A GIAMBELLI FORMULA FOR CLASSICAL G/P SPACES

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Abstract. Let G be a classical complex Lie group, P any parabolic subgroup of G, and G/P the corresponding partial flag variety. We prove an explicit combinatorial Giambelli formula which expresses an arbitrary Schubert class in $\mathrm{H}^*(G/P)$ as a polynomial in certain special Schubert class generators. Our formula extends to one that applies to the torus-equivariant cohomology ring of G/P and to the setting of symplectic and orthogonal degeneracy loci.

0. Introduction

The Giambelli formula [G] is one of the fundamental results concerning Schubert calculus in the cohomology ring of the Grassmannian $\mathfrak{X}$. The variety $\mathfrak{X}$ has a decomposition into Schubert cells, which gives an additive basis of Schubert classes for the cohomology of $\mathfrak{X}$. On the other hand, the ring $\mathrm{H}^*(\mathfrak{X},\mathbb{Z})$ is generated by certain *special* Schubert classes, which are the Chern classes of the universal quotient bundle over $\mathfrak{X}$. The formula of Giambelli expresses a general Schubert class as a determinant of a Jacobi-Trudi matrix with entries given by special classes. One can show that this formula is equivalent to the Pieri rule [P]; see for instance [T4].

The Schubert calculus on $\mathfrak{X}$ can be generalized to any homogeneous space G/P, where G is a complex reductive Lie group and P a parabolic subgroup of G. However, more than a century since the theorems of Pieri and Giambelli were discovered, no combinatorially explicit analogues of these results are known in this generality, unless the Lie group G is of type A. One reason for this is that there is no uniform way to extend the notion of a special Schubert class over all possible Lie types and parabolics. Another serious concern is the more difficult algebro-combinatorial questions that arise in the other Lie types, about which more below.

When G is a *classical* Lie group, one can define special Schubert class generators for the cohomology ring $\mathrm{H}^*(G/P)$ uniformly, as follows. In this situation, the variety G/P parametrizes partial flags of subspaces of a vector space, which in types B, C, and D are required to be isotropic with respect to an orthogonal or symplectic form. First, the special Schubert varieties on any Grassmannian are defined as the locus of (isotropic) linear subspaces which meet a given (isotropic or coisotropic) linear subspace nontrivially, following [BKT1]. The special Schubert classes are the cohomology classes determined by these Schubert varieties. Finally, the special Schubert classes on a partial flag variety G/P are the pullbacks of special Schubert classes on Grassmannians, in agreement with the convention in type A. In most cases, these special classes are equal to the Chern classes of the universal quotient

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bundles over G/P, up to a factor of two. The *Giambelli problem* then is to find an explicit combinatorial formula which writes a general Schubert class in $\mathrm{H}^*(G/P)$ as a polynomial in the special classes. One of our motivations for this is the fact that the known Giambelli formulas expressed in terms of the above special Schubert classes have straightforward – often identical – extensions to the small quantum cohomology ring of G/P; see [Be, CF, FGP, KT2, KT3, BKT3].

The modern formulation of the Giambelli problem is in the setting of an algebraic family of varying partial flag varieties, with applications to *degeneracy loci* of vector bundles. This story also has a long history, from the work of Thom-Porteous and Kempf-Laksov [KL] to the generalizations by Fulton, Pragacz, and others [Fu1, Fu2, Fu3, Fu4, P1, PR2, LP1, KT1, IMN]. In type A, the project culminated with the combinatorial understanding of the polynomials representing quiver loci [BF, BKTY, KMS]. Let T denote a maximal torus and B a Borel subgroup of G with $T \subset B$. Graham [Gra] observed that the degeneracy locus problem for the classical groups from [Fu1, Fu2, Fu3] is equivalent to the Giambelli problem for the Schubert classes in the *T-equivariant cohomology ring* of G/B.

The degeneracy locus formulas we obtain here have a similar shape for all the classical groups, and solve the Giambelli problem for the T-equivariant cohomology of any classical G/P space (when G is an even orthogonal group, the problem is reduced to the main theorem of [BKT4]). It should be noted that in type D there are some unavoidable differences due to the presence of the Euler class [EGr]. For instance, the Chern classes of the universal vector bundles over a non-maximal even orthogonal Grassmannian do not generate its cohomology ring, even with real coefficients; see [T1, BKT1, BKT4] for a detailed analysis.

The case of the complete flag variety G/B, where the cohomology is generated by Schubert divisors, is more amenable to study. Bernstein-Gelfand-Gelfand [BGG] and Demazure [D1, D2] used divided difference operators to construct an algorithm that produces polynomials which represent the Schubert classes on G/B in the Borel presentation [Bo] of the cohomology ring. For the general linear group, Lascoux and Schützenberger [LS1] applied this method to define *Schubert polynomials*, a particularly nice choice of representatives with rich combinatorial properties. A positive combinatorial formula for the coefficients of Schubert polynomials was given by Billey, Jockusch, and Stanley [BJS] – thus resolving the Giambelli question in the case of GL_n/B. There remained combinatorial difficulties to generalize this to a Giambelli formula which holds on any type A partial flag variety. The answer for GL_n/P was obtained by Buch, Kresch, Yong, and the author [BKTY, §5] in the course of their work on the quiver formulas of Buch and Fulton [BF].

Pragacz [P2] solved the Giambelli problem for maximal isotropic orthogonal and symplectic Grassmannians using a Schur Pfaffian [Sch]. At the other extreme, for the full flag varieties G/B in types B, C, and D, a family of Schubert polynomials analogous to the one in type A is not uniquely determined (see [FK2] for a discussion of this phenomenon and [BH, Fu2, LP1, LP2, T2, T3] for examples) and the combinatorics is more challenging. One reason for this is that the strong *stability property* of type A Schubert polynomials under the natural inclusions of the Weyl groups must be understood differently in types B, C, and D. Another is that the images of the special Schubert classes in the stable cohomology rings of isotropic Grassmannians are not algebraically independent.

Giambelli formulas for non-maximal isotropic Grassmannians were discovered only recently [BKT2, BKT4]. Unlike most previously known examples, the nature of these formulas is not determinantal – instead, they are expressed using Young's raising operators [Y]. The corresponding Giambelli polynomials are the *theta* and *eta* polynomials, which play the same role as the Schur polynomials do for the type A Grassmannian. This theory is essential for further progress and will be generalized in the present article. The results provide a combinatorial link between the classical G/P Giambelli problem and the quiver formulas of [BF, BKTY].

We now state one of our main theorems, referring to §1–§4 for the precise definitions. Equip the vector space $E = \mathbb{C}^{2n}$ with a nondegenerate skew-symmetric bilinear form. Fix a sequence $\mathfrak{d}\,:\,d_1 < \cdots < d_p$ of positive integers with $d_p \leq n$. Let $\mathfrak{X}(\mathfrak{d})$ be the variety parametrizing partial flags of subspaces

$$E_\bullet \;:\; 0 \subset E_1 \subset \cdots \subset E_p \subset E \tag{1}$$

with $\dim E_i = d_i$ for each i and E_p isotropic. The Schubert varieties $\mathfrak{X}_w$ in $\mathfrak{X}(\mathfrak{d})$ and their cohomology classes $[\mathfrak{X}_w]$ are indexed by signed permutations w in the Weyl group of Sp_{2n} whose descent positions are included among the d_i. Let $E'_j = E/E_{p+1-j}$ for $1 \leq j \leq p$; the Chern classes of the tautological quotient bundles E'_j are the special Schubert classes in $\mathrm{H}^*(\mathfrak{X}(\mathfrak{d}), \mathbb{Z})$. We then have

$$[\mathfrak{X}_w] = \sum_{\underline{\lambda}} e^w_{\underline{\lambda}}\, \Theta_{\lambda^1}(E'_1) s_{\lambda^2}(E'_2 - E'_1) \cdots s_{\lambda^p}(E'_p - E'_{p-1}) \tag{2}$$

summed over all sequences of partitions $\underline{\lambda} = (\lambda^1, \ldots, \lambda^p)$ with λ^1 k-strict, where $k = n - d_p$. Here Θ_{λ^1} and s_{λ^i} denote theta and Schur polynomials, respectively, and the coefficient $e^w_{\underline{\lambda}}$ is a nonnegative integer which counts the number of p-tuples of leaves of shape $\underline{\lambda}$ in the groves of the transition forest associated to $\mathfrak{d}$ and w.

Let $\mathrm{IG}(n-k, 2n)$ denote the Grassmannian parametrizing isotropic linear subspaces of E of dimension $n-k$. The morphism which sends $E_\bullet$ to E_p realizes $\mathfrak{X}(\mathfrak{d})$ as a fiber bundle over $\mathrm{IG}(n-k, 2n)$ with fiber equal to a type A partial flag variety. The mixed nature of the ingredients in formula (2) is in harmony with this fact. For the type A and orthogonal partial flag varieties, the Giambelli formulas and their underlying combinatorics and geometry are entirely analogous to (2). The definition of the transition forest differs slightly between the types, and the role of the theta polynomial is played by a Schur or an eta polynomial, respectively.

Our proof of (2) and the corresponding formulas for degeneracy loci is mainly combinatorial. We work with the Schubert polynomials of Billey and Haiman [BH], and more generally with their double versions introduced by Ikeda, Mihalcea, and Naruse [IMN]. These objects have most of the combinatorial properties of the type A Schubert polynomials, which are crucial in order for us to obtain explicit positive expressions such as (2); however their connection with the geometry is less clear. This latter problem was recently addressed for the single Schubert polynomials in [T2, T3] and in [IMN] for their double counterparts. Using this theory, our geometric results follow readily by combining the Giambelli formulas for isotropic Grassmannians from [BKT2, BKT4] with new *splitting theorems* for these Schubert polynomials, which are analogues of [BKTY, Thm. 4 and Cor. 3] for the other classical Lie types.

A first step towards splitting the Billey-Haiman Schubert polynomials was formulated by Yong [Yo]. However, his result – and most previous work on these polynomials – does not suffice for the aforementioned applications to geometry.

The point is that the flag of subspaces $E_\bullet$ in (1) need not contain a Lagrangian (i.e., a maximal isotropic) subspace. In terms of the Weyl group, this is simply the fact that the first descent position k of a signed permutation w need not be at *zero*. When $k=0$, the analysis is easier because the cohomology ring of the Lagrangian Grassmannian $\mathrm{LG}(n,2n)$ is more accessible, a result that goes back to Ehresmann [Eh]. The underlying reason for this is that the Schubert classes on LG are indexed by *fully commutative elements* of the Weyl group [St2]. As a consequence, the Schubert calculus on LG is very similar to the classical one.

To complete the algebro-combinatorial picture for any G/P space, we need to extend the established theory to include the k-Grassmannian elements which are not fully commutative, a program initiated in [BKT2, T4]. For our purposes here we introduce the *mixed Stanley functions*, denoted by $J_w(X\,;Y)$ in type C. The function J_w is defined as a sum of products of type A and type C Stanley symmetric functions in two distinct sets of variables. For each fixed $m\geq 0$, it includes among its coefficients the number of reduced words of the signed permutation w such that the last m letters in the word are positive (the letter 0 is used to denote the sign change). If w has no descent positions less than k, then J_w – suitably restricted – is a nonnegative integer linear combination of theta polynomials indexed by k-strict partitions λ. The case when $k=0$ was studied earlier by several authors [H, Kr, L1, B]. It is the *mixed Stanley coefficients* that appear in this expansion which enter into the splitting and degeneracy locus formulas.

Finally, our combinatorial interpretation of the numbers $e^w_{\underline{\lambda}}$ in (2) depends on the *transition equations* of Lascoux and Schützenberger [LS3] and Billey [B]. More precisely, we define in §2.2 and §6 analogues of the Lascoux-Schützenberger transition tree [LS3], which are rooted at suitable elements in the Weyl groups of the symplectic and orthogonal groups. We remark that the difference between the stability property of type A Schubert polynomials and the analogous property in the other classical Lie types is also reflected in the construction of these trees – namely, there is a certain branching rule in [LS3] which has no counterpart in types B, C, and D. It would be interesting to have tableau formulas for the $e^w_{\underline{\lambda}}$, similar to the ones in [BF, BKTY, KMS] for quiver coefficients.

This article is organized so that most of the exposition is in type C; a final section explains the analogous picture in types B and D, and we refer to [BKTY, §4,5] for the results in type A. We review the Schur, theta, and Schubert polynomials, as well as the Stanley symmetric functions we require in §1. The mixed Stanley functions and their basic properties are studied in §2; this includes applications to enumerating reduced words and combinatorial rules for the product of two theta polynomials. In particular we give a short proof of Proposition 4, which includes as a special case the Pieri type products studied by Pragacz and Ratajski [PR1] on non-maximal isotropic Grassmannians. Our splitting theorems for Schubert polynomials are proved in §3, and the applications to symplectic degeneracy loci and Giambelli formulas are deduced in §4 and §5, respectively. We conclude with the Schubert splitting results for the orthogonal groups in §6.

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1. Preliminaries

1.1. **Schur and theta polynomials.** An *integer sequence* is a sequence of integers $\alpha = (\alpha_1, \alpha_2, \ldots)$ only finitely many of which are non-zero. The largest integer $\ell \geq 0$ such that $\alpha_\ell \neq 0$ is called the *length* of α, denoted $\ell(\alpha)$; we will identify an integer sequence of length ℓ with the vector consisting of its first ℓ terms, and set $|\alpha| = \sum \alpha_i$. An integer sequence λ is a *partition* if $\lambda_i \geq \lambda_{i+1} \geq 0$ for all i. We will represent partitions λ by their Young diagram of boxes, let λ' denote the conjugate (or transpose) of λ, and write $\mu \subset \lambda$ for the containment relation between two Young diagrams. Following Young [Y], given any integer sequence α and natural numbers $i < j$, we define

$$R_{ij}(\alpha) = (\alpha_1, \ldots, \alpha_i + 1, \ldots, \alpha_j - 1, \ldots).$$

A raising operator R is any monomial in these R_{ij}'s. If $(u_1, u_2, \ldots)$ is any ordered set of commuting independent variables and α is an integer sequence, we let $u_\alpha = \prod_i u_{\alpha_i}$, with the understanding that $u_0 = 1$ and $u_r = 0$ if $r < 0$. For any raising operator R, let $R\, u_\alpha = u_{R\alpha}$.

Let $c = (c_1, c_2, \ldots)$ and $d = (d_1, d_2, \ldots)$ be two families of commuting variables. Define elements h_r for $r \in \mathbb{Z}$ by the identity of formal power series

$$\sum_{r=-\infty}^{+\infty} h_r t^r = \left(\sum_{i=0}^{\infty} (-1)^i c_i t^i\right)^{-1} \left(\sum_{i=0}^{\infty} (-1)^i d_i t^i\right).$$

Consider the raising operator expression

$$R^0 = \prod_{i<j} (1 - R_{ij})$$

and for any partition λ, define the *Schur polynomial* $s_\lambda(c - d)$ by

$$s_\lambda(c-d) = R^0\, h_\lambda = \det(h_{\lambda_i + j - i})_{i,j}.$$

Let $Y = (y_1, y_2, \ldots)$ and $Z = (z_1, z_2, \ldots)$ be two infinite sets of variables, and define the elementary symmetric functions $e_r(Y)$ by the generating function

$$\prod_{i=1}^{\infty} (1 + y_i t) = \sum_{r=0}^{\infty} e_r(Y) t^r.$$

The *supersymmetric Schur function* $s_\lambda(Y/Z)$ is obtained from $s_\lambda(c-d)$ by setting $c_r = e_r(Y)$ and $d_r = e_r(Z)$ for all $r \geq 1$. The usual Schur S-functions satisfy the identities $s_\lambda(Y) = s_\lambda(Y/Z)|_{Z=0}$ and $s_\lambda(0/Z) = s_\lambda(Y/Z)|_{Y=0} = (-1)^{|\lambda|} s_{\lambda'}(Z)$. In particular, for each integer r, the function $s_r(Y)$ is the complete symmetric function in the variables Y, also denoted $h_r(Y)$.

Fix an integer $k \geq 0$. A partition λ is *k-strict* if all its parts λ_i greater than k are distinct; λ is called *strict* if it is 0-strict. Any such λ determines a raising operator expression R^λ by the prescription

$$R^\lambda = \prod_{i<j} (1 - R_{ij}) \prod_{\substack{i<j \\ \lambda_i + \lambda_j > 2k + j - i}} (1 + R_{ij})^{-1}$$

where the second product is over all pairs $i < j$ such that $\lambda_i + \lambda_j > 2k + j - i$. Define elements g_r for $r \in \mathbb{Z}$ by the identity

$$\sum_{r=-\infty}^{+\infty} g_r t^r = \left(\sum_{i=0}^{\infty} c_i t^i\right)\left(\sum_{i=0}^{\infty} d_i t^i\right)^{-1}$$

and the *theta polynomial* $\Theta_\lambda(c-d)$ by

$$\Theta_\lambda(c-d) = R^\lambda\, g_\lambda. \tag{3}$$

If $X = (x_1, x_2, \ldots)$ is another infinite set of variables, the formal power series $\vartheta_r(X\,;Y)$ for $r \in \mathbb{Z}$ are defined by the equation

$$\prod_{i=1}^{\infty} \frac{1+x_i t}{1-x_i t} \prod_{j=1}^{k}(1+y_j t) = \sum_{r=0}^{\infty} \vartheta_r(X\,;Y) t^r.$$

Following [BKT2], we then set $\Theta_\lambda(X\,;Y) = R^\lambda\, \vartheta_\lambda$. The Θ_λ for λ k-strict form a $\mathbb{Z}$-basis for the ring $\Gamma^{(k)} = \mathbb{Z}[\vartheta_1, \vartheta_2, \ldots]$. The ring $\Gamma = \Gamma^{(0)}$ is the ring of Schur Q-functions (see [M2, III.8] and [Sch]), and in this case $\vartheta_r(X)$ and $\Theta_\lambda(X)$ are denoted by $q_r(X)$ and $Q_\lambda(X)$, respectively.

1.2. **The hyperoctahedral group.** Let W_n denote the hyperoctahedral group of signed permutations on the set $\{1, \ldots, n\}$. We will adopt the notation where a bar is written over an entry with a negative sign. The group W_n is the Weyl group for the root system B_n or C_n, and is generated by the simple transpositions $s_i = (i, i+1)$ for $1 \leq i \leq n-1$ and the sign change $s_0(1) = \overline{1}$. There is a natural embedding $W_n \hookrightarrow W_{n+1}$ defined by adjoining the fixed point $n+1$. The symmetric group S_n is the subgroup of W_n generated by the s_i for $1 \leq i \leq n-1$, and is the Weyl group for the root system A_{n-1}. We let $S_\infty = \cup_n S_n$ and $W_\infty = \cup_n W_n$.

A *reduced word* for $w \in W_n$ is a sequence $a_1 \cdots a_r$ of elements in $\{0, 1, \ldots, n-1\}$ such that $w = s_{a_1} \cdots s_{a_r}$ and r is minimal (so equal to the length $\ell(w)$ of w). Given any $u_1, \ldots, u_p, w \in W_\infty$, we write $u_1 \cdots u_p = w$ if $\ell(u_1) + \cdots + \ell(u_p) = \ell(w)$ and the product of $u_1, \ldots, u_p$ is equal to w. In this case we say that $u_1 \cdots u_p$ is a *reduced factorization* of w. We say that w has a *descent* at position $r \geq 0$ if $w_r > w_{r+1}$, where by definition $w_0 = 0$. An element $w \in W_\infty$ is *compatible* with the sequence $\mathfrak{a} : a_1 < \cdots < a_p$ of nonnegative integers if all descent positions of w are contained in $\mathfrak{a}$. For such w, we say that a reduced factorization $u_1 \cdots u_p = w$ is *compatible with* $\mathfrak{a}$ if $u_j(i) = i$ for all $j > 1$ and $i \leq a_{j-1}$.

We say that a signed permutation $w \in W_\infty$ is *increasing up to k* if it has no descents less than k. This condition is vacuous if $k = 0$, and for positive k it means that $0 < w_1 < w_2 < \cdots < w_k$. An important special case is the *k-Grassmannian* elements, which by definition satisfy $\ell(ws_i) = \ell(w) + 1$ for all $i \neq k$. There is a natural bijection between k-Grassmannian elements of W_∞ and k-strict partitions, obtained as follows. If $w \in W_n$ is k-Grassmannian, there exist unique strict partitions u, ζ, v of lengths k, r, and $n-k-r$, respectively, so that

$$w = (u_k, \ldots, u_1, \overline{\zeta}_1, \ldots, \overline{\zeta}_r, v_{n-k-r}, \ldots, v_1).$$

Define μ_i for $1 \leq i \leq k$ by

$$\mu_i = u_i + i - k - 1 + \#\{j \mid \zeta_j > u_i\}.$$

Then w corresponds to the k-strict partition λ such that the lengths of the first k columns of λ are given by $\mu_1, \ldots, \mu_k$, and the part of λ in columns $k+1$ and higher is given by ζ. Conversely, for any k-strict λ, the corresponding k-Grassmannian element is denoted by w_λ.

1.3. **Schubert polynomials and Stanley symmetric functions.** Following [FS] and [FK1, FK2], we will use the nilCoxeter algebra $\mathcal{W}_n$ of the hyperoctahedral group W_n to define Schubert polynomials and Stanley symmetric functions in types A and C, respectively. $\mathcal{W}_n$ is the free associative algebra with unity generated by the elements $u_0, u_1, \ldots, u_{n-1}$ modulo the relations

$$\begin{array}{rcll} u_i^2 & = & 0 & i \geq 0 \ ; \\ u_i u_j & = & u_j u_i & |i-j| \geq 2 \ ; \\ u_i u_{i+1} u_i & = & u_{i+1} u_i u_{i+1} & i > 0 \ ; \\ u_0 u_1 u_0 u_1 & = & u_1 u_0 u_1 u_0 . & \end{array}$$

For any $w \in W_n$, choose a reduced word $a_1 \cdots a_\ell$ for w and define $u_w = u_{a_1} \ldots u_{a_\ell}$. Since the last three relations listed are the Coxeter relations for W_n, it is clear that u_w is well defined, and that the u_w for $w \in W_n$ form a free $\mathbb{Z}$-basis of $\mathcal{W}_n$. We denote the coefficient of $u_w \in \mathcal{W}_n$ in the expansion of the element $f \in \mathcal{W}_n$ by $\langle f, w \rangle$; thus $f = \sum_{w \in W_n} \langle f, w \rangle \, u_w$ for all $f \in \mathcal{W}_n$.

Let t be an indeterminate and define

$$\begin{gathered} A_i(t) = (1+tu_{n-1})(1+tu_{n-2}) \cdots (1+tu_i) \ ; \\ \tilde{A}_i(t) = (1-tu_i)(1-tu_{i+1}) \cdots (1-tu_{n-1}) \ ; \\ C(t) = (1+tu_{n-1}) \cdots (1+tu_1)(1+2tu_0)(1+tu_1) \cdots (1+tu_{n-1}). \end{gathered}$$

According to [FS, Lemma 2.1] and [FK2, Prop. 4.2], for all commuting variables s, t and indices i, the relations $A_i(s)A_i(t) = A_i(t)A_i(s)$ and $C(s)C(t) = C(t)C(s)$ hold. If $C(X) = C(x_1)C(x_2)\cdots$ and $A(Y) = A_1(y_1)A_1(y_2)\cdots$, we deduce that the functions $F_w(X)$ and $G_\varpi(Y)$ defined for $w \in W_n$ and $\varpi \in S_n$ by

$$F_w(X) = \langle C(X), w \rangle \quad \text{and} \quad G_\varpi(Y) = \langle A(Y), \varpi \rangle$$

are symmetric functions in X and Y, respectively. The G_ϖ and F_w are the type A and type C Stanley symmetric functions, introduced in [Sta] and [BH, FK2, L2]. We have that $F_w = F_{w^{-1}}$.

When G_ϖ is expanded in the basis of Schur functions, one obtains a formula

$$G_\varpi(Y) = \sum_{\lambda \, : \, |\lambda| = \ell(\varpi)} c_\lambda^\varpi s_\lambda(Y) \tag{4}$$

for some nonnegative integers c_λ^ϖ (see [LS2], [EG]). According to [LS2] (see also [M1, (7.22)]), we have $c_\lambda^\varpi = c_{\lambda'}^{\varpi^{-1}}$. Lascoux and Schützenberger [LS3] gave one of the first combinatorial interpretations for the coefficients c_λ^ϖ, as the number of leaves of shape λ in the *transition tree* $T(\varpi)$ they associated to ϖ. Equation (4) may be used to define the double Stanley symmetric functions $G_\varpi(Y/Z)$.

For any $\varpi \in S_n$, the Schubert polynomial $\mathfrak{S}_\varpi$ of Lascoux and Schützenberger is given by

$$\mathfrak{S}_\varpi(Y) = \langle A_1(y_1)A_2(y_2)\cdots A_{n-1}(y_{n-1}), \varpi \rangle \, . \tag{5}$$

The definition (5) is equivalent to the one in [LS1], as is shown in [FS]. Now define

$$\mathfrak{C}_w(X\,; Y, Z) = \left\langle \tilde{A}_{n-1}(z_{n-1}) \cdots \tilde{A}_1(z_1) C(X) A_1(y_1) \cdots A_{n-1}(y_{n-1}), w \right\rangle . \tag{6}$$

If $\mathfrak{C}_w(X\,;Y) := \mathfrak{C}_w(X\,;Y,0)$, then (6) is equivalent to the equation

$$\mathfrak{C}_w(X\,;Y,Z) = \sum_{uv=w} \mathfrak{S}_{u^{-1}}(-Z)\mathfrak{C}_v(X\,;Y) \tag{7}$$

summed over all reduced factorizations $uv = w$ with $u \in S_n$. The polynomials $\mathfrak{C}_w$ in (6) were introduced by Ikeda, Mihalcea, and Naruse [IMN]; they are double versions of the type C Billey-Haiman Schubert polynomials [BH]. Their definition differs from (6), but the equivalence of the two follows by combining (7) with [FK2, §7] and [IMN, Cor. 8.10]. One checks that $\mathfrak{S}_\varpi$ and $\mathfrak{C}_w$ are stable under the natural inclusion of W_n in W_{n+1}, and hence well defined for $\varpi \in S_\infty$ and $w \in W_\infty$, respectively. The $\mathfrak{S}_\varpi(Y)$ for $\varpi \in S_\infty$ form a $\mathbb{Z}$-basis of the polynomial ring $\mathbb{Z}[Y]$, and the $\mathfrak{C}_w(X\,;Y)$ for $w \in W_\infty$ form a $\mathbb{Z}$-basis of $\Gamma[Y]$.

If $\varpi \in S_\infty$ is a Grassmannian permutation with a unique descent at r, then $\mathfrak{S}_\varpi(Y)$ is a Schur polynomial in $(y_1, \ldots, y_r)$. In [BKT2, §6], we obtained the analogue of this result for the $\mathfrak{C}_w(X\,;Y)$: if $w = w_\lambda \in W_\infty$ is the k-Grassmannian permutation associated to the k-strict partition λ, then

$$\mathfrak{C}_{w_\lambda}(X\,;Y) = \Theta_\lambda(X\,;Y). \tag{8}$$

1.4. **Splitting type A Schubert polynomials.** If $r \leq s$ are any two integers, and $P(X,Y,Z)$ is any polynomial or formal power series in the variables x_i, y_j, and z_j, we let $P[r,s]$ denote the power series obtained from $P(X,Y,Z)$ by setting $x_i = 0$ for all i if $0 \notin [r,s]$, $y_j = 0$ if $j \notin [r,s]$, and $z_j = 0$ if $-j \notin [r,s]$. If $0 \in [r,s]$ we set $P^{(r,s)} = P[r,s]$ and $P^{(s)} = P[0,s]$.

If $w \in W_n$ and $v \in S_m$, we define $w \times v \in W_{m+n}$ to be the signed permutation $(w_1, \ldots, w_n, v_1 + n, \ldots, v_m + n)$; in particular, $1_n \times v$ is used to denote the permutation $(1, \ldots, n, v_1 + n, \ldots, v_m + n)$. For $\varpi \in S_\infty$ and $1 \leq r \leq s$, equation (5) immediately gives

$$\mathfrak{S}_\varpi[r,s] = \begin{cases} \mathfrak{S}_v(y_r, \ldots, y_s) & \text{if } \varpi = 1_{r-1} \times v, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Given any sequence $\mathfrak{a} : a_1 < \cdots < a_p$ of positive integers, we furthermore obtain

$$\mathfrak{S}_\varpi(Y) = \sum_{u_1 \cdots u_p = \varpi} \mathfrak{S}_{u_1}[1, a_1]\mathfrak{S}_{u_2}[a_1 + 1, a_2] \cdots \mathfrak{S}_{u_p}[a_{p-1} + 1, a_p]$$

summed over all reduced factorizations $u_1 \cdots u_p = \varpi$.

If ϖ is increasing up to r, then $\mathfrak{S}_\varpi$ is symmetric in $y_1, \ldots, y_r$ and we have

$$\mathfrak{S}_\varpi^{(r)} = G_\varpi^{(r)} = \sum_{\lambda\,:\,|\lambda| = \ell(\varpi)} c_\lambda^\varpi s_\lambda(y_1, \ldots, y_r) \tag{10}$$

with the coefficients c_λ^ϖ as in (4). Suppose now that ϖ is compatible with the sequence $\mathfrak{a}$, and set $Y_i = \{y_{a_{i-1}+1}, \ldots, y_{a_i}\}$ for each i. From the previous considerations, we deduce that $\mathfrak{S}_\varpi$ satisfies the formula

$$\mathfrak{S}_\varpi(Y) = \sum_{u_1 \cdots u_p = \varpi} G_{u_1}(Y_1) \cdots G_{u_p}(Y_p) \tag{11}$$

summed over all reduced factorizations $u_1 \cdots u_p = \varpi$ compatible with $\mathfrak{a}$. Using (4) to refine (11) further gives

$$\mathfrak{S}_\varpi(Y) = \sum_{\underline{\lambda}} c_{\underline{\lambda}}^\varpi s_{\lambda^1}(Y_1) \cdots s_{\lambda^p}(Y_p) \tag{12}$$

summed over all sequences of partitions $\underline{\lambda} = (\lambda^1, \ldots, \lambda^p)$, where

$$c_{\underline{\lambda}}^{\varpi} = \sum_{u_1 \cdots u_p = \varpi} c_{\lambda^1}^{u_1} \cdots c_{\lambda^p}^{u_p},$$

summed over all reduced factorizations $u_1 \cdots u_p = \varpi$ compatible with $\mathfrak{a}$. More general versions of (11), (12) for universal Schubert polynomials [Fu4] and quiver polynomials [BF] are established in [BKTY, KMS]. Equation (12) was used in [BKTY, §5] to obtain Giambelli formulas for type A partial flag varieties.

2. Transition for mixed Stanley functions

2.1. **Mixed Stanley functions.**

Definition 1. Given $w \in W_n$, the (right) *type C mixed Stanley function* $J_w(X\,;Y)$ is defined by the equation

$$J_w(X\,;Y) = \langle C(X)A(Y), w\rangle = \sum_{uv=w} F_u(X)G_v(Y)$$

summed over all reduced factorizations $uv = w$ with $v \in S_n$.

Definition 1 can be easily restated in terms of reduced decompositions and admissible sequences, along the lines of [BH, Eq. (3.2)]. One has a dual notion of a left mixed Stanley function $J'_w(X\,;Y) = \sum_{uv=w} G_{u^{-1}}(Y)F_v(X)$, summed over all reduced factorizations $uv = w$ with $u \in S_n$. This is equivalent to the right version since clearly $J'_w(X\,;Y) = J_{w^{-1}}(X\,;Y)$. Furthermore, observe that $J_w(X\,;Y)$ is well defined for $w \in W_\infty$.

Recall that $J_w^{(k)} = J_w[0,k]$.

Lemma 1. *If $w \in W_\infty$ is increasing up to k, then $\mathfrak{C}_w^{(k)} = J_w^{(k)}$. In particular, if $w = w_\lambda$ is k-Grassmannian, then*

$$J_{w_\lambda}^{(k)}(X\,;Y) = \Theta_\lambda(X\,;Y). \tag{13}$$

Proof. Observe that if $w = uv$ is a reduced factorization, then $\ell(vs_i) = \ell(v)+1$ for all $i < k$, i.e., v is also increasing up to k. It follows that

$$\mathfrak{C}_w^{(k)} = \sum_{uv=w,\, v\in S_\infty} F_u(X)\mathfrak{S}_v^{(k)} = \sum_{uv=w,\, v\in S_\infty} F_u(X)G_v^{(k)} = J_w^{(k)},$$

as claimed. Equation (13) follows from this and (8). □

Example 1. If λ is a k-strict partition, then

$$J_{w_\lambda}(X\,;Y) = \sum_{\mu\subset\lambda} F_{\lambda/\mu}(X)s_{\mu'}(Y) \tag{14}$$

summed over all k-strict partitions $\mu \subset \lambda$. The function $F_{\lambda/\mu}(X)$ in (14) is the skew F-function from [T4, §5], which, when non-zero, is equal to $F_{w_\lambda w_\mu^{-1}}(X)$.

2.2. **Transition equations.** For positive integers $i<j$ we define reflections $t_{ij}\in S_\infty$ and $\overline{t}_{ij},\overline{t}_{ii}\in W_\infty$ by their right actions

$$\begin{aligned}(\ldots,w_i,\ldots,w_j,\ldots)\,t_{ij}&=(\ldots,w_j,\ldots,w_i,\ldots),\\ (\ldots,w_i,\ldots,w_j,\ldots)\,\overline{t}_{ij}&=(\ldots,\overline{w}_j,\ldots,\overline{w}_i,\ldots),\quad\text{and}\\ (\ldots,w_i,\ldots)\,\overline{t}_{ii}&=(\ldots,\overline{w}_i,\ldots).\end{aligned}$$

We let $\overline{t}_{ji}=\overline{t}_{ij}$. According to [B, Thms. 4, 5], the type C Schubert polynomials $\mathfrak{C}_w=\mathfrak{C}_w(X\,;Y)$ satisfy the recursion formula

$$\mathfrak{C}_w=y_r\mathfrak{C}_{wt_{rs}}+\sum_{\substack{1\leq i<r\\ \ell(wt_{rs}t_{ir})=\ell(w)}}\mathfrak{C}_{wt_{rs}t_{ir}}+\sum_{\substack{i\geq 1\\ \ell(wt_{rs}\overline{t}_{ir})=\ell(w)}}\mathfrak{C}_{wt_{rs}\overline{t}_{ir}},\tag{15}$$

where r is the last *positive* descent of w and s is maximal such that $w_s<w_r$. If w is increasing up to k and the last descent r of w satisfies $r>k$, we deduce from (15), Lemma 1, and Lemma 3 below that

$$J_w^{(k)}=\sum_{\substack{1\leq i<r\\ \ell(wt_{rs}t_{ir})=\ell(w)}}J^{(k)}_{wt_{rs}t_{ir}}+\sum_{\substack{i\geq 1\\ \ell(wt_{rs}\overline{t}_{ir})=\ell(w)}}J^{(k)}_{wt_{rs}\overline{t}_{ir}}.\tag{16}$$

The definitions of r and s imply that we have $\ell(wt_{rs})=\ell(w)-1$ in (15) and (16). The next result therefore characterizes the reflections $t=t_{ir}$ and $t=\overline{t}_{ir}$ such that $\ell(wt_{rs}t)=\ell(w)$.

Lemma 2 ([Mo, B]). *Let w be an element of W_∞.*
(a) *If $i<j$, then $\ell(wt_{ij})=\ell(w)+1$ if and only if $w_i<w_j$, and there is no p with $i<p<j$ and $w_i<w_p<w_j$.*
(b) *If $i\leq j$, then $\ell(w\overline{t}_{ij})=\ell(w)+1$ if and only if* (i) *$-w_i<w_j$ and $-w_j<w_i$,* (ii) *in case $i<j$, either $w_i<0$ or $w_j<0$, and* (iii) *there is no $p<i$ such that $-w_j<w_p<w_i$, and no $p<j$ such that $-w_i<w_p<w_j$.*
(c) *If $i<j$ and $\ell(w\overline{t}_{ij})=\ell(w)+1$, then each positive descent of $w\overline{t}_{ij}$ is also a positive descent of w.*

For any $w\in W_\infty$ which is increasing up to k, we construct a rooted tree $T^k(w)$ with nodes given by elements of W_∞ and root w as follows. Let r be the last descent of w. If $w=1$ or $r=k$, then set $T^k(w)=\{w\}$. Otherwise, let $s=\max(i>r\mid w_i<w_r)$ and $\Phi(w)=\Phi_1(w)\cup\Phi_2(w)$, where

$$\begin{aligned}\Phi_1(w)&=\{wt_{rs}t_{ir}\mid 1\leq i<r\ \text{ and }\ \ell(wt_{rs}t_{ir})=\ell(w)\},\\ \Phi_2(w)&=\{wt_{rs}\overline{t}_{ir}\mid i\geq 1\ \text{ and }\ \ell(wt_{rs}\overline{t}_{ir})=\ell(w)\}.\end{aligned}$$

To recursively define $T^k(w)$, we join w by an edge to each $v\in\Phi(w)$, and attach to each $v\in\Phi(w)$ its tree $T^k(v)$.

We call $T^k(w)$ the *k-transition tree* of w. Clearly $T^k(w)$ is a subtree of $T^{k'}(w)$ for any $k'\leq k$. The 0-transition tree of w was studied in [B, BL].

Lemma 3. *The k-transition tree $T^k(w)$ is finite. All the nodes of $T^k(w)$ are increasing up to k, and the leaves of $T^k(w)$ are k-Grassmannian elements.*

Proof. If w is k-Grassmannian then clearly $T^k(w)=\{w\}$. We will show that if w is increasing up to k and not k-Grassmannian, then $\Phi(w)\neq\emptyset$ and all elements of $\Phi(w)$ are increasing up to k. Let $r>k$ be the last descent of w, s be maximal such that $w_s<w_r$, and $v=wt_{rs}$. If $\Phi_1(w)=\emptyset$, then according Lemma 2(a), we must

have $w_i > w_s$ for all $i < r$. If $w_s > 0$ then $v\bar{t}_{rr} \in \Phi_2(w)$. If $w_s < 0$, let $i > 0$ be minimal such that $w_i > -w_s$. Then one can easily check using Lemma 2(b) that $v\bar{t}_{ir} \in \Phi_2(w)$. This proves that $\Phi(w)$ is nonempty.

We next show that all elements of $\Phi(w)$ are increasing up to k. For this, we may assume that $k > 0$. If there exists an $i \leq k$ such that $\ell(vt_{ir}) = \ell(w)$, then Lemma 2(a) implies that $w_i < w_s$ and there is no j with $i < j < r$ and $w_i < w_j < w_s$. It follows that vt_{ir} is increasing up to k. If $\ell(v\bar{t}_{ir}) = \ell(w)$ for some $i \leq k$, then by Lemma 2(b)(ii) we must have $w_s < 0$, since w is increasing up to k and hence $v_i = w_i > 0$. Moreover, by Lemma 2(c), the descent set of $v\bar{t}_{ir}$ is contained in the descent set of v. We deduce that $v\bar{t}_{ir}$ is also increasing up to k.

Finally, it is shown in the proof of [B, Thm. 4] that the recursion defining $T^0(w)$ terminates after a finite number of steps; hence $T^k(w)$ is a finite tree. Moreover, we deduce from loc. cit. that if $w \in W_n$, then all of the nodes of $T^k(w)$ lie in W_{n+r}. □

Definition 2. If $w \in W_\infty$ is increasing up to k and v is a leaf of $T^k(w)$, the *shape* of v is the k-strict partition λ associated to v. For any k-strict partition λ, the *mixed Stanley coefficient* e^w_λ is equal to the number of leaves of $T^k(w)$ of shape λ.

The next result is a type C analogue of equation (10).

Theorem 1. *If $w \in W_\infty$ is increasing up to k, then we have an expansion*

$$\mathfrak{C}^{(k)}_w = J^{(k)}_w = \sum_{\lambda\,:\,|\lambda|=\ell(w)} e^w_\lambda\,\Theta_\lambda \tag{17}$$

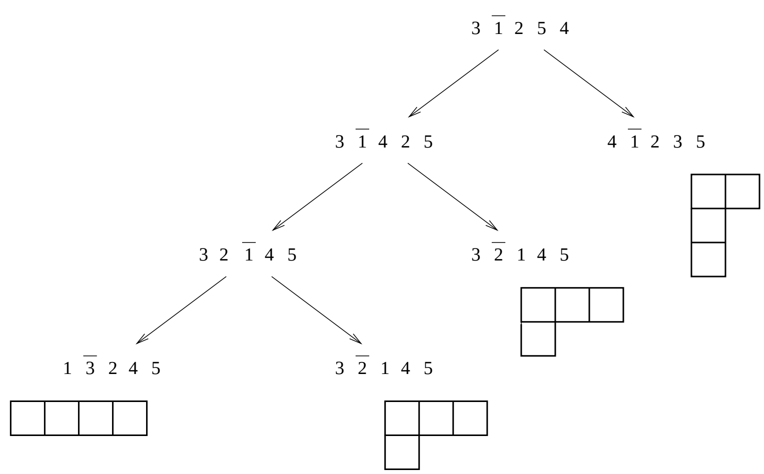
where the sum is over k-strict partitions λ.

Proof. The equality $\mathfrak{C}^{(k)}_w = J^{(k)}_w$ is proved in Lemma 1. We deduce from (16) and the definition of $T^k(w)$ that if w is not k-Grassmannian, then

$$J^{(k)}_w = \sum_{v\in\Phi(w)} J^{(k)}_v.$$

On the other hand, for any k-Grassmannian element $w = w_\lambda$, we have $J^{(k)}_{w_\lambda} = \Theta_\lambda$, by (13). This completes the proof of the theorem. □

Example 2. The 1-transition tree of $w = 3\bar{1}254$ looks as follows.



By Theorem 1 we therefore obtain

$$J^{(1)}_{3\bar{1}254} = \Theta_{(2,1,1)} + 2\,\Theta_{(3,1)} + \Theta_4.$$

When $k=0$, Theorem 1 states that for any $w \in W_\infty$,

$$F_w(X) = \sum_{\lambda\,:\,|\lambda|=\ell(w)} e^w_\lambda\, Q_\lambda(X) \tag{18}$$

summed over strict partitions λ. A transition based formula for the constants e^w_λ in equation (18) was proved by Billey [B]. There are several alternative combinatorial descriptions of these numbers in this case, which include a formula in terms of *Kraśkiewicz tableaux* [Kr, L2] of shape λ. It would be interesting to have an analogous tableau formula for the e^w_λ in the general case where $k>0$. Another natural question is whether the mixed Stanley coefficients can be used to obtain a Littlewood-Richardson type rule for theta polynomials; some positive results in this direction are explained in §2.4.

Example 3. Consider the *double mixed Stanley function*

$$J_w(X\,;Y/Z) = \sum_{\varpi uv = w} G_{\varpi^{-1}}(-Z) F_u(X) G_v(Y)$$

summed over all reduced factorizations $\varpi uv = w$ with $\varpi, v \in S_\infty$. Fix integers $j,k \geq 0$, and suppose that $w \in W_\infty$ is increasing up to k and w^{-1} is increasing up to j. Then $\mathfrak{C}^{(-j,k)}_w(X\,;Y,Z) = J^{(-j,k)}_w(X\,;Y/Z)$. However, the analogue of Theorem 1 fails, at least for fixed k. For an example with $j=k=1$, one checks that $\mathfrak{C}^{(-1,1)}_{231}$ cannot be written as an integer linear combination of $\mathfrak{C}^{(-1,1)}_{2\bar{1}3}$ and $\mathfrak{C}^{(-1,1)}_{312}$, while $2\bar{1}3$ and 312 are the only 1-Grassmannian elements of length two in W_∞.

2.3. **Reduced words and k-bitableaux.** The type A Stanley symmetric functions G_ϖ were used in [Sta] to express the number of reduced words of a permutation ϖ in terms of the numbers f^λ of standard tableaux of shape λ, for the partitions λ which appear in equation (4). Similarly, the type C Stanley symmetric functions F_w can be used to compute the number of reduced words for an element $w \in W_\infty$, as shown in [H, Kr]. We proceed to give an analogue of these results for the mixed Stanley functions J_w.

Let $\mathbf{P}$ denote the ordered alphabet $\{1' < 2' < \cdots < k' < 1 < 2 < \cdots\}$. The symbols $1', \ldots, k'$ are called *marked*, while the rest are *unmarked*. Let λ be a k-strict partition. A *k-bitableau U* of shape λ is a filling of the boxes in λ with elements of $\mathbf{P}$ which is weakly increasing along each row and down each column, such that the marked entries are strictly increasing along each row and the unmarked entries form a *k-tableau T*. We refer to [T4, §5] for the definition of a k-tableau and more details. Each k-bitableau U has an associated multiplicity $r(U)$, which is a nonnegative integer. Let $(xy)^U = \prod_i x_i^{m_i} \prod_j y_j^{n_j}$, where m_i (respectively n_j) denotes the number of times that i (respectively j') appears in U. According to [T4, Thm. 5], we have

$$\Theta_\lambda(X\,;Y) = \sum_U 2^{r(U)} (xy)^U \tag{19}$$

summed over all k-bitableaux U of shape λ.

If a k-bitableau U contains exactly m marked entries, we say that U is of *type* m. U is called *standard* if the entries $1',\ldots,m',1,\ldots,n$ each appear once in U for some m and n; in this case we have $r(U)=n$.

Definition 3. Let λ and μ be k-strict partitions with $\mu\subset\lambda$, and m be a nonnegative integer with $m\leq k$. We denote by $g^{\lambda/\mu}$ the number of standard k-tableaux of skew shape λ/μ, and by h^λ_m the number of standard k-bitableaux of shape λ and type m. We say that a reduced word for $w\in W_\infty$ has *type* m if the last m letters of the word are positive.

Proposition 1. *For any k-strict partition λ and integer $m\leq k$, we have*

$$h^\lambda_m = \sum_{\mu\subset\lambda,\,|\mu|=m} f^\mu g^{\lambda/\mu}$$

where the sum is over all partitions $\mu\subset\lambda$ with $|\mu|=m$.

Proof. Suppose that λ is a k-strict partition. Using e.g. [T4, Prop. 5], one can construct a bijection between the set of standard k-bitableaux of shape λ and type m and the set of reduced words of type m for w_λ. According to [T4, Thm. 6 and Ex. 9], for any k-strict partition μ with $\mu\subset\lambda$ and $\mu_1\leq k$, the number $g^{\lambda/\mu}$ of standard k-tableaux of shape λ/μ is equal to the number of reduced words for $w_\lambda w_\mu^{-1}$. Moreover, the number f^μ of standard tableaux of shape μ equals the number of reduced words for the permutation $w_\mu\in S_\infty$. The result follows. $\square$

The $k=0$ case of the next result is due to Haiman [H] and Kraśkiewicz [Kr].

Proposition 2. *Let $w\in W_\infty$ be increasing up to k and let m be an integer with $0\leq m\leq \min(k,\ell(w))$. Then the number of reduced words of type m for w is equal to $\sum_\lambda e^w_\lambda h^\lambda_m$, where the sum is over all k-strict partitions λ.*

Proof. It is clear that the number of reduced words of type m for w equals 2^{-n} times the coefficient of $x_1\cdots x_n y_1\cdots y_m$ in J_w, where $n=\ell(w)-m$. On the other hand, this coefficient is also equal to $2^n\sum_\lambda e^w_\lambda h^\lambda_m$, by (17) and (19). $\square$

It would be interesting to find a bijective proof of Proposition 2.

2.4. **Multiplication rules.** We show here how Theorem 1 may be used to obtain Littlewood-Richardson type rules for the product of two theta polynomials. In the case of $k=0$, we observe that the transition equations in [B] give a combinatorial rule for the structure constants in the product of *any* two Schur Q-functions. This answers a question of Manivel in the affirmative (compare with [BL, §4]).

We will actually work with the Schur P-functions, which are defined by the equation $P_\lambda=2^{-\ell(\lambda)}Q_\lambda$, for any strict partition λ. Given two strict partitions μ and ν, there are nonnegative integers $f^\lambda_{\mu\nu}$ such that

$$P_\mu P_\nu = \sum_\lambda f^\lambda_{\mu\nu} P_\lambda.$$

The $f^\lambda_{\mu\nu}$ agree with the Schubert structure constants on maximal orthogonal Grassmannians $\mathrm{OG}(n,2n+1)$, when n is sufficiently large. Combinatorial rules for the numbers $f^\lambda_{\mu\nu}$ may be found in [Sa, W, St1, Sh].

Proposition 3. *The coefficient $f^\lambda_{\mu\nu}$ is equal to the number of leaves of $T^0(w_\lambda w_\mu^{-1})$ of shape ν, if $\mu\subset\lambda$, and is equal to zero, otherwise.*

Proof. The structure constants $f^\lambda_{\mu\nu}$ appear in the expansion of *skew Schur Q-functions*

$$Q_{\lambda/\mu} = \sum_\nu f^\lambda_{\mu\nu} Q_\nu \tag{20}$$

in the Q-basis (see for example [M2, III.5 and III.8]). Following [FK2, Thm. 8.2] and [St2, Cor. 6.6], the skew Schur Q-functions are known to be equal to certain type C Stanley symmetric functions. In fact, we have $Q_{\lambda/\mu} = F_{w_\lambda w_\mu^{-1}}$, where w_λ and w_μ are the 0-Grassmannian elements associated to λ and μ (this is a special case of [T4, Thm. 6]). The proposition follows from this, using (18) and (20). $\square$

Now let k be any nonnegative integer. The following result is an analogue of [BL, Lemma 16] for the functions J_w.

Lemma 4. *For $w \in W_n$ and $v \in S_\infty$, we have*

$$J_w J_v = J_{w\times v}.$$

Proof. Since $a_1 \cdots a_\ell$ is a reduced word for $v \in S_m$ if and only if $(a_1+n)\cdots(a_\ell+n)$ is a reduced word for $1_n \times v$, we see that $G_{1_n\times v} = G_v$, $F_{1_n\times v} = F_v$, and $J_v = J_{1_n\times v}$ for each $n \geq 1$. The reduced words for $w \times v$ are all obtained by intertwining a reduced word for w with a reduced word for $1_n \times v$. Moreover, given any reduced factorization $ab = w \times v$, with $b \in S_\infty$, we have $a = a_1 \times a_2$ and $b = b_1 \times b_2$ where $a_1, b_1 \in W_n$ and $a_2, b_1, b_2 \in S_\infty$. We deduce that

$$\begin{aligned}
J_{w\times v}(X\,;Y) &= \sum_{(a_1\times a_2)(b_1\times b_2)=w\times v} F_{a_1\times a_2}(X) G_{b_1\times b_2}(Y) \\
&= \sum_{(a_1\times a_2)(b_1\times b_2)=w\times v} F_{a_1}(X)F_{1_n\times a_2}(X)G_{b_1}(Y)G_{1_n\times b_2}(Y) \\
&= \sum_{a_1b_1=w,\ a_2b_2=v} F_{a_1}(X)G_{b_1}(Y)F_{a_2}(X)G_{b_2}(Y) \\
&= J_w(X\,;Y)J_v(X\,;Y).
\end{aligned}$$

$\square$

We obtain a combinatorial rule for multiplying two theta polynomials, when one of the factors is indexed by a 'small' partition (compare with [BL, Cor. 17]).

Proposition 4. *Let μ and ν be k-strict partitions with $\nu_i \leq k$ for all i, and consider the product expansion*

$$\Theta_\mu \Theta_\nu = \sum_\lambda \varphi^\lambda_{\mu\nu} \Theta_\lambda$$

summed over k-strict partitions λ. Then $\varphi^\lambda_{\mu\nu}$ is equal to the number of leaves of $T^k(w_\mu \times w_\nu)$ of shape λ.

Proof. Observe that $w_\nu \in S_\infty$ if and only if $\nu_i \leq k$ for all i. Lemma 4 therefore applies and gives the equation

$$\Theta_\mu\Theta_\nu = J^{(k)}_{w_\mu} J^{(k)}_{w_\nu} = J^{(k)}_{w_\mu\times w_\nu}.$$

The result now follows from Theorem 1. Recall from [BKT2, §5.4] that in this situation we have $\Theta_\nu = R^0\,\vartheta_\nu = \det(\vartheta_{\nu_i+j-i})_{i,j}$. $\square$

Example 4. 1) For the 1-strict partitions $\mu = (2,1)$ and $\nu = 1$ we have $w_\mu \times w_\nu = 3\bar{1}254 \in W_5$. Example 2 therefore gives $\Theta_{(2,1)}\Theta_1 = \Theta_{(2,1,1)} + 2\,\Theta_{(3,1)} + \Theta_4$.

2) When $k \geq 1$ and $\nu = (1^p)$ for some $p \geq 0$, Proposition 4 gives a 'Pieri type rule' which evaluates the products $\Theta_{(1^p)}\Theta_\mu$ for any k-strict partition μ. A different combinatorial rule for the same Pieri products was obtained by Pragacz and Ratajski [PR1].

3. Splitting type C Schubert polynomials

In this section we give splitting theorems for the single and double type C Schubert polynomials $\mathfrak{C}_w$. For any $k \geq 0$, let $Y_{>k} = (y_{k+1}, y_{k+2}, \ldots)$. The following proposition generalizes the $k = 0$ case from [BH, Thm. 3].

Proposition 5. *If $w \in W_\infty$ is increasing up to k, then*

$$\mathfrak{C}_w(X\,;Y) = \sum_{u(1_k \times v) = w} J_u^{(k)}(X\,;Y)\mathfrak{S}_v(Y_{>k}) \tag{21}$$

where the sum is over all reduced factorizations $u(1_k \times v) = w$ with $v \in S_\infty$. The Schubert polynomials $\mathfrak{C}_w(X\,;Y)$ for $w \in W_\infty$ increasing up to k form a $\mathbb{Z}$-basis for the ring $\Gamma^{(k)}[Y_{>k}] = \Gamma^{(k)}[y_{k+1}, y_{k+2}, \ldots]$.

Proof. From the definition (6) we deduce that for any $w \in W_n$,

$$\mathfrak{C}_w(X\,;Y) = \sum_{u\varpi = w,\, \varpi \in S_\infty} \mathfrak{C}_u[0,k]\mathfrak{S}_\varpi[k+1,n].$$

According to (9), the polynomial $\mathfrak{S}_\varpi[k+1,n]$ is non-zero only if $\varpi = 1_k \times v$ for some $v \in S_\infty$, in which case $\mathfrak{S}_\varpi[k+1,n] = \mathfrak{S}_v(y_{k+1}, \ldots, y_n)$. For all such ϖ, we furthermore note that the element $u = w\varpi^{-1}$ is increasing up to k, and hence $\mathfrak{C}_u[0,k] = J_u[0,k]$ by Theorem 1. This proves equation (21).

Set $\mathfrak{C}_w = \mathfrak{C}_w(X\,;Y)$ and for each $i \geq 0$, let ∂_i denote the divided difference operator from [BH, §2]. Recall that $\partial_i \mathfrak{C}_w = \mathfrak{C}_{ws_i}$, if $w_i > w_{i+1}$, and $\partial_i \mathfrak{C}_w = 0$, otherwise. Since the $\Theta_\lambda = \mathfrak{C}_{w_\lambda}$ for k-strict partitions λ form a $\mathbb{Z}$-basis of $\Gamma^{(k)}$, we deduce that $\partial_i f = 0$ for all $i < k$ and f in the ring $\Gamma^{(k)}[Y_{>k}]$.

Equations (17) and (21) imply that the $\mathfrak{C}_w$ for w increasing up to k are contained in $\Gamma^{(k)}[Y_{>k}]$. Moreover, the $\mathfrak{C}_w$ for $w \in W_\infty$ are known to be a $\mathbb{Z}$-basis of $\Gamma[Y]$ from [BH, Thm. 3]. Given any $f \in \Gamma^{(k)}[Y_{>k}]$, we therefore have $f = \sum_{w \in W_\infty} a_w\, \mathfrak{C}_w$ for some $a_w \in \mathbb{Z}$. Since $\partial_i f = 0$ for all $i < k$, we deduce that only terms $\mathfrak{C}_w$ with w increasing up to k appear in the sum, completing the proof. $\square$

Fix a sequence $\mathfrak{a} : a_1 < \cdots < a_p$ of nonnegative integers and $w \in W_\infty$ which is compatible with $\mathfrak{a}$. Given any sequence of partitions $\underline{\lambda} = (\lambda^1, \ldots, \lambda^p)$ with λ^1 a_1-strict, we define the nonnegative integer

$$e_{\underline{\lambda}}^w = \sum_{u_1 \cdots u_p = w} e_{\lambda^1}^{u_1} c_{\lambda^2}^{u_2} \cdots c_{\lambda^p}^{u_p}, \tag{22}$$

where the sum is over reduced factorizations $u_1 \cdots u_p = w$ compatible with $\mathfrak{a}$ such that $u_2, \ldots, u_p \in S_\infty$, and the integers $c_{\lambda^i}^{u_i}$ and $e_{\lambda^1}^{u_1}$ are as in (4) and (17), respectively.

The number $e_{\underline{\lambda}}^w$ can be described in a more picturesque way as follows. Given a permutation $\varpi \in S_\infty$, let $T(\varpi)$ denote the Lascoux-Schützenberger transition tree associated to ϖ in [LS3, §4]. For any reduced factorization $u_1 \cdots u_p = w$ compatible with $\mathfrak{a}$ such that $u_2, \ldots, u_p \in S_\infty$, the p-tuple of trees $(T^{a_1}(u_1), T(u_2), \ldots, T(u_p))$ is called a *grove*. The collection of all such groves forms the $\mathfrak{a}$-*transition forest* associated to w. The integer $e^w_{\underline{\lambda}}$ is then equal to the number of p-tuples of leaves of shape $\underline{\lambda}$ in the groves of the $\mathfrak{a}$-transition forest associated to w.

Define $Y_i = \{y_{a_{i-1}+1}, \ldots, y_{a_i}\}$ for each $i \geq 1$; in particular $Y_1 = \emptyset$ if $a_1 = 0$.

Theorem 2. *Suppose that $w \in W_\infty$ is compatible with the sequence $\mathfrak{a}$. Then we have*

$$\mathfrak{C}_w(X\,;Y) = \sum_{u_1\cdots u_p = w} J_{u_1}(X\,;Y_1)G_{u_2}(Y_2)\cdots G_{u_p}(Y_p) \tag{23}$$

summed over all reduced factorizations $u_1 \cdots u_p = w$ compatible with $\mathfrak{a}$ such that $u_2, \ldots, u_p \in S_\infty$. Furthermore, we have

$$\mathfrak{C}_w(X\,;Y) = \sum_{\underline{\lambda}} e^w_{\underline{\lambda}}\, \Theta_{\lambda^1}(X\,;Y_1)s_{\lambda^2}(Y_2)\cdots s_{\lambda^p}(Y_p) \tag{24}$$

summed over all sequences of partitions $\underline{\lambda} = (\lambda^1, \ldots, \lambda^p)$ with λ^1 a_1-strict.

Proof. Equation (23) follows from (21) and the type A Schubert splitting formula (11). Moreover, (24) is obtained from (23) by using equations (9), (10), and (17). □

The $k = 0$ case of formula (24) is contained in [Yo, §5]. It is expressed in loc. cit. using a combinatorial interpretation of the coefficients c^{ϖ}_{λ} in (4) due to Fomin and Greene [FG], which was also exploited in [BKTY, Cor. 3].

Fix a second sequence $\mathfrak{b} : 0 = b_1 < \cdots < b_q$ of nonnegative integers such that w^{-1} is compatible with $\mathfrak{b}$. A reduced factorization $u_1 \cdots u_{p+q-1} = w$ is *compatible with* $\mathfrak{a}$, $\mathfrak{b}$ if $u_j(i) = i$ whenever $j < q$ and $i \leq b_{q-j}$ or whenever $j > q$ and $i \leq a_{j-q}$. Given any sequence of partitions $\underline{\lambda} = (\lambda^1, \ldots, \lambda^{p+q-1})$ with λ^q a_1-strict, we define

$$f^w_{\underline{\lambda}} = \sum_{u_1\cdots u_{p+q-1} = w} c^{u_1}_{\lambda^1} \cdots c^{u_{q-1}}_{\lambda^{q-1}} e^{u_q}_{\lambda^q} c^{u_{q+1}}_{\lambda^{q+1}} \cdots c^{u_{p+q-1}}_{\lambda^{p+q-1}}, \tag{25}$$

where the sum is over reduced factorizations $u_1 \cdots u_{p+q-1} = w$ compatible with $\mathfrak{a}$, $\mathfrak{b}$ such that $u_i \in S_\infty$ for all $i \neq q$. Set $Z_j = \{z_{b_{j-1}+1}, \ldots, z_{b_j}\}$ for each j.

Corollary 1. *Suppose that w and w^{-1} are compatible with the sequences $\mathfrak{a}$ and $\mathfrak{b}$, respectively. Then $\mathfrak{C}_w(X\,;Y,Z)$ is equal to*

$$\sum_{u_1\cdots u_{p+q-1} = w} G_{u_1}(0/Z_q)\cdots G_{u_{q-1}}(0/Z_2)J_{u_q}(X\,;Y_1)G_{u_{q+1}}(Y_2)\cdots G_{u_{p+q-1}}(Y_p)$$

summed over all reduced factorizations $u_1 \cdots u_{p+q-1} = w$ compatible with $\mathfrak{a}$, $\mathfrak{b}$ such that $u_i \in S_\infty$ for all $i \neq q$. Furthermore, we have

$$\mathfrak{C}_w(X\,;Y,Z) = \sum_{\underline{\lambda}} f^w_{\underline{\lambda}}\, s_{\lambda^1}(0/Z_q)\cdots \Theta_{\lambda^q}(X\,;Y_1)\cdots s_{\lambda^{p+q-1}}(Y_p) \tag{26}$$

summed over all sequences of partitions $\underline{\lambda} = (\lambda^1, \ldots, \lambda^{p+q-1})$ with λ^q a_1-strict.

Proof. The result follows immediately from equations (7), (11), and Theorem 2. □

According to [IMN, Thm. 8.1], the Schubert polynomials $\mathfrak{C}_w(X\,;Y,Z)$ enjoy the following symmetry property:

$$\mathfrak{C}_w(X\,;Y,Z) = \mathfrak{C}_{w^{-1}}(X\,;-Z,-Y). \tag{27}$$

This also follows immediately from equation (6). By applying Corollary 1 to the right hand side of (27), we obtain dual versions of these splitting equations.

4. Symplectic degeneracy loci

4.1. **Isotropic partial flag bundles.** Let $E \to B$ be a vector bundle of rank $2n$ on an algebraic variety B. Assume that E is a *symplectic* bundle, i.e. E is equipped with an everywhere nondegenerate skew-symmetric form $E \otimes E \to \mathbb{C}$. A subbundle V of E is *isotropic* if the form vanishes when restricted to V; the ranks of isotropic subbundles of E range from 0 to n. Fix a sequence $\mathfrak{a} : a_1 < \cdots < a_p$ of nonnegative integers with $a_p < n$, and set $a_{p+1} = n$ for convenience. We introduce the *isotropic partial flag bundle* $\mathrm{F}^{\mathfrak{a}}(E)$ with its projection map $\rho : \mathrm{F}^{\mathfrak{a}}(E) \to B$. The variety $\mathrm{F}^{\mathfrak{a}}(E)$ parametrizes partial flags

$$E_\bullet \ : \ 0 = E_0 \subset E_1 \subset \cdots \subset E_p \subset E \tag{28}$$

with $\operatorname{rank} E_i = n - a_{p+1-i}$ and E_p isotropic. Here we have identified E with its pullback under the map ρ, and also let (28) denote the tautological partial flag of vector bundles over $\mathrm{F}^{\mathfrak{a}}(E)$.

There is a group monomorphism $\phi : W_n \hookrightarrow S_{2n}$ with image

$$\phi(W_n) = \{\, \varpi \in S_{2n} \mid \varpi(i) + \varpi(2n+1-i) = 2n+1, \quad \text{for all } i \,\}.$$

The map ϕ is determined by setting, for each $w = (w_1, \ldots, w_n) \in W_n$ and $1 \leq i \leq n$,

$$\phi(w)(i) = \begin{cases} n+1-w_{n+1-i} & \text{if } w_{n+1-i} \text{ is unbarred,} \\ n + \overline{w}_{n+1-i} & \text{otherwise.} \end{cases}$$

Let $W^{\mathfrak{a}}$ be the set of signed permutations $w \in W_n$ whose descent positions are listed among the integers $a_1, \ldots, a_p$. These elements are the minimal length coset representatives in $W_n/W_{\mathfrak{a}}$, where $W_{\mathfrak{a}}$ denotes the subgroup of W_n generated by the simple reflections s_i for $i \notin \{a_1, \ldots, a_p\}$.

Fix a flag $0 = F_0 \subset F_1 \subset \cdots \subset F_n \subset E$ of subbundles of E with $\operatorname{rank} F_i = i$ for each i and F_n isotropic. We extend any such flag to a complete flag $F_\bullet$ in E by letting $F_{n+i} = F_{n-i}^{\perp}$ for $1 \leq i \leq n$; we call the completed flag a *complete isotropic flag*. For every $w \in W^{\mathfrak{a}}$ and complete isotropic flag $F_\bullet$ of subbundles of E, we define the *universal Schubert variety* $\mathfrak{X}_w \subset \mathrm{F}^{\mathfrak{a}}(E)$ as the locus of $a \in \mathrm{F}^{\mathfrak{a}}(E)$ such that

$$\dim(E_r(a) \cap F_s(a)) \geq \#\{\, i \leq \operatorname{rank} E_r \mid \phi(w)(i) > 2n - s \,\} \ \ \forall\, r, s. \tag{29}$$

The variety $\mathfrak{X}_w$ is an irreducible subvariety of $\mathrm{F}^{\mathfrak{a}}(E)$ of codimension $\ell(w)$, and may be regarded as a universal degeneracy locus. Formulas for the classes $[\mathfrak{X}_w]$ in the cohomology or Chow ring of $\mathrm{F}^{\mathfrak{a}}(E)$ pull back to identities for corresponding loci whenever one has a symplectic vector bundle V and two flags of isotropic subbundles of V, following [Fu3]. Moreover, they are equivalent to formulas which represent the Schubert classes in the T-equivariant cohomology ring of isotropic partial flag varieties, as observed e.g. in [Gra].

4.2. **Full flag bundles and the geometrization map.** Consider the full flag bundle $\mathrm{F}(E) = \mathrm{F}^{(0,1,2,\ldots,n-1)}(E)$ parametrizing complete isotropic flags of subbundles $E_\bullet$ in E. Let $\mathrm{X} = (\mathrm{x}_1, \ldots, \mathrm{x}_n)$ and $\mathrm{Y} = (\mathrm{y}_1, \ldots, \mathrm{y}_n)$. According to [Fu2, §3], the cohomology (or Chow) ring $\mathrm{H}^*(\mathrm{F}(E), \mathbb{Z})$ is presented as a quotient

$$\mathrm{H}^*(\mathrm{F}(E)) \cong \mathrm{H}^*(B)[\mathrm{X}, \mathrm{Y}]/\mathrm{J}_n, \tag{30}$$

where J_n denotes the ideal generated by the differences $e_i(\mathrm{x}_1^2,\ldots,\mathrm{x}_n^2)-e_i(\mathrm{y}_1^2,\ldots,\mathrm{y}_n^2)$ for $1\leq i\leq n$. The inverse of the isomorphism (30) sends the class of x_i to $-c_1(E_{n+1-i}/E_{n-i})$ and of y_i to $-c_1(F_{n+1-i}/F_{n-i})$ for each i with $1\leq i\leq n$.

The ring $\mathrm{H}^*(\mathrm{F}(E))$ may be used to study the cohomology of any isotropic partial flag bundle $\mathrm{F}^{\mathfrak{a}}(E)$, because the natural surjection $\mathrm{F}(E)\to\mathrm{F}^{\mathfrak{a}}(E)$ induces an injective ring homomorphism $\iota:\mathrm{H}^*(\mathrm{F}^{\mathfrak{a}}(E))\to\mathrm{H}^*(\mathrm{F}(E))$. The tautological vector bundles E_i, F_j, the universal Schubert varieties, and their cohomology classes on $\mathrm{F}^{\mathfrak{a}}(E)$ pull back under ι to the homonymous objects over $\mathrm{F}(E)$.

The Schubert varieties $\mathfrak{X}_w$ on $\mathrm{F}(E)$ are indexed by w in the full Weyl group W_n. Furthermore, the type C double Schubert polynomials $\mathfrak{C}_w(X;Y,Z)$ represent their cohomology classes $[\mathfrak{X}_w]$ in the presentation (30), but only after a certain change of variables. Ikeda, Mihalcea, and Naruse [IMN, §10] provide a different way to connect these Schubert polynomials to geometry, which we will adapt to our current setup. For a closely related approach (which preceded [IMN]) in the case of single Schubert polynomials, see [T2].

The key tool is the following ring homomorphism derived from [IMN], which we call the *geometrization map*:

$$\pi_n:\Gamma[Y,Z]\to\mathrm{H}^*(B)[\mathrm{X},\mathrm{Y}]/\mathrm{J}_n.$$

The homomorphism π_n is determined by setting

$$\pi_n(q_r(X))=\sum_{i=0}^r e_i(\mathrm{X})h_{r-i}(\mathrm{Y})\ \text{ for all } r,$$

$$\pi_n(y_i)=\begin{cases}-\mathrm{x}_i & \text{if } 1\leq i\leq n,\\ 0 & \text{if } i>n,\end{cases}\quad\text{and}\quad \pi_n(z_j)=\begin{cases}\mathrm{y}_j & \text{if } 1\leq j\leq n,\\ 0 & \text{if } j>n.\end{cases}$$

It follows from [Gra, §1] and [IMN, Prop. 7.7 and §10] that for $w\in W_n$, the geometrization map π_n maps $\mathfrak{C}_w(X\,;Y,Z)$ to a polynomial which represents the universal Schubert class $[\mathfrak{X}_w]$ in the presentation (30). Furthermore, we have $\pi_n(\mathfrak{C}_w)\in\mathrm{J}_n$ for $w\in W_\infty\smallsetminus W_n$.

If V and V' are two vector bundles with total Chern classes $c(V)$ and $c(V')$, respectively, we denote $s_\lambda(c(V)-c(V'))$ by $s_\lambda(V-V')$. We similarly denote the class $\Theta_\lambda(c(V)-c(V'))$ by $\Theta_\lambda(V-V')$. To state our main geometric result, let $Q_1=E/E_p, Q_2=E_p/E_{p-1},\ldots,Q_p=E_2/E_1$. Consider a sequence $\mathfrak{b}\,:\,0=b_1<\cdots<b_q$ with $b_q<n$, and set $\widehat{Q}_2=F_n/F_{n-b_2},\ldots,\widehat{Q}_q=F_{n-b_{q-1}}/F_{n-b_q}$.

Theorem 3. *Suppose that $w\in W^{\mathfrak{a}}$ and that w^{-1} is compatible with $\mathfrak{b}$. Then we have*

$$\begin{aligned}[\mathfrak{X}_w]&=\sum_{\underline{\lambda}} f^w_{\underline{\lambda}}\, s_{(\lambda^1)'}(\widehat{Q}_q)\cdots s_{(\lambda^{q-1})'}(\widehat{Q}_2)\Theta_{\lambda^q}(Q_1-F_n)s_{\lambda^{q+1}}(Q_2)\cdots s_{\lambda^{p+q-1}}(Q_p)\\ &=\sum_{\underline{\lambda}} f^w_{\underline{\lambda}}\, s_{\lambda^1}(F_{n+b_{q-1}}-F_{n+b_q})\cdots\Theta_{\lambda^q}(E-E_p-F_n)\cdots s_{\lambda^{p+q-1}}(E_2-E_1)\end{aligned}$$

in $\mathrm{H}^*(\mathrm{F}^{\mathfrak{a}}(E))$, where the sum is over all sequences of partitions $\underline{\lambda}=(\lambda^1,\ldots,\lambda^{p+q-1})$ with λ^q a_1-strict, and the coefficients $f^w_{\underline{\lambda}}$ are given by (25).

Proof. The variables x_i for $1\leq i\leq n$ give the Chern roots of the various vector bundles over $\mathrm{F}^{\mathfrak{a}}(E)$. In particular the Chern roots of Q_1 are $\mathrm{x}_1,\ldots,\mathrm{x}_n,-\mathrm{x}_1,\ldots,-\mathrm{x}_{a_1}$, while those of Q_r for $r\geq 2$ are $-\mathrm{x}_{a_{r-1}+1},\ldots,-\mathrm{x}_{a_r}$. Similarly the Chern roots of F_{n+1-r} are represented by $-\mathrm{y}_r,\ldots,-\mathrm{y}_n$ for each r. With $k=a_1$ we have $\vartheta_r(X\,;Y_1)=\sum_{i=0}^r q_{r-i}(X)e_i(y_1,\ldots,y_{a_1})$ for any $r\geq 0$. Therefore, we obtain

$$\begin{aligned}\pi_n(\vartheta_r(X\,;Y_1)) &= \sum_{i,j\geq 0} e_{r-i-j}(\mathrm{X})h_j(\mathrm{Y})e_i(-\mathrm{x}_1,\ldots,-\mathrm{x}_{a_1}) \\ &= \sum_{j\geq 0} e_{r-j}(\mathrm{x}_1,\ldots,\mathrm{x}_n,-\mathrm{x}_1,\ldots,-\mathrm{x}_{a_1})h_j(\mathrm{Y}) = c_r(Q_1-F_n)\end{aligned}$$

and hence

$$\pi_n(\Theta_\lambda(X\,;Y_1))=\Theta_\lambda(Q_1-F_n)$$

for any a_1-strict partition λ. Moreover, for any partition μ and $r\geq 2$, we have

$$\pi_n(s_\mu(Y_r))=s_\mu(-\mathrm{x}_{a_{r-1}+1},\ldots,-\mathrm{x}_{a_r})=s_\mu(Q_r),$$

while

$$\pi_n(s_\mu(0/Z_r))=s_{\mu'}(-\mathrm{y}_{b_{r-1}+1},\ldots,-\mathrm{y}_{b_r})=s_{\mu'}(\widehat{Q}_r)=s_\mu(F_{n+b_{r-1}}-F_{n+b_r}).$$

We deduce that π_n maps formula (26) onto the desired equality. $\square$

5. Giambelli formulas for symplectic flag varieties

5.1. **Partial isotropic flag varieties.** Equip the vector space $E=\mathbb{C}^{2n}$ with a nondegenerate skew-symmetric bilinear form. Fix a sequence $\mathfrak{a}\,:\,a_1<\cdots<a_p$ of nonnegative integers with $a_p<n$, and set $a_{p+1}=n$. Let $\mathfrak{X}(\mathfrak{a})$ be the variety parametrizing partial flags of subspaces

$$0=E_0\subset E_1\subset\cdots\subset E_p\subset E$$

with $\dim E_i=n-a_{p+1-i}$ and E_p isotropic. The same notation will be used to denote the tautological partial flag $E_\bullet$ of vector bundles over $\mathfrak{X}(\mathfrak{a})$.

A presentation of the cohomology ring of $\mathfrak{X}(\mathfrak{a})$ as a quotient of the symmetric algebra on the characters of a maximal torus in Sp_{2n} is well known [Bo]. We will give here an alternative presentation using the Chern classes of the tautological vector bundles over $\mathfrak{X}(\mathfrak{a})$. Let $Q_1=E/E_p, Q_2=E_p/E_{p-1},\ldots,Q_{p+1}=E_1$ and set $\sigma_i=c_i(Q_1)$ for $1\leq i\leq n+a_1$ and $c_j^r=c_j(Q_r)$ for $2\leq r\leq p+1$ and $1\leq j\leq a_r-a_{r-1}$. We then have the following result

Proposition 6. *The cohomology ring* $\mathrm{H}^*(\mathfrak{X}(\mathfrak{a}),\mathbb{Z})$ *is presented as a quotient of the polynomial ring* $\mathbb{Z}[\sigma_1,\ldots,\sigma_{n+a_1},c_1^2,\ldots,c_{a_2-a_1}^2,\ldots,c_1^{p+1},\ldots,c_{n-a_p}^{p+1}]$ *modulo the relations*

$$\det(\sigma_{1+j-i})_{1\leq i,j\leq r}=(-1)^r\sum_{i_2+\cdots+i_{p+1}=r} c_{i_2}^2\cdots c_{i_{p+1}}^{p+1}\,,\qquad 1\leq r\leq n+a_1$$

and

$$\sigma_r^2+2\sum_{i=1}^{n+a_1-r}(-1)^i\sigma_{r+i}\sigma_{r-i}=0\,,\qquad a_1+1\leq r\leq n. \tag{31}$$

Proof. Let $\mathrm{IG}=\mathrm{IG}(n-a_1,2n)$ be the Grassmannian parametrizing isotropic subspaces of E of dimension $n-a_1$. According to [BKT1, Thm. 1.2], the cohomology ring of IG is isomorphic to the polynomial ring generated by the Chern classes $\sigma_i=c_i(Q_1)$ and the Chern classes of E_p, modulo the relations

$$\det(\sigma_{1+j-i})_{1\leq i,j\leq r}=(-1)^r c_r(E_p)\,,\qquad 1\leq r\leq n+a_1$$

coming from the Whitney sum formula $c(E_p)c(Q_1) = 1$, as well as the relations (31). The map $\mathfrak{X}(\mathfrak{a}) \to \mathrm{IG}$ sending $E_\bullet$ to E_p realizes $\mathfrak{X}(\mathfrak{a})$ as a fiber bundle over IG with fiber a partial SL_{n-a_1} flag variety. We deduce using e.g. [Gr, Thm. 1] that $\mathrm{H}^*(\mathfrak{X}(\mathfrak{a}))$ is isomorphic to the polynomial ring $\mathrm{H}^*(\mathrm{IG})[c^2_1, \ldots, c^2_{a_2-a_1}, \ldots, c^{p+1}_1, \ldots, c^{p+1}_{n-a_p}]$ modulo the relations

$$\sum_{i_2+\cdots+i_{p+1}=r} c^2_{i_2} \cdots c^{p+1}_{i_{p+1}} = c_r(E_p)\,, \qquad 1 \leq r \leq n - a_1.$$

The proposition follows by combining these two facts. □

5.2. **Giambelli formulas.** Our choice of the special Schubert classes on symplectic and orthogonal Grassmannians agrees with the conventions in [BKT1]. Let $F_\bullet$ be a fixed complete isotropic flag of subspaces in $\mathbb{C}^{2n}$. For each $w \in W^{\mathfrak{a}}$, the Schubert variety $\mathfrak{X}_w(F_\bullet)$ in $\mathfrak{X}(\mathfrak{a})$ is defined by the same equation (29) as before. Let $E'_j = E/E_{p+1-j}$ for $1 \leq j \leq p$. The Chern classes $c_i(E'_j)$ for all i, j are the Schubert classes on $\mathfrak{X}(\mathfrak{a})$ which are pullbacks of special Schubert classes on symplectic Grassmannians. By definition, they are the *special Schubert classes* on $\mathfrak{X}(\mathfrak{a})$, and they generate the cohomology ring $\mathrm{H}^*(\mathfrak{X}(\mathfrak{a}))$ by Proposition 6. Specializing Theorem 3 to the case when the base B is a point gives the next result.

Corollary 2. *For every $w \in W^{\mathfrak{a}}$, we have*

$$\begin{aligned}[\mathfrak{X}_w] &= \sum_{\underline{\lambda}} e^w_{\underline{\lambda}}\, \Theta_{\lambda^1}(Q_1) s_{\lambda^2}(Q_2) \cdots s_{\lambda^p}(Q_p) \\ &= \sum_{\underline{\lambda}} e^w_{\underline{\lambda}}\, \Theta_{\lambda^1}(E'_1) s_{\lambda^2}(E'_2 - E'_1) \cdots s_{\lambda^p}(E'_p - E'_{p-1})\end{aligned}$$

in $\mathrm{H}^*(\mathfrak{X}(\mathfrak{a}), \mathbb{Z})$, where the sums are over all sequences of partitions $\underline{\lambda} = (\lambda^1, \ldots, \lambda^p)$ with λ^1 a_1-strict, and the coefficients $e^w_{\underline{\lambda}}$ are given by (22).

6. Splitting orthogonal Schubert polynomials

In this final section we discuss the form of our splitting results for the orthogonal groups; we will work throughout with coefficients in the ring $\mathbb{Z}[\frac{1}{2}]$. For $w \in W_\infty$, let $s(w)$ denote the number of i such that $w(i) < 0$. It follows e.g. from [BH, IMN] that the polynomials $\mathfrak{B}_w = 2^{-s(w)}\mathfrak{C}_w$ represent the Schubert classes in the (equivariant) cohomology ring of odd orthogonal flag varieties. Therefore the solutions to the Schubert polynomial splitting and Giambelli problems for types B and C are essentially the same. We will describe the splitting theorems for the even orthogonal groups below; the story is entirely analogous to the symplectic case.

The elements of the Weyl group $\widetilde{W}_n$ for the root system D_n may be represented by signed permutations, as in e.g. [B, KT1]. The group $\widetilde{W}_n$ is an extension of S_n by an element s_0 which acts on the right by

$$(u_1, u_2, \ldots, u_n)s_0 = (\overline{u}_2, \overline{u}_1, u_3, \ldots, u_n).$$

Let $\widetilde{W}_\infty = \cup_n \widetilde{W}_n$ and $\mathbb{N} = \{0, 1, 2, \ldots\}$. A *reduced word* of $w \in \widetilde{W}_\infty$ is a sequence $a_1 \cdots a_\ell$ of elements in $\mathbb{N}$ such that $w = s_{a_1} \cdots s_{a_\ell}$ and $\ell = \ell(w)$. We say that w has a *descent* at position $r \geq 0$ if $\ell(ws_r) < \ell(w)$, where s_r is the simple reflection indexed by r. If $k \geq 2$, we say that an element $w \in \widetilde{W}_\infty$ is *increasing up to k* if it has no descents less than k; this means that $|w_1| < w_2 < \cdots < w_k$. We also agree that every element of $\widetilde{W}_\infty$ is both increasing up to 0 and increasing up to 1.

For $k \in \mathbb{N} \smallsetminus \{1\}$, an element $w \in \widetilde{W}_\infty$ is k-Grassmannian if $\ell(ws_i) = \ell(w) + 1$ for all $i \neq k$. We say that w is 1-Grassmannian if $\ell(ws_i) = \ell(w) + 1$ for all $i \geq 2$. A *typed k-strict partition* is a pair consisting of a k-strict partition λ together with an integer in $\{0, 1, 2\}$ called the *type* of λ, and denoted $\mathrm{type}(\lambda)$, such that $\mathrm{type}(\lambda) > 0$ if and only if $\lambda_i = k$ for some $i \geq 1$. The geometric significance of the type of λ is explained in [BKT1, §4.5].

Given a k-Grassmannian element $w \in \widetilde{W}_n$, there exist unique strict partitions u, ζ, v of lengths k, r, and $n - k - r$, respectively, so that

$$w = (\widehat{u}_k, \ldots, u_1, \overline{\zeta}_1, \ldots, \overline{\zeta}_r, v_{n-k-r}, \ldots, v_1)$$

where $\widehat{u}_k$ is equal to u_k or $\overline{u}_k$, according to the parity of r. If

$$\mu_i = u_i + i - k - 1 + \#\{j \mid \zeta_j > u_i\},$$

then w corresponds to a typed k-strict partition λ such that the lengths of the first k columns of λ are given by $\mu_1, \ldots, \mu_k$. The part of λ in columns $k + 1$ and higher is given by $(\zeta_1 - 1, \ldots, \zeta_r - 1)$; here it is possible that $\zeta_r = 1$, so that the sequence ends with a zero. Finally, if $\mathrm{type}(\lambda) > 0$, then $\widehat{u}_k$ is unbarred if and only if $\mathrm{type}(\lambda) = 1$. This defines a bijection between the k-Grassmannian elements of $\widetilde{W}_\infty$ and the set of all typed k-strict partitions. We let w_λ denote the element of $\widetilde{W}_\infty$ associated to the typed k-strict partition λ.

Following [L1], we will use the nilCoxeter algebra $\widetilde{\mathcal{W}}_n$ of $\widetilde{W}_n$ to define type D Stanley symmetric functions. $\widetilde{\mathcal{W}}_n$ is the free associative algebra with unity generated by the elements $u_0, u_1, \ldots, u_{n-1}$ modulo the relations

$$\begin{array}{rcll}
u_i^2 & = & 0 & i \geq 0\ ; \\
u_0u_1 & = & u_1u_0 & \\
u_0u_2u_0 & = & u_2u_0u_2 & \\
u_iu_{i+1}u_i & = & u_{i+1}u_iu_{i+1} & i > 0\ ; \\
u_iu_j & = & u_ju_i & j > i+1, \text{ and } (i,j) \neq (0,2).
\end{array}$$

For any $w \in \widetilde{W}_n$, choose a reduced word $a_1 \cdots a_\ell$ for w and define $u_w = u_{a_1} \ldots u_{a_\ell}$. We denote the coefficient of $u_w \in \widetilde{\mathcal{W}}_n$ in the expansion of the element $f \in \widetilde{\mathcal{W}}_n$ by $\langle f, w\rangle$. Let t be an indeterminate and, following [L1, 4.4], define

$$D(t) = (1 + tu_{n-1}) \cdots (1 + tu_2)(1 + tu_1)(1 + tu_0)(1 + tu_2) \cdots (1 + tu_{n-1}).$$

According to [L1, Lemma 4.24], we have $D(s)D(t) = D(t)D(s)$ for any commuting variables s, t. If $D(X) = D(x_1)D(x_2) \cdots$, then the functions $E_w(X)$ defined by

$$E_w(X) = \langle D(X), w\rangle$$

are the type D Stanley symmetric functions, in agreement with [BH, §3].

Next, define

$$\mathfrak{D}_w(X\,; Y, Z) = \left\langle \tilde{A}_{n-1}(z_{n-1}) \cdots \tilde{A}_1(z_1) D(X) A_1(y_1) \cdots A_{n-1}(y_{n-1}), w \right\rangle.$$

The polynomials $\mathfrak{D}_w(X\,; Y) := \mathfrak{D}_w(X\,; Y, 0)$ are the type D Billey-Haiman Schubert polynomials, and the $\mathfrak{D}_w(X\,; Y, Z)$ are their double versions studied in [IMN]. If $w = w_\lambda$ is k-Grassmannian, then $\mathfrak{D}_{w_\lambda}(X\,; Y)$ is equal to an *eta polynomial* $H_\lambda(X\,; Y)$; the H_λ are defined using raising operator expansions analogous to (3) in [BKT4]. When $k = 0$, we have that λ is a strict partition and $H_\lambda(X\,; Y) = E_{w_\lambda}(X) = P_\lambda(X)$ is a Schur P-function.

Definition 4. Given $w \in \widetilde{W}_n$, the *type D mixed Stanley function* $I_w(X\,;Y)$ is defined by the equation

$$I_w(X\,;Y) = \langle D(X)A(Y), w\rangle = \sum_{uv=w} E_u(X)G_v(Y)$$

summed over all reduced factorizations $uv = w$ with $v \in S_n$.

One checks that if w is increasing up to k, then $I_w^{(k)} = \mathfrak{D}_w^{(k)}(X\,;Y)$; in particular, if $w = w_\lambda$ is k-Grassmannian, then $I_{w_\lambda}^{(k)} = H_\lambda$. Furthermore, it follows from [B, Thms. 4, 5] and the type D analogue of Lemma 3 that the I_w for w increasing up to k satisfy the transition equations

$$I_w^{(k)} = \sum_{\substack{1\leq i<r \\ \ell(wt_{rs}t_{ir})=\ell(w)}} I_{wt_{rs}t_{ir}}^{(k)} + \sum_{\substack{i\neq r \\ \ell(wt_{rs}\bar{t}_{ir})=\ell(w)}} I_{wt_{rs}\bar{t}_{ir}}^{(k)}$$

where $r > k$ is the last positive descent of w and s is maximal such that $w_s < w_r$.

For any $w \in \widetilde{W}_\infty$ which is increasing up to k, we construct the k-transition tree $\widetilde{T}^k(w)$ with nodes given by elements of $\widetilde{W}_\infty$ and root w as in §2.2. Let r be the last descent of w. If $w = 1$, or $k \neq 1$ and $r = k$, or $k = 1$ and $r \in \{0,1\}$, then set $\widetilde{T}^k(w) = \{w\}$. Otherwise, let $s = \max(i > r \mid w_i < w_r)$ and $\widetilde{\Phi}(w) = \widetilde{\Phi}_1(w) \cup \widetilde{\Phi}_2(w)$, where

$$\widetilde{\Phi}_1(w) = \{wt_{rs}t_{ir} \mid 1 \leq i < r \ \text{ and } \ \ell(wt_{rs}t_{ir}) = \ell(w)\},$$
$$\widetilde{\Phi}_2(w) = \{wt_{rs}\bar{t}_{ir} \mid i \neq r \ \text{ and } \ \ell(wt_{rs}\bar{t}_{ir}) = \ell(w)\}.$$

To define $\widetilde{T}^k(w)$, we join w by an edge to each $v \in \widetilde{\Phi}(w)$, and attach to each $v \in \widetilde{\Phi}(w)$ its tree $\widetilde{T}^k(v)$.

The assertions of Lemma 3 remain true for $\widetilde{T}^k(w)$, with similar proof. When $k = 0$, this is contained in [B, Thm. 4]. For the case when $r > k > 1$, $\widetilde{\Phi}_1(w) = \emptyset$, and $w_s > 0$, one observes that $wt_{rs}\bar{t}_{1r} \in \widetilde{\Phi}_2(w)$. We deduce that for any $w \in \widetilde{W}_\infty$ which is increasing up to k,

$$\mathfrak{D}_w^{(k)} = I_w^{(k)} = \sum_{\lambda\,:\,|\lambda|=\ell(w)} d_\lambda^w\, H_\lambda \tag{32}$$

where the sum is over typed k-strict partitions λ and d_λ^w denotes the number of leaves of $\widetilde{T}^k(w)$ of shape λ. Moreover, for any such w, we have

$$\mathfrak{D}_w(X\,;Y) = \sum_{u(1_k\times v)=w} I_u^{(k)}(X\,;Y)\mathfrak{S}_v(Y_{>k}) \tag{33}$$

where the sum is over all reduced factorizations $u(1_k \times v) = w$ with $v \in S_\infty$.

Using equations (32) and (33), we obtain splitting theorems for the single and double type D Schubert polynomials $\mathfrak{D}_w$, as in §3. Fix two sequences $\mathfrak{a} : a_1 < \cdots < a_p$ and $\mathfrak{b} : 0 = b_1 < \cdots < b_q$ of nonnegative integers and set $Y_i = \{y_{a_{i-1}+1}, \ldots, y_{a_i}\}$ and $Z_j = \{z_{b_{j-1}+1}, \ldots, z_{b_j}\}$ for each i, j. We say that an element $w \in \widetilde{W}_\infty$ is *compatible* with $\mathfrak{a}$ if all descent positions of w are listed among $0, a_1, \ldots, a_p$, if $a_1 = 1$, or contained in $\mathfrak{a}$, otherwise.

Theorem 4. *Suppose that $w \in \widetilde{W}_\infty$ is compatible with the sequence $\mathfrak{a}$. Then we have*

$$\mathfrak{D}_w(X\,;Y) = \sum_{u_1\cdots u_p = w} I_{u_1}(X\,;Y_1)G_{u_2}(Y_2)\cdots G_{u_p}(Y_p)$$

summed over all reduced factorizations $u_1\cdots u_p = w$ compatible with $\mathfrak{a}$ such that $u_2,\ldots,u_p \in S_\infty$.

Given any sequence of partitions $\underline{\lambda} = (\lambda^1,\ldots,\lambda^{p+q-1})$ with λ^q a_1-strict and typed, and $w \in \widetilde{W}_\infty$ such that w and w^{-1} are compatible with the sequences $\mathfrak{a}$ and $\mathfrak{b}$, respectively, we define

$$g^w_{\underline{\lambda}} = \sum_{u_1\cdots u_{p+q-1}=w} c^{u_1}_{\lambda^1}\cdots c^{u_{q-1}}_{\lambda^{q-1}} d^{u_q}_{\lambda^q} c^{u_{q+1}}_{\lambda^{q+1}}\cdots c^{u_{p+q-1}}_{\lambda^{p+q-1}}.$$

Here the sum is over reduced factorizations $u_1\cdots u_{p+q-1} = w$ compatible with $\mathfrak{a}$, $\mathfrak{b}$ such that $u_i \in S_\infty$ for all $i \neq q$, and the integers $c^{u_i}_{\lambda^i}$ and $d^{u_q}_{\lambda^q}$ are as in (4) and (32), respectively.

Corollary 3. *Suppose that w and w^{-1} are compatible with the sequences $\mathfrak{a}$ and $\mathfrak{b}$, respectively. Then $\mathfrak{D}_w(X\,;Y,Z)$ is equal to*

$$\sum_{u_1\cdots u_{p+q-1}=w} G_{u_1}(0/Z_q)\cdots G_{u_{q-1}}(0/Z_2)I_{u_q}(X\,;Y_1)G_{u_{q+1}}(Y_2)\cdots G_{u_{p+q-1}}(Y_p)$$

summed over all reduced factorizations $u_1\cdots u_{p+q-1} = w$ compatible with $\mathfrak{a}$, $\mathfrak{b}$ such that $u_i \in S_\infty$ for all $i \neq q$. Furthermore, we have

$$\mathfrak{D}_w(X\,;Y,Z) = \sum_{\underline{\lambda}} g^w_{\underline{\lambda}}\, s_{\lambda^1}(0/Z_q)\cdots H_{\lambda^q}(X\,;Y_1)\cdots s_{\lambda^{p+q-1}}(Y_p) \tag{34}$$

summed over all sequences of partitions $\underline{\lambda} = (\lambda^1,\ldots,\lambda^{p+q-1})$ with λ^q a_1-strict and typed.

In the same manner as for the symplectic groups, the above splitting results imply Giambelli and degeneracy locus formulas for the orthogonal groups. We use the Giambelli formula for even orthogonal Grassmannians from [BKT4] in (34).

Table 1 lists the Billey-Haiman Schubert polynomials for the root systems of type C_3 and D_3 indexed by the elements w in the respective Weyl groups which are increasing up to 1. In each case, the polynomial is written as a positive sum of $k = 1$ theta and eta polynomials in the variables (X, y_1) times $s_j(y_2)$ for $j \in \{0, 1\}$. The primed eta polynomials H'_λ are indexed by 1-strict partitions λ of type 2.

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TABLE 1. Schubert polynomials with w increasing up to 1

w	word	$\mathfrak{C}_w(X\,;Y)$	w	word	$\mathfrak{D}_w(X\,;Y)$
123		1	123		1
213	1	Θ_1	213	1	H_1
132	2	$\Theta_1 + y_2$	$\overline{21}3$	0	H_1'
$2\overline{1}3$	01	Θ_2	132	2	$H_1 + H_1' + y_2$
312	21	$\Theta_{(1,1)}$	$\overline{12}3$	01	H_2
231	12	$\Theta_2 + \Theta_1\, y_2$	312	21	$H_{(1,1)}$
$1\overline{2}3$	101	Θ_3	$\overline{31}2$	20	$H'_{(1,1)}$
$3\overline{1}2$	021	$\Theta_{(2,1)}$	231	12	$H_2 + H_1\, y_2$
321	212	$\Theta_{(2,1)} + \Theta_{(1,1)}\, y_2$	$\overline{2}3\overline{1}$	02	$H_2 + H_1'\, y_2$
$23\overline{1}$	012	$\Theta_3 + \Theta_2\, y_2$	$\overline{13}2$	201	H_3
$1\overline{3}2$	2101	Θ_4	$3\overline{21}$	021	$H_{(2,1)}$
$3\overline{2}1$	1021	$\Theta_{(3,1)}$	$\overline{32}1$	120	$H'_{(2,1)}$
$13\overline{2}$	1012	$\Theta_4 + \Theta_3\, y_2$	$\overline{1}3\overline{2}$	012	$H_3 + H_2\, y_2$
$32\overline{1}$	0212	$\Theta_4 + \Theta_{(3,1)} + \Theta_{(2,1)}\, y_2$	321	212	$H_3 + H_{(2,1)} + H_{(1,1)}\, y_2$
$2\overline{3}1$	21021	$\Theta_{(4,1)}$	$\overline{3}2\overline{1}$	202	$H_3 + H'_{(2,1)} + H'_{(1,1)}\, y_2$
$3\overline{21}$	01021	$\Theta_{(3,2)}$	$2\overline{31}$	2021	$H_{(3,1)}$
$12\overline{3}$	21012	$\Theta_5 + \Theta_4\, y_2$	$\overline{23}1$	2120	$H'_{(3,1)}$
$31\overline{2}$	10212	$\Theta_{(4,1)} + \Theta_{(3,1)}\, y_2$	$\overline{1}2\overline{3}$	2012	$H_4 + H_3\, y_2$
$2\overline{31}$	021021	$\Theta_{(4,2)}$	$3\overline{12}$	0212	$H_{(3,1)} + H_{(2,1)}\, y_2$
$21\overline{3}$	210212	$\Theta_{(5,1)} + \Theta_{(4,1)}\, y_2$	$\overline{3}1\overline{2}$	1202	$H'_{(3,1)} + H'_{(2,1)}\, y_2$
$3\overline{12}$	010212	$\Theta_{(4,2)} + \Theta_{(3,2)}\, y_2$	$1\overline{32}$	12021	$H_{(3,2)}$
$1\overline{32}$	1021021	$\Theta_{(4,3)}$	$2\overline{13}$	20212	$H_{(4,1)} + H_{(3,1)}\, y_2$
$2\overline{13}$	0210212	$\Theta_{(5,2)} + \Theta_{(4,2)}\, y_2$	$\overline{2}1\overline{3}$	21202	$H'_{(4,1)} + H'_{(3,1)}\, y_2$
$1\overline{23}$	10210212	$\Theta_{(5,3)} + \Theta_{(4,3)}\, y_2$	$1\overline{23}$	120212	$H_{(4,2)} + H_{(3,2)}\, y_2$

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