. In this way we arrive at the following proposition.

Proposition 2. *For every k with $1 \leq k \leq n$ we have*

$$c_1(\overline{Q}_k) = \sum_{i<k} \Omega_{ik} - \sum_{j>k} \Omega_{kj} - \sum_{p \neq k} \Omega^{pk}$$

and $K_{E_k} = \{\Theta_{\alpha\beta}\}_{1 \leq \alpha,\beta \leq k}$, where

$$\Theta_{\alpha\beta} = -\sum_{j>k} \omega_{\alpha j} \wedge \overline{\omega}_{\beta j} - \sum_{p \neq \alpha,\beta} \omega^{p\alpha} \wedge \overline{\omega}^{p\beta}.$$

Let $\Omega = \bigwedge_{i<j} \Omega_{ij} \wedge \bigwedge_{p<q} \Omega^{pq}$. It follows for instance from [PR, Cor. 5.16] that the class of a point in $\mathfrak{X}$ is Poincaré dual to $\dfrac{1}{2^{n-1}} \displaystyle\prod_{k=1}^{n-1} c_1(\overline{Q}_k^*)^{2n-2k}$. We conclude that $\displaystyle\int_{\mathfrak{X}} \Omega = \prod_{k=1}^{n-1} \frac{2}{(2k)!}$.

4. Arithmetic intersection theory on SO_{2n}/B

4.1. Orthogonal flag varieties over $\operatorname{Spec}\mathbb{Z}$**.** For the rest of this paper, $\mathfrak{X}$ will denote the Chevalley scheme over $\mathbb{Z}$ for the homogeneous space SO_{2n}/B described in §2.1. Over any base field, the scheme $\mathfrak{X}$ parametrizes complete isotropic flags $E_\bullet$ of a $2n$-dimensional vector space E equipped with the skew diagonal symmetric form, with E_n in the same family as $\langle e_1, \ldots, e_n \rangle$. The arithmetic orthogonal flag variety $\mathfrak{X}$ is smooth over $\operatorname{Spec}\mathbb{Z}$, and has a decomposition into Schubert cells induced by the Bruhat decomposition of SO_{2n} (see e.g. [Ja, §13.3] for details).

There is a tautological complete isotropic flag of vector bundles

$$E_\bullet : \ 0 = E_0 \subset E_1 \subset \cdots \subset E_{2n} = E$$

over $\mathfrak{X}$. For each i with $1 \leq i \leq 2n$ we let $\mathcal{E}_i$ denote the short exact sequence

$$\mathcal{E}_i \ : \ 0 \to E_{i-1} \to E_i \to Q_i \to 0.$$

Let $\mathrm{CH}(\mathfrak{X})$ be the Chow ring of algebraic cycles on $\mathfrak{X}$ modulo rational equivalence, with coefficients in the ring A. Since $\mathfrak{X}$ has a cellular decomposition, the class map induces an isomorphism $\mathrm{CH}(\mathfrak{X}) \cong \mathrm{H}^*(\mathfrak{X}(\mathbb{C}), A)$, following [Fu, Ex. 19.1.11] and [KM, Lem. 6].

We deduce that there is a ring isomorphism

$$\mathrm{CH}(\mathfrak{X}) \cong A[\mathrm{X}_n]/J_n.$$

This presentation of $\mathrm{CH}(\mathfrak{X})$ may be understood geometrically as follows. The Whitney sum formula applied to the filtration $E_\bullet$ gives a Chern class equation

$$\prod_{i=1}^{2n} (1 + c_1(Q_i)) = c(E)$$

in $\mathrm{CH}(\mathfrak{X})$, which maps to the identity $\prod_{i=1}^{2n}(1 - \mathrm{x}_i^2) = 1$, since E is a trivial bundle. We thus obtain the relations $e_i(\mathrm{X}_n^2)$ in J_n, for $1 \leq i \leq n-1$. Moreover, the relation $\mathrm{x}_1 \cdots \mathrm{x}_n$ holds because the top Chern class $c_n(E_n)$ vanishes.

We have an isomorphism of abelian groups

$$\mathrm{CH}(\mathfrak{X}) \cong \bigoplus_{w\in\widetilde{W}_n} A\,\mathfrak{D}_w(\mathrm{X}_n)$$

where the polynomial $\mathfrak{D}_w(\mathrm{X}_n)$ represents the class of the codimension $\ell(w)$ Schubert scheme $\mathfrak{X}_w$ in $\mathfrak{X}$. The latter is defined as the closure of the corresponding Schubert cell, the complex points of which are given in §2.1.

4.2. The arithmetic Chow group. For $p \geq 0$ we let $\widehat{\mathrm{CH}}^p(\mathfrak{X})'$ denote the p-th arithmetic Chow group of $\mathfrak{X}$, as defined by Gillet and Soulé [GS1]. As in the case of $\mathrm{CH}(\mathfrak{X})$, we require coefficients in the ring A, so we will work throughout with the groups $\widehat{\mathrm{CH}}^p(\mathfrak{X}) := \widehat{\mathrm{CH}}^p(\mathfrak{X})' \otimes_{\mathbb{Z}} A$. The elements in $\widehat{\mathrm{CH}}^p(\mathfrak{X})$ are represented by arithmetic cycles (Z, g_Z), where Z is a codimension p cycle on $\mathfrak{X}$ and g_Z is a current of type $(p-1, p-1)$ such that the current $dd^c g_Z + \delta_{Z(\mathbb{C})}$ is represented by a smooth differential form on $\mathfrak{X}(\mathbb{C})$. Define $\widehat{\mathrm{CH}}(\mathfrak{X}) = \bigoplus_p \widehat{\mathrm{CH}}^p(\mathfrak{X})$.

Let $\mathcal{A}(\mathfrak{X}(\mathbb{C})) = \bigoplus_p \mathcal{A}^{p,p}(\mathfrak{X}(\mathbb{C}))$ and $\mathcal{A}'(\mathfrak{X}(\mathbb{C})) \subset \mathcal{A}(\mathfrak{X}(\mathbb{C}))$ be the set of forms φ in $\mathcal{A}(\mathfrak{X}(\mathbb{C}))$ which can be written as $\varphi = \partial\eta + \overline{\partial}\eta'$ for some smooth forms η, η'. Define $\widetilde{\mathcal{A}}(\mathfrak{X}(\mathbb{C})) = \mathcal{A}(\mathfrak{X}(\mathbb{C}))/\mathcal{A}'(\mathfrak{X}(\mathbb{C}))$. We let F_∞ be the involution of $\mathfrak{X}(\mathbb{C})$ induced by complex conjugation. Let $\mathcal{A}^{p,p}(\mathfrak{X}_{\mathbb{R}})$ be the subspace of $\mathcal{A}^{p,p}(\mathfrak{X}(\mathbb{C}))$ generated by real forms η such that $F_\infty^*\eta = (-1)^p\eta$; denote by $\widetilde{\mathcal{A}}^{p,p}(\mathfrak{X}_{\mathbb{R}})$ the image of $\mathcal{A}^{p,p}(\mathfrak{X}_{\mathbb{R}})$ in $\widetilde{\mathcal{A}}^{p,p}(\mathfrak{X}(\mathbb{C}))$. Finally, let $\mathcal{A}(\mathfrak{X}_{\mathbb{R}}) = \bigoplus_p \mathcal{A}^{p,p}(\mathfrak{X}_{\mathbb{R}})$ and $\widetilde{\mathcal{A}}(\mathfrak{X}_{\mathbb{R}}) = \bigoplus_p \widetilde{\mathcal{A}}^{p,p}(\mathfrak{X}_{\mathbb{R}})$.

Since the homogeneous space $\mathfrak{X}$ admits a cellular decomposition, it follows as in [KM] that there is an exact sequence

$$0 \longrightarrow \widetilde{\mathcal{A}}(\mathfrak{X}_{\mathbb{R}}) \xrightarrow{a} \widehat{\mathrm{CH}}(\mathfrak{X}) \xrightarrow{\zeta} \mathrm{CH}(\mathfrak{X}) \longrightarrow 0 \tag{5}$$

where the maps a and ζ are defined by

$$a(\eta) = (0, \eta) \qquad \text{and} \qquad \zeta(Z, g_Z) = Z.$$

We equip $E(\mathbb{C})$ with the trivial hermitian metric compatible with the skew diagonal symmetric form $[\ ,\ ]$ on $\mathbb{C}^{2n}$. This metric induces metrics on (the complex points of) all the vector bundles E_i and the line bundles $L_i = E_{n+1-i}/E_{n-i}$, for $1 \leq i \leq n$. We thus obtain hermitian vector bundles $\overline{E}_i$ and line bundles $\overline{L}_i$ and, following [GS2], their arithmetic Chern classes $\widehat{c}_k(\overline{E}_i) \in \widehat{\mathrm{CH}}^k(\mathfrak{X})$ and $\widehat{c}_1(\overline{L}_i) \in \widehat{\mathrm{CH}}^1(\mathfrak{X})$. Set $\widehat{x}_i = -\widehat{c}_1(\overline{L}_i)$ and for any $w \in \widetilde{W}_n$, define

$$\widehat{\mathfrak{D}}_w := \mathfrak{D}_w(\widehat{x}_1, \ldots, \widehat{x}_n) \in \widehat{\mathrm{CH}}^{\ell(w)}(\mathfrak{X}).$$

The unique map of abelian groups

$$\epsilon\,:\,\mathrm{CH}(\mathfrak{X}) \to \widehat{\mathrm{CH}}(\mathfrak{X}) \tag{6}$$

sending the Schubert class $\mathfrak{D}_w(\mathrm{X}_n)$ to $\widehat{\mathfrak{D}}_w$ for all $w \in \widetilde{W}_n$ splits the exact sequence (5). We thus obtain an isomorphism of abelian groups

$$\widehat{\mathrm{CH}}(\mathfrak{X}) \cong \mathrm{CH}(\mathfrak{X}) \oplus \widetilde{\mathcal{A}}(\mathfrak{X}_{\mathbb{R}}). \tag{7}$$

4.3. Computing arithmetic intersections. We now describe an effective procedure for computing arithmetic Chern numbers on the orthogonal flag variety $\mathfrak{X}$, parallel to [T5, §4.3]. Let $c_k(\overline{E}_i)$ and $c_1(\overline{L}_i)$ denote the Chern forms of $\overline{E_i(\mathbb{C})}$ and $\overline{L_i(\mathbb{C})}$, respectively. In the sequel we will identify these with their images in $\widehat{\mathrm{CH}}(\mathfrak{X})$ under the inclusion a. Let $x_i = -c_1(\overline{L}_i)$ for $1 \leq i \leq n$.

We begin with the short exact sequence

$$\overline{\mathcal{E}}_{\mathrm{OG}} \ : \ 0 \to \overline{E}_n \to \overline{E} \to \overline{E}_n^* \to 0$$

where E_n denotes the tautological maximal isotropic subbundle of E over $\mathfrak{X}$. Let $\widetilde{c}(\overline{\mathcal{E}}_{\mathrm{OG}}) \in \widetilde{\mathcal{A}}(\mathfrak{X}_{\mathbb{R}})$ be the *Bott-Chern form* [BC, GS2] associated to $\overline{\mathcal{E}}_{\mathrm{OG}}$ for the total Chern class. This form may be computed using [T1, Prop. 3], which gives

$$\widetilde{c}(\overline{\mathcal{E}}_{\mathrm{OG}}) = \sum_{k=1}^{n-1} (-1)^k \mathcal{H}_k p_k(\overline{E}_n^*). \tag{8}$$

Here $p_r(\overline{E}_n^*) = (-1)^r \mathrm{Tr}((K_{E_n})^r)$ denotes the r-th power sum form of $\overline{E}_n^*$, while $\mathcal{H}_r = 1 + \frac{1}{2} + \cdots + \frac{1}{r}$ is a harmonic number. Furthermore, by [GS2, Thm. 4.8(ii)], we have an equation

$$\widehat{c}(\overline{E}_n)\, \widehat{c}(\overline{E}_n^*) = 1 + \widetilde{c}(\overline{\mathcal{E}}_{\mathrm{OG}}) \tag{9}$$

in $\widehat{\mathrm{CH}}(\mathfrak{X})$.

Consider the hermitian filtration

$$\overline{\mathcal{E}} \ : \ 0 = \overline{E}_0 \subset \overline{E}_1 \subset \cdots \subset \overline{E}_n.$$

Let $\widetilde{c}(\overline{\mathcal{E}}) \in \widetilde{\mathcal{A}}(\mathfrak{X}_{\mathbb{R}})$ be the Bott-Chern form of the hermitian filtration $\overline{\mathcal{E}}$ corresponding to the total Chern class, as defined in [T2]. According to [T2, Thm. 2], we have

$$\prod_{i=1}^{n} (1 - \widehat{x}_i) = \widehat{c}(\overline{E}_n) + \widetilde{c}(\overline{\mathcal{E}}). \tag{10}$$

If $\widetilde{c}(\overline{\mathcal{E}}) = \sum_i \alpha_i$ with $\alpha_i \in \widetilde{\mathcal{A}}^{i,i}(\mathfrak{X}_{\mathbb{R}})$ for each i, then define $\widetilde{c}(\overline{\mathcal{E}}^*) = \sum_i (-1)^{i+1} \alpha_i$. This gives the dual equation

$$\prod_{i=1}^{n} (1 + \widehat{x}_i) = \widehat{c}(\overline{E}_n^*) + \widetilde{c}(\overline{\mathcal{E}}^*). \tag{11}$$

The abelian group $\widetilde{\mathcal{A}}(\mathfrak{X}_{\mathbb{R}}) = \mathrm{Ker}\zeta$ is an ideal of $\widehat{\mathrm{CH}}(\mathfrak{X})$ such that for any hermitian vector bundle $\overline{F}$ over $\mathfrak{X}$ and $\eta, \eta' \in \widetilde{\mathcal{A}}(\mathfrak{X}_{\mathbb{R}})$, we have

$$\widehat{c}_k(\overline{F}) \cdot \eta = c_k(\overline{F}) \wedge \eta \qquad \text{and} \qquad \eta \cdot \eta' = (dd^c \eta) \wedge \eta'. \tag{12}$$

We now multiply (10) with (11) and combine the result with (9) to obtain

$$\prod_{i=1}^{n} (1 - \widehat{x}_i^2) = 1 + \widetilde{c}(\overline{\mathcal{E}}, \overline{\mathcal{E}}^*), \tag{13}$$

where

$$\widetilde{c}(\overline{\mathcal{E}}, \overline{\mathcal{E}}^*) = \widetilde{c}(\overline{\mathcal{E}}_{\mathrm{OG}}) + \widetilde{c}(\overline{\mathcal{E}}) \wedge c(\overline{E}_n^*) + \widetilde{c}(\overline{\mathcal{E}}^*) \wedge c(\overline{E}_n) + (dd^c \widetilde{c}(\overline{\mathcal{E}})) \wedge \widetilde{c}(\overline{\mathcal{E}}^*). \tag{14}$$

By pulling back [T4, Eqn. (6)] to $\mathfrak{X}$, we get the equation

$$\widehat{c}(\overline{E}_n^*) = \frac{1}{2} \mathcal{H}_{n-1} c_{n-1}(\overline{E}_n^*). \tag{15}$$

Equating the top degree terms in (11) and (15) gives

$$\widehat{x}_1\cdots\widehat{x}_n=\frac{1}{2}\mathcal{H}_{n-1}c_{n-1}(\overline{E}_n^*)+\widetilde{c}_n(\overline{\mathcal{E}}^*). \tag{16}$$

In [T2], it is shown that $\widetilde{c}(\overline{\mathcal{E}})$ is a polynomial in the entries of the matrices K_{E_i} and K_{L_i}, $1\leq i\leq n$, with *rational* coefficients. Using this, (8), and (14), we can express the differential form $\widetilde{c}(\overline{\mathcal{E}},\overline{\mathcal{E}}^*)$ as a polynomial in the entries of the matrices K_{E_i} and K_{L_i} with rational coefficients. On the other hand, Proposition 2 gives explicit formulas for all these curvature matrices in terms of $\mathrm{SO}(2n)$-invariant differential forms on $\mathfrak{X}(\mathbb{C})$. Since we are using the skew diagonal symmetric form to define the Lie groups here, the formulas in §3 have to be changed accordingly. The matrix realization of the Lie algebra $\mathfrak{so}(2n,\mathbb{C})$ in this case is given in [GW, §1.2, §2.3], while the basis elements of $\mathfrak{h}$ should be ordered as in [BH, (2.20)]. The indices (i,j) and (p,q) in Proposition 2 are then replaced by $(n+1-j,n+1-i)$ and $(n+1-q,n+1-p)$, respectively. Recalling that $L_i=E_{n+1-i}/E_{n-i}$, we obtain the identities

$$\begin{aligned}
x_1 &= -\Omega_{12}-\Omega_{13}-\cdots-\Omega_{1n}+\Omega^{12}+\Omega^{13}+\cdots+\Omega^{1n}\\
x_2 &= \quad\Omega_{12}-\Omega_{23}-\cdots-\Omega_{2n}+\Omega^{12}+\Omega^{23}+\cdots+\Omega^{2n}\\
&\qquad\vdots\qquad\qquad\quad\vdots\qquad\qquad\quad\vdots\\
x_n &= \quad\Omega_{1n}+\Omega_{2n}+\cdots+\Omega_{n-1,n}+\Omega^{1n}+\Omega^{2n}+\cdots+\Omega^{n-1,n}
\end{aligned}$$

in $\mathcal{A}^{1,1}(\mathfrak{X}_{\mathbb{R}})$. We also deduce the next result.

Proposition 3. *We have* $\widetilde{c}_1(\overline{\mathcal{E}})=\widetilde{c}_1(\overline{\mathcal{E}},\overline{\mathcal{E}}^*)=0$, $\widetilde{c}_2(\overline{\mathcal{E}})=-\sum_{i<j}\Omega_{ij}$, *and*

$$\widetilde{c}_2(\overline{\mathcal{E}},\overline{\mathcal{E}}^*)=-2\sum_{i<j}\Omega_{ij}-2\sum_{p<q}\Omega^{pq}.$$

Proof. The argument is the same as the proof of [T5, Prop. 4]. □

Let $h(\mathrm{X}_n)$ be a homogeneous polynomial in the ideal J_n of §2.1. We give an effective algorithm to compute the arithmetic intersection $h(\widehat{x}_1,\ldots,\widehat{x}_n)$ as a class in $\widetilde{\mathcal{A}}(\mathfrak{X}_{\mathbb{R}})$. First, we decompose h as a sum $h(\mathrm{X}_n)=\sum_i e_i(\mathrm{X}_n^2)f_i(\mathrm{X}_n)+e_n(\mathrm{X}_n)g(\mathrm{X}_n)$ for some polynomials f_i and g. Equation (13) implies that

$$e_i(\widehat{x}_1^2,\ldots,\widehat{x}_n^2)=(-1)^i\,\widetilde{c}_{2i}(\overline{\mathcal{E}},\overline{\mathcal{E}}^*) \tag{17}$$

for $1\leq i\leq n$. Using this, (16), and (12), we see that

$$\begin{aligned}
h(\widehat{x}_1,\widehat{x}_2,\ldots\widehat{x}_n)=&\sum_{i=1}^n(-1)^i\,\widetilde{c}_{2i}(\overline{\mathcal{E}},\overline{\mathcal{E}}^*)\wedge f_i(x_1,\ldots,x_n)\\
&+\left(\frac{1}{2}\mathcal{H}_{n-1}c_{n-1}(\overline{E}_n^*)+\widetilde{c}_n(\overline{\mathcal{E}}^*)\right)\wedge g(x_1,\ldots,x_n)
\end{aligned}$$

in $\widehat{\mathrm{CH}}(\mathfrak{X})$. By the previous analysis, we can write the right hand side of the above equation as a polynomial in the x_i and the entries of the matrices K_{E_i} for $1\leq i\leq n$, with rational coefficients, which is (the class of) an explicit $\mathrm{SO}(2n)$-invariant differential form in $\widetilde{\mathcal{A}}(\mathfrak{X}_{\mathbb{R}})$.

Let $\widehat{\deg}:\widehat{\mathrm{CH}}^{n^2-n+1}(\mathfrak{X})\to\mathbb{R}$ denote the arithmetic degree map of [GS1].

Theorem 2. *For any nonnegative integers $k_1,\ldots,k_n$ with $\sum k_i = n^2-n+1$, the arithmetic Chern number $\widehat{\deg}(\widehat{x}_1^{k_1}\widehat{x}_2^{k_2}\cdots\widehat{x}_n^{k_n})$ is a rational number.*

Proof. Since $\sum k_i = \dim \mathfrak{X} = n^2-n+1$, the monomial $\mathrm{x}_1^{k_1}\cdots\mathrm{x}_n^{k_n}$ lies in the ideal J_n. We therefore obtain

$$\widehat{x}_1^{k_1}\widehat{x}_2^{k_2}\cdots\widehat{x}_n^{k_n} = r\,\Omega$$

for some $r\in\mathbb{Q}$, where Ω is the top invariant form of §3. Using the computation at the end of §3, it follows that

$$\widehat{\deg}(\widehat{x}_1^{k_1}\widehat{x}_2^{k_2}\cdots\widehat{x}_n^{k_n}) = \frac{r}{2}\prod_{k=1}^{n-1}\frac{2}{(2k)!}.$$

□

The flag variety $\mathfrak{X}$ has a natural pluri-Plücker embedding j in projective space. The morphism j is defined as the composite of the natural inclusion of $\mathfrak{X}$ into the variety parametrizing all partial flags

$$0=E_0\subset E_1\subset\cdots\subset E_n\subset E_{2n}=E$$

with $\dim(E_i)=i$ for each i, followed by the pluri-Plücker embedding of the latter type A flag variety into projective space. Let $\overline{\mathcal{O}}(1)$ denote the canonical line bundle over projective space, equipped with its canonical metric (so that $c_1(\overline{\mathcal{O}}(1))$ is the Fubini-Study form). Following [GS1, Fa, BoGS], the *projective height* of $\mathfrak{X}$ relative to $\overline{\mathcal{O}}(1)$ is given by

$$h_{\overline{\mathcal{O}}(1)}(\mathfrak{X}) = \widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{O}}(1))^{n^2-n+1}|\ \mathfrak{X}\right).$$

Using Theorem 2 and arguing as in [T5, §4.6], we conclude that the projective height $h_{\overline{\mathcal{O}}(1)}(\mathrm{SO}_{2n}/B)$ is a rational number. The height formula of Kaiser and Köhler [KK] provides a different proof of this fact. Relating these two approaches to computing the height to each other seems rather difficult; some first steps in this direction are taken in [T3, T4].

4.4. Arithmetic Schubert calculus. For any partition $\lambda\in\mathcal{G}_n$ and $\varpi\in S_n$, define

$$\widehat{\mathfrak{D}}_{\lambda,\varpi} = \mathfrak{D}_{\lambda,\varpi}(\widehat{x}_1,\ldots,\widehat{x}_n).$$

If $\lambda\in\mathcal{G}_n\smallsetminus\mathcal{F}_{n-1}$, let r_λ be the largest repeated part of λ, and let $\overline{\lambda}$ be the partition obtained from λ by deleting two (respectively, one) of the parts r_λ if $r_\lambda<n$ (respectively, if $r_\lambda=n$). If $r_\lambda<n$, then properties (b), (c) in §1.1, (12), and (17) imply that

$$\widehat{\mathfrak{D}}_{\lambda,\varpi} = \widehat{\mathfrak{D}}_{\overline{\lambda},\varpi}\widetilde{P}_{r_\lambda,r_\lambda}(\widehat{x}_1^2,\ldots,\widehat{x}_n^2) = \frac{(-1)^{r_\lambda}}{4}\mathfrak{D}_{\overline{\lambda},\varpi}(x_1,\ldots,x_n)\wedge\widetilde{c}_{2r_\lambda}(\overline{\mathcal{E}},\overline{\mathcal{E}}^*).$$

If $r_\lambda=n$, then property (f) in §1.1, (12), and (16) give

$$\widehat{\mathfrak{D}}_{\lambda,\varpi} = \widehat{\mathfrak{D}}_{\overline{\lambda},\varpi}\widetilde{P}_n(\widehat{x}_1,\ldots,\widehat{x}_n) = \frac{1}{2}\mathfrak{D}_{\overline{\lambda},\varpi}(x_1,\ldots,x_n)\wedge(\frac{1}{2}\mathcal{H}_{n-1}c_{n-1}(\overline{E}_n^*)+\widetilde{c}_n(\overline{\mathcal{E}}^*)).$$

Since $\widehat{\mathfrak{D}}_{\lambda,\varpi}\in a(\widetilde{\mathcal{A}}(\mathfrak{X}_\mathbb{R}))$ whenever $\lambda\in\mathcal{G}_n\smallsetminus\mathcal{F}_{n-1}$, we will denote these classes by $\widetilde{\mathfrak{D}}_{\lambda,\varpi}$. The next theorem uses the basis of orthogonal Schubert polynomials to compute arbitrary arithmetic intersections in $\widehat{\mathrm{CH}}(\mathfrak{X})$ with respect to the splitting (7) induced by (6).

Theorem 3. *Any element of the arithmetic Chow ring $\widehat{CH}(\mathfrak{X})$ can be expressed uniquely in the form* $\sum_{w\in\widetilde{W}_n} a_w\widehat{\mathfrak{D}}_w+\eta$*, where $a_w\in A$ and $\eta\in\widetilde{\mathcal{A}}(\mathfrak{X}_{\mathbb{R}})$. For $u,v\in\widetilde{W}_n$ we have*

$$\widehat{\mathfrak{D}}_u\cdot\widehat{\mathfrak{D}}_v=\sum_{w\in\widetilde{W}_n} d_{uv}^w\,\widehat{\mathfrak{D}}_w+\sum_{\substack{\lambda\in\mathcal{G}_n\smallsetminus\mathcal{F}_{n-1}\\ \varpi\in S_n}} d_{uv}^{\lambda\varpi}\,\widetilde{\mathfrak{D}}_{\lambda,\varpi}, \tag{18}$$

$$\widehat{\mathfrak{D}}_u\cdot\eta=\mathfrak{D}_u(x_1,\ldots,x_n)\wedge\eta,\qquad and\qquad \eta\cdot\eta'=(dd^c\eta)\wedge\eta',$$

where η, $\eta'\in\widetilde{\mathcal{A}}(\mathfrak{X}_{\mathbb{R}})$ and the integers d_{uv}^w, $d_{uv}^{\lambda\varpi}$ are as in (4).

Proof. The first statement is a consequence of the splitting (7). Equation (18) is a consequence of the formal identity (4) and our definitions of $\widehat{\mathfrak{D}}_w$ and $\widetilde{\mathfrak{D}}_{\lambda,\varpi}$. The remaining assertions follow from the structure equations (12). ☐

We remark that one can refine Theorem 3 by replacing $\widehat{\mathrm{CH}}(\mathfrak{X})$ with the invariant arithmetic Chow ring $\widehat{\mathrm{CH}}_{\mathrm{inv}}(\mathfrak{X})$. Following [T5, §4.5], the ring $\widehat{\mathrm{CH}}_{\mathrm{inv}}(\mathfrak{X})$ is obtained by substituting the space $\mathcal{A}(\mathfrak{X}_{\mathbb{R}})$ with a certain subspace of the space of all $\mathrm{SO}(2n)$-invariant differential forms on $\mathfrak{X}(\mathbb{C})$. We leave the details to the reader.

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