DOUBLE SCHUBERT POLYNOMIALS AND DEGENERACY LOCI FOR THE CLASSICAL GROUPS

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Abstract. We propose a theory of double Schubert polynomials $P_w(X,Y)$ for the Lie types B, C, D which naturally extends the family of Lascoux and Schützenberger in type A. These polynomials satisfy positivity, orthogonality and stability properties, and represent the classes of Schubert varieties and degeneracy loci of vector bundles. When w is a maximal Grassmannian element of the Weyl group, $P_w(X,Y)$ can be expressed in terms of Schur-type determinants and Pfaffians, in analogy with the type A formula of Kempf and Laksov. An example, motivated by quantum cohomology, shows there are no Chern class formulas for degeneracy loci of 'isotropic morphisms' of bundles.

0. Introduction

In recent years there has been interest in finding natural polynomials that represent the classes of Schubert varieties and degeneracy loci of vector bundles (see [Tu] and [FP] for expositions). Our aim here is to define and study polynomials which we propose as type B, C and D double Schubert polynomials. Special cases of these polynomials provide orthogonal and symplectic analogues of the determinantal formula of Kempf and Laksov [KL].

For the general linear group the corresponding objects are the double Schubert polynomials $\{\mathfrak{S}_\varpi(X,Y),\ \varpi \in S_n\}$ of Lascoux and Schützenberger [LS] [L]. These type A polynomials possess a series of remarkable properties, and it is desirable to have a theory for the other types with as many of them as possible. Fomin and Kirillov [FK] have shown that a theory of (single) Schubert polynomials in types B, C and D cannot satisfy all of the type A properties simultaneously. The theory developed here has qualities which are desirable from both the geometric and combinatorial points of view. Our three families of double Schubert polynomials have a 'common core', consisting of those polynomials which correspond to elements of the symmetric group S_n, sitting inside the respective Weyl groups. The polynomials in this core have positive integer coefficients and are obtained by specializing type A double Schubert polynomials. This has a natural geometric interpretation, using the inclusion of the corresponding isotropic flag bundle into the partial SL_{2n}-flag bundle obtained by omitting the isotropicity condition.

When restricted to maximal Grassmannian elements of the Weyl group, the single versions of our polynomials are the $\widetilde{P}$- and $\widetilde{Q}$-polynomials of Pragacz and Ratajski [PR2]. The latter objects are polynomials in the Chern roots of the tautological vector bundles over maximal isotropic Grassmannians, which represent the

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Schubert classes in the cohomology (or Chow ring) of the base variety. In this sense, they play a role in types B, C and D analogous to that of Schur's S-functions in type A (note that Schur's Q-functions, introduced in [S] to study projective representations of the symmetric and alternating groups, do not have the same geometric property [P2, 6.11]). The utility of the $\widetilde{P}$- and $\widetilde{Q}$-polynomials in the description of (relative) Schubert calculus and degeneracy loci was established in [PR2]. Moreover, according to [Ta] and [KT1], the multiplication of $\widetilde{Q}$-polynomials describes both *arithmetic* and *quantum* Schubert calculus on the Lagrangian Grassmannian (see also [KT2]). Thus, the double Schubert polynomials in this paper are closely related to natural families of representing polynomials.

Our motivation for this work was the search for an explicit general formula for Lagrangian and orthogonal degeneracy loci; to place this problem in context we first recall the relevant results in the setting of type A. Let Q and V be vector bundles of ranks n and $N = m + n$ respectively on an algebraic variety $\mathfrak{X}$, and consider a morphism of vector bundles $\psi\colon V \to Q$. Assume that we have a complete filtration

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_N = V$$

of V by subbundles such that $\dim V_i = i$ for all i. We are also given a rank sequence $0 < r_1 < r_2 < \cdots < r_m \leqslant N$ which corresponds to an integer partition $\lambda = (\lambda_i)$ with $\lambda_i = n + i - r_i$, and let $|\lambda|$ denote the sum of the parts of λ. There is the *degeneracy locus*

$$\mathfrak{X}'_\lambda = \{\, x \in \mathfrak{X} \mid \operatorname{rk}(V_{r_i}(x) \xrightarrow{\psi} Q(x)) \leqslant r_i - i \quad \text{for all } i\,\}. \tag{1}$$

Assume for simplicity that $\mathfrak{X}$ is smooth and that $\mathfrak{X}'_\lambda$ has codimension $|\lambda|$ in $\mathfrak{X}$. According to [KL], the class $[\mathfrak{X}'_\lambda]$ of this locus in the Chow ring $CH(\mathfrak{X})$ is given by a Schur determinant:

$$[\mathfrak{X}'_\lambda] = \det(c_{\lambda_i+j-i}(Q - V_{r_i}))_{i,j} \tag{2}$$

where each $c_k(E-F)$ is defined by the Chern class equation $c(E-F) = c(E)c(F)^{-1}$. Suppose now that ψ is surjective, and consider the exact sequence

$$0 \longrightarrow S \longrightarrow V \xrightarrow{\psi} Q \longrightarrow 0.$$

In this case $\mathfrak{X}'_\lambda$ coincides with the locus

$$\mathfrak{X}_\lambda = \{\, x \in \mathfrak{X} \mid \dim(S(x) \cap V_{r_i}(x)) \geqslant i \quad \text{for all } i\,\}. \tag{3}$$

We would like to consider a direct analogue of (1) in types B, C and D (see Section 4.1 for the precise definition). Unfortunately, as we shall see by example (Section 4), there are no Chern class formulas for degeneracy loci in this level of generality. We consider instead the analogue of the locus (3) for the other types, and begin with the Lagrangian case. Here V is a rank $2n$ symplectic vector bundle over $\mathfrak{X}$, and E and F are Lagrangian subbundles of V. The bundle F comes with a complete filtration $F_\bullet$ by subbundles $F_i \subset F$, $1 \leqslant i \leqslant n$. For each (strict) partition λ with ℓ nonzero parts the degeneracy locus $\mathfrak{X}_\lambda \subset \mathfrak{X}$ is defined by

$$\mathfrak{X}_\lambda = \{\, x \in \mathfrak{X} \mid \dim(E(x) \cap F_{n+1-\lambda_i}(x)) \geqslant i \quad \text{for } \ 1 \leqslant i \leqslant \ell\,\}.$$

Assuming that $\mathfrak{X}_\lambda$ has codimension $|\lambda|$ in $\mathfrak{X}$, our main geometric result (Corollary 4) is a determinantal formula for the the class $[\mathfrak{X}_\lambda] \in CH^{|\lambda|}(\mathfrak{X})$ as a polynomial in the Chern classes of the bundles E and F_i, which is a type C analogue of (2). Corollaries 8 and 10 solve the analogous problem in the two orthogonal cases. These results answer a question of Fulton and Pragacz [FP, Section 9.5]; note that a priori it is not clear why $[\mathfrak{X}_\lambda]$ should be expressed as *any* polynomial in the Chern classes of the bundles. Our formulas generalize those obtained by Pragacz and Ratajski [PR2] for some special cases of these loci.

The main ingredients used in the proofs are the geometric work of Fulton [F2] [F3] and Graham [Gra] and the algebraic tools developed by Lascoux, Pragacz and Ratajski [PR2] [LP1] [LP2]. The degeneracy locus formula (2) for $\mathfrak{X}'_\lambda$ was generalized to maps of flagged vector bundles by Fulton [F1], using the work of Bernstein, Gelfand, Gelfand [BGG] and Demazure [D1] [D2] and the double Schubert polynomials of Lascoux and Schützenberger [LS] [L]. Fulton later extended the geometric part of this story to the other classical groups [F2] [F3]; however there is no general degeneracy locus formula for *morphisms* between bundles in types B, C and D (note that there are such formulas for morphisms with symmetries; see [HT] [JLP] [P1] [F3] [PR2] [LP3]).

To elaborate further, let V be a vector bundle on $\mathfrak{X}$ equipped with a nondegenerate symplectic or orthogonal form. For each element w in the corresponding Weyl group there is a degeneracy locus $\mathfrak{X}_w$, defined using the attitude of an isotropic flag $E_\bullet$ with respect to a fixed complete isotropic flag $F_\bullet$ of subbundles of V (see [F3] [FP] or Sections 2, 3). In general one has an algorithm to write $[\mathfrak{X}_w]$ as a polynomial P_w in the first Chern classes X, Y of the quotient line bundles of the flags, by applying divided difference operators to a kernel $P_{w_0}(X,Y)$ (which corresponds to the longest Weyl group element w_0). The polynomial P_{w_0} represents the class of the diagonal in the flag bundle of V, and is defined only modulo an ideal of relations. Each specific choice of P_{w_0} produces a different family of representing polynomials P_w, which are candidates for *double Schubert polynomials*. The above construction ensures that the resulting theory of polynomials has direct geometric significance.

As explained earlier, unlike the situation in type A, it is no longer clear which kernel will give the most desirable theory. For (single) Schubert polynomials there have been several works in this direction [BH] [FK] [PR2] [LP1] [LP2]. Candidates for double Schubert polynomials in types B, C, D are implicit in the works [F2] [F3] and [LP1, Appendix B3]. The double Schubert polynomials proposed in this article differ from both of these sources. We choose a kernel $P_{w_0}(X,Y)$ leading to polynomials $P_w(X,Y)$ (where $P\in\{\mathfrak{B},\mathfrak{C},\mathfrak{D}\}$ depends on the Lie type) which have the following two main properties:

(i) (Positivity) When w is a permutation in the symmetric group S_n, $P_w(X,Y)$ is equal (in types B and C) or closely related (in type D) to the type A double Schubert polynomial $\mathfrak{S}_{\varpi_0 w\varpi_0}(\varpi_0 X,-Y)$, where ϖ_0 denotes the permutation of longest length in S_n. In particular, $\mathfrak{B}_w(X,Y)$ and $\mathfrak{C}_w(X,Y)$ have positive integer coefficients.

(ii) (Maximal Grassmannian) When w is a maximal Grassmannian element of the Weyl group, $P_w(X,Y)$ can be expressed by an explicit formula involving Schur-type determinants and Pfaffians.

In type C our polynomials specialize (when $Y=0$) to the *symplectic Schubert polynomials* $\mathfrak{C}_w(X)$ of [PR2] [LP1], and, for maximal Grassmannian elements w, to the $\widetilde{Q}$-polynomials of [PR2]. Properties (i), (ii) above combined with the results of [PR2] [LP1] [LP2] can be used to push the theory further, and obtain

(iii) (Orthogonality) Products of the single Schubert polynomials $P_w(X) = P_w(X, 0)$ give a (positive coefficient) orthonormal basis for the full polynomial ring, as a module over the ring of Weyl group invariants. This product basis corresponds in geometry to a split form of the fibration of the isotropic flag bundle over the maximal Grassmannian bundle, with fibers isomorphic to the type A flag variety (see Table 1 in Section 2.4).

(iv) (Stability) The single Schubert polynomials $P_w(X)$ satisfy a stability property under the natural inclusions of the Weyl groups. The double Schubert polynomials $P_w(X, Y)$ are stable for certain special Weyl group elements, related to the loci for morphisms with symmetries referred to earlier.

The hyperoctahedral group is a semidirect product of S_n with $(\mathbb{Z}_2)^n$. A central theme of this work is that the restrictions of double Schubert polynomials in types B, C, D to the two factors in this product (with the second realized by the maximal Grassmannian elements) are amenable to study and should be related to previously known families of polynomials. In fact, our polynomials are characterized (Proposition 3) by specifying the values of their single versions for the maximal elements in each factor, assuming the kernel P_{w_0} satisfies a 'Cauchy formula' (Corollary 2).

Our interest in these questions originated in an application to quantum cohomology. In the SL_N case, Bertram [Be] used (2) to prove a 'quantum Giambelli formula' for the Schubert classes in the small quantum cohomology ring of the Grassmannian. The degeneracy locus problem in loc. cit. occurs on a Quot scheme [Gro] and it is crucial that (2) holds for an arbitrary morphism ψ. There is a type C analogue of the Quot scheme, and one can ask whether natural Lagrangian analogues of $[\mathfrak{X}'_\lambda]$ can be expressed in terms of the Chern classes of the bundles involved. In Section 4 we show that *no such formula exists* (Proposition 8) by analyzing the structure of the Quot scheme $LQ_1(2, 4)$, which compactifies the space of degree 1 maps from $\mathbb{P}^1$ to the Lagrangian Grassmannian $LG(2, 4)$. We hope this example is of independent interest.

Here is a brief outline of this article. In Section 1 we introduce the type C double Schubert polynomials $\mathfrak{C}_w(X, Y)$ and prove their basic algebraic properties. Section 2 connects this work to the geometry of symplectic and Lagrangian degeneracy loci. The double Schubert polynomials and loci for the orthogonal types B and D are studied in Section 3. Finally, Section 4 presents the example of the Lagrangian Quot scheme $LQ_1(2, 4)$.

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1. Type C double Schubert polynomials

1.1. **Initial definitions.** Let us begin with some combinatorial preliminaries: for the most part we will follow the notational conventions of [M3, Section 1]; although not strictly necessary, the reader may find it helpful to identify an integer partition

$\lambda = (\lambda_1, \ldots, \lambda_r)$ with its Young diagram of boxes. The sum $\sum \lambda_i$ of the parts of λ is the *weight* $|\lambda|$ and the number of (nonzero) parts is the *length* $\ell(\lambda)$ of λ. We set $\lambda_r = 0$ for any $r > \ell(\lambda)$. Define the containment relation $\mu \subset \lambda$ for partitions by the inclusion of their respective diagrams. The union $\lambda \cup \mu$, intersection $\lambda \cap \mu$ and set-theoretic difference $\lambda \smallsetminus \mu$ are defined using the (multi)sets of parts of λ and μ. A partition is *strict* if all its nonzero parts are different; we define $\rho_n = (n, n-1, \ldots, 1)$ and let $\mathcal{D}_n$ be the set of strict partitions λ with $\lambda \subset \rho_n$. For $\lambda \in \mathcal{D}_n$, the *dual* partition $\lambda' = \rho_n \smallsetminus \lambda$ is the strict partition whose parts complement the parts of λ in the set $\{1, \ldots, n\}$.

Define the *excess* $e(\lambda)$ of a strict partition λ by

$$e(\lambda) = |\lambda| - (1 + \cdots + \ell(\lambda)) = |\lambda| - \ell(\lambda)(\ell(\lambda)+1)/2.$$

More generally, given $\alpha, \beta \in \mathcal{D}_n$ with $\alpha \cap \beta = \emptyset$, define an *intertwining number* $e(\alpha, \beta)$ as follows: for each i set

$$m_i(\alpha, \beta) = \#\{\, j \mid \alpha_i > \beta_j > \alpha_{i+1} \,\}$$

and

$$e(\alpha, \beta) = \sum_{i \geqslant 1} i\, m_i(\alpha, \beta).$$

Note that for any $\lambda \in \mathcal{D}_n$ we have $e(\lambda, \lambda') = e(\lambda)$.

We will use multiindex notation for sequences of commuting independent variables; in particular for any k with $1 \leqslant k \leqslant n$ let $X_k = (x_1, \ldots, x_k)$ and $Y_k = (y_1, \ldots, y_k)$; also set $X = X_n$ and $Y = Y_n$. Following Pragacz and Ratajski [PR2], for each partition λ with $\lambda_1 \leqslant n$, we define a symmetric polynomial $\widetilde{Q}_\lambda \in \mathbb{Z}[X]$ as follows: set $\widetilde{Q}_i(X) = e_i(X)$ to be the i-th elementary symmetric polynomial in the variables X. For i, j nonnegative integers let

$$\widetilde{Q}_{i,j}(X) = \widetilde{Q}_i(X)\widetilde{Q}_j(X) + 2\sum_{k=1}^{j} (-1)^k \widetilde{Q}_{i+k}(X)\widetilde{Q}_{j-k}(X).$$

If $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r \geqslant 0)$ is a partition with r even (by putting $\lambda_r = 0$ if necessary), set

$$\widetilde{Q}_\lambda(X) = \mathrm{Pfaffian}[\widetilde{Q}_{\lambda_i,\lambda_j}(X)]_{1 \leqslant i < j \leqslant r}.$$

These $\widetilde{Q}$-polynomials are modelled on Schur's Q-polynomials [S] and were used in [PR2] to describe certain Lagrangian and orthogonal degeneracy loci; we will generalize these results in the following sections. We also need the reproducing kernel

$$\widetilde{Q}(X, Y) = \sum_{\lambda \in \mathcal{D}_n} \widetilde{Q}_\lambda(X)\widetilde{Q}_{\lambda'}(Y).$$

For more information on these polynomials we refer to [PR2] [LP1].

Let S_n denote the symmetric group of permutations of the set $\{1, \ldots, n\}$ whose elements σ are written in single-line notation as $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ (as usual we will write all mappings on the left of their arguments.) S_n is the Weyl group for the root system A_{n-1} and is generated by the simple transpositions s_i for $1 \leqslant i \leqslant n-1$, where s_i interchanges i and $i+1$ and fixes all other elements of $\{1, \ldots, n\}$.

The hyperoctahedral group W_n is the Weyl group for the root systems B_n and C_n; the elements of W_n are permutations with a sign attached to each entry. We will adopt the notation where a bar is written over an element with a negative sign (as in [LP1, Section 1]); the latter are the *barred* entries, and their positive counterparts are *unbarred*. W_n is an extension of S_n by an element s_0 which acts on the right by

$$(u_1, u_2, \ldots, u_n)s_0 = (\overline{u}_1, u_2, \ldots, u_n).$$

The elements of maximal length in S_n and W_n are

$$\varpi_0 = (n, n-1, \ldots, 1) \quad \text{and} \quad w_0 = (\overline{1}, \overline{2}, \ldots, \overline{n})$$

respectively. Let $\lambda \subset \rho_n$ be a strict partition, $\ell = \ell(\lambda)$ and $k = n - \ell = \ell(\lambda')$. The barred permutation

$$w_\lambda = (\overline{\lambda}_1, \ldots, \overline{\lambda}_\ell, \lambda'_k, \ldots, \lambda'_1)$$

is the *maximal Grassmannian element* of W_n corresponding to λ (for details, see [LP1, Prop. 1.7] and [BGG] [D1] [D2]).

The group W_n acts on the ring $A[X]$ of polynomials in X with coefficients in any commutative ring A: the transposition s_i interchanges x_i and x_{i+1} for $1 \leqslant i \leqslant n-1$, while s_0 replaces x_1 by $-x_1$ (all other variables remain fixed). The ring of invariants $A[X]^{W_n}$ is the ring of polynomials in $A[X]$ symmetric in the $X^2 = (x_1^2, \ldots, x_n^2)$.

Assume that 2 is not a zero divisor in A. Following [BGG] and [D1] [D2], there are divided difference operators $\partial_i \colon A[X] \to A[X]$. For $1 \leqslant i \leqslant n-1$ they are defined by

$$\partial_i(f) = (f - s_i f)/(x_i - x_{i+1})$$

while

$$\partial_0(f) = (f - s_0 f)/(2x_1),$$

for any $f \in A[X]$. For each $w \in W_n$, define an operator ∂_w by setting

$$\partial_w = \partial_{a_1} \circ \cdots \circ \partial_{a_r}$$

if $w = s_{a_1} \cdots s_{a_r}$ is a reduced decomposition of w (so $r = \ell(w)$). One can show that ∂_w is well-defined, using the fact that the operators ∂_i satisfy the same set of Coxeter relations as the generators s_i, $0 \leqslant i \leqslant n-1$. In the applications here and in the next section we will take $A = \mathbb{Z}[Y]$. For each i with $1 \leqslant i \leqslant n-1$ we set $\partial'_i = -\partial_i$, while $\partial'_0 := \partial_0$. For each $w \in W_n$ we use ∂'_w to denote the divided difference operator where we use the ∂'_i's instead of the ∂_i's.

Following Lascoux and Schützenberger [LS] [L] we define, for each $\varpi \in S_n$, a type A *double Schubert polynomial* $\mathfrak{S}_\varpi(X,Y) \in \mathbb{Z}[X,Y]$ by

$$\mathfrak{S}_\varpi(X,Y) = \partial_{\varpi^{-1}\varpi_0}\left(\Delta(X,Y)\right)$$

where

$$\Delta(X,Y) = \prod_{i+j\leqslant n} (x_i - y_j).$$

Our main references for these polynomials are [M1] [M2]. The type A (single) *Schubert polynomial* is $\mathfrak{S}_\varpi(X) := \mathfrak{S}_\varpi(X,0)$. The double and single polynomials are related by the formula [M2, (6.3)]

$$\mathfrak{S}_\varpi(X,Y) = \sum_{u,v} \mathfrak{S}_u(X)\mathfrak{S}_v(-Y), \tag{4}$$

summed over all $u, v \in S_n$ such that $u = v\varpi$ and $\ell(\varpi) = \ell(u) + \ell(v)$.

1.2. **Main theorems.** We intend to give analogues of the polynomials $\mathfrak{S}_\varpi(X,Y)$ for the other classical groups, and begin with the symplectic group. Our definition is motivated by the properties that follow and the applications to the geometry of degeneracy loci in Section 2.

Definition 1. *For every $w \in W_n$ the* type C double Schubert polynomial $\mathfrak{C}_w(X,Y)$ *is given by*

$$\mathfrak{C}_w(X,Y) = (-1)^{n(n-1)/2}\partial'_{w^{-1}w_0}\left(\Delta(X,Y)\widetilde{Q}(X,Y)\right).$$

The polynomials $\mathfrak{C}_w(X) := \mathfrak{C}_w(X,0)$ have already been defined by Lascoux, Pragacz and Ratajski [PR2] [LP1, Appendix A], where they are called *symplectic Schubert polynomials*. It is shown in loc. cit. that the $\mathfrak{C}_w(X)$ enjoy 'maximal Grassmannian' and 'stability' properties. We will see that the type C double Schubert polynomials $\mathfrak{C}_w(X,Y)$ satisfy an analogue of the former but are not stable under the natural embedding $W_{n-1} \hookrightarrow W_n$ in general. Note that the $\mathfrak{C}_w(X,Y)$ differ from the polynomials found in [LP1, Appendix B3], because the divided difference operators in loc. cit. (which agree with those in [F3]) differ from the ones used in Definition 1. [1] For the precise connection of the $\mathfrak{C}_w(X,Y)$ with geometry, see the proof of Theorem 3.

Let $\mathfrak{C}_\lambda(X,Y) = \mathfrak{C}_{w_\lambda}(X,Y)$ for each partition $\lambda \in \mathcal{D}_n$. In type A, when $\varpi \in S_n$ is a Grassmannian permutation ϖ_λ, the Schubert polynomial $\mathfrak{S}_{\varpi_\lambda}(X,Y)$ is a *multi-Schur function* (see [M2, (6.14)]). We may thus regard the $\mathfrak{C}_\lambda(X,Y)$ as type C analogues of multi-Schur determinants. Our first theorem gives a determinantal expression for these functions.

Theorem 1 (Maximal Grassmannian)**.** *For any strict partition $\lambda \in \mathcal{D}_n$ the double Schubert polynomial $\mathfrak{C}_\lambda(X,Y)$ is equal to*

$$(-1)^{e(\lambda)+|\lambda'|}\sum_\alpha \widetilde{Q}_\alpha(X)\sum_\beta(-1)^{e(\alpha,\beta)+|\beta|}\widetilde{Q}_{(\alpha\cup\beta)'}(Y)\det(e_{\beta_i-\lambda'_j}(Y_{n-\lambda'_j})),$$

where the first sum is over all $\alpha \in \mathcal{D}_n$ and the second over $\beta \in \mathcal{D}_n$ with $\beta \supset \lambda'$, $\ell(\beta) = \ell(\lambda')$ and $\alpha \cap \beta = \emptyset$.

Proof. We argue along the lines of the proof of [LP1, Theorem A.6]. Let $\ell = \ell(\lambda)$, $k = n - \ell$ and observe that $w_\lambda = w_0\,\tau_\lambda\,\varpi_k\,\delta_k^{-1}$, where

$$\tau_\lambda = (\lambda'_k,\ldots,\lambda'_1,\lambda_1,\ldots,\lambda_\ell)$$
$$\varpi_k = (k,\ldots,2,1,k+1,\ldots,n)$$
$$\delta_k = (s_\ell\cdots s_1s_0)(s_{\ell+1}\cdots s_1s_0)\cdots(s_{n-1}\cdots s_1s_0);$$

it follows that

$$\partial_{w_\lambda^{-1}w_0} = \partial_{\delta_k}\circ\partial_{\varpi_k}\circ\partial_{\tau_\lambda^{-1}}.$$

[1]Lascoux and Pragacz have informed us that the polynomials $\mathfrak{C}_w(X,Y)$ were defined in a preliminary version of [LP1].

Now compute

$$\partial_{\tau_\lambda^{-1}}(\Delta(X,Y)) = \mathfrak{S}_{\varpi_0\tau_\lambda}(X,Y) \tag{5}$$

$$= \prod_{j=1}^{k}\prod_{p=1}^{n-\lambda'_j}(x_{k+1-j}-y_p) \tag{6}$$

$$= \sum_{\gamma}(-1)^{|\gamma|-|\lambda'|}\prod_{j=1}^{k} x_{k+1-j}^{n-\gamma_j}e_{\gamma_j-\lambda'_j}(Y_{n-\lambda'_j}). \tag{7}$$

In the above calculation equality (6) holds because the permutation

$$\varpi_0\tau_\lambda = (n+1-\lambda'_k,\ldots,n+1-\lambda'_1,n+1-\lambda_1,\ldots,n+1-\lambda_\ell)$$

is *dominant* [M2, (6.14)], and the sum (7) is over all k-tuples $\gamma=(\gamma_1,\ldots,\gamma_k)$ of nonnegative integers.

Observe that

$$\partial'_{\delta_k}\circ\partial'_{\varpi_k} = (\nabla_1)^k$$

where $\nabla_1 = \partial'_{n-1}\cdots\partial'_1\partial_0$. The action of $(\nabla_1)^k$ relevant to our computation is explained by Lascoux and Pragacz; if $\nu_i \leqslant n-i$ for $1\leqslant i\leqslant k$ then [LP1, Lemma 5.10]:

$$(\nabla_1)^k(x_1^{\nu_1}\cdots x_k^{\nu_k}\cdot f) = \nabla_1(x_1^{\nu_k}\cdots\nabla_1(x_1^{\nu_2}\cdot\nabla_1(x_1^{\nu_1}\cdot f))\cdots)$$

for any function f symmetric in the X variables. Moreover [LP1, Theorem 5.1] states that if $\lambda\in\mathcal{D}_n$ and $0<r\leqslant n$ then $\nabla_1(x_1^{n-r}\widetilde{Q}_\lambda(X))$ vanishes unless $\lambda_p=r$ for some p, in which case

$$\nabla_1(x_1^{n-r}\widetilde{Q}_\lambda(X)) = (-1)^{p-1}\widetilde{Q}_{\lambda\smallsetminus r}(X).$$

Using the above facts and equation (7) we deduce that

$$\partial'_{w_\lambda^{-1}w_0}(\Delta(X,Y)\,\widetilde{Q}(X,Y)) = (\nabla_1)^k\left(\widetilde{Q}(X,Y)\partial'_{\tau_\lambda^{-1}}(\Delta(X,Y))\right)$$

$$= (\nabla_1)^k\left(\widetilde{Q}(X,Y)\sum_{\substack{\gamma\in\mathcal{D}_n\\ \ell(\gamma)=k}}(-1)^{|\gamma|-|\lambda'|+\ell(\tau_\lambda)}\sum_{\sigma\in S_k}\prod_{j=1}^{k}x_{k+1-j}^{n-\gamma_{\sigma(j)}}\,e_{\gamma_{\sigma(j)}-\lambda'_j}(Y_{n-\lambda'_j})\right)$$

$$= (-1)^{r(\lambda)}\sum_{\alpha}\widetilde{Q}_\alpha(X)\sum_{\beta}(-1)^{e(\alpha,\beta)+|\beta|}\widetilde{Q}_{(\alpha\cup\beta)'}(Y)\det(e_{\beta_i-\lambda'_j}(Y_{n-\lambda'_j}))$$

where the ranges of summation are as in the statement of the theorem and $r(\lambda)=\ell(\tau_\lambda)+k(k-1)/2+|\lambda'|$. Finally, note that

$$\ell(\tau_\lambda) = \binom{n+1}{2}+\binom{\ell}{2}-\binom{k+1}{2}-|\lambda|$$

and the signs fit to complete the proof. □

In Section 2 we apply Theorem 1 to obtain formulas for Lagrangian degeneracy loci.

Let $*\colon S_n\to S_n$ be the length preserving involution defined by

$$\varpi^* = \varpi_0\varpi\varpi_0.$$

The next theorem shows that for $\varpi\in S_n$, the polynomial $\mathfrak{C}_\varpi(X,Y)$ is closely related to a type A double Schubert polynomial, with the X variables in reverse order. See Section 2.3 for a geometric interpretation.

Theorem 2 (Positivity). *For every $\varpi \in S_n$,*

$$\mathfrak{C}_\varpi(X,Y) = \mathfrak{S}_{\varpi^*}(\varpi_0 X, -Y).$$

In particular, $\mathfrak{C}_\varpi(X,Y)$ has nonnegative integer coefficients.

Proof. Let $v_0 = \varpi_0 w_0$; then the analysis in [LP1, Sect. 4] gives

$$\partial'_{v_0}(\mathfrak{S}_u(X)\widetilde{Q}(X,Y)) = (-1)^{\ell(u)}\mathfrak{S}_u(\varpi_0 X) \tag{8}$$

for every $u \in S_n$. We now use (4) and (8) to compute

$$\begin{aligned}\mathfrak{C}_{\varpi_0}(X,Y) &= (-1)^{n(n-1)/2}\partial'_{v_0}(\Delta(X,Y)\widetilde{Q}(X,Y)) \\ &= (-1)^{\ell(\varpi_0)}\partial'_{v_0}\left(\sum_{u,v}\mathfrak{S}_u(X)\mathfrak{S}_v(-Y)\widetilde{Q}(X,Y)\right) \\ &= \sum_{u,v}(-1)^{\ell(v)}\mathfrak{S}_u(\varpi_0 X)\mathfrak{S}_v(-Y) \\ &= \mathfrak{S}_{\varpi_0}(\varpi_0 X, -Y),\end{aligned}$$

where the above sums are over all $u, v \in S_n$ with $u = v\varpi_0$ and $\ell(u)+\ell(v) = \ell(\varpi_0)$. It follows that for any $\varpi \in S_n$,

$$\mathfrak{C}_\varpi(X,Y) = \partial'_{\varpi^{-1}\varpi_0}(\varpi_0\mathfrak{S}_{\varpi_0}(X,-Y)) \tag{9}$$

$$= \varpi_0\partial_{\varpi_0\varpi^{-1}}\sum_{u\in S_n}\mathfrak{S}_u(X)\mathfrak{S}_{u\varpi_0}(Y). \tag{10}$$

Applying [M2, (4.3)] gives

$$\partial_{\varpi_0\varpi^{-1}}\mathfrak{S}_u(X) = \begin{cases} \mathfrak{S}_{u\varpi\varpi_0}(X) & \text{if } \ell(u)+\ell(\varpi)+\ell(u\varpi) = 2\ell(\varpi_0), \\ 0 & \text{otherwise.}\end{cases} \tag{11}$$

We now use (11) in (10) and apply (4) to obtain

$$\mathfrak{C}_\varpi(X,Y) = \varpi_0\mathfrak{S}_{\varpi_0\varpi\varpi_0}(X,-Y),$$

which is the desired result. □

Combining Theorem 2 with the known results for type A double Schubert polynomials gives corresponding ones for the $\mathfrak{C}_\varpi(X,Y)$. For example, we get

Corollary 1. *For every $\varpi \in S_n$ we have $\mathfrak{C}_\varpi(X) = \mathfrak{S}_{\varpi^*}(\varpi_0 X)$. Moreover*

$$\mathfrak{C}_\varpi(X,Y) = \sum_{u,v}\mathfrak{C}_u(X)\mathfrak{C}_v(\varpi_0 Y)$$

summed over all $u, v \in S_n$ with $u = v\varpi^*$ and $\ell(u)+\ell(v) = \ell(\varpi)$.

1.3. **Product basis and orthogonality.** Following Lascoux and Pragacz [LP1, Sect. 1] (up to a sign!), we define a $\mathbb{Z}[X]^{W_n}$-linear scalar product

$$\langle\ ,\ \rangle : \mathbb{Z}[X]\times\mathbb{Z}[X] \longrightarrow \mathbb{Z}[X]^{W_n} \tag{12}$$

by

$$\langle f, g\rangle = \partial_{w_0}(fg) \tag{13}$$

for any $f, g \in \mathbb{Z}[X]$. For each partition $\lambda \in \mathcal{D}_n$ let $\mathfrak{C}_\lambda(X) = \mathfrak{C}_\lambda(X,0) = \widetilde{Q}_\lambda(X)$. We deduce from Corollary 1 and [M1, (4.11)] that the set $\{\mathfrak{C}_\varpi(X)\}_{\varpi\in S_n}$ is a free $\mathbb{Z}$-basis for the additive subgroup of $\mathbb{Z}[X]$ spanned by the monomials $\{x_1^{\alpha_1}\cdots x_n^{\alpha_n}\}$, with $\alpha_i \leqslant i-1$ for each i. It follows that the $\{\mathfrak{C}_\varpi(X)\}_{\varpi\in S_n}$ form a basis of $\mathbb{Z}[X]$ as a $\mathbb{Z}[X]^{S_n}$-module. In addition, we know from [PR2, Prop. 4.7] that $\{\mathfrak{C}_\lambda(X)\}_{\lambda\in\mathcal{D}_n}$ is a basis of $\mathbb{Z}[X]^{S_n}$ as a $\mathbb{Z}[X]^{W_n}$-module.

Proposition 1 (Orthogonality)**.** (i) *The products* $\{\mathfrak{C}_\varpi(X)\mathfrak{C}_\lambda(X)\}$ *for* $\varpi\in S_n$ *and* $\lambda\in\mathcal{D}_n$ *form a basis for the polynomial ring* $\mathbb{Z}[X]$ *as a* $\mathbb{Z}[X]^{W_n}$*-module. Moreover, we have the orthogonality relation*

$$\Big\langle \mathfrak{C}_u(X)\mathfrak{C}_\lambda(X), \mathfrak{C}_{v\varpi_0}(-\varpi_0 X)\mathfrak{C}_{\mu'}(X)\Big\rangle = \delta_{u,v}\delta_{\lambda,\mu} \tag{14}$$

for all $u,v\in S_n$ *and* $\lambda,\mu\in\mathcal{D}_n$*.*

(ii) *Let* $\mathfrak{C}_{\varpi,\lambda}(X)=\mathfrak{C}_\varpi(X)\mathfrak{C}_\lambda(X)$ *and suppose* $u,v\in S_n$ *and* $\lambda,\mu\in\mathcal{D}_n$ *are such that* $\ell(u)+\ell(v)=n(n-1)/2$*. Then*

$$\Big\langle \mathfrak{C}_{u,\lambda}(X), \mathfrak{C}_{v,\mu}(X)\Big\rangle = \begin{cases} 1 & \text{if} \quad v=\varpi_0 u \quad \text{and} \quad \mu=\lambda', \\ 0 & \text{otherwise.}\end{cases} \tag{15}$$

Proof. It is shown in [LP1, Sect. 1] that $\mathbb{Z}[X]$ is a free $\mathbb{Z}[X]^{W_n}$-module with basis $\{\mathfrak{S}_\varpi(X)\widetilde{Q}_\lambda(X)\}$, for $\varpi\in S_n$, $\lambda\in\mathcal{D}_n$. Furthermore, this basis has the orthogonality property [LP1, p. 12]

$$\Big\langle \mathfrak{S}_u(X)\widetilde{Q}_\lambda(X), \mathfrak{S}_{v\varpi_0}(-\varpi_0 X)\widetilde{Q}_{\mu'}(X)\Big\rangle = (-1)^{\ell(\varpi_0)}\delta_{u,v}\delta_{\lambda,\mu}.$$

Part (i) follows immediately by applying Corollary 1. To prove (ii), factor $\partial_{w_0}=\partial_{v_0}\partial_{\varpi_0}$ (where $v_0=w_0\varpi_0$) and observe that

$$\Big\langle \mathfrak{C}_{u,\lambda}(X), \mathfrak{C}_{v,\mu}(X)\Big\rangle = \partial_{\varpi_0}(\mathfrak{C}_u(X)\mathfrak{C}_v(X))\cdot\partial_{v_0}(\mathfrak{C}_\lambda(X)\mathfrak{C}_\mu(X)). \tag{16}$$

Now use (16) and the corresponding properties of the inner products defined by ∂_{ϖ_0} and ∂_{v_0} (stated in [M1, (5.4)] and [LP1, (10)], respectively). □

Note that the polynomials in the basis $\{\mathfrak{C}_{\varpi,\lambda}(X)\}$ have positive coefficients, but do not represent the Schubert cycles in the complete symplectic flag variety $Sp(2n)/B$. The orthogonality relations (14) and (15) are however direct analogues of the ones for type A Schubert polynomials [M1, (5.4) (5.5) (5.8)]. The divided difference operator in (13) for the maximal length element corresponds to a Gysin homomorphism in geometry (see [BGG] [D1] [AC]).

Let $\{\mathfrak{C}^{\varpi,\lambda}(X)\}_{(\varpi,\lambda)\in S_n\times\mathcal{D}_n}$ be the $\mathbb{Z}[X]^{W_n}$-basis of $\mathbb{Z}[X]$ adjoint to the basis $\{\mathfrak{C}_{\varpi,\lambda}(X)\}$ relative to the scalar product (12). By Proposition 1 we have

$$\mathfrak{C}^{\varpi,\lambda}(X)=\mathfrak{C}_{\varpi\varpi_0}(-\varpi_0 X)\mathfrak{C}_{\lambda'}(X). \tag{17}$$

We now obtain type C analogues of [M2, (5.7) and (5.9)]:

Corollary 2 (Cauchy formula)**.** *We have*

$$\begin{aligned}\mathfrak{C}_{w_0}(X,Y) &= \sum_{\varpi,\lambda}\mathfrak{C}_{\varpi\varpi_0}(-\varpi_0 X)\mathfrak{C}_{\lambda'}(X)\mathfrak{C}_\varpi(\varpi_0 Y)\mathfrak{C}_\lambda(Y) \\ &= \sum_{\varpi,\lambda}\mathfrak{C}^{\varpi,\lambda}(X)\mathfrak{C}_{\varpi,\lambda}(\varpi_0 Y),\end{aligned}$$

summed over all $\varpi\in S_n$ *and* $\lambda\in\mathcal{D}_n$*.*

Proof. We use (4) and the definitions in Section 1.1 to get

$$\begin{aligned}\mathfrak{C}_{w_0}(X,Y) &= \mathfrak{S}_{\varpi_0}(Y,X)\widetilde{Q}(X,Y)\\ &= \sum_{\varpi\in S_n}\mathfrak{S}_{\varpi}(Y)\mathfrak{S}_{\varpi\varpi_0}(-X)\sum_{\lambda\in\mathcal{D}_n}\widetilde{Q}_\lambda(X)\widetilde{Q}_{\lambda'}(Y).\end{aligned}$$

The result now follows from Corollary 1 and (17). □

1.4. **Examples.** 1) The type C double Schubert polynomials for $n=2$ are

$$\mathfrak{C}_{(\overline{1},\overline{2})} = (y_1-x_1)\widetilde{Q}(X_2,Y_2) = (y_1-x_1)(x_1+x_2+y_1+y_2)(x_1x_2+y_1y_2)$$

$$\mathfrak{C}_{(1,\overline{2})} = -x_1^2x_2+(x_2^2-y_2^2)y_1+x_2y_1^2 \qquad \mathfrak{C}_{(\overline{2},\overline{1})} = \widetilde{Q}(X_2,Y_2)$$

$$\mathfrak{C}_{(\overline{2},1)} = x_1x_2+(x_1+x_2)y_1+y_1^2 \qquad \mathfrak{C}_{(2,\overline{1})} = x_2^2+x_2(y_1+y_2)+y_1y_2$$

$$\mathfrak{C}_{(2,1)} = x_2+y_1 \qquad \mathfrak{C}_{(\overline{1},2)} = x_1+x_2+y_1+y_2$$

$$\mathfrak{C}_{(1,2)} = 1.$$

2) The type C double Schubert polynomials $\mathfrak{C}_\varpi$ for $n=3$ and $\varpi\in S_3$ are

$$\mathfrak{C}_{(3,2,1)} = (x_2+y_1)(x_3+y_1)(x_3+y_2)$$

$$\mathfrak{C}_{(2,3,1)} = (x_3+y_1)(x_3+y_2) \qquad \mathfrak{C}_{(3,1,2)} = (x_2+y_1)(x_3+y_1)$$

$$\mathfrak{C}_{(2,1,3)} = x_2+x_3+y_1+y_2 \qquad \mathfrak{C}_{(1,3,2)} = x_3+y_1$$

$$\mathfrak{C}_{(1,2,3)} = 1.$$

3) For fixed n and $i=0,1,\ldots,n-1$ we have

$$\mathfrak{C}_{s_i}(X,Y) = x_{i+1}+\cdots+x_n+y_1+\cdots+y_{n-i}.$$

4) Consider the partition $\lambda=\rho_k$ for some $k\leqslant n$. Then Theorem 1 gives

$$\mathfrak{C}_{\rho_k}(X,Y) = \sum_{\alpha\in\mathcal{D}_k}\widetilde{Q}_\alpha(X)\widetilde{Q}_{\rho_k\smallsetminus\alpha}(Y). \tag{18}$$

Further special cases of Theorem 1 are given in Corollaries 5 and 6.

1.5. **Stability.** For each $m<n$ let $i=i_{m,n}\colon W_m\hookrightarrow W_n$ be the embedding via the first m components. The maps $i_{m,n}$ are used to define the infinite hyperoctahedral group $W_\infty=\bigcup_n W_n$. We say that a type C double Schubert polynomial $\mathfrak{C}_w(X_m,Y_m)$ with $w\in W_m$ is *stable* if

$$\mathfrak{C}_{i(w)}(X_n,Y_n)\big|_{x_{m+1}=\cdots=x_n=y_{m+1}=\cdots=y_n=0} = \mathfrak{C}_w(X_m,Y_m)$$

for all $n>m$. Theorems 1 and 2 show that this stability property fails in general, even for maximal Grassmannian elements. For example when $m=2$, $n=3$ and $w=(\overline{2},1)$ we have

$$\mathfrak{C}_{i(w)}(X_3,Y_3)\big|_{x_3=y_3=0} = x_1x_2+(x_1+x_2)(y_1+y_2)+y_1^2+y_1y_2+y_2^2$$

which differs from $\mathfrak{C}_w(X_2,Y_2)$, given in Example 1. In addition, Theorem 2 shows that the only permutation $\varpi\in S_n$ for which $\mathfrak{C}_\varpi(X,Y)$ is stable is $\varpi=1$.

For $u,v\in W_\infty$ we say that u precedes v in the weak Bruhat order, and write $u\leq v$, if there are generators $s_{a_1},\ldots,s_{a_r}\in W_\infty$ with $v=us_{a_1}\cdots s_{a_r}$ and $r=\ell(v)-\ell(u)\geqslant 0$. For each $k>0$ let w_{ρ_k} be the maximal Grassmannian permutation corresponding to ρ_k, that is

$$w_{\rho_k} = [\overline{k},\overline{k-1},\ldots,\overline{1},k+1,k+2,\ldots]\in W_\infty.$$

Proposition 2 (Stability). *If $w \leq w_{\rho_k}$ for some $k>0$ then $\mathfrak{C}_w(X,Y)$ is stable.*

Proof. Equation (18) above shows that $\mathfrak{C}_w(X,Y)$ is stable when $w = w_{\rho_k}$. It follows from the definition of the polynomials $\mathfrak{C}_w(X,Y)$ by divided difference operators that

$$\partial_i' \mathfrak{C}_w = \begin{cases} \mathfrak{C}_{ws_i} & \text{if } \ ws_i \leq w, \\ 0 & \text{otherwise.} \end{cases}$$

This implies that if $\mathfrak{C}_w(X,Y)$ is stable, then so is $\mathfrak{C}_u(X,Y)$ for any $u \leq w$. □

The elements $w = (w_1, w_2, \ldots) \in W_\infty$ which satisfy the hypothesis of Proposition 2 are characterized by the following three properties, which should hold for all $i<j$: (i) if w_i and w_j are unbarred then $w_i < w_j$, (ii) if w_i and w_j are barred then $|w_i| > |w_j|$, (iii) the barred entries are smaller than the unbarred entries in absolute value. One sees that the only maximal Grassmannian elements w_λ with these properties are the w_{ρ_k} themselves.

Lascoux, Pragacz and Ratajski have shown in [LP1, Thm. A.2] that the symplectic Schubert polynomials $\mathfrak{C}_w(X)$ are stable, for any $w \in W_m$, in the sense that

$$\mathfrak{C}_{i(w)}(X_n)\big|_{x_{m+1}=\cdots=x_n=0} \;=\; \mathfrak{C}_w(X_m).$$

When w is a maximal Grassmannian element w_λ, the stability of $\mathfrak{C}_w(X) = \widetilde{Q}_\lambda(X)$ is clear. Stability for $\mathfrak{C}_\varpi(X)$ when $\varpi \in S_n$ is less obvious, and is equivalent (by Corollary 1) to the following 'shift' property of type A single Schubert polynomials, which can be checked directly.

Corollary 3. *If $\varpi \in S_n$ with $\varpi(1)=1$ and τ is the n-cycle $(n \ldots 21)$, then*

$$\mathfrak{S}_\varpi(\tau X)\big|_{x_n=0} = \mathfrak{S}_{\tau\varpi\tau^{-1}}(X).$$

1.6. **Further properties.** The next result uses the Cauchy formula of Corollary 2 to give a characterization of the type C double Schubert polynomials.

Proposition 3. *Suppose that we are given a homogeneous polynomial $P_{w_0}(X,Y) \in \mathbb{Z}[X,Y]$, and for each $w \in W_n$ define $P_w(X,Y) = \partial'_{w^{-1}w_0}(P_{w_0}(X,Y))$. Let $P_\varpi(X) = P_\varpi(X,0)$ and $P_\lambda(X) = P_{w_\lambda}(X,0)$ for every pair $(\varpi,\lambda) \in S_n \times \mathcal{D}_n$. Assume that these polynomials satisfy the following properties:*

(i) $P_{w_0}(X,Y) = \sum_{\varpi,\lambda} P_{\varpi\varpi_0}(-\varpi_0 X) P_{\lambda'}(X) P_\varpi(\varpi_0 Y) P_\lambda(Y)$

(ii) $P_{\varpi_0}(X) = \mathfrak{S}_{\varpi_0}(\varpi_0 X) = x_2 x_3^2 \cdots x_n^{n-1}$ *and* $P_{\rho_n}(X) = \widetilde{Q}_{\rho_n}(X)$.

Then $P_w(X,Y) = \mathfrak{C}_w(X,Y)$ for all $w \in W_n$.

Proof. Recall that if a polynomial f is homogeneous of degree d, then $\partial_i' f$ is homogeneous of degree $d-1$ for each i. Hence, the definition of the polynomials P_w and their normalization in (ii) imply that $P_w(X)$ is homogeneous of degree $\ell(w)$. In particular $P_w(0)=0$ unless $w=1$, when $P_1(X)=1$. Properties (i) (with $Y=0$) and (ii) thus give

$$P_{w_0}(X) = P_{\varpi_0}(-\varpi_0 X) P_{\rho_n}(X) = (-1)^{n(n-1)/2} \mathfrak{S}_{\varpi_0}(X) \widetilde{Q}_{\rho_n}(X) = \mathfrak{C}_{w_0}(X).$$

We deduce that $P_w(X) = \partial'_{w^{-1}w_0}(P_{w_0}(X)) = \mathfrak{C}_w(X)$ for all $w \in W_n$. The result now follows from (i) and Corollary 2. □

We close this section with a vanishing property, which reflects the fact that the top polynomial $\mathfrak{C}_{w_0}(X,Y)$ represents the class of the diagonal in flag bundles (see [Gra, Thm. 1.1], [Bi, Sect. 8] and Section 2).

Proposition 4 (Vanishing). *We have*

$$\mathfrak{C}_{w_0}(X, wX) = \begin{cases} 2^n x_1 \cdots x_n \prod_{i>j}(x_i^2 - x_j^2) & \text{if } \ w = \varpi_0 \in S_n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The vanishing property for $\widetilde{Q}(X,Y)$ from [LP1, Sect. 2] gives

$$\widetilde{Q}(X, wX) = \begin{cases} \prod_{i \geqslant j}(x_i + x_j) & \text{if } \ w \in S_n, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, $\Delta(X, \varpi X) = 0$ for all $\varpi \in S_n \setminus \{\varpi_0\}$. □

2. Symplectic degeneracy loci

2.1. **Main representation theorem.** Let V be a vector bundle of rank $2n$ on an algebraic variety $\mathfrak{X}$. Assume that V is a *symplectic* bundle, i.e. V is equipped with an everywhere nondegenerate skew-symmetric form $V \otimes V \to \mathbb{C}$. Consider two Lagrangian (i.e., maximal isotropic) subbundles E, F of V together with flags of subbundles

$$\begin{gathered} 0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_n = E \subset V \\ 0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = F \subset V, \end{gathered}$$

where $\dim E_i = \dim F_i = i$ for all i. These can be extended to complete flags $E_\bullet$, $F_\bullet$ in V by defining $E_{n+i} = E_{n-i}^\perp$ for $1 \leqslant i \leqslant n$, and similarly for $F_\bullet$. For each i with $1 \leqslant i \leqslant n$ let

$$x_i = -c_1(E_{n+1-i}/E_{n-i}) \quad \text{and} \quad y_i = -c_1(F_i/F_{i-1}).$$

There is a group monomorphism $\phi\colon W_n \hookrightarrow S_{2n}$ with image

$$\phi(W_n) = \{\, \sigma \in S_{2n} \mid \sigma(i) + \sigma(2n+1-i) = 2n+1, \ \ \text{for all} \ \ i \,\}.$$

The map ϕ is determined by setting, for each $w = (w_1, \ldots, w_n) \in W_n$,

$$\phi(w)(i) = \begin{cases} n + 1 - w_{n+1-i} & \text{if } w_{n+1-i} \text{ is unbarred,} \\ n + \overline{w}_{n+1-i} & \text{otherwise.} \end{cases} \tag{19}$$

For every $w \in W_n$ define the *degeneracy locus* $\mathfrak{X}_w \subset \mathfrak{X}$ as the locus of $a \in \mathfrak{X}$ such that

$$\dim(E_r(a) \cap F_s(a)) \geqslant \#\{\, i \leqslant r \mid \phi(w)(i) > 2n - s \,\} \ \ \text{for} \ \ 1 \leqslant r \leqslant n,\ 1 \leqslant s \leqslant 2n.$$

The precise definition of $\mathfrak{X}_w$ as a subscheme of $\mathfrak{X}$ is obtained by pulling back from the universal case, which takes place on the bundle $F(V) \to \mathfrak{X}$ of isotropic flags in V, following [F3]; see also [FP, Sect. 6.2 and App. A]. If $G_\bullet$ denotes the universal flag over $F(V)$, then the flag $E_\bullet$ corresponds to a section $\sigma\colon \mathfrak{X} \to F(V)$ of $F(V) \to \mathfrak{X}$ such that $\sigma^*(G_\bullet) = E_\bullet$. The degeneracy locus $\mathfrak{X}_w \subset \mathfrak{X}$ is defined as the inverse image $\sigma^{-1}(\widetilde{\mathfrak{X}}_w)$ of the *universal Schubert variety* $\widetilde{\mathfrak{X}}_w$ in $F(V)$.

Assume that $\mathfrak{X}_w$ is of pure codimension $\ell(w)$ and $\mathfrak{X}$ is Cohen-Macaulay. In this case, we have a formula for the fundamental class $[\mathfrak{X}_w]$ of $\mathfrak{X}_w$ in the Chow group $CH^{\ell(w)}(\mathfrak{X})$ of codimension $\ell(w)$ algebraic cycles on $\mathfrak{X}$ modulo rational equivalence, expressed as a polynomial in the Chern roots $X = \{x_i\}$ and $Y = \{y_i\}$ of the vector bundles E and F. The exact result is

Theorem 3 (Locus Representation). $[\mathfrak{X}_w] = \mathfrak{C}_w(X,Y)$ *in* $CH^{\ell(w)}(\mathfrak{X})$.

Proof. We will show that the above statement is equivalent to the corresponding result in [F3] (Theorem 1). Note that, as in loc. cit., the proof is obtained by pulling back the corresponding equality on the flag bundle $F(V)$. In order to avoid notational confusion define new alphabets $U = (u_1, \ldots, u_n)$ and $Z = (z_1, \ldots, z_n)$ with

$$u_i = -c_1(E_i/E_{i-1}) = x_{n+1-i} \quad \text{and} \quad z_i = -c_1(F_{n+1-i}/F_{n-i}) = y_{n+1-i},$$

in agreement with the conventions in [F3]. Identify the Weyl group W_n with its image $\phi(W_n) \subset S_{2n}$ and let $\partial_1^u, \ldots, \partial_n^u$ denote the divided difference operators with respect to the u variables, defined as in loc. cit. For example $\partial_n^u(f) = (f - s_n f)/(2u_n)$, where s_n sends u_n to $-u_n$. Theorem 1 of [F3] gives

$$[\mathfrak{X}_w] = \partial^u_{\phi(w^{-1}w_0)}\left(\Delta(U,Z)\,F(U,Z)\right), \tag{20}$$

where

$$F(U,Z) = \det\left(e_{n+1+j-2i}(U) + e_{n+1+j-2i}(Z)\right)_{i,j}.$$

Using the criterion in [Gra, Theorem 1.1], we see that one can replace the kernel $\Delta(U,Z)\,F(U,Z)$ in (20) with

$$(-1)^{n(n-1)/2}\prod_{i>j}(u_i - y_j)\,F(U,Y) = (-1)^{n(n-1)/2}\varpi_0(\Delta(U,Y))\,F(U,Y), \tag{21}$$

where the permutation ϖ_0 acts on the u variables (compare with the analysis in [Gra, Section 5]). Next we apply [LP1, Proposition 2.4] to replace $F(U,Y)$ in (21) by $\widetilde{Q}(U,Y)$, as these two polynomials are congruent modulo the relations in the Chow ring of the flag bundle of V. Note also that we have the identities

$$\partial^u_{n-i}\varpi_0 = -\varpi_0\partial^u_i, \quad 1 \leqslant i \leqslant n-1, \quad \text{and} \quad \partial^u_n\varpi_0 = \varpi_0\partial^u_0, \tag{22}$$

where we define $\partial_0^u(f) = (f - s_0 f)/(2u_1)$. The map ϕ interchanges s_i with s_{n-i} for all i; it follows from (20), (21) and (22) that

$$[\mathfrak{X}_w] = (-1)^{n(n-1)/2}\varpi_0(\partial^u_{w^{-1}w_0})'\left(\Delta(U,Y)\,\widetilde{Q}(U,Y)\right) = \mathfrak{C}_w(X,Y),$$

as required. □

2.2. **Lagrangian degeneracy loci for subbundles.** We turn next to the special case of Theorem 3 which describes Lagrangian degeneracy loci. Note that the variables $X = \{x_1, \ldots, x_n\}$ are the Chern roots of E^* and for each i, $Y_i = \{y_1, \ldots, y_i\}$ are the Chern roots of F_i^*. For each strict partition $\lambda \in \mathcal{D}_n$ define the *degeneracy locus* $\mathfrak{X}_\lambda \subset \mathfrak{X}$ as

$$\mathfrak{X}_\lambda = \{\, a \in \mathfrak{X} \mid \dim(E(a) \cap F_{n+1-\lambda_i}(a)) \geqslant i \text{ for } 1 \leqslant i \leqslant \ell(\lambda)\,\}. \tag{23}$$

The precise scheme-theoretic definition $\mathfrak{X}_\lambda$ is obtained as before, by pulling back the corresponding universal locus $\widetilde{\mathfrak{X}}_{w_\lambda}$ by the section σ. There is an equality $\mathfrak{X}_\lambda = \mathfrak{X}_{w_\lambda}$ of schemes, hence the analysis in Section 2.1 applies. Assuming that $\mathfrak{X}_\lambda$ is of pure codimension $|\lambda|$ and $\mathfrak{X}$ is Cohen-Macaulay, Theorem 3 says that the fundamental class $[\mathfrak{X}_\lambda]$ equals $\mathfrak{C}_\lambda(X,Y)$ in the Chow group $CH^{|\lambda|}(\mathfrak{X})$. Now Theorem 1 gives

Corollary 4 (Lagrangian Degeneracy Loci)**.** *We have*

$$[\mathfrak{X}_\lambda] = (-1)^{e(\lambda)+|\lambda'|}\sum_\alpha \widetilde{Q}_\alpha(E^*)\sum_\beta (-1)^{e(\alpha,\beta)+|\beta|}\widetilde{Q}_{(\alpha\cup\beta)'}(F^*)\det(c_{\beta_i-\lambda'_j}(F^*_{n-\lambda'_j}))$$

in $CH^{|\lambda|}(\mathfrak{X})$, where the first sum is over all $\alpha \in \mathcal{D}_n$ and the second over $\beta \in \mathcal{D}_n$ with $\beta \supset \lambda'$, $\ell(\beta) = \ell(\lambda')$ and $\alpha \cap \beta = \emptyset$.

We can apply Corollary 4 to recover the results of [PR2] which compute the class $[\mathfrak{X}_\lambda]$ for some special partitions λ. For instance, when $\lambda = k$ for some $k \leqslant n$ the locus $\mathfrak{X}_\lambda$ is given by a single Schubert condition. Let s_r denote the Schur polynomial (or complete homogeneous function) corresponding to r.

Corollary 5 ([PR2], Proposition 6.1)**.** *We have*

$$[\mathfrak{X}_k] = \sum_{p=0}^{k} c_p(E^*) s_{k-p}(F^*_{n+1-k}).$$

Proof. For each $p < k$ there is an equality of $(k-p) \times (k-p)$ matrices

$$\{e_{1+j-i}(Y_{n-k+j})\} = \{e_{1+j-i}(Y_{n-k+1})\} \cdot \{e_{j-i}(Y_{n-k+j} \smallsetminus Y_{n-k+1})\}. \tag{24}$$

Taking determinants in (24) shows that

$$\det(c_{1+j-i}(F^*_{n-k+j}))_{1 \leqslant i,j \leqslant k-p} = s_{k-p}(F^*_{n-k+1}). \tag{25}$$

Now apply Corollary 4 and use (25) to obtain

$$[\mathfrak{X}_k] = \sum_{p=1}^{k} c_p(E^*) s_{k-p}(F^*_{n+1-k}) + \sum_{q=1}^{k} (-1)^{q-1} c_q(F^*) s_{k-q}(F^*_{n+1-k}).$$

The proof is completed by noting that

$$\sum_{q=1}^{k} (-1)^{q-1} e_q(Y) s_{k-q}(Y_{n+1-k}) = \det(e_{1+j-i}(Y_{n+1-i}))_{1 \leqslant i,j \leqslant k} = s_k(Y_{n+1-k}),$$

which follows from (25) and the Laplace expansion of a determinant. □

When $\lambda = \rho_k$, Corollary 4 becomes Theorem 9.1 of [PR2]. More generally, we have the following result:

Corollary 6. *Let $\lambda = \rho_k \smallsetminus j$ for an integer j with $1 \leqslant j \leqslant k$. Then*

$$[\mathfrak{X}_\lambda] = \sum_{p=j}^{k} (-1)^{k-p} c_{p-j}(F^*_{n-j}) \sum_{\substack{\alpha \in \mathcal{D}_k \\ \alpha \cap p = \emptyset}} (-1)^{e(\alpha,p)} \widetilde{Q}_\alpha(E^*) \widetilde{Q}_{\rho_k \smallsetminus (\alpha \cup p)}(F^*)$$

where $e(\alpha, p) = \#\{\, i \mid \alpha_i > p \,\}$.

Remark. The formula for Lagrangian degeneracy loci in Corollary 4 (and its predecessor from Theorem 1) interpolates between two extreme cases, where it assumes a simple form. These correspond to the partitions $\lambda = \rho_k$ (Example 4 in Section 1) and $\lambda = k$ (Corollary 5), and were derived in [PR2] using purely geometric methods. Note that a very different formula for the $\lambda = \rho_k$ locus is given in [F2] [F3]; it would be interesting to have an analogous statement for the remaining maximal Grassmannian loci.

2.3. **Restriction of type A loci.** We now give a geometric interpretation of Theorem 2, in the setting of flag bundles. Let $F'(V)$ be the partial SL_{2n}-flag bundle of which parametrizes flags of subbundles

$$0 = E'_0 \subset E'_1 \subset E'_2 \subset \cdots \subset E'_n \subset V$$

with $\dim E'_i = i$ for all i; by abuse of notation let $E'_\bullet$ also denote the universal flag of vector bundles over $F'(V)$. Suppose that

$$0 = F'_0 \subset F'_1 \subset F'_2 \subset \cdots \subset F'_{2n} = V$$

is a fixed complete flag of subbundles of V, and set

$$x'_i = -c_1(E'_i/E'_{i-1}) \quad \text{and} \quad y'_j = -c_1(F'_{2n+1-j}/F'_{2n-j})$$

for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant 2n$. For any permutation $\varpi \in S_n$, we have a universal codimension $\ell(\varpi)$ Schubert variety $\mathfrak{X}'_\varpi$ in $F'(V)$, defined as the locus of $a \in F'(V)$ such that

$$\dim(E'_r(a) \cap F'_s(a)) \geqslant \#\,\{\, i \leqslant r \mid \varpi(i) > 2n-s \,\} \ \text{ for } \ 1 \leqslant r \leqslant n,\, 1 \leqslant s \leqslant 2n.$$

Identify ϖ with a permutation in S_{2n} by using the embedding $S_n \hookrightarrow S_{2n}$ which fixes the last n entries. Then the class of $\mathfrak{X}'_\varpi$ in $CH^{\ell(\varpi)}(F'(V))$ is given by

$$[\mathfrak{X}'_\varpi] = \mathfrak{S}_\varpi(X', Y')$$

where $X' = (x'_1, \ldots, x'_n, 0, \ldots, 0)$ and $Y' = (y'_1, \ldots, y'_{2n})$. This follows from [F1, Prop. 8.1] by dualizing; to navigate through the changes in notation, we have found [FP, Sect. 2.2] helpful.

Let $\theta \colon F(V) \hookrightarrow F'(V)$ be the natural inclusion and $\theta^* \colon CH(F'(V)) \to CH(F(V))$ the induced homomorphism on Chow groups. Define the complete flag $F'_\bullet$ by setting $F'_j = F_j$ for $1 \leqslant j \leqslant 2n$. We then have $\theta^*(x'_i) = x_{n+1-i}$ for $1 \leqslant i \leqslant n$, and hence can identify $\theta^*(X')$ with $\varpi_0 X$, where ϖ_0 denotes the longest element in S_n. The symplectic form on V induces isomorphisms $V/(F_j^\perp) \cong F_j^*$ for each j; it follows that $\theta^*(y'_j) = -y_j$ for $1 \leqslant j \leqslant n$. Using the stability property of type A double Schubert polynomials [M1, (6.5)] and [F1, Cor. 2.11] we see that

$$\theta^*(\mathfrak{S}_\varpi(X', Y')) = \mathfrak{S}_\varpi(\varpi_0 X, -Y) = \mathfrak{C}_{\varpi^*}(X, Y), \tag{26}$$

where the last equality is Theorem 2.

On the other hand, by comparing the defining conditions for $\mathfrak{X}'_\varpi$ and $\widetilde{\mathfrak{X}}_{\varpi^*}$ one checks that

$$\theta^{-1}(\mathfrak{X}'_\varpi) = F(V) \cap \mathfrak{X}'_\varpi = \widetilde{\mathfrak{X}}_{\varpi^*}$$

and that the intersection is proper and generically transverse along $\widetilde{\mathfrak{X}}_{\varpi^*}$. It follows that

$$\theta^*[\mathfrak{X}'_\varpi] = [\widetilde{\mathfrak{X}}_{\varpi^*}] \tag{27}$$

and thus Theorem 2 corresponds in geometry to the restriction of universal Schubert classes (27).

2.4. **Schubert calculus.** We now suppose that V is a symplectic vector space and let $\mathfrak{X} = F_{Sp}(V)$ denote the variety of complete isotropic flags in V. There is a tautological isotropic flag $E_\bullet$ of vector bundles over $\mathfrak{X}$, and we let $F_\bullet$ be a fixed isotropic flag in V. For each $w \in W_n$ the locus $\mathfrak{X}_w$ is a Schubert variety in $\mathfrak{X}$; let $\sigma_w = [\mathfrak{X}_w] \in CH^{\ell(w)}(\mathfrak{X})$ be the corresponding Schubert class. The classes $\{\sigma_w\}_{w \in W_n}$ form an integral basis for the Chow ring $CH(\mathfrak{X})$. Hence, for each $u, v, w \in W_n$, we can define *type C structure constants* $f(u,v,w)$ by the equation

$$\sigma_u \sigma_v = \sum_{w \in W_n} f(u,v,w) \sigma_w. \tag{28}$$

The $f(u,v,w)$ are nonnegative integers.

Theorem 3 implies that $\sigma_w = \mathfrak{C}_w(X)$, that is, the symplectic Schubert polynomials represent the Schubert classes. We deduce that the $\{\mathfrak{C}_w(X)\}_{w \in W_n}$ form a $\mathbb{Z}$-basis for the quotient ring $\mathbb{Z}[X]/I_n$, where I_n denotes the ideal of positive degree Weyl group invariants in $\mathbb{Z}[X]$ (using the well known isomorphism of the latter with $CH(\mathfrak{X})$; see e.g. [Bo]). Recall that the type A Schubert polynomials $\mathfrak{S}_\varpi(X)$ for $\varpi \in S_n$ form a $\mathbb{Z}$-basis for the additive subgroup H_n of $\mathbb{Z}[X]$ spanned by the monomials x^α, $\alpha \subset \rho_{n-1}$ [M1, (4.11)]. Using Corollary 1 we obtain

Corollary 7. *Fix a permutation $\varpi \in S_n$. Suppose that* (i) *the polynomial $P_\varpi \in \mathbb{Z}[X]$ represents the Schubert class σ_ϖ in $CH(\mathfrak{X}) \cong \mathbb{Z}[X]/I_n$ and* (ii) $P_\varpi \in \varpi_0 H_n$. *Then $P_\varpi(X) = \mathfrak{C}_\varpi(X) = \mathfrak{S}_{\varpi^*}(\varpi_0 X)$.*

We will apply the results of Section 1 to obtain some information about the structure constants $f(u,v,w)$. For this we need the following

Definition 2. *Let $u, v \in S_n$ be two permutations. We say that the pair (u,v) is* n-stable *if the product $\mathfrak{S}_u(X)\mathfrak{S}_v(X)$ is contained in H_n.*

The previous remarks show that (u,v) is n-stable exactly when there is an equation of type A Schubert polynomials

$$\mathfrak{S}_u(X)\mathfrak{S}_v(X) = \sum_{w \in S_n} c(u,v,w)\mathfrak{S}_w(X) \tag{29}$$

for some integers $c(u,v,w)$ (which are type A structure constants).

Examples. 1) If $u \in S_k$ and $v \in S_\ell$ then (u,v) is n-stable for any $n \geqslant k + \ell - 1$.

2) Let $\leq$ denote the weak Bruhat order. The Leibnitz rule for divided differences

$$\partial_i(fg) = (\partial_i f)g + (s_i f)(\partial_i g) \tag{30}$$

implies that if (u,v) is n-stable and $u' \leq u$, $v' \leq v$ then (u',v') is also n-stable.

3) For each $i > 0$ and $\varpi \in S_\infty$ let $d_i(\varpi) = \deg_{x_i}(\mathfrak{S}_\varpi(X))$. Then (u,v) is n-stable if and only if $d_i(u) + d_i(v) \leqslant \max\{n-i, 0\}$ for all i. One sees that $d_1(\varpi)$ equals the number of nonempty columns in the diagram of ϖ, or alternatively

$$d_1(\varpi) = \#\{\, j \mid \exists\, i < j \;:\; \varpi(i) > \varpi(j)\,\}.$$

This follows e.g. from the combinatorial algorithm for constructing Schubert polynomials due to Bergeron [M1, Chap. 4]. In particular, $d_1(\varpi) \geqslant \varpi(1) - 1$ for all ϖ. We deduce that if $u(1) + v(1) > n + 1$ then the pair (u,v) is not n-stable.

4) For any partition λ (not necessarily strict) and integer $r \geqslant \ell(\lambda)$ there is a Grassmannian permutation $\varpi_{\lambda,r} \in S_\infty$, determined by $\varpi_{\lambda,r}(i) = \lambda_{r+1-i} + i$ for $i \leqslant r$ and $\varpi_{\lambda,r}(i) < \varpi_{\lambda,r}(i+1)$ for $i > r$. One knows [M2, (4.7)] that (for n sufficiently large) the Schubert polynomial $\mathfrak{S}_{\varpi_{\lambda,r}}(X_n)$ is equal to the Schur function $s_\lambda(x_1, \ldots, x_r)$; moreover $\deg_{x_i}(s_\lambda) = \lambda_1$ for each i. Let μ be a second partition and assume that the integer s satisfies $\ell(\mu) \leqslant s \leqslant r$. We deduce that the pair $(\varpi_{\lambda,r}, \varpi_{\mu,s})$ is n-stable if and only if $n \geqslant \lambda_1 + \max\{r, \mu_1 + s\} - 1$.

5) If $\varpi \in S_n$ we denote by $1 \times \varpi$ the permutation $(1, \varpi(1)+1, \ldots, \varpi(n)+1)$ in S_{n+1}. We then have the following

Proposition 5. *Suppose that $u, v \in S_n$. Then (u,v) is n-stable if and only if $(1 \times u, 1 \times v)$ is $(n+1)$-stable.*

Proof. Given the Schubert polynomial $\mathfrak{S}_{1\times u}(X)$ one can recover $\mathfrak{S}_u(X)$ by using Corollary 3. It follows easily from this that if $(1 \times u, 1 \times v)$ is $(n+1)$-stable then (u,v) is n-stable. For the converse, apply [M1, (4.22)] to obtain

$$\mathfrak{S}_{1\times u}(X) = \partial_1 \cdots \partial_{n-1}(x_1 \cdots x_{n-1}\mathfrak{S}_u(X)) \tag{31}$$

Fibration	$F_{Sp}(2n) \to LG(n,2n) = LG$	$F_{SL}(n) \times LG \to LG$
Basis	$\{\mathfrak{C}_w(X)\}_{w\in W_n}$	$\{\mathfrak{C}_{\varpi,\lambda}(X)\}_{(\varpi,\lambda)\in S_n\times\mathcal{D}_n}$
Group	$S_n \ltimes (\mathbb{Z}_2)^n = W_n$	$S_n \times (\mathbb{Z}_2)^n$

TABLE 1

for every $u \in S_n$. Assuming that (u,v) is n-stable, we deduce that $(1\times u, 1\times v)$ is $(n+1)$-stable from (30), (31) and a straightforward induction. □

Extend the involution $*\colon S_n \to S_n$ of Section 1.2 to all of W_n by letting $w^* = \varpi_0 w \varpi_0$, for each $w \in W_n$.

Proposition 6. *Assume that $(u,v) \in S_n \times S_n$ is n-stable. Then for each $w \in W_n$,*

$$f(u^*,v^*,w^*) = \begin{cases} c(u,v,w) & \text{if } \ w \in S_n, \\ 0 & \text{otherwise.} \end{cases} \tag{32}$$

Proof. We apply the permutation ϖ_0 to the polynomials in (29), then use Corollary 1 to obtain

$$\mathfrak{C}_{u^*}(X)\mathfrak{C}_{v^*}(X) = \sum_{w\in S_n} c(u,v,w)\mathfrak{C}_{w^*}(X),$$

and compare this with (28). □

Remarks. 1) If $u,v \in S_m$ then (u,v) is n-stable for any $n \geqslant 2m-1$, thus (32) is true for all such n. Hence every type A structure constant $c(u,v,w)$ appears as the stable limit of type C constants $f(u^*,v^*,w^*)$, with the involution $*$ defined using the rank of the corresponding hyperoctahedral group.

2) Poincaré duality for the Schubert classes in the complete SL_n-flag variety implies that

$$c(u,v,w) = c(u,\varpi_0 w, \varpi_0 v) = c(u^*,v^*,w^*)$$

for all $u,v,w \in S_n$. Moreover, the structure constants $c(u,v,w)$ and $f(u,v,w)$ are stable under the inclusions of the respective Weyl groups. These properties are compatible because of the general identity

$$c(1\times u, 1\times v, 1\times w) = c(u,v,w), \tag{33}$$

valid for all $u,v,w \in S_\infty$. Note that (33) is a special case of [BS, Cor. 4.5.6].

3) The special case of the Pieri rule of [PR1] displayed after Example 2.3 of loc. cit. and the Giambelli-type formula in [PR1, Cor. 2.4] are consequences of Proposition 6, applied to (non-maximal) isotropic Grassmannians. To recover these formulas from Proposition 6, notice that if $\varpi_\lambda \in S_n$ is a Grassmannian permutation corresponding to the partition λ, then $\varpi_0\varpi_\lambda\varpi_0$ is also Grassmannian and corresponds to the conjugate (or transpose) of λ.

Table 1 illustrates the connections between some of the geometric, algebraic and combinatorial objects in this paper. There is a fibration of the symplectic flag variety $F_{Sp}(2n) = Sp(2n)/B_1$ over the Lagrangian Grassmannian $LG(n,2n) = Sp(2n)/P_n$, with fiber equal to the SL_n-flag variety $F_{SL}(n) = SL(n)/B_2$. The right column in Table 1 corresponds to the trivial product fibration $F_{SL}(n) \times LG(n, 2n)$. The symplectic Schubert polynomials give a $\mathbb{Z}$-basis for the cohomology (or Chow) ring in either column, as shown. When restricted to the parameter spaces S_n and $\mathcal{D}_n$, the Schubert polynomial and product bases coincide. The correspondence between the set $\mathcal{D}_n$ and the group $(\mathbb{Z}_2)^n = \{\pm 1\}^n$ is given as follows: the strict partition $\lambda \in \mathcal{D}_n$ corresponds to the vector $(\epsilon_r) \in \{\pm 1\}^n$ with

$$\epsilon_r = \begin{cases} -1 & \text{if } r \text{ is a part of } \lambda, \\ 1 & \text{otherwise.} \end{cases}$$

3. Orthogonal Schubert polynomials and degeneracy loci

In this section we describe the analogues of the main results of Sections 1 and 2 for the orthogonal groups of types B_n and D_n. The alphabets X_k and Y_k for $1 \leqslant k \leqslant n$ are defined as before, with $X = X_n$ and $Y = Y_n$. For each $\lambda \in \mathcal{D}_n$ define the $\widetilde{P}$-polynomial

$$\widetilde{P}_\lambda(X) = 2^{-\ell(\lambda)} \widetilde{Q}_\lambda(X)$$

and, for $k \in \{n-1, n\}$, the reproducing kernel

$$\widetilde{P}_k(X, Y) = \sum_{\lambda \in \mathcal{D}_k} \widetilde{P}_\lambda(X) \widetilde{P}_{\rho_k \smallsetminus \lambda}(Y).$$

3.1. Type B double Schubert polynomials and loci.

The Weyl group of the root system B_n is the same as that for C_n, as are the divided difference operators ∂_i for $1 \leqslant i \leqslant n-1$. However the operator ∂_0 differs from the one in type C by a factor of 2:

$$\partial_0(f) = (f - s_0 f)/x_1$$

for any $f \in A[X]$. In this and the next subsection we will take $A = \mathbb{Z}[1/2][Y]$. The ∂_i are used as before to define operators ∂'_w for all barred permutations $w \in W_n$.

Definition 3. *For every $w \in W_n$ the* type B double Schubert polynomial $\mathfrak{B}_w(X, Y)$ *is given by*

$$\mathfrak{B}_w(X, Y) = (-1)^{n(n-1)/2} \partial'_{w^{-1} w_0} \left(\Delta(X, Y) \widetilde{P}_n(X, Y) \right).$$

For each partition $\lambda \in \mathcal{D}_n$ let $\mathfrak{B}_\lambda(X, Y) = \mathfrak{B}_{w_\lambda}(X, Y)$. Using the results of [LP2, Appendix] and the same arguments as in Section 1 produces the following two theorems.

Theorem 4 (Maximal Grassmannian)**.** *For any strict partition $\lambda \in \mathcal{D}_n$ the double Schubert polynomial $\mathfrak{B}_\lambda(X, Y)$ is equal to*

$$(-1)^{e(\lambda)+|\lambda'|} \sum_\alpha \widetilde{P}_\alpha(X) \sum_\beta (-1)^{e(\alpha,\beta)+|\beta|} \widetilde{P}_{(\alpha\cup\beta)'}(Y) \det(e_{\beta_i - \lambda'_j}(Y_{n-\lambda'_j})),$$

where the first sum is over all $\alpha \in \mathcal{D}_n$ and the second over $\beta \in \mathcal{D}_n$ with $\beta \supset \lambda'$, $\ell(\beta) = \ell(\lambda')$ and $\alpha \cap \beta = \emptyset$.

Theorem 5 (Positivity)**.** *For every $\varpi \in S_n$,*

$$\mathfrak{B}_\varpi(X, Y) = \mathfrak{S}_{\varpi^*}(\varpi_0 X, -Y).$$

In particular, $\mathfrak{B}_\varpi(X, Y)$ has nonnegative integer coefficients.

Consider an orthogonal vector bundle V of rank $2n+1$ on an algebraic variety $\mathfrak{X}$, so that V is equipped with an everywhere nondegenerate quadratic form. One is given flags of isotropic subbundles of V as before, with $E = E_n$ and $F = F_n$. These are extended to complete flags $E_\bullet$ and $F_\bullet$ by setting $E_{n+i} = E_{n+1-i}^\perp$ and $F_{n+i} = F_{n+1-i}^\perp$ for $1 \leqslant i \leqslant n+1$. The classes $x_i, y_i \in CH^1(\mathfrak{X})$ are defined as in Section 2. In this case we have a monomorphism $\psi\colon W_n \hookrightarrow S_{2n+1}$ with image

$$\psi(W_n) = \{\, \sigma \in S_{2n+1} \mid \sigma(i) + \sigma(2n+2-i) = 2n+2, \quad \text{for all} \;\; i \,\},$$

determined by the equalities

$$\psi(w)(i) = \begin{cases} n+1-w_{n+1-i} & \text{if } w_{n+1-i} \text{ is unbarred,} \\ n+1+\overline{w}_{n+1-i} & \text{otherwise} \end{cases}$$

for each $w = (w_1, \ldots, w_n) \in W_n$. The *degeneracy locus* $\mathfrak{X}_w$ is the locus of $a \in \mathfrak{X}$ such that

$$\dim(E_r(a) \cap F_s(a)) \geqslant \#\, \{\, i \leqslant r \mid \psi(w)(i) > 2n+1-s \,\}$$

for $1 \leqslant r \leqslant n$, $1 \leqslant s \leqslant 2n$. Assuming that $\mathfrak{X}_w$ is of pure codimension $\ell(w)$ and $\mathfrak{X}$ is Cohen-Macaulay, we have

Theorem 6 (Locus Representation)**.** $[\mathfrak{X}_w] = \mathfrak{B}_w(X,Y)$ *in* $CH^{\ell(w)}(\mathfrak{X})$.

The proof of this theorem is the same as its analogue in Section 2, using the corresponding results of [F3], [Gra] and [LP2]. The maximal isotropic degeneracy loci $\mathfrak{X}_\lambda$ for $\lambda \in \mathcal{D}_n$ are defined by the same inequalities (23) as in the symplectic case. By combining the two previous theorems we immediately obtain a formula for the class $[\mathfrak{X}_\lambda]$ in $CH^{|\lambda|}(\mathfrak{X})$:

Corollary 8. *We have*

$$[\mathfrak{X}_\lambda] = (-1)^{e(\lambda)+|\lambda'|} \sum_\alpha \widetilde{P}_\alpha(E^*) \sum_\beta (-1)^{e(\alpha,\beta)+|\beta|} \widetilde{P}_{(\alpha\cup\beta)'}(F^*) \det(c_{\beta_i - \lambda'_j}(F^*_{n-\lambda'_j})),$$

where the first sum is over all $\alpha \in \mathcal{D}_n$ and the second over $\beta \in \mathcal{D}_n$ with $\beta \supset \lambda'$, $\ell(\beta) = \ell(\lambda')$ and $\alpha \cap \beta = \emptyset$.

3.2. **Type D double Schubert polynomials and loci.** The situation here differs significantly from that of the previous sections, so we will give more details. The Weyl group $\widetilde{W}_n$ of type D is an extension of S_n by an element $s_\square$ which acts on the right by

$$(u_1, u_2, \ldots, u_n)s_\square = (\overline{u}_2, \overline{u}_1, u_3, \ldots, u_n).$$

$\widetilde{W}_n$ may be realized as a subgroup of W_n by sending $s_\square$ to $s_0 s_1 s_0$. The barred permutation $\widetilde{w}_0 \in \widetilde{W}_n$ of maximal length is given by

$$\widetilde{w}_0 = \begin{cases} (\overline{1}, \ldots, \overline{n}) & \text{if } n \text{ is even,} \\ (1, \overline{2}, \ldots, \overline{n}) & \text{if } n \text{ is odd.} \end{cases}$$

For $1 \leqslant i \leqslant n-1$ the action of the generators s_i on the polynomial ring $A[X]$ and the divided difference operators ∂_i are the same as before. We let $s_\square$ act by sending (x_1, x_2) to $(-x_2, -x_1)$ and fixing the remaining variables, while

$$\partial_\square(f) := (f - s_\square f)/(x_1 + x_2)$$

for all $f \in A[X]$. Define operators $\partial'_i = -\partial_i$ for each i with $1 \leqslant i \leqslant n-1$ and set $\partial'_\square := \partial_\square$; these are used as above to define ∂_w and ∂'_w for all $w \in \widetilde{W}_n$.

Definition 4. *For every $w \in \widetilde{W}_n$ the* type D double Schubert polynomial $\mathfrak{D}_w(X,Y)$ *is given by*

$$\mathfrak{D}_w(X,Y) = (-1)^{n(n-1)/2}\partial'_{w^{-1}\widetilde{w}_0}\left(\Delta(X,Y)\widetilde{P}_{n-1}(X,Y)\right).$$

The maximal Grassmannian elements w_λ in $\widetilde{W}_n$ are parametrized by partitions $\lambda \in \mathcal{D}_{n-1}$. For each such λ we set $\ell = \ell(\lambda)$; then

$$w_\lambda = (\overline{\lambda_1+1}, \ldots, \overline{\lambda_\ell+1}, \widehat{1}, \mu_{n-\ell-1}, \ldots, \mu_1)$$

where $\mu = \rho_n \smallsetminus (\lambda_1+1, \ldots, \lambda_\ell+1, 1)$ and $\widehat{1}$ is equal to 1 or $\overline{1}$ according to the parity of ℓ. Let $\lambda' = \rho_{n-1} \smallsetminus \lambda$ be the dual partition of λ and

$$k = \begin{cases} n-\ell & \text{if } n = \ell \pmod 2, \\ n-\ell-1 & \text{if } n \neq \ell \pmod 2. \end{cases}$$

Let $\mathcal{D}^+_{n-1}$ be the set of strictly decreasing sequences β of elements of the set $\{0,1,\ldots,n-1\}$, with length $\ell(\beta)$ equal to the number of terms in the sequence. We think of $\mathcal{D}^+_{n-1}$ as consisting of partitions in $\mathcal{D}_{n-1}$, possibly with an 'extra zero part'. If $\alpha \in \mathcal{D}_{n-1}$ and $\beta \in \mathcal{D}^+_{n-1}$ let

$$n_i(\alpha,\beta) = \#\{\, j \mid \alpha_i > \beta_j \geqslant \alpha_{i+1} \,\} \quad \text{and} \quad f(\alpha,\beta) = \sum_{i=1}^{\ell(\alpha)} i\, n_i(\alpha,\beta).$$

Also define

$$f(\lambda) = \begin{cases} e(\lambda)+\ell(\lambda) & \text{if } n = \ell \pmod 2, \\ e(\lambda) & \text{if } n \neq \ell \pmod 2 \end{cases}$$

and note that $f(\lambda) = f(\lambda,\lambda')$, provided that λ' is identified with an element in $\mathcal{D}^+_{n-1}$ *of length* k. Let $\mathfrak{D}_\lambda(X,Y) = \mathfrak{D}_{w_\lambda}(X,Y)$; we can now state

Theorem 7 (Maximal Grassmannian I)**.** *For any strict partition $\lambda \in \mathcal{D}_{n-1}$ the double Schubert polynomial $\mathfrak{D}_\lambda(X,Y)$ is equal to*

$$(-1)^{f(\lambda)+|\lambda'|}\sum_\alpha \widetilde{P}_\alpha(X)\sum_\beta(-1)^{f(\alpha,\beta)+|\beta|}\widetilde{P}_{(\alpha\cup\beta)'}(Y)\det(e_{\beta_i-\lambda'_j}(Y_{n-1-\lambda'_j})),$$

where the first sum is over all $\alpha \in \mathcal{D}_{n-1}$ and the second over $\beta \in \mathcal{D}^+_{n-1}$ with $\beta \supset \lambda'$, $\ell(\beta) = k$ and $\alpha \cap \beta = \emptyset$.

Proof. Given $\lambda \in \mathcal{D}_{n-1}$ define ℓ and k as above and associate to λ a partition ν of length k as follows:

$$\nu = \begin{cases} \rho_n \smallsetminus (\lambda_1+1, \ldots, \lambda_\ell+1) & \text{if } n = \ell \pmod 2, \\ \rho_n \smallsetminus (\lambda_1+1, \ldots, \lambda_\ell+1, 1) & \text{if } n \neq \ell \pmod 2. \end{cases}$$

Note that $w_\lambda = \widetilde{w}_0\tau_\lambda\,\varpi_k\,\delta_k^{-1}$, where

$$\tau_\lambda = \begin{cases} (\nu_k, \ldots, \nu_1, \lambda_1+1, \ldots, \lambda_\ell+1) & \text{if } n = \ell \pmod 2, \\ (\nu_k, \ldots, \nu_1, \lambda_1+1, \ldots, \lambda_\ell+1, 1) & \text{if } n \neq \ell \pmod 2 \end{cases}$$

$$\varpi_k = (k, \ldots, 2, 1, k+1, \ldots, n)$$

$$\delta_k = (s_{n-k}\cdots s_2 s_1 s_{n-k+1}\cdots s_2 s_\square)\cdots(s_{n-2}\cdots s_2 s_1 s_{n-1}\cdots s_2 s_\square);$$

hence

$$\partial_{w_\lambda^{-1}\widetilde{w}_0} = \partial_{\delta_k}\circ\partial_{\varpi_k}\circ\partial_{\tau_\lambda^{-1}}.$$

We compute as in the proof of Theorem 1:

$$\partial_{\tau_\lambda^{-1}}(\Delta(X,Y)) = \prod_{j=1}^{k}\prod_{p=1}^{n-\nu_j}(x_{k+1-j}-y_p) \tag{34}$$

$$= \sum_{\gamma}(-1)^{|\gamma|-|\nu|}\prod_{j=1}^{k} x_{k+1-j}^{n-\gamma_j}e_{\gamma_j-\nu_j}(Y_{n-\nu_j}) \tag{35}$$

where the sum (35) is over all k-tuples $\gamma=(\gamma_1,\ldots,\gamma_k)$ of nonnegative integers.

Observe that

$$\partial'_{\delta_k}\partial'_{\varpi_k} = \partial_{\delta_k}\partial'_{\varpi_k} = (-1)^{k/2}\partial_{\delta_k}\partial_{\varpi_k}$$

since $k(k-1)/2 \equiv k/2 \pmod 2$. The operator ∂_{ϖ_k} is a Jacobi symmetrizer; hence for any partition $\theta=(\theta_i)$ we have $\partial_{\varpi_k}(x_1^{\theta_1}\cdots x_k^{\theta_k})=0$ unless $\theta_1>\theta_2>\cdots>\theta_k$, when

$$\partial_{\varpi_k}(x_1^{\theta_1}\cdots x_k^{\theta_k}) = s_{\theta_1-k+1,\theta_2-k+2,\ldots,\theta_k}(x_1,\ldots,x_k)$$

is a Schur S-polynomial. In the latter case we can apply [LP2, Theorem 11] and deduce that for any partition $\lambda\in\mathcal{D}_{n-1}$,

$$(-1)^{k/2}\partial_{\delta_k}\partial_{\varpi_k}(x_1^{\theta_1}\cdots x_k^{\theta_k}\widetilde{P}_\lambda(X)) = 0$$

unless each part of $\overline{\theta}=(n-1-\theta_k,\ldots,n-1-\theta_1)\in\mathcal{D}^+_{n-1}$ occurs in λ. In this case, the image is $(-1)^{f(\alpha,\overline{\theta})}\widetilde{P}_\alpha(X)$ where $\alpha=\lambda\smallsetminus\overline{\theta}$ (note that the operator $\partial_\square$ in [LP2] differs from ours by a sign). Moreover, if $(m_1,\ldots,m_k)$ is a k-tuple of distinct nonnegative integers and $\sigma\in S_k$ is such that $m_{\sigma(1)}>\cdots>m_{\sigma(k)}$ then

$$\partial_{\varpi_k}(x_1^{m_1}\cdots x_k^{m_k}) = \mathrm{sgn}(\sigma)\partial_{\varpi_k}(x_1^{m_{\sigma(1)}}\cdots x_k^{m_{\sigma(k)}}),$$

hence the previous analysis applies, up to $\mathrm{sgn}(\sigma)$.

Noting that $\nu_j-1=\lambda'_j$ for $1\leqslant j\leqslant k$ and $|\nu|\equiv|\lambda'| \pmod 2$, use (35) to compute

$$\partial'_{w_\lambda^{-1}\widetilde{w}_0}(\Delta(X,Y)\,\widetilde{P}_{n-1}(X,Y)) = \partial'_{\delta_k}\partial'_{\varpi_k}\left(\widetilde{P}_{n-1}(X,Y)\partial'_{\tau_\lambda^{-1}}(\Delta(X,Y))\right)$$

$$= \partial'_{\delta_k}\partial'_{\varpi_k}\left(\widetilde{P}_{n-1}(X,Y)\sum_{\substack{\gamma\in\mathcal{D}_n\\ \ell(\gamma)=k}}(-1)^{|\gamma|-|\nu|+\ell(\tau_\lambda)}\sum_{\sigma\in S_k}\prod_{j=1}^{k}x_{k+1-j}^{n-\gamma_{\sigma(j)}}e_{\gamma_{\sigma(j)}-\nu_j}(Y_{n-\nu_j})\right)$$

$$= (-1)^{r(\lambda)}\sum_{\alpha}\widetilde{P}_\alpha(X)\sum_{\beta}(-1)^{f(\alpha,\beta)+|\beta|}\widetilde{P}_{(\alpha\cup\beta)'}(Y)\det(e_{\beta_i-\lambda'_j}(Y_{n-1-\lambda'_j}))$$

where the ranges of summation are as in the statement of the theorem and $r(\lambda)=\ell(\tau_\lambda)+k(k-1)/2+|\lambda'|$. Finally, observe that

$$\ell(\tau_\lambda) = |\nu|+\binom{\ell}{2}-\binom{k+1}{2} \equiv f(\lambda)+\binom{k}{2}+\binom{n}{2} \pmod 2$$

so the signs fit to complete the proof. $\square$

The analogue of Theorem 2 in type D_n differs from the one in type B_n. Let $H\cong(\mathbb{Z}_2)^{n-1}$ be the normal subgroup of $\widetilde{W}_n$ consisting of those elements equal to $(1,\ldots,n)$ in absolute value. For each $\varpi\in S_n$, define the parameter space

$$L(\varpi) = \{w\in\varpi H \mid \ell(w)=\ell(\varpi)\}.$$

Theorem 8 (Positivity)**.** *For every $\varpi \in S_n$,*

$$\sum_{w \in L(\varpi)} \mathfrak{D}_w(X,Y) = \mathfrak{S}_{\varpi^*}(\varpi_0 X, -Y). \tag{36}$$

Proof. We argue along the lines of the proof of Theorem 2, using the operator

$$\Pi = (\partial_\square + \partial_1')\,\partial_2'\,(\partial_\square + \partial_1')\,(\partial_3'\partial_2')\,\cdots\,(\partial_\square + \partial_1')\,(\partial_{n-1}'\cdots\partial_2')\,(\partial_\square + \partial_1')$$

in place of ∂'_{v_0}. We claim that the image of $\Pi\colon \mathbb{Z}[1/2][X] \to \mathbb{Z}[1/2][X]$ is a free $\mathbb{Z}[1/2][X]^{\widetilde{W}_n}$-module with basis $\{\mathfrak{S}_\varpi(\varpi_0 X)\}_{\varpi\in S_n}$. This statement is a type D analogue of [LP1, Prop. 4.1], and the proof is similar. The only difference is that here we use the relation $(\partial_\square - \partial_1')\,\Pi = 0$, and thus $\partial'_{w_\lambda s_\square}(\partial_\square - \partial_1')\,\Pi = 0$, for each $\lambda \in \mathcal{D}_{n-1}$. Moreover, [LP2, Cor. 16] implies that

$$\partial'_{w_\lambda s_\square}(\partial_\square - \partial_1')(\widetilde{P}_\mu(X)) = \begin{cases} 1 & \text{if } \ \mu = \lambda, \\ 0 & \text{if } \ |\mu| = |\lambda| \ \text{ but } \ \mu \neq \lambda \end{cases}$$

(note that $\partial'_{w_\lambda s_\square}\partial_\square = \partial'_{w_\lambda}$). Now the vanishing property for $\widetilde{P}_{n-1}(X,Y)$ from [LP2, Prop. 2] and the same argument as in [LP1, Sect. 4] show that

$$\Pi(\mathfrak{S}_u(X)\widetilde{P}_{n-1}(X,Y)) = (-1)^{\ell(u)}\mathfrak{S}_u(\varpi_0 X)$$

for every $u \in S_n$.

Observe, since $L(\varpi_0) = \varpi_0 H$, that $\Pi = \sum_{w\in L(\varpi_0)} \partial'_{w^{-1}w_0}$. We deduce as in the proof of Theorem 2 that

$$\sum_{w\in L(\varpi_0)} \mathfrak{D}_w(X,Y) = (-1)^{\ell(\varpi_0)}\Pi(\Delta(X,Y)\widetilde{P}_{n-1}(X,Y)) = \mathfrak{S}_{\varpi_0}(\varpi_0 X, -Y). \tag{37}$$

For any permutation ϖ, apply the operator $\partial'_{\varpi^{-1}\varpi_0}$ to both sides of equation (37). We have seen that

$$\partial'_{\varpi^{-1}\varpi_0}(\mathfrak{S}_{\varpi_0}(\varpi_0 X, -Y)) = \mathfrak{S}_{\varpi^*}(\varpi_0 X, -Y),$$

so it remains to show that

$$\partial'_{\varpi^{-1}\varpi_0} \sum_{w\in L(\varpi_0)} \mathfrak{D}_w(X,Y) = \sum_{w\in L(\varpi)} \mathfrak{D}_w(X,Y)$$

for all $\varpi \in S_n$. To prove this, note that for each $w \in L(\varpi_0)$, the operator $\partial'_{\varpi^{-1}\varpi_0}\partial'_{w^{-1}w_0}$ either vanishes or equals $\partial'_{u^{-1}w_0}$, for a unique $u \in L(\varpi)$. Moreover, different elements w lead to different elements u. $\square$

Note that the number of terms in the sum (36) equals $2^{h(\varpi)}$, where $h(\varpi)$ is defined to be the number of $j \geqslant 2$ such that $\varpi(i) > \varpi(j)$ for all $i < j$, that is, the number of 'new lows' in the sequence $\varpi(1), \ldots, \varpi(n)$.

Corollary 9. *If $\varpi \in S_n$ satisfies $\varpi(1) = 1$, then*

$$\mathfrak{D}_\varpi(X,Y) = \mathfrak{S}_{\varpi^*}(\varpi_0 X, -Y).$$

In particular, $\mathfrak{D}_\varpi(X,Y)$ has nonnegative integer coefficients.

Consider now a vector bundle V of rank $2n$ on an algebraic variety $\mathfrak{X}$ with a quadratic form and rank n isotropic subbundles E and F with complete flags of subbundles as in the type C setting. We assume that E and F are in the same family, that is $\dim(E(a) \cap F(a)) \equiv n \pmod 2$ for every $a \in \mathfrak{X}$. Define the classes $x_i, y_i \in CH^1(\mathfrak{X})$ as before.

There is a monomorphism $\phi\colon \widetilde{W}_n \hookrightarrow S_{2n}$ whose image consists of those permutations $\sigma \in S_{2n}$ such that $\sigma(i)+\sigma(2n+1-i) = 2n+1$ for all i and the number of $i \leqslant n$ such that $\sigma(i) > n$ is even. The map ϕ is defined by the same equation (19) as in the type C case. Set $\delta(w) = 0$ if $\overline{1}$ is a part of w, and $\delta(w) = 1$ otherwise, and define $\widetilde{\phi}\colon \widetilde{W}_n \hookrightarrow S_{2n}$ by

$$\widetilde{\phi}(w) = s_n^{\delta(w)}\phi(w).$$

The map $\widetilde{\phi}$ is a modification of ϕ so that in the sequence of values of $\widetilde{\phi}(w)$, $n+1$ always comes before n. We need also the alternate complete flag $\widetilde{F}_\bullet$, with $\widetilde{F}_i = F_i$ for $i \leqslant n-1$ but completed with a maximal isotropic subbundle $\widetilde{F}_n$ in the opposite family from E. Define

$$F_\bullet^\delta = \begin{cases} F_\bullet & \text{if } n = \delta \pmod 2, \\ \widetilde{F}_\bullet & \text{if } n \neq \delta \pmod 2. \end{cases}$$

For $w \in \widetilde{W}_n$, the *degeneracy locus* $\mathfrak{X}_w$ is the locus of $a \in \mathfrak{X}$ such that

$$\dim(E_r(a) \cap F_s^{\delta(w)}(a)) \geqslant \#\{\, i \leqslant r \mid \widetilde{\phi}(w)(i) > 2n-s \,\}$$

for $1 \leqslant r \leqslant n$, $1 \leqslant s \leqslant 2n$. Recall that the flag $E_\bullet$ corresponds to a section $\sigma\colon \mathfrak{X} \to F(V)$ of the bundle $F(V) \to \mathfrak{X}$ of isotropic flags in V. The subscheme $\mathfrak{X}_w$ of $\mathfrak{X}$ is then the inverse image under σ of the closure of the locus of $y \in F(V)$ such that

$$\dim(E_r(y) \cap F_s(y)) = \#\{\, i \leqslant r \mid \phi(w_0ww_0)(i) > 2n-s \,\}$$

for $1 \leqslant r \leqslant n-1$, $1 \leqslant s \leqslant 2n$. This relates the present formalism to that in [F3]. With the same assumptions on $\mathfrak{X}$ and the codimension of $\mathfrak{X}_w$ as before, and with the same arguments, we obtain

Theorem 9 (Locus Representation)**.** $[\mathfrak{X}_w] = \mathfrak{D}_w(X,Y)$ *in* $CH^{\ell(w)}(\mathfrak{X})$.

The maximal isotropic degeneracy locus $\mathfrak{X}_\lambda$ for $\lambda \in \mathcal{D}_{n-1}$ is defined as

$$\mathfrak{X}_\lambda = \{\, a \in \mathfrak{X} \mid \dim(E(a) \cap F_{n-\lambda_i}(a)) \geqslant i \quad \text{for} \quad 1 \leqslant i \leqslant \ell(\lambda) \,\}. \tag{38}$$

Theorems 7 and 9 imply the following formula for the class $[\mathfrak{X}_\lambda]$ in $CH^{|\lambda|}(\mathfrak{X})$:

Corollary 10. *We have*

$$[\mathfrak{X}_\lambda] = (-1)^{f(\lambda)+|\lambda'|}\sum_\alpha \widetilde{P}_\alpha(E^*)\sum_\beta (-1)^{f(\alpha,\beta)+|\beta|}\widetilde{P}_{(\alpha\cup\beta)'}(F^*)\det(c_{\beta_i-\lambda'_j}(F^*_{n-1-\lambda'_j})),$$

where the first sum is over all $\alpha \in \mathcal{D}_{n-1}$ *and the second over* $\beta \in \mathcal{D}^+_{n-1}$ *with* $\beta \supset \lambda'$, $\ell(\beta) = k$ *and* $\alpha \cap \beta = \emptyset$.

3.3. **Further results.** The Propositions, Corollaries and Examples in Sections 1 and 2 have orthogonal analogues. In particular there are geometric interpretations of Theorems 5 and 8 (as in Section 2.3) and connections to the formulas of [PR2] for types B and D. We omit most of them here because their statements and proofs are straightforward, following the type C case. The specialization $Y = 0$ produces type B and D *Schubert polynomials*

$$\mathfrak{B}_w(X) = \mathfrak{B}_w(X,0) \qquad \text{and} \qquad \mathfrak{D}_w(X) = \mathfrak{D}_w(X,0). \tag{39}$$

Note that these polynomials differ from the orthogonal Schubert polynomials defined in [LP2] by a sign, which depends on the degree. It is however still true (arguing in the same way as [LP1, Thm. A.2]) that the polynomials (39) have a stability property: for any $w \in W_m$,

$$\mathfrak{B}_{i(w)}(X_n)\big|_{x_{m+1}=\cdots=x_n=0} = \mathfrak{B}_w(X_m)$$

where $i\colon W_m \hookrightarrow W_n$ is the natural embedding. In addition, the set $\{\mathfrak{B}_\varpi(X)\mathfrak{B}_\lambda(X)\}$ for $\varpi \in S_n$, $\lambda \in \mathcal{D}_n$ forms an orthogonal product basis of $A[X]$ with respect to the $A[X]^{W_n}$-linear scalar product

$$A[X] \times A[X] \longrightarrow A[X]^{W_n}$$

defined by the maximal divided difference operator, as in (13) and Proposition 1 (here $A = \mathbb{Z}[1/2]$). Similarly, the $\mathfrak{D}_w(X)$ are stable under the inclusion $\widetilde{W}_m \hookrightarrow \widetilde{W}_n$, and the products $\{\widehat{\mathfrak{D}}_\varpi(X)\mathfrak{D}_\lambda(X)\}$ for $\varpi \in S_n$, $\lambda \in \mathcal{D}_{n-1}$ form an orthogonal basis for $A[X]$ as a $A[X]^{\widetilde{W}_n}$-module, where $\widehat{\mathfrak{D}}_\varpi(X) := \sum_{w \in L(\varpi)} \mathfrak{D}_w(X)$.

We next discuss some properties special to the type D double Schubert polynomials. Fix an integer $\ell > 0$, let $\widetilde{Y}_n = (y_1, \ldots, y_{n-1}, -y_n)$ and define

$$\widehat{Y} = \begin{cases} \widetilde{Y}_n & \text{if } n = \ell \pmod 2, \\ Y_n & \text{if } n \neq \ell \pmod 2. \end{cases}$$

Theorem 10 (Maximal Grassmannian II)**.** *For each strict partition $\lambda \in \mathcal{D}_{n-1}$ of length ℓ, the double Schubert polynomial $\mathfrak{D}_\lambda(X,Y)$ is equal to*

$$(-1)^{e(\lambda)+|\lambda'|} \sum_\alpha \widetilde{P}_\alpha(X) \sum_\gamma (-1)^{e(\alpha,\gamma)+|\gamma|} \widetilde{P}_{(\alpha\cup\gamma)'}(\widehat{Y}) \det(e_{\gamma_i - \lambda'_j}(Y_{n-1-\lambda'_j})),$$

where the first sum is over all $\alpha \in \mathcal{D}_{n-1}$ and the second over $\gamma \in \mathcal{D}_{n-1}$ with $\gamma \supset \lambda'$, $\ell(\gamma) = \ell(\lambda')$ and $\alpha \cap \gamma = \emptyset$.

Proof. If $n \neq \ell \pmod 2$, then the claim follows directly from Theorem 7. Assume that $n = \ell \pmod 2$, fix a partition $\alpha \in \mathcal{D}_{n-1}$ and equate the coefficients of $\widetilde{P}_\alpha(X)$ in the sums that occur in Theorems 7 and 10:

$$(-1)^\ell \sum_\beta (-1)^{f(\alpha,\beta)+|\beta|} \widetilde{P}_{(\alpha\cup\beta)'}(Y_n) \det(e_{\beta_i - \lambda'_j}(Y_{n-1-\lambda'_j})) = \tag{40}$$

$$\sum_\gamma (-1)^{e(\alpha,\gamma)+|\gamma|} \widetilde{P}_{(\alpha\cup\gamma)'}(\widetilde{Y}_n) \det(e_{\gamma_i - \lambda'_j}(Y_{n-1-\lambda'_j})). \tag{41}$$

To prove this equality, we expand each determinant $\det(e_{\beta_i - \lambda'_j}(Y_{n-1-\lambda'_j}))$ in (40) along the last (kth) column, and compare with (41). The result then follows by using the identity in the next Proposition (for varying μ and r).

Proposition 7. *For each partition $\mu \in \mathcal{D}_{n-1}$ of length r, we have*

$$\sum_{i=1}^r (-1)^{i-1} \widetilde{P}_{\mu \smallsetminus \mu_i}(Y_n)\, e_{\mu_i}(Y_{n-1}) = (-1)^{r+1} \widetilde{P}_\mu(Y_n) + \widetilde{P}_\mu(\widetilde{Y}_n).$$

The proof of Proposition 7, while elementary, uses additional algebraic formalism, and is given in [KT2, Appendix]. □

Corollary 11. *For each strict partition $\lambda \in \mathcal{D}_{n-1}$ of length ℓ, the maximal isotropic degeneracy locus $\mathfrak{X}_\lambda$ of* (38) *satisfies*

$$[\mathfrak{X}_\lambda] = (-1)^{e(\lambda)+|\lambda'|} \sum_\alpha \widetilde{P}_\alpha(E^*) \sum_\gamma (-1)^{e(\alpha,\gamma)+|\gamma|} \widetilde{P}_{(\alpha\cup\gamma)'}(\widehat{F}^*) \det(e_{\gamma_i - \lambda'_j}(F^*_{n-1-\lambda'_j})),$$

where the first sum is over all $\alpha \in \mathcal{D}_{n-1}$, the second is over $\gamma \in \mathcal{D}_{n-1}$ with $\gamma \supset \lambda'$, $\ell(\gamma) = \ell(\lambda')$ and $\alpha \cap \gamma = \emptyset$, while

$$\widehat{F} = \begin{cases} \widetilde{F}_n & \text{if } n = \ell \pmod 2, \\ F_n & \text{if } n \neq \ell \pmod 2. \end{cases}$$

Examples. 1) We have

$$\mathfrak{D}_{s_\square}(X,Y) = (1/2)(x_1 + \ldots + x_n + y_1 + \ldots + y_{n-1} \pm y_n),$$

with the sign of y_n positive (resp. negative) if n is even (resp. odd). Also,

$$\mathfrak{D}_{s_1}(X,Y) = (1/2)(-x_1 + \ldots + x_n + y_1 + \ldots + y_{n-1} \mp y_n),$$

with the opposite sign convention for y_n. Note that $\mathfrak{D}_{s_\square} + \mathfrak{D}_{s_1} = \mathfrak{B}_{s_1} = \mathfrak{C}_{s_1}$ while $\mathfrak{D}_{s_i} = \mathfrak{B}_{s_i} = \mathfrak{C}_{s_i}$ for $i > 1$.

2) Theorem 10 gives

$$\mathfrak{D}_{\rho_\ell}(X,Y) = \sum_{\alpha \in \mathcal{D}_\ell} \widetilde{P}_\alpha(X) \widetilde{P}_{\rho_\ell \smallsetminus \alpha}(\widehat{Y}).$$

It is clear from the above examples that the type D double Schubert polynomials do not satisfy the same stability property as in types B and C. However, we see that for each $\ell > 0$, the polynomial $\mathfrak{D}_{\rho_\ell}(X, \widehat{Y})$ is stable, in the sense of §1.5.

4. Example: the Lagrangian Quot scheme $LQ_1(2,4)$

In this section we study the problem of extending the formula for degeneracy loci from Corollary 4 to degeneracy loci of morphisms of vector bundles satisfying isotropicity conditions, in analogy with the work of Kempf and Laksov [KL] in type A. We provide an example showing that a direct analogue of the Kempf-Laksov result fails in type C. This example hinges on two ingredients: Quot schemes and degeneracy loci for a *morphism* from a flagged symplectic vector bundle to a vector bundle. Both of these, while classical for type A, have not received attention in the other Lie types. We begin with a description of degeneracy loci (we shall see that there are two reasonable definitions, although our example will be for a particular $\lambda \in \mathcal{D}_n$ for which the two definitions agree), and later introduce the Lagrangian Quot scheme $LQ_1(2,4)$ which serves as a compactification of the moduli space of degree 1 maps $\mathbb{P}^1 \to LG(2,4)$.

4.1. **Lagrangian degeneracy loci for isotropic morphisms.** Let $\mathfrak{X}$ be an algebraic variety over any ground field. Let V be a symplectic vector bundle of rank $2n$ over $\mathfrak{X}$ with complete isotropic flag of subbundles F_i, $1 \leq i \leq n$, and let Q be a vector bundle of rank n over $\mathfrak{X}$. We say that a morphism of vector bundles $\psi\colon V \to Q$ is *isotropic* if the composite $Q^* \to V^* \to V \to Q$ is zero, where the middle map is the isomorphism coming from the symplectic form on V. For such an isotropic morphism ψ and for $\lambda \in \mathcal{D}_n$, we define the *Lagrangian degeneracy loci* $\mathfrak{X}'_\lambda$ and $\mathfrak{X}''_\lambda$ by

$$\mathfrak{X}'_\lambda = \{\, a \in \mathfrak{X} \mid \operatorname{rk}(F_{n+1-\lambda_i}(a) \xrightarrow{\psi} Q(a)) \leqslant n + 1 - i - \lambda_i \text{ for } 1 \leqslant i \leqslant \ell(\lambda) \,\} \tag{42}$$

$$\mathfrak{X}''_\lambda = \{\, a \in \mathfrak{X} \mid \operatorname{rk}(F^\perp_{n+1-\lambda_i}(a) \xrightarrow{\psi} Q(a)) \leqslant n - i \text{ for } 1 \leqslant i \leqslant \ell(\lambda) \,\}. \tag{43}$$

When ψ is surjective, conditions (42) and (43) are equivalent to (23) (with $E = \operatorname{Ker}\psi$), and $\mathfrak{X}'_\lambda = \mathfrak{X}''_\lambda$. If ψ is not everywhere of full rank, we only have $\mathfrak{X}'_\lambda \subset \mathfrak{X}''_\lambda$ in general. However, when $\lambda = \rho_k$ for some k, the two definitions above yield the same scheme, and we may speak without ambiguity of the Lagrangian degeneracy locus of the morphism ψ. For instance, when $\lambda = (1)$, both (42) and (43) are the same as a type A degeneracy locus, so the Kempf-Laksov formula (2) dictates $[\mathfrak{X}'_1] = c_1(Q) - c_1(F_n)$.

Proposition 8. *Fix $n = 2$ and $\lambda = (2,1)$, and consider a general smooth variety $\mathfrak{X}$ with vector bundles V, Q, F_1, F_2 and isotropic morphism $\psi\colon V \to Q$ as above. Then there is no polynomial in the Chern classes of these bundles whose value equals $[\mathfrak{X}'_\lambda]$ in $CH^*(\mathfrak{X})$ whenever $\mathfrak{X}'_\lambda$ has codimension 3.*

We will work over $\mathbb{C}$ and consider the special case where the ambient bundle V as well as the flag of isotropic subbundles $F_\bullet$ are trivial. We therefore have $V = \mathcal{O}_{\mathfrak{X}} \otimes W$ and $F_i = \mathcal{O}_{\mathfrak{X}} \otimes W_i$ for some symplectic vector space W of dimension $N = 2n$, with a fixed flag of isotropic subspaces $W_1 \subset \cdots \subset W_n$. If ψ is surjective (and $E = \operatorname{Ker}\psi$ as above) then for any $\lambda \in \mathcal{D}_n$, only the leading term of the Lagrangian degeneracy locus formula in Corollary 4 survives, hence

$$[\mathfrak{X}_\lambda] = \widetilde{Q}_\lambda(E^*) \tag{44}$$

in $CH^*(\mathfrak{X})$, provided the degeneracy locus $\mathfrak{X}_\lambda$ of (23) has the expected dimension.

It is straightforward to give examples of the failure of (44) for isotropic morphisms of vector bundles. For instance, consider $\mathfrak{X} = \mathbb{P}^3$, $\lambda = (2,1)$, $Q = \mathcal{O}(1) \oplus \mathcal{O}(1)$ and the morphism

$$\psi\colon \mathcal{O}_{\mathfrak{X}} \otimes W \xrightarrow{\begin{pmatrix} a & b & c & d \\ 0 & 0 & -b & a \end{pmatrix}} Q. \tag{45}$$

Here $W = \mathbb{C}^4$ with the standard symplectic form

$$\langle u, v\rangle = u_1v_3 - u_3v_1 + u_2v_4 - u_4v_2 \tag{46}$$

and $[a : b : c : d]$ are the homogeneous coordinates on $\mathbb{P}^3$. To prove Proposition 8, we will produce an example of an isotropic morphism of vector bundles $V^* \to E^*$ on a nonsingular projective variety $\mathfrak{X}$, surjective at the generic point, such that the degeneracy locus $\mathfrak{X}'_\lambda$ is smooth of the expected codimension, but whose fundamental class is not equal to *any* polynomial in the Chern classes of E. The example was motivated by the theory of quantum cohomology, and we explain this connection next.

4.2. **Quantum cohomology of Grassmannians.** The theory of degeneracy loci for morphisms of vector bundles has a direct application to the study of the quantum cohomology of flag manifolds in type A (see e.g. [Be] [FP, App. J] [C-F]). In the case of the SL_N-Grassmannian G, Bertram [Be] used the formula (2) of Kempf-Laksov [KL] to prove a 'quantum Giambelli formula'. The quantum Giambelli formula calculates the class of a Schubert variety in the (small) quantum cohomology ring $QH^*(G)$, with respect to a given presentation of this ring in terms of generators and relations.

It is natural to copy Bertram's arguments and work towards a quantum Giambelli formula for the type C (as well as type D) maximal Grassmannian varieties, contingent upon having a formula such as (44) in the situation of a morphism $V^* \to E^*$, which is generically the projection to a Lagrangian quotient bundle. Unfortunately, this approach would dictate the *wrong* quantum Giambelli formula. The simplest example is provided by the Lagrangian Grassmannian $LG = LG(2,4)$ of isotropic 2-planes in $\mathbb{C}^4$.

The (small) quantum cohomology ring $QH^*(LG)$ is generated by the special Schubert classes σ_1 and σ_2, together with a formal variable q of degree 3. The structure constants of $QH^*(LG)$ are the numbers of rational curves on LG satisfying incidence conditions (also known as Gromov-Witten invariants). Lines on LG are parametrized by the $\mathbb{P}^3$ of linear subspaces ℓ of $\mathbb{C}^4$:

$$\ell \subset \mathbb{C}^4 \quad \longleftrightarrow \quad \{\, \Sigma \mid \ell \subset \Sigma \subset \ell^{\perp} \,\}.$$

Fixing the line in LG determined by $\ell_0 \subset \mathbb{C}^4$, as well as a point $\Sigma_0 \in LG$ in general position, i.e., satisfying $\ell_0 \not\subset \Sigma_0$, we ask how many lines on LG are incident to this line and this point. Clearly we have conditions $\ell \subset \Sigma_0$ for the line corresponding to ℓ to pass through the point, and $\ell \subset \ell_0^{\perp}$ for the line to meet the given line; thus there is a unique line on LG incident to a line and a point in general position. Consequently, the quantum product of the Schubert classes σ_1 and σ_2 receives a quantum correction term with coefficient 1:

$$\sigma_1 * \sigma_2 = \sigma_{2,1} + q \tag{47}$$

(here $\sigma_{2,1}$ is dual to the class of a point in LG and equals the classical cup product $\sigma_1 \cup \sigma_2$). Using the arguments of [Be], a formula such as (44) which is valid for morphisms would predict *no* quantum correction term in (47). This discrepancy leads to a counterexample to (44) for isotropic morphisms, which we study in detail for the remainder of this section.

The loci $\mathfrak{X}'_\lambda$ of (42) on *Lagrangian Quot schemes* (defined below in the situation that we require) are the 'correct' ones for analysis of the quantum cohomology of Lagrangian Grassmannians in analogy with [Be]. Our particular analysis will involve $\lambda = \rho_2$, for which the loci $\mathfrak{X}'_\lambda$, $\mathfrak{X}''_\lambda$ coincide anyway. For more information on the quantum cohomology of Lagrangian Grassmannians, including the full quantum Giambelli formula for $LG(n,2n)$, see [KT1].

4.3. **Quot schemes: a review.** Grothendieck's Quot schemes [Gro] parametrize quotients with given Hilbert polynomial of a fixed coherent sheaf on an algebraic variety. If $G(m,N)$ denotes the Grassmannian of m-dimensional subspaces of $W \cong \mathbb{C}^N$, then a morphism $\mathbb{P}^1 \to G(m,N)$ is equivalent to a map $\mathcal{O}_{\mathbb{P}^1} \otimes W \to Q$ to a rank $n := N - m$ quotient bundle Q. The Quot scheme Q_d parametrizing quotient *sheaves* of $\mathcal{O}_{\mathbb{P}^1} \otimes W$ with Hilbert polynomial $nt+n+d$ is a smooth projective variety which compactifies the moduli space of degree d maps $\mathbb{P}^1 \to G(m,N)$. On $Q_d \times \mathbb{P}^1$ there is a universal quotient map $\psi\colon \mathcal{O} \otimes W \to T_d$. While T_d is not locally free in general, the kernel S_d of ψ is locally free. The intersection-theoretic ingredient in [Be] is the Kempf-Laksov formula (2) applied to the (nonsurjective) dual morphism

$$\mathcal{O}_{Q_d \times \mathbb{P}^1} \otimes W^* \to S_d^*.$$

On our way to defining the Lagrangian Quot scheme, we review the $m=1$, $N=2$ case of Quot scheme just described, namely the Quot scheme compactification of the parameter space PGL_2 of maps $\mathbb{P}^1 \to \mathbb{P}^1$. Let W be a vector space of dimension 2. The Quot scheme parametrizes short exact sequences

$$0 \longrightarrow S \longrightarrow W \otimes \mathcal{O}_{\mathbb{P}^1} \longrightarrow Q \longrightarrow 0$$

with Q of rank 1 and degree 1. So S must be isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1)$, and the Quot scheme Q_1 is the space of nontrivial maps of bundles $\mathcal{O}_{\mathbb{P}^1}(-1) \to W \otimes \mathcal{O}_{\mathbb{P}^1}$, up to multiplication by a global scalar.

Choosing a basis for W and a basis $\{x, y\}$ of $\operatorname{Hom}(\mathcal{O}(-1), \mathcal{O})$, we have $Q_1 \cong \mathbb{P}^3$ with universal sheaf sequence

$$0 \longrightarrow \mathcal{O}(-1,-1) \xrightarrow{\begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}} \mathcal{O} \oplus \mathcal{O} \longrightarrow Q \longrightarrow 0$$

on $\mathbb{P}^3 \times \mathbb{P}^1$, where $[a : b : c : d]$ are the homogeneous coordinates on $\mathbb{P}^3$.

4.4. **Quot scheme of maps to $LG(2,4)$.** We now take $W = \mathbb{C}^4$, endowed with the standard symplectic form (46). We introduce the *Lagrangian Quot scheme* $LQ_1 = LQ_1(2,4)$. Consider the functor which parametrizes rank 2 degree 1 quotients of $\mathcal{O}_{\mathbb{P}^1} \otimes W$ which are (generically) Lagrangian: the symplectic form defines an isomorphism $\mathcal{O} \otimes W \to \mathcal{O} \otimes W^*$, and the Lagrangian condition on an exact sequence

$$0 \longrightarrow S \longrightarrow \mathcal{O}_{\mathbb{P}^1} \otimes W \longrightarrow Q \longrightarrow 0 \tag{48}$$

is that the composite

$$S \longrightarrow \mathcal{O} \otimes W \longrightarrow \mathcal{O} \otimes W^* \longrightarrow S^*$$

is the zero map. Since the Lagrangian condition is a closed condition, such quotients are parametrized by a closed subscheme LQ_1 of the usual Quot scheme Q_1.

A typical affine chart of the Lagrangian Quot scheme looks like

$$0 \longrightarrow \mathcal{O} \oplus \mathcal{O}(-1) \xrightarrow{\begin{pmatrix} 1 & 0 \\ q & x+dy \\ r & ex+fy \\ s & gx+hy \end{pmatrix}} \mathcal{O} \otimes W \longrightarrow Q \longrightarrow 0$$

on $(\operatorname{Spec} k[d,e,f,g,h,q,r,s]/(e+qg-s, f+qh-sd)) \times \mathbb{P}^1$. Such charts cover LQ_1, hence LQ_1 is smooth.

When $S \to \mathcal{O}_{\mathbb{P}^1} \otimes W$ is of full rank, there corresponds a morphism from $\mathbb{P}^1$ to the Lagrangian Grassmannian $LG(2,4)$. Since $LG(2,4)$ is a quadric hypersurface in a four-dimensional projective space, and since lines on $LG(2,4)$ are parametrized by $\mathbb{P}^3$, we expect the parameter space of degree 1 maps $\mathbb{P}^1 \to LG(2,4)$ to be a PGL_2-bundle over $\mathbb{P}^3$, compactified by the Lagrangian Quot scheme.

Since it is needed below, we record the sheaf sequence corresponding to the compactification of the space of degree 1 maps $\mathbb{P}^1 \to LG(2,4)$ whose image passes through a fixed point in LG. The parameter space is a copy of $\mathbb{P}^3 \times \mathbb{P}^1$, which we endow with homogeneous coordinates $[a : b : c : d]$ and $[s : t]$ on respective factors. The sheaf sequence is the following sequence on $\mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1$:

$$0 \longrightarrow \mathcal{O}(0,-1,0) \oplus \mathcal{O}(-1,-1,-1) \xrightarrow{\begin{pmatrix} s & 0 \\ t & s(ax+by) \\ 0 & -t(cx+dy) \\ 0 & s(cx+dy) \end{pmatrix}} \mathcal{O} \otimes W \longrightarrow Q \longrightarrow 0. \tag{49}$$

We note the geometry: lines through a fixed point on LG sweep out a quadric cone which is a singular hyperplane section of LG under its fundamental (Plücker) embedding. The $[s : t]$ coordinates select the line. For each fixed $[s : t]$, the compactification of the parametrized maps to this line is a copy of $\mathbb{P}^3$, just as in the previous subsection.

Proposition 9. *The Lagrangian Quot scheme $LQ_1(2,4)$ compactifying the space of degree 1 maps from $\mathbb{P}^1$ to $LG(2,4)$ is isomorphic to the projectivization of the bundle $R \oplus R$ on $\mathbb{P}^3$, where R is a rank 2 vector bundle which fits into an exact sequence*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \longrightarrow \Omega^1_{\mathbb{P}^3}(1) \longrightarrow R \longrightarrow 0. \tag{50}$$

More naturally, in fact, $LQ_1 \simeq P(R \otimes \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}))$. Recall from Section 4.2 that lines on $LG(2,4)$ are parametrized by the $\mathbb{P}^3$ of 1-dimensional subspaces $\ell \subset W$. To each such ℓ we associate the set of isotropic 2-dimensional subspaces of W containing ℓ, or equivalently, the set of 2-planes containing ℓ and contained in $\ell^\perp$. Thus any map $\mathbb{P}^1 \to LG(2,4)$ determines a one-dimensional subspace $\ell \subset W$.

We describe what happens on the level of points, and then we prove Proposition 9 by phrasing everything in terms of universal bundles. In any exact sequence (48), the splitting type of S must be $\mathcal{O} \oplus \mathcal{O}(-1)$. As in the previous subsection, it is easier to describe the vector bundle S with injective morphism of its sheaf of sections to $\mathcal{O} \otimes W$, than it is to describe the quotient sheaf Q. The line ℓ associated to a morphism $\mathbb{P}^1 \to LG(2,4)$ can be recovered from the sheaf sequence (48) as the image of the map on global sections

$$\Gamma(S) \longrightarrow W = \Gamma(\mathcal{O}_{\mathbb{P}^1} \otimes W).$$

This makes sense even if $S \to \mathcal{O} \otimes W$ is not everywhere of full rank. So to every point of LQ_1 we can associate a line in W.

The bundle R is constructed to have fiber $\ell^\perp/\ell$ at the point of $\mathbb{P}^3$ corresponding to $\ell \subset W$. An element of $(\ell^\perp/\ell) \otimes \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}))$ determines a sheaf morphism

$$\mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^1} \otimes (\ell^\perp/\ell). \tag{51}$$

What we seek is a morphism

$$\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^1} \otimes W \tag{52}$$

with image contained in $\mathcal{O}_{\mathbb{P}^1} \otimes \ell^\perp$ and whose map on global sections has image ℓ. Points of the projectivization of $(\ell^\perp/\ell) \otimes \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}))$ correspond exactly to morphisms (51) up to global automorphism of $\mathcal{O}(-1)$, i.e., a global scale factor. Such a morphism (51) in turn determines, uniquely, a morphism (52) which sends global sections into ℓ, up to global automorphism of $\mathcal{O} \oplus \mathcal{O}(-1)$.

Proof of Proposition 9. Let $U \simeq \mathcal{O}(-1)$ be the universal subbundle on $\mathbb{P}^3$, and let $R = U^\perp/U$, where $\perp$ refers to the standard skew-symmetric form $\langle\ ,\ \rangle$ on W, and hence also on $\mathcal{O}_{\mathbb{P}^3} \otimes W$. The first claim is that R fits into an exact sequence (50). This is clear, since the (dualized, twisted) Euler sequence on $\mathbb{P}^3$

$$0 \longrightarrow \Omega^1(1) \longrightarrow \mathcal{O}^4 \longrightarrow U^* \longrightarrow 0$$

identifies $\Omega^1(1)$ with $U^\perp$.

On $\mathbb{P}^3 \times \mathbb{P}^1$ there is the natural map of vector bundles

$$R \otimes \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_{\mathbb{P}^1}) \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow R.$$

Let $P := P(R \otimes \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_{\mathbb{P}^1})) \to \mathbb{P}^3$ be the projectivization of the indicated vector bundle. We follow the convention of [F4]: points of P are points of the base together with one-dimensional subspaces of the fiber. Let $S = \mathcal{O}_P(-1)$ be the universal subbundle of $R \otimes \mathrm{Hom}(\mathcal{O}(-1), \mathcal{O}))$ on P. Then we have a morphism of bundles on $P \times \mathbb{P}^1$

$$S \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow R = U^{\perp}/U \subset (\mathcal{O} \otimes W)/U.$$

Over any affine open subset of $\mathbb{P}^3$, this lifts to a morphism

$$\mathcal{O} \oplus (S \otimes \mathcal{O}_{\mathbb{P}^1}(-1)) \longrightarrow \mathcal{O} \otimes W, \tag{53}$$

such that, fiberwise, the map on global sections has image U. To phrase this mathematically, let φ denote the projection $P \times \mathbb{P}^1 \to P$, and equally, the restriction over an affine open of $\mathbb{P}^3$. Then φ_* applied to (53) should be an isomorphism onto $U \subset \mathcal{O} \otimes W$. This requirement determines the lift, uniquely up to automorphisms of the fibers of $\mathcal{O} \oplus (S \otimes \mathcal{O}_{\mathbb{P}^1}(-1))$.

Since the Quot functor is a sheaf, these maps, defined locally, patch to give a vector bundle E and injective map of sheaves $E \to \mathcal{O} \otimes W$ on $P \times \mathbb{P}^1$, such that the quotient sheaf is flat over P and on every fiber of $P \times \mathbb{P}^1 \to P$ has rank 1 and degree 1. The resulting morphism $P \to LQ_1$ is an isomorphism since it is a birational morphism of smooth varieties, which is one-to-one on geometric points. $\square$

The natural map $\varphi^*\varphi_* E \to E$ identifies a rank 1 subbundle of E with the pullback to $P \times \mathbb{P}^1$ of the universal subbundle on $\mathbb{P}^3$. If we let π denote the composite projection $P \times \mathbb{P}^1 \to P \to \mathbb{P}^3$, then the cokernel of $\pi^* U \to E$, restricted to $\pi^{-1}(A)$ for any affine open set $A \subset \mathbb{P}^3$, is identified with $S \otimes \mathcal{O}_{\mathbb{P}^1}(-1)$. This identification is unique up to scaling by an invertible regular function on $\pi^{-1}(A)$. Hence, there is an exact sequence

$$0 \longrightarrow \pi^* U \longrightarrow E \longrightarrow S \otimes \pi^* L \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow 0$$

for some line bundle L on $\mathbb{P}^3$. For any line Λ in $\mathbb{P}^3$, the restriction of E to $\pi^{-1}(\Lambda)$ must be the subsheaf in the sequence (49), up to isomorphism, so we deduce that $L = \mathcal{O}_{\mathbb{P}^3}(-1)$. Thus we have

Corollary 12. *Under the isomorphism $P(R \oplus R) \simeq LQ_1$ from Proposition 9, the universal subsheaf E of $\mathcal{O} \otimes W$ on $LQ_1 \times \mathbb{P}^1$ fits into an exact sequence*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \longrightarrow E \longrightarrow S \otimes \mathcal{O}_{\mathbb{P}^3}(-1) \otimes \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow 0,$$

where S is the universal subbundle on $P(R \oplus R)$.

4.5. **Chern class computations on LQ_1.** Consider, on $\mathfrak{X} := LQ_1 \times \mathbb{P}^1$, the trivial rank 4 bundle $V = \mathcal{O}_{\mathfrak{X}} \otimes W$, with standard symplectic form (46) and flag of isotropic subbundles $F_\bullet = \{F_i\}$ (where F_i is the subbundle of sections with nonzero entries only among the first i coordinates). The kernel of the universal quotient map on $\mathfrak{X}$ is the vector bundle E from Section 4.4. Dualizing, we get a morphism

$$\psi \colon V^* \to E^* \tag{54}$$

which is isotropic. On $LQ_1 \simeq P(R \oplus R)$ we let h denote the hyperplane class on the $\mathbb{P}^3$ base of the projectivized vector bundle, and we let z denote the first Chern class of $\mathcal{O}_{P(R\oplus R)}(1)$ (so $z = -c_1(S)$, where S is the universal subbundle on $P(R \oplus R)$). Let p be the class of a point on $\mathbb{P}^1$. Since R has Chern polynomial $1 + h^2t^2$, the Chern polynomial of $R \oplus R$ is $1 + 2h^2t^2$. Hence, the Chow ring (or cohomology ring) of $\mathfrak{X}$ is given by

$$CH(\mathfrak{X}) = \mathbb{Z}[h, z, p]/(h^4, z^4 + 2h^2z^2, p^2).$$

By Corollary 12, we have

$$\begin{aligned} c_1(E^*) &= 2h + z + p, \\ c_2(E^*) &= h^2 + hz + hp, \\ c_1(E^*)c_2(E^*) &= 2h^3 + 3h^2z + 3h^2p + hz^2 + 2hzp. \end{aligned}$$

Recall that we have defined Lagrangian degeneracy loci $\mathfrak{X}'_\lambda$, $\mathfrak{X}''_\lambda$ associated to an isotropic morphism to a flagged vector bundle. We consider $\lambda = (2,1)$ and the morphism (54); since $\lambda = \rho_2$, we have $\mathfrak{X}'_\lambda = \mathfrak{X}''_\lambda$. Translating the condition (42) into conditions on E, we have

$$\mathfrak{X}'_\lambda = \{\, a \in \mathfrak{X} \mid \operatorname{rk}(E(a) \to (V/F_2)(a)) = 0 \,\}.$$

If $L \subset \mathbb{P}^3$ is the line which parametrizes one-dimensional subspaces of F_2, then $\mathfrak{X}'_\lambda$ is the hypersurface in $\pi^{-1}(L) \simeq \mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1$ (where π is the projection $\mathfrak{X} \to \mathbb{P}^3$ as in the previous subsection) defined by the homogeneous equation $cx + dy = 0$ in the coordinates of (49). So $\mathfrak{X}'_\lambda$ has the expected codimension 3 (and, in fact, is smooth). Its class in the Chow ring is

$$[\mathfrak{X}'_\lambda] = h^2z + h^2p \neq c_1(E^*)c_2(E^*) = \widetilde{Q}_{2,1}(E^*). \tag{55}$$

The above continues to be an inequality upon restriction to $LQ_1 \times \{[1:0]\}$. In fact, restricting (54) to a section of $LQ_1 \times \{[1:0]\} \simeq P(R \oplus R) \to \mathbb{P}^3$ reproduces the example (45) of Section 4.1. Observe now that the class (55) is not equal to any polynomial in the Chern classes of E. In particular, this proves Proposition 8.

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