Bott-Chern Forms and Arithmetic Intersections

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Abstract

Let $\overline{\mathscr{E}}: \ 0 \to \overline{S} \to \overline{E} \to \overline{Q} \to 0$ be a short exact sequence of hermitian vector bundles with metrics on S and Q induced from that on E. We compute the Bott-Chern form $\widetilde{\phi}(\overline{\mathscr{E}})$ corresponding to any characteristic class ϕ, assuming $\overline{E}$ is projectively flat. The result is used to obtain a new presentation of the Arakelov Chow ring of the arithmetic Grassmannian.

1 Introduction

Arakelov theory is an intersection theory for varieties over rings $\mathcal{O}_F$ of algebraic integers, analogous to the usual one over fields. The fundamental idea is that in order to have a good theory of intersection numbers, one has to include information at the infinite primes.

The work of Arakelov in dimension two has been generalized by Gillet and Soulé to higher dimensional *arithmetic varieties* X, by which we mean regular, projective and flat schemes over Spec$\mathbb{Z}$. They define an *arithmetic Chow ring* $\widehat{CH}(X)_{\mathbb{Q}}$ whose elements are represented by cycles on X together with Green currents on $X(\mathbb{C})$. The theory is a blend of arithmetic, algebraic geometry and complex hermitian geometry. For example, the Faltings height of an arithmetic variety X is realized as an 'arithmetic degree' with respect to a hermitian line bundle over X.

A *hermitian vector bundle* $\overline{E} = (E, h)$ over X is an algebraic vector bundle E on X together with a hermitian metric h on the corresponding holomorphic vector bundle $E(\mathbb{C})$ on the complex manifold $X(\mathbb{C})$. To such an object one associates arithmetic Chern classes $\widehat{c}(\overline{E})$ with values in $\widehat{CH}(X)$.

These satisfy most of the usual properties of Chern classes, with one exception: given a short exact sequence of hermitian vector bundles

$$\overline{\mathscr{E}}:\ 0 \to \overline{S} \to \overline{E} \to \overline{Q} \to 0 \tag{1}$$

the class $\widehat{c}(\overline{S})\widehat{c}(\overline{Q}) - \widehat{c}(\overline{E})$ vanishes when $\overline{E}$ is the orthogonal direct sum of $\overline{S}$ and $\overline{Q}$. In general however this difference is non-zero and is the image in $\widehat{CH}(X)$ of a differential form on $X(\mathbb{C})$, the *Bott-Chern form* associated to the exact sequence $\overline{\mathscr{E}}$.

These secondary characteristic classes were originally defined by Bott and Chern [BC] with applications to value distribution theory. They later occured in the work of Donaldson [Do] on Hermitian-Einstein metrics. Bismut, Gillet and Soulé [BiGS] gave a new axiomatic definition for Bott-Chern forms, suitable for use in arithmetic intersection theory. Given an exact sequence of hermitian holomorphic vector bundles as in (1), we have $c(\overline{S})c(\overline{Q}) - c(\overline{E}) = dd^c\eta$ for some form η; the Bott-Chern form of $\overline{\mathscr{E}}$ is a natural choice of such an η.

Calculating these forms is important because they give relations in the arithmetic Chow ring of an arithmetic variety. No systematic work has appeared on this; rather one finds scattered calculations throughout the literature (see for example [BC], [C1], [D], [GS2], [GSZ], [Ma], [Mo]). We confine ourselves to the case where the metrics on S and Q are induced from the one on E. Our goal is to give explicit formulas for the Bott-Chern forms corresponding to *any* characteristic class, when they can be expressed in terms of the characteristic classes of the bundles involved. This is not always possible as these forms are not closed in general; however the situation is completely understood when E is a projectively flat bundle. The results build on the work of Bott, Chern, Cowen, Deligne, Gillet, Soulé and Maillot. Some of our calculations overlap with previous work, but with simpler proofs.

The main application we give to arithmetic intersection theory is a new presentation of the Arakelov Chow ring of the Grassmannian over Spec$\mathbb{Z}$. Maillot [Ma] gave a presentation of this ring and formulated an 'arithmetic Schubert calculus'. We hope our work contributes towards a better understanding of these intersections.

This paper is organized as follows. Section 2 is a review of some basic material on invariant and symmetric functions. In §3 we recall the hermitian geometry we will need, including the definition of Bott-Chern forms. The basic tool for calculating these forms is reviewed in §4, with some applications that have appeared before in the literature. §7 is mainly an exposition of the arithmetic intersection theory that we require. The rest of the paper is new. In sections 5 and 6 we derive formulas for computing Bott-Chern forms of short exact sequences (with the induced metrics) for any characteristic class when $\overline{E}$ is flat or more generally projectively flat. We emphasize the central role played by the *power sum forms* in the results; to our knowledge this phenomenon has not been observed before. The combinatorial identities involving harmonic numbers that we encounter are also interesting. Sections 2-6 contain results in hermitian complex geometry and may be read without prior knowledge of Arakelov theory. §8 applies our calculations to obtain a presentation of the Arakelov Chow ring of the arithmetic Grassmannian.

This should be regarded as a companion paper to [T]; both papers will be part of the author's 1997 University of Chicago thesis. I wish to thank my advisor William Fulton for many useful conversations and exchanges of ideas.

2 Invariant and symmetric functions

The symmetric group S_n acts on the polynomial ring $\mathbb{Z}[x_1, x_2, \ldots, x_n]$ by permuting the variables, and the ring of invariants $\Lambda(n) = \mathbb{Z}[x_1, x_2, \ldots, x_n]^{S_n}$ is the ring of symmetric polynomials. For $B = \mathbb{Q}$ or $\mathbb{C}$, let $\Lambda(n, B) = \Lambda(n) \otimes_{\mathbb{Z}} B$.

Let $e_k(x_1, \ldots, x_n)$ be the k-th elementary symmetric polynomial in the variables $x_1, \ldots, x_n$ and $p_k(x_1, \ldots, x_n) = \sum_i x_i^k$ the k-th power sum. The fundamental theorem on symmetric functions states that $\Lambda(n) = \mathbb{Z}[e_1, \ldots, e_n]$ and that $e_1, \ldots, e_n$ are algebraically independent. For λ a partition, i.e. a decreasing sequence $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m$ of nonnegative integers, define $p_\lambda := \prod_{i=1}^m p_{\lambda_i}$. It is well known that the p_λ's form an additive $\mathbb{Q}$-basis for the ring of symmetric polynomials (cf. [M], §2). The two bases are related by Newton's identity:

$$p_k - e_1 p_{k-1} + e_2 p_{k-2} - \cdots + (-1)^k k e_k = 0. \tag{2}$$

Another important set of symmetric functions related to the cohomology ring of Grassmannians are the Schur polynomials. For a partition λ as above, the Schur polynomial s_λ is defined by

$$s_\lambda(x_1, \ldots, x_n) = \frac{1}{\Delta} \cdot \det(x_i^{\lambda_j + n - j})_{1 \leqslant i,j \leqslant n},$$

where $\Delta = \prod_{1 \leqslant i < j \leqslant n} (x_i - x_j)$ is the Vandermonde determinant. The s_λ for all λ of length $m \leqslant n$ form a $\mathbb{Z}$-basis of $\Lambda(n)$ (cf. [M], §I.3).

Let $\mathbb{C}[T_{ij}]$ $(1 \leqslant i, j \leqslant n)$ be the coordinate ring of the space $M_n(\mathbb{C})$ of $n \times n$ matrices. $GL_n(\mathbb{C})$ acts on matrices by conjugation, and we let $I(n) = \mathbb{C}[T_{ij}]^{GL_n(\mathbb{C})}$ denote the corresponding graded ring of invariants. There is an isomorphism $\tau : I(n) \to \Lambda(n, \mathbb{C})$ given by evaluating an invariant polynomial ϕ on the diagonal matrix $\mathrm{diag}(x_1, \ldots, x_n)$. We will often identify ϕ with the the symmetric polynomial $\tau(\phi)$. We will need to consider invariant polynomials with rational coefficients; let $I(n, \mathbb{Q}) \simeq \mathbb{Q}[x_1, x_2, \ldots, x_n]^{S_n}$ be the corresponding ring.

Given $\phi \in I(n)$, let ϕ' be a k-multilinear form on $M_n(\mathbb{C})$ such that

$$\phi'(gA_1g^{-1}, \ldots, gA_kg^{-1}) = \phi'(A_1, \ldots, A_k)$$

for $g \in GL(n, \mathbb{C})$ and $\phi(A) = \phi'(A, A, \ldots, A)$. Such forms are most easily constructed for the power sums p_k by setting

$$p'_k(A_1, A_2, \ldots, A_k) = \mathrm{Tr}(A_1 A_2 \cdots A_k).$$

For p_λ we can take $p'_\lambda = \prod p'_{\lambda_i}$. Since the p_λ's are a basis of $\Lambda(n, \mathbb{Q})$, it follows that one can use the above constructions to find multilinear forms ϕ' for any $\phi \in I(n)$.

An explicit formula for ϕ' is given by polarizing ϕ:

$$\phi'(A_1, \ldots, A_k) = \frac{(-1)^k}{k!} \sum_{j=1}^{k} \sum_{i_1 < \ldots < i_j} (-1)^j \phi(A_{i_1} + \ldots + A_{i_j}).$$

Although above formula for ϕ' is symmetric in $A_1, \ldots, A_k$, this property is not needed for the applications that follow.

3 Hermitian differential geometry

Let X be a complex manifold, E a rank n holomorphic vector bundle over X. Denote by $A^k(X, E)$ the C^∞ sections of $\Lambda^k T^*X \otimes E$, where T^*X denotes the cotangent bundle of X. In particular $A^k(X)$ is the space of smooth complex k-forms on X. Let $A^{p,q}(X)$ the space of smooth complex forms of type (p, q) on X and $A(X) := \bigoplus_p A^{p,p}(X)$. The decomposition $A^1(X, E) = A^{1,0}(X, E) \bigoplus A^{0,1}(X, E)$ induces a decomposition $D = D^{1,0} + D^{0,1}$ of each connection D on E. Let $d = \partial + \overline{\partial}$ and $d^c = (\partial - \overline{\partial})/(4\pi i)$.

Assume now that E is equipped with a hermitian metric h. The pair (E, h) is called a *hermitian vector bundle*. The metric h induces a canonical connection $D = D(h)$ such that $D^{0,1} = \overline{\partial}_E$ and D is *unitary*, i.e.

$$d\,h(s,t) = h(Ds,t) + h(s,Dt), \text{ for all } s,t \in A^0(X,E).$$

The connection D is called the *hermitian holomorphic connection* of (E, h). D can be extended to E-valued forms by using the Leibnitz rule:

$$D(\omega \otimes s) = d\omega \otimes s + (-1)^{\deg \omega} \omega \otimes Ds.$$

The composite

$$K = D^2 : A^0(X,E) \to A^2(X,E)$$

is $A^0(X)$-linear; hence $K \in A^2(X, \mathrm{End}(E))$. In fact $K = D^{1,1} \in A^{1,1}(X, \mathrm{End}(E))$, because $D^{0,2} = \overline{\partial}_E^2 = 0$, so $D^{2,0}$ also vanishes by unitarity. K is called the *curvature* of D.

Given a hermitian vector bundle $\overline{E} = (E, h)$ and an invariant polynomial $\phi \in I(n)$ there is an associated differential form $\phi(\overline{E}) := \phi(\frac{i}{2\pi}K)$, defined locally by identifying $\mathrm{End}(E)$ with $M_n(\mathbb{C})$; $\phi(\overline{E})$ makes sense globally on X since ϕ is invariant by conjugation. These differential forms are d and d^c closed and have the following properties (cf. [BC]):
(i) The de Rham cohomology class of $\phi(\overline{E})$ is independent of the metric h and coincides with the usual characteristic class from topology.
(ii) For every holomorphic map $f : X \to Y$ of complex manifolds,

$$f^*(\phi(E,h)) = \phi(f^*E, f^*h).$$

One thus obtains the *Chern forms* $c_k(\overline{E})$ with $c_k = e_k(x_1, \dots, x_n)$, the *power sum forms* $p_k(\overline{E})$, the *Chern character form* $ch(\overline{E})$ with $ch(x_1, \dots, x_n) = \sum_i \exp(x_i) = \sum_k \frac{p_k}{k!}$, etc.

We fix some more notation: A direct sum $\overline{E}_1 \bigoplus \overline{E}_2$ of hermitian vector bundles will always mean the orthogonal direct sum $(E_1 \bigoplus E_2, h_1 \oplus h_2)$. Let $\widetilde{A}(X)$ be the quotient of $A(X)$ by $\mathrm{Im}\partial + \mathrm{Im}\overline{\partial}$. If $\omega \in A(X)$ is a closed form, the cup product $\wedge\omega : \widetilde{A}(X) \to \widetilde{A}(X)$ and the operator $dd^c : \widetilde{A}(X) \to A(X)$ are well defined.

Let $\mathscr{E} : \ 0 \to S \to E \to Q \to 0$ be an exact sequence of holomorphic vector bundles on X. Choose arbitrary hermitian metrics h_S, h_E, h_Q on S, E, Q respectively. Let

$$\overline{\mathscr{E}} = (\mathscr{E}, h_S, h_E, h_Q) : \ 0 \to \overline{S} \to \overline{E} \to \overline{Q} \to 0.$$

Note that we do not in general assume that the metrics h_S or h_Q are induced from h_E. We say that $\overline{\mathscr{E}}$ is *split* when $(E, h_E) = (S \bigoplus Q, h_S \oplus h_Q)$ and $\mathscr{E}$ is the obvious exact sequence. Following [GS2], we have the following

Theorem 1 *Let $\phi \in I(n)$ be any invariant polynomial. There is a unique way to attach to every exact sequence $\overline{\mathscr{E}}$ a form $\widetilde{\phi}(\overline{\mathscr{E}})$ in $\widetilde{A}(X)$ in such a way that:*
(i) $dd^c\widetilde{\phi}(\overline{\mathscr{E}}) = \phi(\overline{S} \bigoplus \overline{Q}) - \phi(\overline{E})$,
(ii) *For every map $f : X \to Y$ of complex manifolds, $\widetilde{\phi}(f^*(\overline{\mathscr{E}})) = f^*\widetilde{\phi}(\overline{\mathscr{E}})$,*
(iii) *If $\overline{\mathscr{E}}$ is split, then $\widetilde{\phi}(\overline{\mathscr{E}}) = 0$.*

In [BC], Bott and Chern solved the equation $dd^c\widetilde{\phi}(\overline{\mathscr{E}}) = \phi(\overline{S} \bigoplus \overline{Q}) - \phi(\overline{E})$ when the metrics on S and Q are induced from the metric on E. In [BiGS] a new axiomatic definition of these forms was given, more generally for an acyclic complex of holomorphic vector bundles on X.

The following useful calculation is an immediate consequence of the definition ([GS2], Prop. 1.3.1):

Proposition 1 *Let ϕ and ψ be two invariant polynomials. Then*

$$\widetilde{\phi + \psi}(\overline{\mathscr{E}}) = \widetilde{\phi}(\overline{\mathscr{E}}) + \widetilde{\psi}(\overline{\mathscr{E}}).$$

$$\widetilde{\phi\psi}(\overline{\mathscr{E}}) = \widetilde{\phi}(\overline{\mathscr{E}})\psi(\overline{E}) + \phi(\overline{S} \oplus \overline{Q})\widetilde{\psi}(\overline{\mathscr{E}}) = \widetilde{\phi}(\overline{\mathscr{E}})\psi(\overline{S} \oplus \overline{Q}) + \phi(\overline{E})\widetilde{\psi}(\overline{\mathscr{E}}).$$

Proof. One checks that right hand side of these identities satisfies the three properties of Theorem 1 that characterize the left hand side. □

We will also need to know the behaviour of $\widetilde{c}$ when $\overline{\mathscr{E}}$ is twisted by a line bundle. The following is a consequence of [GS2], Prop. 1.3.3:

Proposition 2 *For any hermitian line bundle $\overline{L}$,*

$$\widetilde{c_k}(\overline{\mathscr{E}} \otimes \overline{L}) = \sum_{i=1}^{k} \binom{n-i}{k-i} \widetilde{c_i}(\overline{\mathscr{E}}) c_1(\overline{L})^{k-i}.$$

4 Calculating Bott-Chern Forms

In this section we will consider an exact sequence

$$\overline{\mathscr{E}} : \ 0 \to \overline{S} \to \overline{E} \to \overline{Q} \to 0.$$

where the metrics on $\overline{S}$ and $\overline{Q}$ are induced from the metric on E. Let r, n be the ranks of the bundles S and E. Let $\phi \in I(n)$ be homogeneous of degree k. We will formulate a theorem for calculating the Bott-Chern form $\widetilde{\phi}(\overline{\mathscr{E}})$. This result follows from the work of Bott-Chern, M. Cowen, J. Bismut and Gillet-Soulé.

Let ϕ' be defined as in §2. For any two matrices $A, B \in M_n(\mathbb{C})$ set

$$\phi'(A;B) := \sum_{i=1}^{k} \phi'(A, A, \ldots, A, B_{(i)}, A, \ldots, A),$$

where the index i means that B is in the i-th position.

Choose a local orthonormal frame $s = (s_1, s_2, \ldots, s_n)$ of E such that the first r elements generate S, and let $K(\overline{S})$, $K(\overline{E})$ and $K(\overline{Q})$ be the curvature matrices of $\overline{S}$, $\overline{Q}$ and $\overline{E}$ with respect to s. Let $K_S = \frac{i}{2\pi}K(\overline{S})$, $K_E = \frac{i}{2\pi}K(\overline{E})$ and $K_Q = \frac{i}{2\pi}K(\overline{Q})$. The matrix K_E has the form

$$K_E = \left(\begin{array}{c|c} K_{11} & K_{12} \\ \hline K_{21} & K_{22} \end{array}\right)$$

where K_{11} is an $r \times r$ submatrix. Also consider the matrices

$$K_0 = \left(\begin{array}{c|c} K_S & 0 \\ \hline K_{21} & K_Q \end{array}\right) \quad \text{and} \quad J_r = \left(\begin{array}{c|c} Id_r & 0 \\ \hline 0 & 0 \end{array}\right).$$

Let u be a formal variable and $K(u) := uK_E + (1-u)K_0$. Finally, let $\phi^!(u) = \phi'(K(u); J_r)$. We then have the following

Theorem 2

$$\widetilde{\phi}(\overline{\mathscr{E}}) = \int_0^1 \frac{\phi^!(u) - \phi^!(0)}{u}\, du. \tag{3}$$

Proof. We prove that $\widetilde{\phi}(\overline{\mathscr{E}})$ as defined above satisfies axioms (i)-(iii) of Theorem 1. The main step is the first axiom; this was essentially done in [BC] §4, when $\phi = c$ is the total Chern class. In the form (3) (again for the total Chern class), the equation was given by M. Cowen in [C1] and [C2], while simplifying Bott and Chern's proof. We follow both sources in sketching a proof of this more general result.

Let h and h_Q denote the metrics on E and Q respectively. Define the orthogonal projections $P_1 : \overline{E} \to \overline{S}$ and $P_2 : \overline{E} \to \overline{Q}$ and put

$$h_u(v, v') = uh(P_1 v, P_1 v') + h(P_2 v, P_2 v')$$

for $v, v' \in E_x$ and $0 < u \leqslant 1$. Then h_u is a hermitian norm, $h_1 = h$ and $h_u \to h_Q$ as $u \to 0$. Let $K(E, h_u)$ be the curvature matrix of (E, h_u) relative to the holomorphic frame s defined above. Proposition 3.1 of [C2] proves that $\frac{i}{2\pi}K(E,h_u) = K(u)$. It follows from Proposition 3.28 of [BC] that for $0 < t \leqslant 1$,

$$\phi(E, h_t) - \phi(E, h) = dd^c \int_t^1 \frac{\phi'(K(u); J_r)}{u}\, du.$$

If we could set $t = 0$ we would be done; however, the integral will not be convergent in general. Note that $K(u) = K_0 + uK_1$, where $K_1 \in A^{1,1}(X, \mathrm{End}(E))$ is independent of u. Therefore it will suffice to show that $\phi'(K_0; J_r)$ is a closed form, so that it can be deleted from the integral. For this we may assume that $\phi = p_\lambda$ is a product of power sums, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ a partition. Then

$$p'_\lambda(K_0; J_r) = \sum_{i=1}^m \mathrm{Tr}(K_S)^{\lambda_i - 1} \prod_{j \neq i} (\mathrm{Tr}(K_S) + \mathrm{Tr}(K_Q))^{\lambda_j} = \sum_{i=1}^m p_{\lambda_i - 1}(\overline{S}) \prod_{j \neq i} p_{\lambda_j}(\overline{S} \oplus \overline{Q})$$

is certainly a closed form.

This proves axioms (i) and (iii); axiom (ii) is easily checked as well. □

Remark. A similar deformation to the one in [C2] was used by Deligne in [D], 5.11 for a calculation involving the Chern character form. Special cases of Theorem 2 have been used in the literature before, see for example [GS2] Prop. 5.3, [GSZ] 2.2.3 and [Ma] Theorem 3.3.1.

We deduce some simple but useful calculations:

Corollary 1 (a) $\widetilde{c_1^k}(\overline{\mathscr{E}}) = 0$ *for all* $k \geqslant 1$ *and* $\widetilde{c}_m(\overline{\mathscr{E}}) = 0$ *for all* $m > \mathrm{rk}E$.
(b) $\widetilde{p_2}(\overline{\mathscr{E}}) = 2(\mathrm{Tr}K_{11} - c_1(\overline{S}))$ *and* $\widetilde{c_2}(\overline{\mathscr{E}}) = c_1(\overline{S}) - \mathrm{Tr}K_{11}$.

Proof. (a) $c_1^!(u)$ is independent of u; hence $\widetilde{c_1}(\overline{\mathscr{E}}) = 0$. The result for higher powers of c_1 follows from Proposition 1. In addition, $\widetilde{c}_m(\overline{\mathscr{E}}) = 0$ for $m > \mathrm{rk}E$ is an immediate consequence of the definition.
(b) Using the bilinear form p'_2 described previously, we find

$$p_2^!(u) = 2(u\mathrm{Tr}K_{11} + (1-u)c_1(\overline{S})),$$

thus

$$\widetilde{p_2}(\overline{\mathscr{E}}) = 2\int_0^1 \frac{u\mathrm{Tr}K_{11} + (1-u)c_1(\overline{S}) - c_1(\overline{S})}{u}\, du = 2(\mathrm{Tr}K_{11} - c_1(\overline{S})).$$

To calculate $\widetilde{c_2}(\overline{\mathscr{E}})$, use the identity $2c_2 = c_1^2 - p_2$. $\square$

Corollary 1(b) agrees with an important calculation of Deligne's in [D], 10.1, which we now describe: Using the C^∞ splitting of $\mathscr{E}$, we can write the $\overline{\partial}$ operator for E in matrix form:

$$\overline{\partial}_E = \begin{pmatrix} \overline{\partial}_S & \alpha \\ 0 & \overline{\partial}_Q \end{pmatrix}, \quad \text{for some } \alpha \in A^{0,1}(X, \mathrm{Hom}(Q,S)).$$

Let $\alpha^* \in A^{1,0}(X, \mathrm{Hom}(S,Q))$ be the transpose of α, defined using complex conjugation of forms and the metrics h_S and h_Q. If ∇ is the induced connection on $\mathrm{Hom}(Q,S)$, we have

$$K_E = \left(\begin{array}{c|c} K_S - \frac{i}{2\pi}\alpha\alpha^* & \nabla^{1,0}\alpha \\ \hline -\nabla^{0,1}\alpha^* & K_Q - \frac{i}{2\pi}\alpha^*\alpha \end{array}\right).$$

Thus Corollary 1(b) implies that

$$\widetilde{c_2}(\overline{\mathscr{E}}) = -\frac{1}{2\pi i}\mathrm{Tr}(\alpha\alpha^*) = \frac{1}{2\pi i}\mathrm{Tr}(\alpha^*\alpha),$$

and we have recovered Deligne's result. In this form the calculation was used by A. Moriwaki and C. Soulé to obtain a Bogomolov-Gieseker type inequality and a Kodaira vanishing theorem on arithmetic surfaces, respectively (see [Mo] and [S]).

The calculation of $\widetilde{c_2}$ shows that in general Bott-Chern forms are not closed. In fact, calculating $\widetilde{c_k}$ for $k \geqslant 3$ leads to much more complicated formulas, involving traces of products of curvature matrices, for which a clear geometric interpretation is lacking (unlike the matrix α above, whose negative transpose $-\alpha^*$ is the second fundamental form of $\overline{\mathscr{E}}$). In the next two sections we shall see that when $\overline{E}$ is a projectively flat bundle, the Bott-Chern forms are closed and can be calculated explicitly for any $\phi \in I(n)$.

5 $0 \to \overline{S} \to \overline{E} \to \overline{Q} \to 0$ with $\overline{E}$ flat

Throughout this section we will assume that the hermitian vector bundle $\overline{E}$ is *flat*, i.e. that $K_E = 0$. As before, the metrics h_S and h_Q will be induced from the metric on E. Define the *harmonic numbers* $\mathcal{H}_k = \sum_{i=1}^{k} \frac{1}{i}$, $\mathcal{H}_0 = 0$.

Let λ be a partition of k (we denote this by $\lambda \vdash k$). Recall that the polynomials $\{p_\lambda : \lambda \vdash k\}$ form a $\mathbb{Q}$-basis for the vector space of symmetric homogeneous polynomials in $x_1, \dots, x_n$ of degree k. The following result computes the Bott-Chern form corresponding to any such invariant polynomial:

Theorem 3 *The Bott-Chern class $\widetilde{p_\lambda}(\overline{\mathscr{E}})$ in $\widetilde{A}(X)$ is the class of*
(i) $k\mathcal{H}_{k-1}p_{k-1}(\overline{Q})$, *if* $\lambda = k = (k, 0, 0, \dots, 0)$
(ii) *0, otherwise.*

Proof. Let us first compute $\widetilde{p_k}(\overline{\mathscr{E}})$ for $p_k(A) = \mathrm{Tr}(A^k)$. Since $K_E = 0$, the deformed matrix $K(u) = (1-u)K_{S\oplus Q}$, where $K_{S\oplus Q} = \left(\begin{array}{c|c} K_S & 0 \\ \hline 0 & K_Q \end{array}\right)$. Since

$$\int_0^1 \frac{(1-u)^{k-1}-1}{u}\,du = -\int_0^1 \frac{t^{k-1}-1}{t-1}\,dt = -\mathcal{H}_{k-1},$$

we obtain

$$\widetilde{p_k}(\overline{\mathscr{E}}) = -\mathcal{H}_{k-1}p'_k(K_{S\oplus Q}; J_r) = -k\mathcal{H}_{k-1}\mathrm{Tr}(K_S^{k-1}) = -k\mathcal{H}_{k-1}p_{k-1}(\overline{S}).$$

Now since $p_k(\overline{S}\bigoplus\overline{Q}) - p_k(\overline{E})$ is exact, $p_k(\overline{E}) = 0$ and $p_k(\overline{S}\bigoplus\overline{Q}) = p_k(\overline{S}) + p_k(\overline{Q})$, we conclude that $p_k(\overline{S}) = -p_k(\overline{Q})$ in $\widetilde{A}(X)$, for each $k \geqslant 1$. This proves (i).

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ be a partition ($m \geqslant 2$). Proposition 1 implies that

$$\widetilde{p_\lambda}(\overline{\mathscr{E}}) = \widetilde{p_{\lambda_1}}(\overline{\mathscr{E}})p_{\lambda_2}\cdots p_{\lambda_m}(\overline{S}\bigoplus\overline{Q}).$$

But $\widetilde{p_{\lambda_1}}(\overline{\mathscr{E}})$ is a closed form (by (i)), and $p_{\lambda_2}\cdots p_{\lambda_m}(\overline{S}\bigoplus\overline{Q})$ is an exact form. Thus $\widetilde{p_\lambda}(\overline{\mathscr{E}})$ is exact, and so vanishes in $\widetilde{A}(X)$. $\square$

It follows from Theorem 3 that for any $\phi \in I(n)$, the Bott-Chern form $\widetilde{\phi}(\overline{\mathscr{E}})$ is a linear combination of homogeneous components of the Chern character form $ch(\overline{Q})$. In [Ma], Theorem 3.4.1 we find the calculation

$$\widetilde{c_k}(\overline{\mathscr{E}}) = \mathcal{H}_{k-1}\sum_{i=0}^{k-1} ic_i(\overline{S})c_{k-1-i}(\overline{Q}) \tag{4}$$

for the Chern forms $\widetilde{c_k}$. Our result gives the following

Proposition 3 $\widetilde{c_k}(\overline{\mathscr{E}}) = (-1)^{k-1}\mathcal{H}_{k-1}p_{k-1}(\overline{Q})$.

Proof. By Newton's identity (2) we have

$$\widetilde{p_k} - \widetilde{c_1 p_{k-1}} + \widetilde{c_2 p_{k-2}} - \cdots + (-1)^k k\widetilde{c_k} = 0. \tag{5}$$

Reasoning as in Theorem 3, we see that if ϕ and ψ are two homogeneous invariant polynomials of positive degree, then $\widetilde{\phi\psi}(\overline{\mathscr{E}}) = 0$ in $\widetilde{A}(X)$. Thus (5) gives $\widetilde{c_k}(\overline{\mathscr{E}}) = \dfrac{(-1)^{k-1}}{k}\widetilde{p_k}(\overline{\mathscr{E}}) = (-1)^{k-1}\mathcal{H}_{k-1}p_{k-1}(\overline{Q})$. □

Remark. The result of Proposition 3 agrees with (4), i.e.

$$(-1)^k p_k(\overline{Q}) = \sum_{i=0}^{k} ic_i(\overline{S})c_{k-i}(\overline{Q})$$

in $\widetilde{A}(X)$. To see this, let $h(t) = \sum c_i(\overline{S})t^i$, $g(t) = \sum c_j(\overline{Q})t^j$, and $f(t) = \sum ic_i(\overline{S})t^i$. Then $h(t)g(t) = 1$ in $\widetilde{A}(X)[t]$, and $f(t) = th'(t)$. Choose formal variables $\{x_\alpha\}_{1\leqslant\alpha\leqslant r}$ and set $c_i(\overline{S}) = e_i(x_1,\ldots,x_r)$, so that $h(t) = \prod_\alpha(1 + x_\alpha t)$. Then $f(t) = \sum_\alpha tx_\alpha \prod_{\beta\neq\alpha}(1 + x_\beta t)$. Thus

$$f(t)g(t) = \frac{f(t)}{h(t)} = \sum_\alpha \frac{x_\alpha t}{1 + x_\alpha t} = r - \sum_\alpha \frac{1}{1 + x_\alpha t} =$$

$$r - \sum_{\alpha,i}(-1)^i x_\alpha^i t^i = r - \sum_i (-1)^i p_i(\overline{S})t^i = r + \sum_i (-1)^i p_i(\overline{Q})t^i.$$

Comparing coefficients of t^k on both sides gives the result.

We can use Theorem 3 to calculate $\widetilde{\phi}(\overline{\mathscr{E}})$ for $\phi \in I(n)_k$: it is enough to find the coefficient of the power sum p_k when ϕ is expressed as a linear combination of $\{p_\lambda\}_{\lambda\vdash k}$ in $\Lambda(n,\mathbb{Q})$. For example, we have

Corollary 2 $\widetilde{ch}(\overline{\mathscr{E}}) = \sum_k \mathcal{H}_k ch_k(\overline{Q})$, *where* ch_k *denotes the* k*-th homogeneous component of the Chern character form.*

Corollary 3 *Let* λ *be a partition of* k *and* s_λ *the corresponding Schur polynomial in* $\Lambda(n,\mathbb{Q})$. *Then* $\widetilde{s_\lambda}(\overline{\mathscr{E}}) = 0$ *unless* λ *is a hook* $\lambda_i = (i,1,1,\ldots,1)$, *in which case* $\widetilde{s_{\lambda_i}}(\overline{\mathscr{E}}) = (-1)^{k-i}\mathcal{H}_{k-1}p_{k-1}(\overline{Q})$.

Proof. The proof is based on the Frobenius formula

$$s_\lambda = \frac{1}{k!}\sum_{\sigma\in S_k}\chi_\lambda(\sigma)p_{(\sigma)}$$

where (σ) denotes the partition of k determined by the cycle structure of σ (cf. [M], §I.7). By the above remark, $\widetilde{s_\lambda}(\overline{\mathscr{E}}) = \chi_\lambda((12\ldots k))\mathcal{H}_{k-1}p_{k-1}(\overline{Q})$. Using the combinatorial rule for computing χ_λ found in [M], p. 117, Example 5, we obtain

$$\chi_\lambda((12\ldots k)) = \begin{cases} (-1)^{k-i}, & \text{if } \lambda = \lambda_i \text{ is a hook} \\ 0, & \text{otherwise.} \end{cases}$$

□

The most natural instance of a sequence $\overline{\mathscr{E}}$ with $\overline{E}$ flat is the classifying sequence over the Grassmannian $G(r,n)$. As we shall see in §8, the calculation of Bott-Chern forms for this sequence leads to a presentation of arithmetic intersection ring of the arithmetic Grassmannian over Spec$\mathbb{Z}$.

6 Calculations when $\overline{E}$ is projectively flat

We will now generalize the results of the last section to the case where E is *projectively flat*, i.e. the curvature matrix K_E of $\overline{E}$ is a multiple of the identity matrix: $K_E = \omega Id_n$. This is true if $E = \overline{L}^{\oplus n}$ for some hermitian line bundle $\overline{L}$, with $\omega = c_1(\overline{L})$ the first Chern form of $\overline{L}$.

The Bott-Chern forms (for the induced metrics) are always closed in this case as well, and will be expressed in terms of characteristic classes of the bundles involved. However this seems to be the most general case where this phenomenon occurs.

The key observation is that for projectively flat bundles, the curvature matrix $K_E = \omega Id_n$ in *any* local trivialization. Thus we have

$$K(u) = \left(\begin{array}{c|c} (1-u)K_S + u\omega Id_r & 0 \\ \hline 0 & (1-u)K_Q + u\omega Id_s \end{array}\right)$$

where $s = n - r$ denotes the rank of Q. Now Theorem 2 gives

$$\widetilde{p_k}(\overline{\mathscr{E}}) = k\int_0^1 \frac{1}{u}\mathrm{Tr}[(u\omega Id_r + (1-u)K_S)^{k-1} - K_S^{k-1}]\,du =$$

$$-k\mathcal{H}_{k-1}p_{k-1}(\overline{S}) + k\sum_{j=1}^{k-1}\binom{k-1}{j}\mathrm{Tr}(\omega^j K_S^{k-1-j})\int_0^1 u^{j-1}(1-u)^{k-j-1}\,du.$$

Integrating by parts gives $\int_0^1 u^m(1-u)^n\,du = \frac{1}{m+n+1}\binom{m+n}{n}^{-1}$, thus

$$\frac{1}{k}\widetilde{p_k}(\overline{\mathscr{E}}) = -\mathcal{H}_{k-1}p_{k-1}(\overline{S}) + \sum_{j=1}^{k-1}\frac{\omega^j}{j}p_{k-1-j}(\overline{S}). \tag{6}$$

We can rewrite this as an equation involving power sums of the quotient bundle: since $p_k(\overline{S}) + p_k(\overline{Q}) - p_k(\overline{L}^{\oplus n}) = 0$ in $\widetilde{A}(X)$, we have $p_k(\overline{S}) = n\omega^k - p_k(\overline{Q})$. Thus (6) becomes

$$\frac{1}{k}\widetilde{p_k}(\overline{\mathscr{E}}) = \mathcal{H}_{k-1}p_{k-1}(\overline{Q}) - \sum_{j=1}^{k-1}\frac{\omega^j}{j}p_{k-1-j}(\overline{Q}). \tag{7}$$

Theorem 4 *Let X be a complex manifold, $\overline{E}$ a projectively flat hermitian vector bundle over X. Let $0 \to \overline{S} \to \overline{E} \to \overline{Q} \to 0$ a short exact sequence of vector bundles over X with metrics on S, Q induced from $\overline{E}$. Then for any invariant polynomial $\phi \in I(n)$, $\phi(\overline{S} \bigoplus \overline{Q}) = \phi(\overline{E})$ as differential forms on X.*

Proof. Since the p_λ form an additive basis for $I(n)$, it suffices to prove the result when $\phi = p_\lambda$. The above calculation shows that $\widetilde{p_k}$ is a closed form. This combined with Proposition 1 shows that $\widetilde{p_\lambda}$ is closed for any partition λ. Thus

$$p_\lambda(\overline{S} \bigoplus \overline{Q}) - p_\lambda(\overline{E}) = dd^c\widetilde{p_\lambda} = 0.$$

$\square$

Remark. If E is a trivial vector bundle, this result follows by pulling back the exact sequence $\overline{\mathscr{E}}$ from the classifying sequence on the Grassmannian. The forms are equal there because they are invariant with respect to the $U(n)$ action, so harmonic.

The Bott-Chern forms $\widetilde{p_\lambda}$ for a general partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ can be computed by using Proposition 1. If $|\lambda| = \sum \lambda_i = k$ then we have

$$\widetilde{p_\lambda}(\overline{\mathscr{E}}) = \sum_{i=1}^{m}\widetilde{p_{\lambda_i}}(\overline{\mathscr{E}})\prod_{j\neq i}p_{\lambda_j}(\overline{E}) = n^{m-1}\sum_{i=1}^{m}\omega^{k-\lambda_i}\widetilde{p_{\lambda_i}}(\overline{\mathscr{E}}). \tag{8}$$

In principle equations (7) and (8) can be used to compute $\widetilde{\phi}(\overline{\mathscr{E}})$ for any characteristic class ϕ.

We now find a more explicit formula for the Bott-Chern forms of Chern classes. The computation is not as straightforward, as the argument of

Proposition 3 does not apply. Since by Theorem 2 the calculation depends only on the curvature matrices K_E, K_S and K_Q, we may assume

$$\overline{\mathscr{E}} \ : \ \ 0 \to \overline{S} \to \overline{L} \otimes \mathbb{C}^n \to \overline{Q} \to 0$$

is our chosen sequence, and define a new sequence

$$\overline{\mathscr{E}'} = \overline{\mathscr{E}} \otimes \overline{L}^* \ : \ 0 \to \overline{S} \otimes \overline{L}^* \to \mathbb{C}^n \to \overline{Q} \otimes \overline{L}^* \to 0.$$

The metrics on the bundles in $\overline{\mathscr{E}'}$ are induced from the trivial metric on $\mathbb{C}^n$. Using Propositions 2 and 3 now gives

$$\widetilde{c}_k(\overline{\mathscr{E}}) = \widetilde{c}_k(\overline{\mathscr{E}'} \otimes \overline{L}) = \sum_{i=1}^{k} \binom{n-i}{k-i} \widetilde{c}_i(\overline{\mathscr{E}'}) c_1(\overline{L})^{k-i} =$$

$$\sum_{i=1}^{k} \binom{n-i}{k-i} (-1)^{i-1} \mathcal{H}_{i-1} p_{i-1}(\overline{Q} \otimes \overline{L}^*) \omega^{k-i} =$$

$$\sum_{i=1}^{k} \sum_{j=0}^{i-1} (-1)^j \binom{n-i}{k-i} \binom{i-1}{j} \mathcal{H}_{i-1} \omega^{k-1-j} p_j(\overline{Q}) =$$

$$\sum_{j=0}^{k-1} (-1)^j d_j \omega^{k-1-j} p_j(\overline{Q}),$$

where

$$d_j = \sum_{i=j+1}^{k} \binom{n-i}{k-i} \binom{i-1}{j} \mathcal{H}_{i-1}.$$

To find a closed form for the sum d_j, we can use the general identity

$$\sum_{i=q-s}^{n-p} \binom{n-i}{p} \binom{s+i}{q} \mathcal{H}_{s+i} = \binom{n+s+1}{p+q+1} (\mathcal{H}_{n+s+1} - \mathcal{H}_{p+q+1} + \mathcal{H}_p). \tag{9}$$

This is identity (10) in [Sp]. In passing we note that writing equation (9) without the harmonic number terms:

$$\sum_{i=q-s}^{n-p} \binom{n-i}{p} \binom{s+i}{q} = \binom{n+s+1}{p+q+1}$$

gives a well known identity among binomial coefficients. Applying (9) to d_j and replacing k by $k+1$ and j by $k-i$ we arrive at the formula

$$\widetilde{c_{k+1}}(\overline{\mathscr{E}}) = \sum_{i=0}^{k} (-1)^{k-i} \binom{n}{i} \mathscr{H}_i \omega^i p_{k-i}(\overline{Q}),$$

where $\mathscr{H}_i = \mathcal{H}_n - \mathcal{H}_{n-i} + \mathcal{H}_{k-i}$. As remarked previously, this calculation is valid for any projectively flat bundle $\overline{E}$ with $c_1(\overline{E}) = n\omega$.

Of course one can use the above method to compute the Bott-Chern form $\widetilde{p_k}(\overline{\mathscr{E}})$ as well; however this leads to a more complicated formula than (7). Equating the two proves(!) the following interesting combinatorial identity (compare [Sp], identity (30)):

$$\sum_{i=0}^{s}(-1)^{i+1}\binom{n}{i, s-i}\mathcal{H}_{n-s+i} = \frac{1}{s} \qquad (n \geqslant s). \tag{10}$$

Here $\binom{n}{i,j}$ is a trinomial coefficient.

The following summarizes the calculations of this section:

Theorem 5 *Let X be a complex manifold, $\overline{E}$ a projectively flat hermitian vector bundle over X, with $c_1(\overline{E}) = n\omega$. Let $0 \to \overline{S} \to \overline{E} \to \overline{Q} \to 0$ be a short exact sequence of vector bundles over X with metrics on S, Q induced from $\overline{E}$. Then*

$$\widetilde{p_{k+1}}(\overline{\mathscr{E}}) = (k+1)\mathcal{H}_k p_k(\overline{Q}) - (k+1)\sum_{i=1}^{k}\frac{\omega^i}{i}p_{k-i}(\overline{Q})$$

$$\widetilde{c_{k+1}}(\overline{\mathscr{E}}) = \sum_{i=0}^{k}(-1)^{k-i}\binom{n}{i}\mathscr{H}_i\omega^i p_{k-i}(\overline{Q})$$

where $\mathscr{H}_i = \mathcal{H}_n - \mathcal{H}_{n-i} + \mathcal{H}_{k-i}$.

Note that the formulas in Theorem 5 reduce to the ones of the previous section when $\omega = 0$!

7 Arithmetic intersection theory

We recall here the generalization of Arakelov theory to higher dimensions due to H. Gillet and C. Soulé. Our main references are [GS1], [GS2] and the exposition in [SABK]. For A an abelian group, $A_{\mathbb{Q}}$ denotes $A \otimes_{\mathbb{Z}} \mathbb{Q}$. Let X be an *arithmetic scheme over* $\mathbb{Z}$, by which we mean a regular scheme, projective and flat over Spec$\mathbb{Z}$. For $p \geqslant 0$, let $X^{(p)}$ be the set of integral subschemes of X of codimension p and $Z^p(X)$ be the group of codimension p cycles on X. The p-th Chow group of X: $CH^p(X) := Z^p(X)/R^p(X)$, where $R^p(X)$ is the subgroup of $Z^p(X)$ generated by the cycles div$f$, $f \in k(x)^*$, $x \in X^{(p-1)}$.

Let $CH(X) = \bigoplus_p CH^p(X)$. If X is smooth over Spec$\mathbb{Z}$, then the methods of [F] can be used to give $CH(X)$ the structure of a commutative ring. In general one has a product structure on $CH(X)_{\mathbb{Q}}$ after tensoring with $\mathbb{Q}$.

Let $D^{p,p}(X(\mathbb{C}))$ denote the space of complex currents of type (p,p) on $X(\mathbb{C})$, and $F_\infty : X(\mathbb{C}) \to X(\mathbb{C})$ the involution induced by complex conjugation. Let $D^{p,p}(X_{\mathbb{R}})$ (resp. $A^{p,p}(X_{\mathbb{R}})$) be the subspace of $D^{p,p}(X(\mathbb{C}))$ (resp. $A^{p,p}(X(\mathbb{C}))$) generated by real currents (resp. forms) T such that $F_\infty^* T = (-1)^p T$; denote by $\widetilde{D}^{p,p}(X_{\mathbb{R}})$ and $\widetilde{A}^{p,p}(X_{\mathbb{R}})$ the respective images in $\widetilde{D}^{p,p}(X(\mathbb{C}))$ and $\widetilde{A}^{p,p}(X(\mathbb{C}))$.

An *arithmetic cycle* on X of codimension p is a pair (Z, g_Z) in the group $Z^p(X) \bigoplus \widetilde{D}^{p-1,p-1}(X_{\mathbb{R}})$, where g_Z is a *Green current* for $Z(\mathbb{C})$, i.e. a current such that $dd^c g_Z + \delta_{Z(\mathbb{C})}$ is represented by a smooth form. The group of arithmetic cycles is denoted by $\widehat{Z}^p(X)$. If $x \in X^{(p-1)}$ and $f \in k(x)^*$, we let $\widehat{\mathrm{div}} f$ denote the arithmetic cycle $(\mathrm{div} f, [-\log |f_{\mathbb{C}}|^2 \cdot \delta_{x(\mathbb{C})}])$.

The *p-th arithmetic Chow group of* X: $\widehat{CH}^p(X) := \widehat{Z}^p(X) / \widehat{R}^p(X)$, where $\widehat{R}^p(X)$ is the subgroup of $\widehat{Z}^p(X)$ generated by the cycles $\widehat{\mathrm{div}} f$, $f \in k(x)^*$, $x \in X^{(p-1)}$. Let $\widehat{CH}(X) = \bigoplus_p \widehat{CH}^p(X)$.

We have the following canonical morphisms of abelian groups:

$$\zeta : \widehat{CH}^p(X) \longrightarrow CH^p(X), \quad [(Z, g_Z)] \longmapsto [Z],$$

$$\omega : \widehat{CH}^p(X) \longrightarrow \mathrm{Ker} d \cap \mathrm{Ker} d^c \ \ (\subset A^{p,p}(X_{\mathbb{R}})), \quad [(Z, g_Z)] \longmapsto dd^c g_Z + \delta_{Z(\mathbb{C})},$$

$$a : \widetilde{A}^{p-1,p-1}(X_{\mathbb{R}}) \longrightarrow \widehat{CH}^p(X), \quad \eta \longmapsto [(0, \eta)].$$

One can define a pairing $\widehat{CH}^p(X) \otimes \widehat{CH}^q(X) \to \widehat{CH}^{p+q}(X)_{\mathbb{Q}}$ which turns $\widehat{CH}(X)_{\mathbb{Q}}$ into a commutative graded unitary $\mathbb{Q}$-algebra. The maps ζ, ω are $\mathbb{Q}$-algebra homomorphisms. If X is smooth over $\mathbb{Z}$ one does not have to tensor with $\mathbb{Q}$. The definition of this pairing is difficult; the construction uses the *star product* of Green currents, which in turn relies upon Hironaka's resolution of singularities to get to the case of divisors. The functor $\widehat{CH}^p(X)$ is contravariant in X, and covariant for proper maps which are smooth on the generic fiber.

Choose a Kähler form ω_0 on $X(\mathbb{C})$ such that $F_\infty^* \omega_0 = -\omega_0$ (this is equivalent to requiring that the corresponding Kähler metric is invariant under F_∞). It is natural to utilize the theory of harmonic forms on X in the study of Green currents on $X(\mathbb{C})$. Following [GS1], we call the pair $\overline{X} = (X, \omega_0)$ an *Arakelov variety.* By the Hodge decomposition theorem, we have $A^{p,p}(X_{\mathbb{R}}) = \mathcal{H}^{p,p}(X_{\mathbb{R}}) \oplus \mathrm{Im} d \oplus \mathrm{Im} d^*$, where $\mathcal{H}^{p,p}(X_{\mathbb{R}}) = \mathrm{Ker}\Delta \subset A^{p,p}(X)$ denotes the space of harmonic (with respect to ω_0) (p,p) forms α on $X(\mathbb{C})$ such that $F_\infty^* \alpha = (-1)^p \alpha$. The subgroup $CH^p(\overline{X}) := \omega^{-1}(\mathcal{H}^{p,p}(X_\mathbb{R}))$ of $\widehat{CH}^p(X)$ is called the *p-th Arakelov Chow group of* X. Let $CH(\overline{X}) = \bigoplus_{p \geqslant 0} CH^p(\overline{X})$. $CH^p(\overline{X})$ is a direct summand of $\widehat{CH}^p(X)$, and there is an exact sequence

$$CH^{p,p-1}(X) \xrightarrow{\rho} \mathcal{H}^{p-1,p-1}(X_\mathbb{R}) \xrightarrow{a} CH^p(\overline{X}) \xrightarrow{\zeta} CH^p(X) \longrightarrow 0. \quad (11)$$

In the above sequence the group $CH^{p,p-1}(X)$ is defined as the $E_2^{p,1-p}$ term of a certain spectral sequence used by Quillen to calculate the higher algebraic K-theory of X, and the map ρ coincides with the Beilinson regulator map (cf. [G] and [GS1], 3.5).

If $\mathcal{H}(X_\mathbb{R}) = \bigoplus_p \mathcal{H}^{p,p}(X_\mathbb{R})$ is a subring of $\bigoplus_p A^{p,p}(X_\mathbb{R})$, for example if $X(\mathbb{C})$ is a curve, an abelian variety or a hermitian symmetric space (e.g. a Grassmannian), then $CH(\overline{X})_\mathbb{Q}$ is a subring of $\widehat{CH}(X)_\mathbb{Q}$. This is not the case in general; for example it fails to be true for the complete flag varieties.

Arakelov [A] introduced the group $CH^1(\overline{X})$, where $\overline{X} = (X, g_0)$ is an arithmetic surface with the metric g_0 on the Riemann surface $X(\mathbb{C})$ given by $\frac{i}{2g} \sum \omega_j \wedge \overline{\omega}_j$. Here g is the genus of $X(\mathbb{C})$ and $\{\omega_j\}$ for $1 \leqslant j \leqslant g$ is an orthonormal basis of the space of holomorphic one forms on $X(\mathbb{C})$.

A *hermitian vector bundle* $\overline{E} = (E, h)$ on an arithmetic scheme X is an algebraic vector bundle E on X such that the induced holomorphic vector bundle $E(\mathbb{C})$ on $X(\mathbb{C})$ has a hermitian metric h, which is invariant under complex conjugation, i.e. $F_\infty^*(h) = h$.

To any hermitian vector bundle one can attach characteristic classes $\widehat{\phi}(\overline{E}) \in \widehat{CH}(X)_\mathbb{Q}$, for any $\phi \in I(n, \mathbb{Q})$, where $n = \mathrm{rk} E$. For example, we have *arithmetic Chern classes* $\widehat{c}_k(\overline{E}) \in \widehat{CH}^k(X)$. Some basic properties of these classes are:
(1) $\widehat{c}_0(\overline{E}) = 1$ and $\widehat{c}_k(\overline{E}) = 0$ for $k > \mathrm{rk} E$.
(2) The form $\omega(\widehat{c}_k(\overline{E})) = c_k(\overline{E}) \in A^{k,k}(X_\mathbb{R})$ is the k-th Chern form of the hermitian bundle $\overline{E(\mathbb{C})}$.
(3) $\zeta(\widehat{c}_k(\overline{E})) = c_k(E) \in CH^k(X)$.
(4) $f^* \widehat{c}_k(\overline{E}) = \widehat{c}_k(f^* \overline{E})$, for every morphism $f : X \to Y$ of regular schemes, projective and flat over $\mathbb{Z}$.
(5) If $\overline{L}$ is a hermitian line bundle, $\widehat{c}_1(\overline{L})$ is the class of $(\mathrm{div}(s), -\log \|s\|^2)$ for any rational section s of L.

Analogous properties are satisfied by $\widehat{\phi}$ for any $\phi \in I(n, \mathbb{Q})$ (see [GS2], Th. 4.1). The most relevant property of these characteristic classes is their behaviour in short exact sequences: if

$$\overline{\mathscr{E}} : \; 0 \to \overline{S} \to \overline{E} \to \overline{Q} \to 0$$

is such a sequence of hermitian vector bundles over X, then

$$\widehat{\phi}(\overline{S}\oplus\overline{Q})-\widehat{\phi}(\overline{E})=a(\widetilde{\phi}(\overline{\mathscr{E}})). \tag{12}$$

Relation (12) is the main tool for calculating intersection products of classes in $\widehat{CH}(X)$ that come from characteristic classes of vector bundles. Combining it with the results of §4 and §5 gives immediate consequences for such intersections. For example, we have

Corollary 4 *Let $\overline{\mathscr{E}}:0\to\overline{S}\to\overline{E}\to\overline{Q}\to 0$ be a short exact sequence of hermitian vector bundles over an arithmetic scheme X. Assume that the metrics on $S(\mathbb{C})$, $Q(\mathbb{C})$ are induced from that on $E(\mathbb{C})$.*
(a) If $\overline{E(\mathbb{C})}$ is flat, then

$$(1)\quad \widehat{p_\lambda}(\overline{S}\oplus\overline{Q})=\widehat{p_\lambda}(\overline{E}),\quad \textit{if } \lambda \textit{ has length } >1, \textit{ and}$$

$$(2)\quad \widehat{p_k}(\overline{S})+\widehat{p_k}(\overline{Q})-\widehat{p_k}(\overline{E})=k\mathcal{H}_{k-1}a(p_{k-1}(\overline{Q})),\ \forall k\geqslant 1,$$

in the arithmetic Chow group $\widehat{CH}(X)_{\mathbb{Q}}$.
(b) If $\overline{E}=\overline{L}^{\oplus n}$ for some hermitian line bundle $\overline{L}$ and $\omega=c_1(\overline{L(\mathbb{C})})$, then

$$\widehat{c}(\overline{S})\widehat{c}(\overline{Q})-\widehat{c}(\overline{E})=\sum_{i,j}(-1)^j\binom{n}{i}(\mathcal{H}_n-\mathcal{H}_{n-i}+\mathcal{H}_j)a(\omega^i p_j(\overline{Q})),$$

in the arithmetic Chow group $\widehat{CH}(X)$.

8 Arakelov Chow rings of Grassmannians

In this section $G=G(r,n)$ will denote the Grassmannian over Spec$\mathbb{Z}$. Over any field k, G parametrizes the r-dimensional linear subspaces of a vector space over k. Let

$$\overline{\mathscr{E}}:\ 0\to\overline{S}\to\overline{E}\to\overline{Q}\to 0 \tag{13}$$

denote the universal exact sequence of vector bundles over G. Here the trivial bundle $E(\mathbb{C})$ is given the trivial metric and the tautological subbundle $S(\mathbb{C})$ and quotient bundle $Q(\mathbb{C})$ the induced metrics. The homogeneous space $G(\mathbb{C})\simeq U(n)/(U(r)\times U(n-r))$ is a complex manifold. $G(\mathbb{C})$ is endowed with a natural $U(n)$-invariant metric coming from the Kähler form $\eta_G=c_1(\overline{Q(\mathbb{C})})$.

G is a smooth arithmetic scheme and $G(\mathbb{C})$ with the metric coming from η_G is a hermitian symmetric space, so we have an Arakelov Chow ring

$CH(\overline{G})$. Note that since the hermitian vector bundles in (13) are invariant under the action of $U(n)$, their Chern forms are harmonic, and thus the arithmetic characteristic classes obtained are all elements of $CH(\overline{G})$. V. Maillot [Ma] found a presentation of $CH(\overline{G})$, using the above observation and the short exact sequence (11). We wish to offer another description of this ring, based on the calculations in this paper.

First recall the geometric picture: for the ordinary Chow ring we have

$$CH(G) = \frac{\mathbb{Z}[c(S), c(Q)]}{\langle c(S)c(Q) = 1 \rangle}. \tag{14}$$

(see for instance [F], Example 14.6.6). If $x_1, \ldots, x_r$ are the Chern roots of S, $y_1, \ldots, y_s$ are the Chern roots of Q, $H = S_r \times S_{n-r}$ is the product of two symmetric groups, and t is a formal variable, then (14) can be rewritten

$$CH(G) = \frac{\mathbb{Z}[x_1, \ldots, x_r, y_1, \ldots, y_s]^H}{\left\langle \prod_i (1 + x_i t) \prod_j (1 + y_j t) = 1 \right\rangle}. \tag{15}$$

Maillot's presentation of $CH(\overline{G})$ is an analogue of (14); ours will be an analogue of (15). We introduce $2n$ variables

$$\widehat{x}_1, \ldots, \widehat{x}_r, \widehat{y}_1, \ldots, \widehat{y}_s, x_1, \ldots, x_r, y_1, \ldots, y_s$$

and consider the rings

$$A = \mathbb{Z}[\widehat{x}_1, \ldots, \widehat{x}_r, \widehat{y}_1, \ldots, \widehat{y}_s]^H \quad \text{and} \quad B = \mathbb{R}[x_1, \ldots, x_r, y_1, \ldots, y_s]^H$$

and the ring homomorphism $\omega : A \to B$ defined by $\omega(\widehat{x}_i) = x_i$ and $\omega(\widehat{y}_j) = y_j$. A ring structure is defined on the abelian group $A \oplus B$ by setting

$$(\widehat{x}, x') * (\widehat{y}, y') = (\widehat{x}\widehat{y}, \omega(\widehat{x})y' + x'\omega(\widehat{y})).$$

We will adopt the convention that $\widehat{\alpha}$ denotes $(\widehat{\alpha}, 0)$, β denotes $(0, \beta)$, and any product $\prod x_i y_j$ denotes $(0, \prod x_i y_j)$; the multiplication $*$ is thus characterized by the properties $\widehat{\alpha} * \beta = \alpha\beta$ and $\beta_1 * \beta_2 = 0$. We now define two sets of relations in $(A \oplus B)[t]$:

$$\mathcal{R}_1 : \prod_i (1 + x_i t) \prod_j (1 + y_j t) = 1,$$

$$\mathcal{R}_2 : \prod_i (1 + \widehat{x}_i t) * \prod_j (1 + \widehat{y}_j t) * \left(1 + t \sum_j \frac{\log(1 + y_j t)}{1 + y_j t} \right) = 1.$$

and let $\mathcal{A}$ denote the quotient of the graded ring $A \oplus B$ by these relations. Using this notation we can state

Theorem 6 *There is a unique ring isomorphism* $\Phi : \mathcal{A} \to CH(\overline{G})$ *such that*

$$\Phi(\prod_i (1+\widehat{x}_i t^i)) = \sum_i \widehat{c}_i(\overline{S}) t^i, \qquad \Phi(\prod_j (1+\widehat{y}_j t^j)) = \sum_j \widehat{c}_j(\overline{Q}) t^j,$$

$$\Phi(\prod_i (1+x_i t^i)) = \sum_i a(c_i(\overline{S})) t^i, \qquad \Phi(\prod_j (1+y_j t^j)) = \sum_j a(c_j(\overline{Q})) t^j.$$

Proof. The isomorphism Φ of $\mathcal{A}$ with $CH(\overline{G})$ is obtained exactly as in [Ma], Theorem 4.0.5. The key fact is that since G has a cellular decomposition (in the sense of [F], Ex. 1.9.1.), it follows that $CH^{p,p-1}(G) = 0$ for all p (using the excision exact sequence for the groups $CH^{*,*}(G)$; cf. [G], §8). Summing the sequence (11) over all p gives

$$0 \longrightarrow \mathcal{H}(G_{\mathbb{R}}) \xrightarrow{a} CH(\overline{G}) \xrightarrow{\zeta} CH(G) \longrightarrow 0. \tag{16}$$

We can now use our knowledge of the rings $\mathcal{H}(G_{\mathbb{R}})$ and $CH(G)$ together with the five lemma, as in loc. cit. The multiplication $*$ is a consequence of the general identity $a(x)y = a(x\omega(y))$ in $\widehat{CH}(G)$. To complete the argument we must show that the relation $\widehat{c}(\overline{S})\widehat{c}(\overline{Q}) = 1 + a(\widetilde{c}(\overline{\mathscr{E}}))$ translates to the relation $\mathcal{R}_2$ above.

Let $p_i(y)$ be the i-th power sum in the variables $y_1 \ldots, y_s$, identified under Φ with the class $a(p_i(\overline{Q}))$ in $CH(\overline{G})$ (we will use such identifications freely in the sequel). We also define $p_a(t) = \sum_{i=0}^{\infty} (-1)^{i+1}\mathcal{H}_i p_i(y) t^{i+1}$. Proposition 3 implies that

$$\widehat{c}_t(\overline{S})\widehat{c}_t(\overline{Q}) = 1 + a(\widetilde{c}_t(\overline{\mathscr{E}})) = 1 - p_a(t), \tag{17}$$

where the subscript t denotes the corresponding Chern polynomial. Multiplying both sides of (17) by $1 + p_a(t)$ and using the properties of multiplication in $\mathcal{A}$ gives the equivalent form

$$\widehat{c}_t(\overline{S}) * \widehat{c}_t(\overline{Q}) * (1 + p_a(t)) = 1. \tag{18}$$

We now note that the *harmonic number generating function*

$$\sum_{i=0}^{\infty} \mathcal{H}_i t^i = \frac{t}{1-t} + \frac{1}{2}\frac{t^2}{1-t} + \frac{1}{3}\frac{t^3}{1-t} + \cdots = \frac{\log(1-t)}{t-1}.$$

It follows that

$$p_a(-t) = \sum_{i=0}^{\infty} \mathcal{H}_i p_i(y) t^{i+1} = t \sum_{j=1}^{s} \sum_{i=0}^{\infty} \mathcal{H}_i (y_j t)^i = -t \sum_{j=1}^{s} \frac{\log(1-y_j t)}{1-y_j t}$$

and thus

$$p_a(t) = t \sum_{j=1}^{s} \frac{\log(1+y_j t)}{1+y_j t}.$$

Substituting this in equation (18) gives relation $\mathcal{R}_2$. $\square$

Theorem 6 shows that the relations in the Arakelov Chow ring of G are the classical geometric ones perturbed by a new 'arithmetic factor' of $1+p_a(t)$. While this factor is closely related to the power sums $p_i(\overline{Q})$, the most natural basis of symmetric functions for doing calculations in $CH(G)$ is the basis of Schur polynomials (corresponding to the Schubert classes; see for example [F], §14.7). The arithmetic analogues of the special Schubert classes involve the power sum perturbation above; multiplication formulas are thus quite complicated (see [Ma]).

In geometry the Chern roots x_i and y_j all 'live' on the complete flag variety above G. There are certainly natural line bundles on the flag variety whose first Chern classes correspond to the roots in Theorem 6. However on flag varieties the situation is more complicated and our knowledge is not as complete. We refer the reader to [T] for more details.

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