GIAMBELLI AND DEGENERACY LOCUS FORMULAS FOR CLASSICAL G/P SPACES

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Abstract. Let G be a classical complex Lie group, P any parabolic subgroup of G, and $X = G/P$ the corresponding homogeneous space, which parametrizes (isotropic) partial flags of subspaces of a fixed vector space. In the mid 1990s, Fulton, Pragacz, and Ratajski [F3, PR2, FP] asked for global formulas which express the cohomology classes of the universal Schubert varieties in flag bundles – when the space X varies in an algebraic family – in terms of the Chern classes of the vector bundles involved in their definition. This has applications to the theory of degeneracy loci of vector bundles and is closely related to the Giambelli problem for the torus-equivariant cohomology ring of X. In this article, we explain the answer to these questions which was obtained in [T7], in terms of combinatorial data coming from the Weyl group.

0. Introduction

The theory of degeneracy loci of vector bundles has its roots in the 19th century, motivated by questions in elimination theory and enumerative algebraic geometry. The modern subject began with the work of Thom and Porteous in topology, which was generalized and extended to the algebraic setting by Kempf, Laksov, and Lascoux [KL, L1, F5]. The simplest example involves two complex vector bundles E, F on a smooth algebraic variety M. Given a generic map of vector bundles $f : E \to F$ and r any integer, the locus M_r of points $m \in M$ where $\mathrm{rank}(f_m) \leq r$ is called a *degeneracy locus*. Thom [Th] showed that the homology class of M_r must be Poincaré dual to a universal polynomial in the Chern classes of the vector bundles E and F, and Porteous [Po] later found this representing polynomial. Such degeneracy loci arise frequently in problems of algebraic geometry and singularity theory, therefore explicit Chern class formulas for these loci can be quite useful. We refer to [Tu, P3, FP, FR2, Ka2] for surveys, and to [Be, DP, EvG, FR1, F3, FL, HT, JLP, Ka1, KT2, PP, P1, PR2, Sa, SdS, T1, T3] for an incomplete list of applications.

In a series of papers in the 1990s, Fulton [F1, F2, F3] generalized the work of Kempf-Laksov further, to a map of flagged vector bundles, and studied an analogue of the same problem for the other classical Lie groups. This involved degeneracy loci given by incidence relations between a pair of isotropic flags of subbundles of a fixed vector bundle, which is equipped with a symplectic or orthogonal form. In all cases, the *Schubert polynomials* representing the cohomology classes of the loci

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were defined by an algorithm using divided difference operators (stemming from [BGG, D1, D2, LS1]) applied to a 'top polynomial' which represented the class of the diagonal (the locus of points on the base variety where the two flags coincide). Related computations were performed at much the same time by Pragacz and Ratajski [PR2], and other competing theories of Schubert polynomials in the Lie types B, C, and D were discovered [BH, FK3, LP1]. In [F3] and [FP, §9.5], Fulton and Pragacz asked for combinatorially explicit, global formulas for the cohomology classes of degeneracy loci, which have a similar shape for all the classical groups, and are determinantal whenever possible. The aim of this article is to describe the answer to this question which was obtained in [T7], building on a series of earlier works, in terms of data coming from the Weyl group.

Graham [Gr] recast the above degeneracy locus problem using the language of Lie theory, and studied the universal case when the structure group G of the fibre bundles involved is any complex reductive group (see also [Br1, §6.6]). He observed that the degeneracy locus question of [F3] is essentially equivalent to the problem of obtaining a formula for the *equivariant Schubert classes* in the torus-equivariant cohomology ring of the flag variety G/B (when G is a classical group, there is also the twisted case, when the bilinear form takes values in a line bundle). Indeed, from the point of view of a Lie theorist, there seems to be no reason to exclude the exceptional groups from the degeneracy locus story. We will suggest two reasons below why the classical groups appear to be special for this question.

In type A, the double Schubert polynomials of Lascoux and Schützenberger [L2, LS1] were characterized as the unique polynomials that satisfy the general degeneracy locus formula of [F1]. Fomin and Kirillov [FK3] observed that this strong uniqueness property breaks down in type B, where in fact there is a plethora of theories of (single) Schubert polynomials. However, the Schubert polynomials of Billey and Haiman [BH] impressed us as the most combinatorially explicit theory among those available in the other classical Lie types. These polynomials enjoyed most of the properties of the type A single Schubert polynomials, but their translation (as given in [BH]) into Chern class formulas in $\mathrm{H}^*(G/B)$ involved a change of variables and an ensuing loss of combinatorial control. This problem was first addressed by the author [T3, T4], using a more natural and geometric substitution of the variables, with applications to arithmetic intersection theory. Ikeda, Mihalcea, and Naruse [IMN] later introduced double versions of the Billey-Haiman Schubert polynomials and extended the substitution of [T3, T4] to this setting – expressing it in a better way, as a ring homomorphism (the *geometrization maps* of §7.3). With this work, the search for a satisfactory analogue of the Lascoux-Schützenberger theory in the other classical Lie types was finally over.

Although the decision of which theory of Schubert polynomials to use is clearly important, by construction they only provide formulas in terms of the *Chern roots* of the vector bundles involved. When the initial degeneracy locus problem carries the symmetries of a parabolic subgroup P of G, we seek an answer which manifestly exhibits the same symmetries. This should generalize the Jacobi-Trudi determinants and Schur Pfaffians that appear when the Schubert polynomials are evaluated on (maximal) Grassmannian elements of the Weyl group, as in [KL, F3, PR2, FP, KT1]. In other words, we desire formulas that are *native to* G/P, i.e., expressed in terms of Schubert classes that live in the cohomology ring of the homogeneous space G/P. It turned out that a precise understanding of the

Giambelli problem for $\mathrm{H}^*(G/P)$, which is closely related to the degeneracy locus formulas above, was necessary for further progress.

The cohomology of $X = G/P$ is a free abelian group on the basis of Schubert classes, the cohomology classes of the Schubert varieties. When G is a classical Lie group, there are certain *special Schubert classes* among these, which generate the ring $\mathrm{H}^*(X)$. This is one place where the fact that G is classical is important: at present, we do not know how to define special classes for the exceptional groups. For classical G, one has a good definition of special Schubert varieties, which is uniform across the four types. In this case, the variety X parametrizes partial flags of subspaces of a vector space, which in types B, C, and D are required to be isotropic with respect to an orthogonal or symplectic form. If X is an (isotropic) Grassmannian, then the special Schubert varieties are defined as the locus of (isotropic) linear subspaces which meet a given (isotropic or coisotropic) linear subspace nontrivially, following [Pi, BKT1]. The special Schubert varieties on any partial flag variety X are the inverse images of the special Schubert varieties on the Grassmannians to which X projects. The special Schubert classes are the cohomology classes of the special Schubert varieties; in most examples, they are equal to the Chern classes of the universal *quotient* bundles over X, up to a factor of two.

The Giambelli problem for $\mathrm{H}^*(X)$ challenges us to write a general Schubert class as an explicit polynomial in the above special classes. The papers [G1, BKT2, BKT4] addressed this question for all (isotropic) Grassmannians, and [BKTY1, T7] extended the answer to any classical G/P space. To do this, we had to go beyond the known hermitian symmetric, fully commutative examples, and invent a considerable body of new combinatorics. The Schubert classes are indexed by (typed) *k-strict partitions*, the Giambelli formulas are expressed using Young's *raising operators* [Y, Li] and studied using a new calculus of these operators [BKT2, T5], the Schur polynomials are extended to *theta* and *eta polynomials*, and instead of Young tableaux, we count paths in *k-transition trees*. Ultimately, all of these objects can be understood purely in terms of the combinatorics of the Weyl group of (signed) permutations.

The degeneracy locus problem is equivalent to the Giambelli problem when the space X varies in an algebraic family, and thus would appear to be more difficult. Indeed, in most cases where determinantal formulas for the double Schubert polynomials representing the loci were known, these formulas were significantly more complicated than their single versions – which address the Giambelli problem in that case. The type A paper [BKTY1] changed that paradigm: it established the surprising fact that if one uses the language of *quiver polynomials*, then the answer to the degeneracy locus problem has the same shape as that for the Giambelli problem, and indeed, a near identical proof! This picture was generalized to all classical types in [T7], in a synthesis which used all of the above ingredients, and added some new ones. The results were combinatorial *splitting formulas* for the Schubert polynomials of [BH, IMN], and direct translations of these into degeneracy locus formulas, with the symmetries native to the appropriate G/P space.

The goal of this paper is to explain the above story. The narrative combines elements from algebraic geometry, Lie theory, and combinatorics, and we have strived to keep the exposition as self-contained as possible. We include one original contribution: a new proof of the main result of [IMN], which states that the double Schubert polynomials in types B, C, and D represent the Schubert classes. The setup in [IMN] uses localization in equivariant cohomology, which we do not require here. The key idea – exploited in [T5, T7, T8] – is to use the elegant approach to Schubert polynomials via the nilCoxeter algebra and the Yang-Baxter equation, pioneered in [FS, FK2, FK3]. One of the advantages of this approach is that the Schubert polynomials are defined simply and directly in terms of reduced decompositions in the Weyl group, without requiring the use of a 'top polynomial'. From this point of view, one can also understand why the *stability* property of Schubert polynomials is needed: it is only in the stable equivariant cohomology ring that compatibility with divided differences alone (both left and right! – an important insight of [IMN]) is enough to characterize the universal Schubert classes, up to a scalar factor. Anderson and Fulton [AF] have recently also given a different proof of the main theorem of [IMN], within the framework of degeneracy loci, using a geometric argument which employs Kazarian's multi-Schur Pfaffians [Ka1].

We have made no attempt to write a survey, and in particular the extensive literature on the Schubert calculus and the equivariant cohomology of homogeneous spaces is barely touched upon. In special cases, there are alternatives to the combinatorial formulas shown here; the reader may consult [AF, Ar, Bi2, BJS, BKT2, BKT4, FP, I, IN1, Ka1, KT1, LRS, L4, Mi, T5, T8, TW] for examples of what is known, and the papers [Br1, BKT1, BKTY1, IMN, T5, T7] for further references to related research. Throughout this article, we work with cohomology groups, at times with rational coefficients. However, from these, one can deduce results for cohomology with integer coefficients, and also in the algebraic category, for the Chow groups of algebraic cycles modulo rational equivalence. The necessary modifications to achieve this are explained in detail in [Br1, EGr, F3, Gr].

This article is organized as follows. We begin in §1 and §2 with a discussion of Giambelli formulas for Grassmannians, expressing them using the language of raising operators. Section 3 contains general facts about the cohomology of G/P spaces and the Giambelli problem in this context. The combinatorial data coming from the Weyl group and the algebraic objects necessary to state the general degeneracy locus formulas are given in §4 and §5, respectively. In particular, §5.3 contains splitting formulas for Schubert polynomials, which admit direct translations in §6 to Chern class formulas for degeneracy loci. Section 7 outlines the proofs of the main theorems, and §8 contains some questions for the future.

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1. The Giambelli formula of classical Schubert calculus

The main object of study in classical Schubert calculus is the Grassmannian $X = \mathrm{G}(m, n)$, which is the set of all m-dimensional complex linear subspaces of $V = \mathbb{C}^n$. Given any subset H of V, we let $\langle H\rangle$ denote the $\mathbb{C}$-linear span of H. Let $e_1, \ldots, e_n$ denote the canonical basis of $\mathbb{C}^n$, and $d = n - m$ be the codimension of the subspaces in X. The general linear group $\mathrm{GL}_n(\mathbb{C})$ acts transitively on X, and the stabilizer of the point $\langle e_1, \ldots, e_m\rangle$ under this action can be identified with the subgroup P of matrices in $\mathrm{GL}_n(\mathbb{C})$ of the block form

$$\left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right),$$

where the 0 in the lower left corner denotes a $d \times m$ zero matrix. In this way we get a description of X as a coset space

$$X = \mathrm{GL}_n(\mathbb{C})/P$$

from which one can deduce that X is a complex manifold of dimension md. The subgroup P is a *maximal parabolic subgroup* of $\mathrm{GL}_n(\mathbb{C})$. A similar analysis shows that the manifold X is isomorphic to $\mathrm{U}(n)/(\mathrm{U}(m)\times\mathrm{U}(d))$, and hence is a compact manifold. In fact, X is a projective algebraic variety, and may be described by a system of quadratic polynomial equations, known as the Plücker relations. For further details on this and other aspects of this section, we refer to [F4, Ma].

In the latter half of the 19th century, Hermann Schubert gave a first systematic treatment of enumerative projective geometry [Sc1], in which the Grassmannian X played a prominent part. The course of his study led him to introduce certain natural closed algebraic subsets of X, later known as the Schubert varieties [Sc2]. To define them, set $F_i = \langle e_1, \ldots, e_i\rangle$ for each integer $i \in [1, n]$, and consider the complete flag of subspaces

$$F_\bullet \ : \ 0 = F_0 \subset F_1 \subset \cdots \subset F_n = V.$$

The stabilizer $B \subset \mathrm{GL}_n(\mathbb{C})$ of $F_\bullet$ is the *Borel subgroup* of upper triangular matrices in $\mathrm{GL}_n(\mathbb{C})$. In modern language, the Schubert varieties are the closures of the B-orbits in X. Each B-orbit in X is called a Schubert cell; there are finitely many such cells, and they induce a cell decomposition of the manifold X.

We call a subset $\mathcal{P} \subset [1, n]$ of cardinality m an *index set*. Any point $\Sigma \in X$ defines an index set $\mathcal{P}(\Sigma)$ by

$$\mathcal{P}(\Sigma) := \{p \in [1, n] \mid \Sigma \cap F_p \supsetneq \Sigma \cap F_{p-1}\}.$$

Observe that $\mathcal{P}(\Sigma') = \mathcal{P}(\Sigma)$ for any point Σ' in the orbit $B.\Sigma \subset X$. On the other hand, given any subspace $\Sigma \subset V$, one can easily construct a basis $\{g_1, \ldots, g_n\}$ of V such that $F_i = \langle g_1, \ldots, g_i\rangle$ for each i and $\Sigma = \langle \{g_1, \ldots, g_n\} \cap \Sigma\rangle$. It follows from this that any point $\Sigma' \in X$ such that $\mathcal{P}(\Sigma') = \mathcal{P}(\Sigma)$ must be in the orbit $B.\Sigma$. In other words, the B-orbits (or Schubert cells) in X correspond 1-1 to the index sets $\mathcal{P}$. We let $X^\circ_{\mathcal{P}}(F_\bullet)$ denote the Schubert cell given by $\mathcal{P}$, that is,

$$X^\circ_{\mathcal{P}}(F_\bullet) := \{\Sigma \in X \mid \mathcal{P}(\Sigma) = \mathcal{P}\}. \tag{1}$$

The definition implies that we have a cell decomposition

$$\mathrm{G}(m, n) = \coprod_{\mathcal{P}} X^\circ_{\mathcal{P}}(F_\bullet).$$

Suppose that $\mathcal{P} = \{p_1 < \cdots < p_m\}$ is an index set. Any subspace $\Sigma \in X^\circ_{\mathcal{P}}(F_\bullet)$ is spanned by the rows of a unique $m \times n$ matrix $A = \{a_{ij}\}$ in a special reduced row echelon form: there is a pivot entry 1 in position (i, p_i), all other entries in the ith row after the pivot are zero, and all entries below the pivot entries are zero. If $j < p_i$ and $j \neq p_r$ for all $r < i$, then a_{ij} is a free variable, and gives an affine coordinate for the Schubert cell $X^\circ_{\mathcal{P}}$. For example, if $m=4$, $n=10$, and $\mathcal{P}=\{3,5,6,9\}$, then

$$A=\begin{pmatrix} * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & 1 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & 0 & 1 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & 0 & 0 & * & * & 1 & 0 \end{pmatrix}.$$

From this description we see that the B-orbit $X^\circ_{\mathcal{P}}$ is isomorphic to affine space $\mathbb{C}^{|\mathcal{P}|}$, where $|\mathcal{P}|$ is the number of $*$'s in the reduced row echelon form of matrices in the cell, namely

$$|\mathcal{P}|=\sum_{j=1}^m (p_j-j).$$

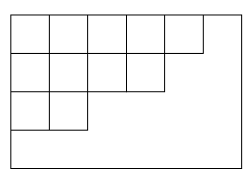
At this point it is convenient to introduce a different parametrization for the Schubert cells which makes their *codimension* apparent: let

$$\lambda_j := d+j-p_j, \quad 1\leq j\leq m.$$

It is clear that there is a 1-1 correspondence between the vectors $\lambda=(\lambda_1,\ldots,\lambda_m)$ and index sets $\mathcal{P}$ (for fixed m and n); for example $\mathcal{P}=\{3,5,6,9\}$ corresponds to $\lambda=(4,3,3,1)$. The conditions on the index set $\mathcal{P}$ imply that

$$d\geq\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_m\geq 0,$$

equivalently, that $\lambda=(\lambda_1,\ldots,\lambda_m)$ is a *partition* whose first part λ_1 is at most d and number of nonzero parts λ_j is at most m. Recall that any partition λ can be represented by a Young diagram of boxes, arranged in left-justified rows, with λ_j boxes in the jth row. The above conditions state that the diagram of λ is contained in an $m\times d$ rectangle, which is the Young diagram of the partition $(d^m)=(d,\ldots,d)$. The example shown below corresponds to a Schubert cell in $\mathrm{G}(4,10)$ indexed by the partition $\lambda=(5,4,2)$.



We identify a partition with its Young diagram; an inclusion $\lambda\subset\mu$ of partitions corresponds to the containment of their respective diagrams. The *weight* of λ, denoted $|\lambda|$, is the total number of boxes in λ, hence $|\lambda|=\sum_{j=1}^m\lambda_j$, and λ is a partition of the integer $|\lambda|$. For each λ as above, we have a Schubert cell X°_λ, which is equal to $X^\circ_{\mathcal{P}}$ for the index set $\mathcal{P}$ corresponding to λ. X°_λ has (complex) dimension

$$\sum_{j=1}^m(d-\lambda_j)=md-|\lambda|=\dim X-|\lambda|,$$

and therefore codimension $|\lambda|$ in X.

The Schubert variety $X_\lambda(F_\bullet)$ is the closure of the Schubert cell $X^\circ_\lambda(F_\bullet)$; it is an algebraic variety also of codimension $|\lambda|$ in X. We have

$$X_\lambda(F_\bullet)=\coprod_{\mu\supset\lambda}X^\circ_\mu(F_\bullet)=\{\Sigma\in X\ |\ \dim(\Sigma\cap F_{d+j-\lambda_j})\geq j,\ 1\leq j\leq m\}.$$

For each partition λ contained in (d^m), let $[X_\lambda]\in\mathrm{H}^{2|\lambda|}(X,\mathbb{Z})$ denote the cohomology class Poincaré dual to the cycle defined by $X_\lambda(F_\bullet)$. If $F'_\bullet$ is another complete flag, then there is an element g in $\mathrm{GL}_n(\mathbb{C})$ such that $g \cdot F_\bullet = F'_\bullet$. It follows that $[X_\lambda(F_\bullet)] = [X_\lambda(F'_\bullet)]$, and therefore that the *Schubert class* $[X_\lambda]$ only depends on the partition λ, and not on the flag $F_\bullet$. The cell decomposition of X implies that the classes of the Schubert varieties give a $\mathbb{Z}$-basis for $\mathrm{H}^*(X,\mathbb{Z})$. In other words, there is a direct sum decomposition

$$\mathrm{H}^*(X,\mathbb{Z}) = \bigoplus_{\lambda\subset(d^m)} \mathbb{Z}\,[X_\lambda].$$

Of course, the cohomology $\mathrm{H}^*(X,\mathbb{Z})$ is also a commutative ring under the cup product, and dually under the intersection product of homology cycles. It follows that the structure of this ring is determined by intersecting Schubert varieties in general position. The simplest such varieties are the *special Schubert varieties*

$$X_r(F_\bullet) = \{\Sigma \in X \mid \Sigma \cap F_{d+1-r} \neq 0\}$$

for $1 \leq r \leq d$ (here the index r is identified with the partition $(r, 0, \ldots, 0)$). Historically, it was natural to focus on the X_r since these spaces are the easiest to work with geometrically. The corresponding classes $[X_r]$ are the *special Schubert classes.* These cohomology classes can be realized as characteristic classes of certain universal vector bundles over $\mathrm{G}(m,n)$. Let E' denote the tautological rank m vector bundle over X, E the trivial rank n vector bundle, and $E'' = E/E'$ the rank d quotient bundle, so that we have a short exact sequence

$$0 \to E' \to E \to E'' \to 0 \tag{2}$$

of vector bundles over X. Then $[X_r]$ is by definition the r-th *Segre class* of E', or equivalently, the r-th *Chern class* of E'', denoted $c_r(E'')$.

The work of Pieri [Pi] and Giambelli [G1] established that the special classes $c_r = c_r(E'')$ generate the cohomology ring $\mathrm{H}^*(X,\mathbb{Z})$. Giambelli proved the following explicit formula which writes a general Schubert class $[X_\lambda]$ as a polynomial in special classes:

$$[X_\lambda] = \det(c_{\lambda_i+j-i}(E''))_{1\leq i,j\leq m}. \tag{3}$$

In equation (3) and in the remainder of this paper, our convention is that $c_0 = 1$ and $c_r = 0$ whenever $r < 0$. Observe that there are relations among the c_r in $\mathrm{H}^*(X,\mathbb{Z})$, so that the right hand side of formula (3) is not unique. However, the natural inclusion $\mathrm{G}(m,n) \hookrightarrow \mathrm{G}(m+1,n+1)$ induces a surjection

$$\mathrm{H}^*(\mathrm{G}(m+1,n+1),\mathbb{Z}) \to \mathrm{H}^*(\mathrm{G}(m,n),\mathbb{Z}).$$

For a fixed partition λ and codimension d, the Giambelli polynomial in (3) is the unique one that is preserved under the above map, for all m greater than or equal to the number of (nonzero) parts of λ.

For our purposes here it will be important to rewrite formula (3) using A. Young's raising operators [Y]. An *integer sequence* is a sequence of integers $\alpha = (\alpha_1, \alpha_2, \ldots)$ only finitely many of which are nonzero. Given any integer sequence α and natural numbers $i < j$, we define

$$R_{ij}(\alpha) := (\alpha_1, \ldots, \alpha_i + 1, \ldots, \alpha_j - 1, \ldots).$$

A *raising operator* R is any monomial in these R_{ij}'s. If $(c_1, c_2, \ldots)$ is any ordered set of commuting independent variables, we let $c_\alpha := \prod_{i\geq 1} c_{\alpha_i}$, with the understanding that $c_0 = 1$ and $c_r = 0$ if $r < 0$. For any raising operator R, set $R\,c_\alpha := c_{R\alpha}$ (note that we slightly abuse the notation here and consider that the raising operator R

acts on the index α, and not on the monomial c_α itself). Consider the raising operator expression

$$R^0 := \prod_{i<j}(1-R_{ij}).$$

If we expand the infinite product R^0 as a formal power series in the R_{ij} and apply the result to c_α, only finitely many of the summands are nonzero. Therefore $R^0\, c_\alpha$ is a well defined polynomial in the variables c_r, and in fact we have

$$R^0\, c_\alpha = \det(c_{\alpha_i+j-i})_{i,j}. \tag{4}$$

Equation (4) is a formal consequence of the Vandermonde identity

$$\prod_{1\leq i<j\leq m}(\mathrm{x}_i-\mathrm{x}_j) = \det(\mathrm{x}_i^{m-j})_{1\leq i,j\leq m}\ ;$$

for a proof of this see for example [T6]. It follows that we may rewrite (3) as

$$[X_\lambda] = R^0\, c_\lambda(E''), \tag{5}$$

where $c_\lambda(E'') = \prod_i c_{\lambda_i}(E'')$ denotes a monomial in the Chern classes of E'', and R^0 is applied to c_λ as above.

Example 1. We have

$$\begin{gathered}
[X_{(5,4,2)}] = (1-R_{12})(1-R_{13})(1-R_{23})\, c_{(5,4,2)} \\
= (1-R_{12}-R_{13}-R_{23}+R_{12}R_{13}+R_{12}R_{23}+R_{13}R_{23}-R_{12}R_{13}R_{23})\, c_{(5,4,2)} \\
= c_{(5,4,2)} - c_{(6,3,2)} - c_{(6,4,1)} - c_{(5,5,1)} + c_{(7,3,1)} + c_{(6,4,1)} + c_{(6,5,0)} - c_{(7,4,0)} \\
= c_5c_4c_2 - c_6c_3c_2 - c_5^2c_1 + c_7c_3c_1 + c_6c_5 - c_7c_4 = \begin{vmatrix} c_5 & c_6 & c_7 \\ c_3 & c_4 & c_5 \\ 1 & c_1 & c_2 \end{vmatrix}.
\end{gathered}$$

Soon after he proved (3), Giambelli published a second paper [G2] where he studied a parallel formalism in the theory of symmetric polynomials. For any integer r, let $e_r(Y_{(d)})$ denote the r-th elementary symmetric polynomial in the commuting variables $Y_{(d)} = (y_1,\ldots,y_d)$. Given a partition μ with at most d nonzero parts, consider

$$s_\mu(Y_{(d)}) = \det(y_i^{\mu_i+d-j})_{1\leq i,j\leq d}\Big/ \det(y_i^{d-j})_{1\leq i,j\leq d}. \tag{6}$$

The $s_\mu(Y_{(d)})$ for varying μ may be identified with the polynomial characters of the general linear group $\mathrm{GL}_d(\mathbb{C})$; this had been established a few years earlier by Schur in his 1901 thesis [S1] (in fact, equation (6) is a special case of the Weyl character formula). For any partition λ, let $\widetilde{\lambda}$ be the conjugate partition, whose Young diagram is the transpose of the diagram of λ. Then Jacobi [J] and Trudi proved that the *Schur polynomial* $s_{\widetilde{\lambda}}(Y_{(d)})$ satisfies

$$s_{\widetilde{\lambda}}(Y_{(d)}) = R^0\, e_\lambda(Y_{(d)}) = \det(e_{\lambda_i+j-i}(Y_{(d)}))_{i,j} \tag{7}$$

for any $\lambda \subset (d^m)$, where $e_\lambda := \prod_i e_{\lambda_i}$. We may thus consider $s_{\widetilde{\lambda}}$ as a polynomial in the algebraically independent variables e_r, for $1\leq r\leq d$. In the theory of characteristic classes, the variables $y_1,\ldots,y_d$ represent the Chern roots of the quotient vector bundle E'', and $e_r(Y_{(d)})$ is identified with the r-th Chern class $c_r(E'')$. Using $c(E'')$ to denote the total Chern class $1+c_1(E'')+\cdots+c_d(E'')$ of E'', we obtain the following restatement of equations (3) and (5).

Theorem 1 (Classical Giambelli, [G1]). *For any partition λ whose diagram fits inside an $m \times (n-m)$ rectangle, we have*

$$[X_\lambda] = s_{\widetilde{\lambda}}(c(E'')) \tag{8}$$

in the cohomology ring of $\mathrm{G}(m, n)$.

For more on the connection between the representation theory of the general linear group and the classical Schubert calculus, see [Bel, BK, T2].

2. Giambelli formulas for isotropic Grassmannians

The study of homogeneous spaces of Lie groups was extended further during the first half of the twentieth century by the work of Élie Cartan [C1, C2] and Ehresmann [E]. They considered the irreducible compact hermitian symmetric spaces, which generalize the Grassmannian $\mathrm{G}(m, n)$, and began exploring their cohomology rings. Rather than proceeding along the lines of the classical Schubert calculus, this work used Cartan's theory of invariant differential forms. It was only in the 1980s that analogues of Pieri's rule and Giambelli's formula were obtained for all hermitian symmetric Grassmannians, in the work of Hiller and Boe [HB] and Pragacz [P2]. More recently, Pragacz and Ratajski [PR1, PR3] proved Pieri type rules and Buch, Kresch, and the author [BKT1, BKT2, BKT4] generalized both the Pieri and Giambelli formulas of [HB, P2] to arbitrary symplectic and orthogonal Grassmannians, using different notions of special Schubert classes. We will follow the references [BKT2, BKT4] in this section.

Let $V = \mathbb{C}^N$ and equip V with a nondegenerate skew-symmetric or symmetric bilinear form $(\ , \ )$. A subspace Σ of V is called *isotropic* if the restriction of $(\ , \ )$ to Σ vanishes identically. Since the form is nondegenerate, the dimension of any isotropic subspace is at most $N/2$. Given a nonnegative integer $m \leq N/2$, we let X denote the complex manifold which parametrizes all the isotropic subspaces of dimension m in V. This space has a transitive action of the group $G = \mathrm{Sp}(V)$ or $G = \mathrm{SO}(V)$ of linear automorphisms preserving the form on V, unless $m = N/2$ and the form is symmetric. In the latter case the space of isotropic subspaces has two isomorphic connected components, each a single $\mathrm{SO}(V)$ orbit.

An *isotropic flag* $F_\bullet$ is a complete flag

$$0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_N = V$$

of subspaces of V such that $F_i = F_j^\perp$ whenever $i + j = N$; in particular, F_i is an isotropic subspace for all $i \leq N/2$. Let $B \subset G$ denote the Borel subgroup which is the stabilizer of the flag $F_\bullet$. The Schubert cells in X relative to the flag $F_\bullet$ are the orbit closures for the natural action of B on X. We call a subset $\mathcal{P}$ of $[1, N]$ of cardinality m an *index set* if for all $i, j \in \mathcal{P}$ we have $i + j \neq N + 1$. A point Σ in X defines an index set $\mathcal{P}(\Sigma)$ by the prescription

$$\mathcal{P}(\Sigma) := \{p \in [1, N] \mid \Sigma \cap F_p \supsetneq \Sigma \cap F_{p-1}\},$$

since no vector in $F_j \smallsetminus F_{j-1}$ is orthogonal to a vector in $F_{N+1-j} \smallsetminus F_{N-j}$, for each j. In the same manner as in §1, equation (1) establishes a one to one correspondence between Schubert cells $X^\circ_{\mathcal{P}}(F_\bullet)$ relative to $F_\bullet$ and index sets $\mathcal{P}$.

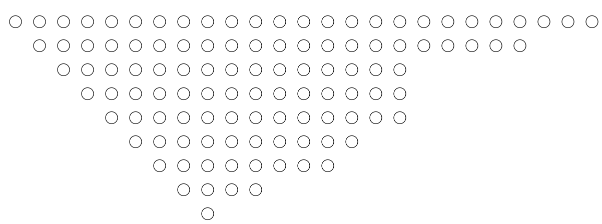
The closures of the Schubert cells are the Schubert varieties $X_{\mathcal{P}}(F_\bullet)$, and their classes $[X_{\mathcal{P}}]$ in $\mathrm{H}^*(X, \mathbb{Z})$ are the Schubert classes, which form an additive basis of $\mathrm{H}^*(X, \mathbb{Z})$. Moreover, there are special Schubert varieties, consisting of the locus of subspaces Σ in X which meet a given subspace F_j non-trivially, and corresponding special Schubert classes, which generate the cohomology ring of X. In the following sections, we will see that the Schubert varieties and classes may equivalently be indexed by *k-strict partitions* and *typed k-strict partitions*. As in §1, this is a convention which makes their codimension (or cohomological degree) apparent, and we will require in order to state the Giambelli formulas of this section.

2.1. **Symplectic Grassmannians.** Suppose that $N = 2n$ is even and the form $(\ ,\ )$ is skew-symmetric, so that $X = \mathrm{IG}(m, 2n)$ is a *symplectic Grassmannian*. Write $m = n-k$ for some k with $0 \leq k \leq n-1$. If $k = 0$, then X is the Lagrangian Grassmannian $\mathrm{LG}(n, 2n)$, and if $k = n-1$, then X is projective $2n-1$ space $\mathbb{P}^{2n-1}$, since every line through the origin in V is isotropic. These are the only hermitian symmetric examples. The space $\mathrm{IG}(n-k, 2n)$ may be identified with a quotient $\mathrm{Sp}_{2n}(\mathbb{C})/P_k$ of the symplectic group $\mathrm{Sp}_{2n}(\mathbb{C})$ by a maximal parabolic subgroup P_k. For instance, the subgroup P_0 is known as the *Siegel parabolic* and consists of those matrices of the symplectic group whose lower left quadrant is an $n \times n$ zero matrix (when we choose the standard symplectic basis for $\mathbb{C}^{2n}$, as in §7.3).

Consider the infinite set of pairs

$$\Delta^\circ = \{(i,j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq i < j\}$$

and define a partial order on Δ° by agreeing that $(i',j') \leq (i,j)$ if $i' \leq i$ and $j' \leq j$. A subset D of Δ° is an order ideal if $(i,j) \in D$ implies $(i',j') \in D$ for all $(i',j') \in \Delta^\circ$ with $(i',j') \leq (i,j)$. In the next figure, the pairs (i,j) in a typical finite order ideal are displayed as positions in a matrix above the main diagonal.



A partition λ is *k-strict* if no part λ_j greater than k is repeated; if $k = 0$ this means that λ is a strict partition, i.e., has distinct nonzero parts. To any k-strict partition λ we associate the order ideal

$$\mathcal{C}(\lambda) := \{(i,j) \in \Delta^\circ \mid \lambda_i + \lambda_j > 2k + j - i\}.$$

The set $\mathcal{C}(\lambda)$ is an important invariant of the partition λ which appears in the definitions of both the index set $\mathcal{P}(\lambda)$ and the raising operator expression R^λ associated to λ (equations (9) and (12) below; compare also with (33)).

The Schubert varieties on $\mathrm{IG}(n-k, 2n)$ are indexed by k-strict partitions whose diagrams fit in an $(n-k) \times (n+k)$ rectangle. Any such λ corresponds to an index set

$$\mathcal{P}(\lambda) = \{p_1(\lambda) < \cdots < p_m(\lambda)\}$$

given by the prescription

$$p_j(\lambda) := n + k + j - \lambda_j - \#\{i < j \mid (i,j) \in \mathcal{C}(\lambda)\}. \tag{9}$$

The reader is invited to show that equation (9) gives a bijection between index sets and k-strict partitions as claimed (a proof is provided in [BKT1, §4.1]). If $F_\bullet$ is a fixed isotropic flag of subspaces in V, we obtain the Schubert cell

$$X^\circ_\lambda(F_\bullet) := \{\Sigma \in X \mid \mathcal{P}(\Sigma) = \mathcal{P}(\lambda)\} \tag{10}$$

and the Schubert variety $X_\lambda(F_\bullet)$ is the closure of this cell. One can show that

$$X_\lambda(F_\bullet) = \{\Sigma \in X \mid \dim(\Sigma \cap F_{p_j(\lambda)}) \geq j \quad \forall\, 1 \leq j \leq \ell(\lambda)\}\,, \tag{11}$$

where the *length* $\ell(\lambda)$ is the number of (nonzero) parts of λ; see for example [BKT1, §4.2] and [BKT4, App. A]. This variety has codimension $|\lambda|$ and defines, using Poincaré duality, a Schubert class $[X_\lambda]$ in $\mathrm{H}^{2|\lambda|}(\mathrm{IG},\mathbb{Z})$.

The *special Schubert varieties* are given by

$$X_r(F_\bullet) = \{\Sigma \in X \mid \Sigma \cap F_{n+k+1-r} \neq 0\}$$

for $1 \leq r \leq n+k$, and their classes $[X_r]$ are the *special Schubert classes*. Let E' denote the tautological rank $(n-k)$ vector bundle over $\mathrm{IG}(n-k,2n)$, E the trivial rank $2n$ vector bundle, and $E'' = E/E'$ the quotient bundle. As in the example of the type A Grassmannian in §1, we have $[X_r] = c_r(E'')$ for $1 \leq r \leq n+k$.

For any k-strict partition λ, we define the operator

$$R^\lambda := \prod_{i<j}(1-R_{ij}) \prod_{(i,j)\in\mathcal{C}(\lambda)} (1+R_{ij})^{-1}, \tag{12}$$

where the first product is over all pairs $i<j$ and second product is over pairs $i<j$ such that $\lambda_i+\lambda_j > 2k+j-i$. Note that equation (12) is a multiplicative analogue of equation (9). If $c = 1 + c_1t + c_2t^2 + \cdots$ is any formal power series in commuting variables c_r, we define the *theta polynomial* Θ_λ by

$$\Theta_\lambda(c) := R^\lambda\, c_\lambda. \tag{13}$$

Theorem 2 (Giambelli for IG, [BKT2])**.** *For any k-strict partition λ whose diagram fits inside an $(n-k)\times(n+k)$ rectangle, we have*

$$[X_\lambda] = \Theta_\lambda(c(E'')) \tag{14}$$

in the cohomology ring of $\mathrm{IG}(n-k,2n)$.

Example 2. Let $k=1$ and λ be the 1-strict partition $(3,1,1)$, with $\mathcal{C}(\lambda) = \{(1,2)\}$. Then the following computation holds in the ring $\mathrm{H}^*(\mathrm{IG}(4,10),\mathbb{Z})$:

$$[X_{(3,1,1)}] = \frac{1-R_{12}}{1+R_{12}}(1-R_{13})(1-R_{23})\, c_{(3,1,1)} = (1-2R_{12}+2R_{12}^2)(1-R_{13}-R_{23})\, c_{(3,1,1)}$$

$$= c_{(3,1,1)} - 2c_{(4,0,1)} - c_{(4,1)} + 2c_5 - c_{(3,2)} + 2c_{(4,1)} - 2c_5 = c_3c_1^2 - c_4c_1 - c_3c_2.$$

Comparing (14) with (5) and (8), we see that the polynomials Θ_λ play the role of the Schur polynomials for the Giambelli problem on $\mathrm{IG}(n-k,2n)$. An important difference with the story for the type A Grassmannian is that there are *relations* among the c_r which persist even as $n\to\infty$, namely:

$$\frac{1-R_{12}}{1+R_{12}}\, c_{(r,r)} = c_r^2 + 2\sum_{i=1}^{r}(-1)^i c_{r+i}c_{r-i} = 0, \quad \text{for all } r > k. \tag{15}$$

Another difference is that the raising operator expressions R^λ which enter in (13) depend on the partition λ (see also Example 3 below).

We next specialize the above to the Lagrangian Grassmannian $\mathrm{LG}(n,2n)$, which is the case where $k=0$. The Schubert classes in $\mathrm{H}^*(\mathrm{LG}(n,2n),\mathbb{Z})$ are indexed by strict partitions λ whose diagrams fit inside a square of side n. The theta polynomial (13) specializes to a *Q-polynomial*

$$Q_\lambda(c) := \prod_{i<j}\frac{1-R_{ij}}{1+R_{ij}}\, c_\lambda. \tag{16}$$

The Giambelli formula of Theorem 2 for LG becomes

$$[X_\lambda] = Q_\lambda(c(E'')). \tag{17}$$

Following Pragacz [P2], formula (17) may be expressed using a Schur Pfaffian, as follows. For partitions $\lambda = (a, b)$ with only two parts, we have

$$[X_{(a,b)}] = \frac{1-R_{12}}{1+R_{12}}\, c_{(a,b)} = c_a c_b - 2c_{a+1}c_{b-1} + 2c_{a+2}c_{b-2} - \cdots \tag{18}$$

while for λ with 3 or more parts,

$$[X_\lambda] = \text{Pfaffian}([X_{(\lambda_i,\lambda_j)}])_{1\le i<j\le 2\ell'} \tag{19}$$

where ℓ' is the least positive integer such that $2\ell' \geq \ell(\lambda)$.

As we alluded to above, the identities (18) and (19) go back to the work of Schur on the projective representations of symmetric groups [HH, Jo, S2, St1], where he introduced a family of symmetric functions $\{Q_\lambda(X)\}$ known as Schur Q-functions. We let $X = (x_1, x_2, \ldots)$ be a list of variables, define $q_r(X)$ by the equation

$$\prod_{i=1}^{\infty} \frac{1+x_i t}{1-x_i t} = \sum_{r=0}^{\infty} q_r(X) t^r$$

and then use the same relations (18) and (19) with $q_r(X)$ in place of c_r to define $Q_{(a,b)}(X)$ and then $Q_\lambda(X)$, for each strict partition λ. Once more we emphasize that there are relations among the q_r, the simplest being $q_1^2 = 2q_2$; hence, the above polynomials which define $Q_\lambda(X)$ are not uniquely determined. However, the equivalence of the raising operator and Pfaffian definitions of Q_λ is a formal consequence of the following Pfaffian identity from [S2]:

$$\prod_{1\le i<j\le 2\ell'} \frac{\mathrm{x}_i - \mathrm{x}_j}{\mathrm{x}_i + \mathrm{x}_j} = \text{Pfaffian}\left(\frac{\mathrm{x}_i - \mathrm{x}_j}{\mathrm{x}_i + \mathrm{x}_j}\right)_{1\le i,j\le 2\ell'},$$

which holds in the quotient field of $\mathbb{Z}[\mathrm{x}_1, \ldots, \mathrm{x}_{2\ell'}]$.

Remark 1. Although the definition (16) is not standard, it is in direct analogy with the usage of the term 'Schur polynomial' in type A. We reserve the name 'Q-polynomial' for the polynomial in the variables c_r given in (16), and also for its principal specialization, when c_r is replaced by $q_r(X)$ for each integer r. This nomenclature extends to the theta polynomials; compare (13) with (37) in §5.2.

Example 3. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a k-strict partition. If $\lambda_i \leq k$ for each i, then

$$\Theta_\lambda(c) = \prod_{i<j}(1 - R_{ij})\, c_\lambda = \det(c_{\lambda_i+j-i})_{i,j}, \tag{20}$$

while if $\lambda_i > k$ for all nonzero parts λ_i, then

$$\Theta_\lambda(c) = \prod_{i<j} \frac{1-R_{ij}}{1+R_{ij}}\, c_\lambda = \text{Pfaffian}\left(\frac{1-R_{12}}{1+R_{12}}\, c_{\lambda_i,\lambda_j}\right)_{i<j}. \tag{21}$$

We deduce that as λ varies, the polynomial $\Theta_\lambda(c)$ interpolates between the Jacobi-Trudi determinant (20) and the Schur Pfaffian (21). In general, the inclusion of a pair (i, j) in the set $\mathcal{C}(\lambda)$ which specifies the denominators in (12) depends not only on the size of λ_i and λ_j, but also on their relative position in the sequence λ.

2.2. **Orthogonal Grassmannians.** Consider the case where $m < N/2$ and the form $(\ , \ )$ is symmetric, so that $X = \mathrm{OG}(m, N)$ is an *orthogonal Grassmannian.* If $N = 2n+1$ is odd, then the Schubert varieties in $\mathrm{OG}(m, 2n+1)$ are indexed by the same set of k-strict partitions that index the Schubert varieties in $\mathrm{IG}(m, 2n)$. For any k-strict partition λ, let $\ell_k(\lambda)$ denote the number of parts λ_i of λ which are strictly greater than k. Let E''_{IG} and E''_{OG} be the universal quotient vector bundles over $\mathrm{IG}(n-k, 2n)$ and $\mathrm{OG}(n-k, 2n+1)$, respectively. One then knows (see for example [BS, §3.1]) that the map which sends $c_r(E''_{\mathrm{IG}})$ to $c_r(E''_{\mathrm{OG}})$ for all r extends to an isomorphism of graded rings

$$\mathrm{H}^*(\mathrm{IG}(n-k, 2n), \mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^*(\mathrm{OG}(n-k, 2n+1), \mathbb{Q}),$$

which sends a Schubert class $[X_\lambda]$ on IG to $2^{\ell_k(\lambda)}$ times the corresponding Schubert class on OG. This isomorphism shows that the Schubert calculus on symplectic and odd orthogonal Grassmannians coincides, up to well determined powers of two. In particular, one can easily transfer the Giambelli formula of §2.1 to $\mathrm{OG}(m, 2n+1)$.

We next assume that $N = 2n$ is even, so that $m = n-k$ with $k > 0$. A *typed k-strict partition* is a pair consisting of a k-strict partition λ together with an integer in $\{0, 1, 2\}$ called the *type* of λ, and denoted $\mathrm{type}(\lambda)$, such that $\mathrm{type}(\lambda) > 0$ if and only if $\lambda_i = k$ for some $i \geq 1$. We usually omit the type from the notation for the pair $(\lambda, \mathrm{type}(\lambda))$. To any typed k-strict partition λ we associate the order ideal

$$\mathcal{C}'(\lambda) := \{(i,j) \in \Delta^\circ \mid \lambda_i + \lambda_j \geq 2k + j - i\}$$

in Δ°. The Schubert cells in the cohomology of the even orthogonal Grassmannian $X = \mathrm{OG}(n-k, 2n)$ are indexed by the typed k-strict partitions λ whose diagrams are contained in an $(n-k) \times (n+k-1)$ rectangle. For any such λ, define the index function $p_j = p_j(\lambda)$ by

$$p_j(\lambda) := n + k + j - \lambda_j - \#\{\, i < j \mid (i,j) \in \mathcal{C}'(\lambda)\,\} \\ - \begin{cases} 1 & \text{if } \lambda_j > k, \text{ or } \lambda_j = k < \lambda_{j-1} \text{ and } n + j + \mathrm{type}(\lambda) \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

We obtain an index set $\mathcal{P}(\lambda)$ associated to any typed k-strict partition λ as above, and a Schubert cell $X^\circ_{\mathcal{P}}(F_\bullet)$ defined by (10).

The Schubert variety $X_{\mathcal{P}}(F_\bullet)$ is best defined as the closure of $X^\circ_{\mathcal{P}}(F_\bullet)$, since a geometric description of $X_{\mathcal{P}}(F_\bullet)$ analogous to (11) involves subtle parity conditions (see [BKT4, App. A]). We say that two maximal isotropic subspaces E and F of V are in the same family if

$$\dim(E \cap F) \equiv n \pmod{2}.$$

Fix a maximal isotropic subspace L of V, so that $\dim(L) = n$. The cohomology classes $[X_{\mathcal{P}}]$ of the Schubert varieties $X_{\mathcal{P}}(F_\bullet)$ in OG are are independent of the choice of isotropic flag $F_\bullet$ as long as F_n is in the same family as L. If $\mathcal{P}$ corresponds to λ, then the associated Schubert class $[X_\lambda] = [X_{\mathcal{P}}]$ in $\mathrm{H}^{2|\lambda|}(X, \mathbb{Z})$ is said to have a type which agrees with the type of λ.

The special Schubert varieties in $\mathrm{OG}(m, 2n)$ can be defined as before by a single Schubert condition, as the locus of $\Sigma \in X$ which intersect a given isotropic subspace or its orthogonal complement non-trivially (see [BKT1, §3.2]). The corresponding special Schubert classes

$$\tau_1, \ldots, \tau_{k-1}, \tau_k, \tau'_k, \tau_{k+1}, \ldots, \tau_{n+k-1} \tag{22}$$

are indexed by the typed k-strict partitions with a single nonzero part, and generate the cohomology ring $\mathrm{H}^*(X,\mathbb{Z})$. Here $\mathrm{type}(\tau_k)=1$, $\mathrm{type}(\tau'_k)=2$, and if

$$0 \to E' \to E \to E'' \to 0$$

denotes the universal sequence of vector bundles over X, then we have

$$c_r(E'') = \begin{cases} \tau_r & \text{if } r<k, \\ \tau_k+\tau'_k & \text{if } r=k, \\ 2\tau_r & \text{if } r>k. \end{cases} \tag{23}$$

We set $c_\alpha = \prod_i c_{\alpha_i}$. Given any typed k-strict partition λ, we define the operator

$$R^\lambda := \prod (1-R_{ij}) \prod_{(i,j)\in\mathcal{C}'(\lambda)} (1+R_{ij})^{-1} \tag{24}$$

where the first product is over all pairs $i<j$ and the second product is over pairs $i<j$ such that $\lambda_i+\lambda_j \geq 2k+j-i$. Let R be any finite monomial in the operators R_{ij} which appears in the expansion of the formal power series R^λ in (24). If $\mathrm{type}(\lambda)=0$, then set $R\star c_\lambda := c_{R\lambda}$. Suppose that $\mathrm{type}(\lambda)>0$, let d be the least index such that $\lambda_d=k$, and set

$$\widehat{\alpha} := (\alpha_1,\ldots,\alpha_{d-1},\alpha_{d+1},\ldots,\alpha_\ell)$$

for any integer sequence α of length ℓ. If R involves any factors R_{ij} with $i=d$ or $j=d$, then let $R\star c_\lambda := \frac{1}{2}\, c_{R\lambda}$. If R has no such factors, then let

$$R\star c_\lambda := \begin{cases} \tau_k\, c_{\widehat{R\lambda}} & \text{if } \mathrm{type}(\lambda)=1, \\ \tau'_k\, c_{\widehat{R\lambda}} & \text{if } \mathrm{type}(\lambda)=2. \end{cases}$$

We define the *eta polynomial* H_λ by

$$H_\lambda(c) := 2^{-\ell_k(\lambda)} R^\lambda \star c_\lambda. \tag{25}$$

Note that $H_\lambda(c)$ is really a polynomial in the variables τ_r for $r\geq 1$ and τ'_k, which are related to the variables c_r by the formal equations (23).

Theorem 3 (Giambelli for OG, [BKT4])**.** *For every typed k-strict partition λ whose diagram fits inside an $(n-k)\times(n+k-1)$ rectangle, we have*

$$[X_\lambda] = H_\lambda(c(E'')) \tag{26}$$

in the cohomology ring of $\mathrm{OG}(n-k,2n)$.

Example 4. Consider the typed 1-strict partition $\lambda=(3,1,1)$ with $\mathrm{type}(\lambda)=2$. The corresponding Schubert class in $\mathrm{H}^*(\mathrm{OG}(4,10),\mathbb{Z})$ satisfies

$$\begin{aligned} [X_{(3,1,1)}] &= \frac{1}{2}\frac{1-R_{12}}{1+R_{12}}\,\frac{1-R_{13}}{1+R_{13}}\,(1-R_{23})\star c_{(3,1,1)} \\ &= \frac{1}{2}(1-2R_{12}+2R_{12}^2)(1-2R_{13}-R_{23})\star c_{(3,1,1)} \\ &= \tau_3\tau'_1(\tau_1+\tau'_1) - 2\tau_4\tau'_1 - \tau_3\tau_2 + \tau_5. \end{aligned}$$

In general, the Giambelli formula (26) expresses the Schubert class $[X_\lambda]$ as a polynomial in the special Schubert classes (22) with *integer* coefficients.

To conclude this section, we consider the space of all maximal isotropic subspaces of an even dimensional orthogonal vector space V. This set is a disjoint union of two connected components, each giving one family (or $\mathrm{SO}(V)$-orbit) of such subspaces. The (irreducible) algebraic variety $\mathrm{OG} = \mathrm{OG}(n, 2n)$ is defined by choosing one of these two components. If H is a hyperplane in V on which the restriction of the symmetric form is nondegenerate, then the map $\Sigma \mapsto \Sigma \cap H$ gives an isomorphism between $\mathrm{OG}(n, 2n)$ and $\mathrm{OG}(n-1, 2n-1)$. The analysis for the (type B) odd orthogonal Grassmannian in the maximal isotropic case therefore applies to OG. In particular, the Schubert classes X_λ may be indexed by strict partitions λ with $\lambda_1 \leq n-1$, and the Giambelli formula for OG reads

$$[X_\lambda] = P_\lambda(c(E'')),$$

where $E'' \to \mathrm{OG}$ is the universal quotient bundle and the P-polynomial P_λ is related to Q_λ by the equation $P_\lambda = 2^{-\ell(\lambda)} Q_\lambda$.

3. Cohomology of G/P spaces

In this section G will denote a connected complex reductive Lie group. The main examples we will consider are the classical groups: the general linear group $\mathrm{GL}_n(\mathbb{C})$, the symplectic group $\mathrm{Sp}_{2n}(\mathbb{C})$, and the (odd and even) orthogonal groups $\mathrm{SO}_N(\mathbb{C})$. A closed algebraic subgroup P of G is called a parabolic subgroup if the quotient $X = G/P$ is compact, or equivalently, a projective algebraic variety. We proceed to study the topology of the complex manifold X by generalizing the constructions found in the previous sections.

The Borel subgroups B of G are the maximal connected solvable subgroups; a subgroup P of G is parabolic if and only if it contains a Borel subgroup. Fix a Borel subgroup B, let $T \cong (\mathbb{C}^*)^r$ be a maximal torus in B, and $W = N_G(T)/T$ be the Weyl group of G. The simple reflections s_α, which generate W, are indexed by the set Δ of positive simple roots α, and are also in one-to-one correspondence with the vertices of the *Dynkin diagram* D associated to the root system. The parabolic subgroups P containing B are in bijection with the subsets Δ_P of Δ, or the subsets of the vertices of D, as shown in the figures of §6. In particular each root $\alpha \in \Delta$ corresponds to a maximal parabolic subgroup, which is associated to the subset $\Delta \smallsetminus \{\alpha\}$. We let W_P denote the subgroup of W generated by all the simple reflections in Δ_P.

The *length* $\ell(w)$ of an element w in W is equal to the least number of simple reflections whose product is w. We denote by w_0 the element of longest length in W. It is known that every coset in W/W_P has a unique representative w of minimal length; we denote the set of all minimal length W_P-coset representatives by W^P. The manifold X has complex dimension equal to the length of the longest element in W^P. The set W^P (or the coset space W/W_P) indexes the Schubert cells, varieties, and classes on X as follows. First, the Bruhat decomposition $G = \coprod_{w \in W} BwB$ of the group G induces a cell decomposition

$$G/P = \coprod_{w \in W^P} BwP/P$$

of the homogeneous space $X = G/P$. The cell BwP/P is isomorphic to the affine space $\mathbb{C}^{\ell(w)}$. Since we prefer the length of the indexing element to equal the codimension of the cell in X, we define the Schubert cell X_w° to be Bw_0wP/P. The

Schubert variety X_w is defined as the closure of the Schubert cell X_w°, and its cohomology class $[X_w]$ lies in $\mathrm{H}^{2\ell(w)}(X,\mathbb{Z})$. This completes the additive description of the cohomology of X, as a free abelian group on the basis of Schubert classes $[X_w]$:

$$\mathrm{H}^*(X,\mathbb{Z}) = \bigoplus_{w\in W^P} \mathbb{Z}\,[X_w].$$

3.1. **The Giambelli problem.** We seek to generalize the classical Schubert calculus of §1 to the G/P spaces in their natural cell decompositions described above. By this we mean to understand as explicitly as possible the multiplicative structure of the cohomology ring $\mathrm{H}^*(G/P,\mathbb{Z})$, expressed in the basis of Schubert classes. However, it is clear that the extension of the classical Pieri and Giambelli formulas to G/P depends on a choice of *special Schubert classes* which generate the cohomology ring. For arbitrary G, it is fortunate that there is one example of a parabolic P where the choice of generating set is clear: the case when $P = B$ is a Borel subgroup of G, and $X = G/B$ is the (complete) flag variety of G. In this case, the cohomology ring $\mathrm{H}^*(X,\mathbb{Q})$ is generated by the classes of the Schubert divisors X_{s_α}, one for each simple root α. Equivalently, we can form a generating set by taking any $\mathbb{Z}$-basis of $\mathrm{H}^2(X,\mathbb{Z})$, which is related to the Schubert divisor basis by a linear change of variables. In a seminal paper, Borel [Bo], using the theory of group characters and associated characteristic classes of line bundles over G/B, gave an invariant, group theoretic approach to this question, which we recall below.

A *character* of the group B is a homomorphism of algebraic groups $B \to \mathbb{C}^*$. We denote the abelian group of characters of B by $\widehat{B}$. Observe that any character χ of B is uniquely determined by its restriction to T, since $B = T \ltimes U$ is the semidirect product of T and the unipotent subgroup U of B, and regular invertible functions on U are constant. It follows that the character group $\widehat{T}$ of T is isomorphic to $\widehat{B}$.

If χ is a character of B, then we get an induced free action of B on the product $G \times \mathbb{C}$ by $b\cdot(g,z) = (gb^{-1},\chi(b)z)$. The quotient space $L_\chi = (G\times\mathbb{C})/B$, also denoted by $G\times^B \mathbb{C}$, projects to the flag manifold G/B by sending the orbit of (g,z) to gB. This makes L_χ into the total space of a holomorphic line bundle over G/B, which is the homogeneous line bundle associated to the weight χ. Let $\mathrm{Pic}(G/B)$ be the Picard group of isomorphism classes of line bundles on G/B, with the group operation given by the tensor product. Then the map $\chi \mapsto L_\chi$ is a group homomorphism $\widehat{B} \to \mathrm{Pic}(G/B)$. Composing this with the first Chern class map $c_1 : \mathrm{Pic}(G/B) \to \mathrm{H}^2(G/B,\mathbb{Z})$ gives a group homomorphism

$$\begin{gathered}\widehat{B} \to \mathrm{H}^2(G/B,\mathbb{Z}),\\ \chi \mapsto c_1(L_\chi).\end{gathered}$$

If $S(\widehat{B})$ denotes the symmetric algebra of the $\mathbb{Z}$-module $\widehat{B}$, then the above map extends to a homomorphism of graded rings

$$c : S(\widehat{B}) \to \mathrm{H}^*(G/B,\mathbb{Z})$$

called the *characteristic homomorphism*.

Let $\mathcal{S} := S(\widehat{B})\otimes_{\mathbb{Z}}\mathbb{Q}$. Borel [Bo] proved that the morphism c is surjective, after tensoring with $\mathbb{Q}$, and that the kernel of c is the ideal generated by the W-invariants of positive degree in $\mathcal{S}$, denoted $\langle \mathcal{S}_+^W\rangle$. We thus obtain the classical Borel presentation of the cohomology ring of G/B,

$$\mathrm{H}^*(G/B,\mathbb{Q}) \cong \mathcal{S}/\langle \mathcal{S}_+^W\rangle. \tag{27}$$

Furthermore, for any parabolic subgroup $P \supset B$, the projection map $G/B \to G/P$ induces an injection $\mathrm{H}^*(G/P) \hookrightarrow \mathrm{H}^*(G/B)$, and this inclusion is realized in the presentation (27) by taking W_P invariants on the right hand side:

$$\mathrm{H}^*(G/P, \mathbb{Q}) \cong \mathcal{S}^{W_P}/\langle \mathcal{S}_+^W \rangle. \tag{28}$$

One knows that if the algebraic group G is *special* in the sense of [SeC], then the isomorphisms (27) and (28) hold with $\mathbb{Z}$-coefficients. For the classical groups, this is the case if G is the general linear group GL_n or the symplectic group Sp_{2n}.

Example 5. Suppose that $G = \mathrm{GL}_n(\mathbb{C})$, and choose the Borel subgroup B of upper triangular matrices, as in §1. The flag manifold G/B then parametrizes complete flags of subspaces

$$0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_n = \mathbb{C}^n$$

in $\mathbb{C}^n$. The maximal torus $T \subset B$ is the group of invertible diagonal matrices, and the characters of T are the maps

$$\mathrm{diag}(t_1, \ldots, t_n) \mapsto t_1^{\alpha_1} \cdots t_n^{\alpha_n},$$

where $\alpha_1, \ldots, \alpha_n$ are integers. In this way, the multiplicative group of characters $\widehat{T}$ (and $\widehat{B}$) is identified with the additive group $\mathbb{Z}^n$, and we have

$$S(\widehat{T}) = S(\widehat{B}) = \mathbb{Z}[\mathrm{x}_1, \ldots, \mathrm{x}_n].$$

The Weyl group W is the symmetric group S_n, and (27) implies that $\mathrm{H}^*(G/B, \mathbb{Z})$ is isomorphic to $\mathbb{Z}[\mathrm{x}_1, \ldots, \mathrm{x}_n]/\mathrm{I}$, where I is the ideal generated by the non-constant elementary symmetric polynomials in $\mathrm{x}_1, \ldots, \mathrm{x}_n$.

In the case of the Grassmannian $\mathrm{G}(m, n) = \mathrm{GL}_n(\mathbb{C})/P$, the associated parabolic subgroup of $W = S_n$ is $W_P = S_m \times S_{n-m}$, embedded in S_n in the obvious way. The W_P-coset representatives of minimal length are the permutations $\varpi \in S_n$ such that $\varpi_1 < \cdots < \varpi_m$ and $\varpi_{m+1} < \cdots < \varpi_n$. If $\lambda = (\lambda_1, \ldots, \lambda_m)$ is a partition which indexes a Schubert variety X_λ in $\mathrm{G}(m, n)$, then the minimal length coset representative ϖ corresponding to λ is determined by the equations

$$\varpi_j = \lambda_{m+1-j} + j, \quad \text{for } 1 \leq j \leq m.$$

The ring presentation (28) assumes the form

$$\mathrm{H}^*(\mathrm{G}(m, n), \mathbb{Z}) \cong \left(\mathbb{Z}[\mathrm{x}_1, \ldots, \mathrm{x}_m]^{S_m} \otimes \mathbb{Z}[\mathrm{x}_{m+1}, \ldots, \mathrm{x}_n]^{S_{n-m}}\right)/\mathrm{I}$$

and the two groups of variables $\mathrm{x}_1, \ldots, \mathrm{x}_m$ and $\mathrm{x}_{m+1}, \ldots, \mathrm{x}_n$ are the Chern roots of the vector bundles E' and E'' in the universal exact sequence (2), respectively. The relations generating the ideal I translate into the Whitney sum formula

$$c(E')c(E'') = c(E) = 1.$$

Although the presentation (28) is very natural from a Lie-theoretic point of view, in general there will be more than one way to identify special Schubert classes which generate the cohomology ring of G/P among the W_P-invariants in $\mathcal{S}$. However, when G is a *classical group*, there is a good uniform choice of special Schubert class generators of $\mathrm{H}^*(G/P)$, as explained in the introduction. Initially, the special Schubert classes on any Grassmannian are defined exactly as in §1 and §2. If P is arbitrary, then the partial flag variety G/P admits projection maps to various Grassmannians G/P_r, defined by omitting all but one of the subspaces in the flags parametrized by G/P. The special Schubert classes on G/P are defined to be the pullbacks of the special Schubert classes from the Grassmannians G/P_r. The

Giambelli problem then asks for an explicit combinatorial formula which writes a general Schubert class in $\mathrm{H}^*(G/P)$ as a polynomial in these special classes.

3.2. **Equivariant cohomology and degeneracy loci.** The definition of equivariant cohomology begins with the construction of a contractible space EG on which G acts freely. If $BG = EG/G$ denotes the quotient space, which is a classifying space for the group G, then the map $EG \to BG$ is a universal principal G-bundle. In the topological category, this means that if $\mathcal{U} \to M$ is any principal G-bundle, then there is a morphism $f : M \to BG$, which is unique up to homotopy, such that $\mathcal{U} \cong f^*EG$. If X is any topological space endowed with a G-action, then G acts diagonally on $EG \times X$ (that is, $g(e,x) := (ge,gx)$) and the quotient

$$EG \times^G X := (EG \times X)/G$$

exists. The G-equivariant cohomology ring $\mathrm{H}^*_G(X)$ of X is then defined by

$$\mathrm{H}^*_G(X) := \mathrm{H}^*(EG \times^G X),$$

where we usually take cohomology with $\mathbb{Q}$-coefficients. Note that any subgroup of G also acts freely on EG, therefore we can similarly define classifying spaces $BB := EG/B$, $BT := EG/T$, etc., and equivariant cohomology rings $\mathrm{H}^*_B(X)$, $\mathrm{H}^*_T(X)$, respectively. The natural map $EG \times^T X \to EG \times^B X$ induces an isomorphism

$$\mathrm{H}^*_T(X) \cong \mathrm{H}^*_B(X). \tag{29}$$

For more on equivariant cohomology, see for example [Hs, Br2].

A *flag bundle* is a fibration with fibers isomorphic to the flag variety $X = G/B$. Motivated by the theory of degeneracy loci of vector bundles, Fulton, Pragacz, and Ratajski [F2, F3, PR2] studied the question of obtaining explicit Chern class formulas for the classes of universal Schubert varieties in flag bundles for the classical Lie groups. In effect, this is the Giambelli problem of §3.1 when the homogeneous space X varies in a family. We give a description of the problem here assuming that we are in Lie type A, B, or C for simplicity. Suppose that $E \to M$ is a vector bundle over a variety M, which in type B or C comes equipped with a nondegenerate symmetric or skew-symmetric bilinear form $E \otimes E \to \mathbb{C}$, respectively. Assume that $E_\bullet$ and $F_\bullet$ are two complete flags of subbundles of E, taken to be isotropic in types B and C. For any w in the Weyl group, we have the degeneracy locus

$$\mathfrak{X}_w := \{b \in M \mid \dim(E_r(b) \cap F_s(b)) \geq d_w(r,s) \ \forall\, r,s\}, \tag{30}$$

where $d_w(r,s)$ is a function taking values in the nonnegative integers. The inequalities in (30) are exactly those which define the Schubert variety $X_w(F_\bullet)$ in the flag variety X (they are given explicitly in §6). Assuming that M is smooth and that the locus $\mathfrak{X}_w$ has pure codimension $\ell(w)$ in M (hypotheses which can both be relaxed), one seeks a formula for the cohomology class $[\mathfrak{X}_w] \in \mathrm{H}^*(M)$ in terms of the Chern classes of the vector bundles which appear in (30).

Graham [Gr] studied the above problem by placing it in a more general Lie-theoretic framework, as follows. The morphism $BB \to BG$ is a flag bundle, and he showed that the fiber product space $BB \times_{BG} BB$ is a classifying space for the question posed by Fulton et. al., because any other example will pull back from it. Therefore, all the desired formulas for $[\mathfrak{X}_w]$ occur in $\mathrm{H}^*(BB \times_{BG} BB)$. Furthermore, it is known (see for example [Br2, §1]) that $\mathrm{H}^*(BB) = \mathcal{S}$, $\mathrm{H}^*(BG) = \mathcal{S}^W$, and the Leray-Hirsch theorem implies that there is a natural isomorphism

$$\mathrm{H}^*(BB \times_{BG} BB) \cong \mathcal{S} \otimes_{\mathcal{S}^W} \mathcal{S}. \tag{31}$$

This point of view on degeneracy loci and the polynomials which represent their cohomology classes will be used in §7.3, and we refer there for more details.

Another key observation in [Gr] is that the degeneracy locus problem is equivalent to the question of obtaining an *equivariant Giambelli formula* in $\mathrm{H}^*_T(G/B)$. Indeed, there is a $(B\times B)$-equivariant isomorphism

$$EG \times G \to EG \times_{BG} EG$$

sending (e,g) to (e,ge), and passing to the quotient spaces gives a natural isomorphism

$$EG \times^B (G/B) \cong BB \times_{BG} BB \tag{32}$$

which maps $EG\times^B X_w$ to $\mathfrak{X}_w$. In view of the isomorphism (32), we will call $BB\times_{BG} BB$ the *Borel mixing space* associated to G/B. Taking the cohomology of both sides of (32) and using (29), we obtain an isomorphism

$$\mathrm{H}^*_T(G/B) = \mathrm{H}^*(EG\times^B (G/B)) \cong \mathrm{H}^*(BB\times_{BG} BB)$$

which sends the *equivariant Schubert class* $[X_w]^T := [EG\times^B X_w]$ to $[\mathfrak{X}_w]$. Finally, we remark that an equivariant Giambelli formula for $[X_w]^T$ in $\mathrm{H}^*_T(G/B)$ specializes to a Giambelli formula for $[X_w]$ in $\mathrm{H}^*(G/B)$ when we set all the variables coming from the linear T-action equal to zero.

Remark 2. The interpretation of the Giambelli problem for the cohomology of G/P given here is our own; compare with the discussion in [Hi, Chp. III §3]. We note that there are other kinds of 'Giambelli formulas' which appear in the literature. The subject of degeneracy loci also has a long history (see for example [FP]) and includes varieties such as those discussed in the introduction, defined by imposing rank conditions on generic *morphisms* between vector bundles. The degeneracy loci considered here, which are defined by intersecting subbundles of a fixed vector bundle as in (30), were introduced and studied in [F3, PR2].

4. Weyl groups, Grassmannian elements, and transition trees

In this section we explain the combinatorial objects which enter into the algebraic and geometric formulas of later sections. We begin with a discussion of the Weyl groups for the classical root systems of type A, B, C, and D.

4.1. **Weyl groups.**

Let W_n denote the *hyperoctahedral group* of signed permutations on the set $\{1,\ldots,n\}$, which is the semidirect product $S_n \ltimes \mathbb{Z}_2^n$ of the symmetric group S_n with $\mathbb{Z}_2^n$. We adopt the notation where a bar is written over an entry with a negative sign; thus $w=(3,\overline{1},2)$ maps $(1,2,3)$ to $(3,-1,2)$. The group W_n is the Weyl group for the root system B_n or C_n, and is generated by the simple transpositions $s_i=(i,i+1)$ for $1\leq i\leq n-1$ and the sign change $s_0(1)=\overline{1}$. The *symmetric group* S_n is the subgroup of W_n generated by the s_i for $1\leq i\leq n-1$, and is the Weyl group for the root system A_{n-1}. The Weyl group $\widetilde{W}_n$ for the root system D_n is the subgroup of W_n consisting of all signed permutations with an even number of sign changes. The group $\widetilde{W}_n$ is an extension of S_n by the element $s_\Box = s_0s_1s_0$, which acts on the right by

$$(w_1,w_2,\ldots,w_n)s_\Box = (\overline{w}_2,\overline{w}_1,w_3,\ldots,w_n).$$

There are natural embeddings $W_n\hookrightarrow W_{n+1}$ and $\widetilde{W}_n\hookrightarrow\widetilde{W}_{n+1}$ defined by adjoining the fixed point $n+1$. We let $S_\infty := \cup_n S_n$, $W_\infty := \cup_n W_n$, and $\widetilde{W}_\infty := \cup_n \widetilde{W}_n$.

Consider the two sets $\mathbb{N}_0 := \{0, 1, \ldots\}$ and $\mathbb{N}_\Box := \{\Box, 1, \ldots\}$ whose members index the simple reflections. A *reduced word* of an element w in W_∞ (respectively $\widetilde{W}_\infty$) is a sequence $a_1 \cdots a_\ell$ of elements of $\mathbb{N}_0$ (respectively $\mathbb{N}_\Box$) such that $w = s_{a_1} \cdots s_{a_\ell}$ and ℓ is minimal, so (by definition) equal to the length $\ell(w)$ of w. We say that w has *descent* at position r if $\ell(ws_r) < \ell(w)$, where s_r is the simple reflection indexed by r. For $r \in \mathbb{N}_0$, this is equivalent to the condition $w_r > w_{r+1}$, where we set $w_0 = 0$ (so w has a descent at position 0 if and only if $w_1 < 0$).

4.2. **Grassmannian elements.** A permutation $\varpi \in S_\infty$ is *Grassmannian* if there exists an $m \geq 1$ such that $\varpi_i < \varpi_{i+1}$ for all $i \neq m$. The *shape* of such a Grassmannian permutation ϖ is the partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ with $\lambda_{m+1-j} = \varpi_j - j$ for $1 \leq j \leq m$. Notice that there are infinitely many permutations of a given shape λ. However, for each fixed m and $n > m$, we obtain a bijection between the set of permutations in S_n with at most one descent at position m and the set of partitions λ whose diagram fits inside an $m \times (n - m)$ rectangle.

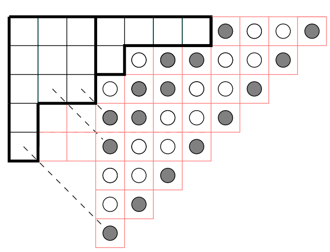
Fix a nonnegative integer k. An element $w = (w_1, w_2, \ldots)$ in W_∞ is called *k-Grassmannian* if and only if we have $\ell(ws_i) = \ell(w) + 1$ for all $i \neq k$. When $k = 0$, this says that w is increasing: $w_1 < w_2 < \cdots$, while when $k > 0$, then w is k-Grassmannian if and only if

$$0 < w_1 < \cdots < w_k \quad \text{and} \quad w_{k+1} < w_{k+2} < \cdots .$$

There is an explicit bijection between k-Grassmannian elements w of W_∞ and k-strict partitions λ, under which the elements in W_n correspond to those partitions whose diagram fits inside an $(n - k) \times (n + k)$ rectangle. This bijection is obtained as follows. The absolute value of the negative entries in w form a (possibly empty) strict partition μ whose diagram gives the part of the diagram of λ which lies in columns $k + 1$ and higher. The boxes in these columns which lie outside of the diagram of μ are partitioned into south-west to north-east *diagonals*. For each i between 1 and k, let d_i be the diagonal among these which contains w_i boxes. Then the bottom box B_i of λ in column $k + 1 - i$ is *k-related* to d_i, for $1 \leq i \leq k$. Here 'k-related' means that the directed line segment joining B_i with the lowest box of the diagonal d_i is north-west to south-east (this notion was used in [BKT1, BKT2]). We will denote the Weyl group element associated to λ by w_λ.

Example 6. (a) The 3-Grassmannian element $w = (3, 5, 8, \overline{4}, \overline{1}, 2, 6, 7)$ corresponds to the 3-strict partition $\lambda = (7, 4, 3, 1, 1)$.

λ =



(b) A permutation $\varpi \in S_\infty$ with a unique descent at position m may be viewed as an m-Grassmannian element of W_∞, via the natural inclusion of S_∞ in W_∞. In this case, the m-strict partition associated to ϖ is the conjugate of the shape of ϖ as a Grassmannian permutation.

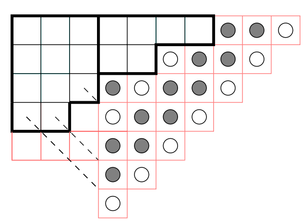
Fix a $k \in \mathbb{N}_\Box$ with $k \neq 1$. An element $w \in \widetilde{W}_\infty$ is called k-Grassmannian if $\ell(ws_i) = \ell(w)+1$ for all $i \neq k$. We say that w is 1-Grassmannian if $\ell(ws_i) = \ell(w)+1$ for all $i \geq 2$. The $\Box$-Grassmannian elements are therefore the ones whose values strictly increase, while if $k > 0$, then w is k-Grassmannian if and only if

$$|w_1| < w_2 < \cdots < w_k \quad \text{and} \quad w_{k+1} < w_{k+2} < \cdots .$$

There is a bijection between the k-Grassmannian elements w of $\widetilde{W}_\infty$ and typed k-strict partitions λ, under which the elements in $\widetilde{W}_n$ correspond to typed partitions whose diagram fits inside an $(n-k) \times (n+k-1)$ rectangle, obtained as follows. Subtract one from the absolute value of each negative entry w_{k+j}, for $j \geq 1$, to obtain the parts of a strict partition μ. As above, the partition μ gives the part of λ which lies in columns $k+1$ and higher, and the boxes in these columns outside of μ are partitioned into south-west to north-east diagonals. Choose $d_1, \ldots, d_k$ among these diagonals such that the number of boxes in d_i is given by $|w_i| - 1$, for $1 \leq i \leq k$. The bottom boxes of λ in the first k columns are then *k'-related* to these k diagonals, with bottom box B_i in column $k+1-i$ related to diagonal d_i. Here 'k'-related' differs from 'k-related' by a shift up by one unit, as shown in the next figure. Finally, we have $\mathrm{type}(\lambda) > 0$ if and only if $|w_1| > 1$, and in this case $\mathrm{type}(\lambda)$ is equal to 1 or 2 depending on whether $w_1 > 0$ or $w_1 < 0$, respectively. We will denote the Weyl group element associated to λ by w_λ, as above.

Example 7. The 3-Grassmannian element $w = (\overline{2}, 6, 7, \overline{5}, \overline{3}, \overline{1}, 4, 8)$ corresponds to the 3-strict partition $\lambda = (7, 5, 3, 2)$ of type 2.

$\lambda \ =$



To each k-Grassmannian element w in W_∞ or $\widetilde{W}_\infty$, we attach a finite order ideal $\mathcal{C}(w)$ in Δ° by the prescription

$$\mathcal{C}(w) := \{(i,j) \in \Delta^\circ \mid w_{k+i} + w_{k+j} < 0\}. \tag{33}$$

If λ is the k-strict partition or typed k-strict partition corresponding to w, then $w = w_\lambda$ and we say that λ is the *shape* of w. In either case, it is easy to check that $\mathcal{C}(w)$ is equal to the ideal $\mathcal{C}(\lambda)$ or $\mathcal{C}'(\lambda)$ defined in §2, respectively.

4.3. **Transition trees and Stanley coefficients.** The notion of a transition was introduced by Lascoux and Schützenberger in [LS1, LS3], where it was applied to achieve efficient, recursive computations of type A Schubert polynomials and a new form of the Littlewood-Richardson rule. Analogues of their transition equations for the other classical Lie types were obtained by Billey [Bi1], using the Schubert polynomials found in [BH]. We will require the extension of this theory given in [T7], which includes the k-Grassmannian elements for all $k > 0$. For positive integers $i<j$ we define reflections $t_{ij}\in S_\infty$ and $\bar{t}_{ij},\bar{t}_{ii}\in W_\infty$ by their right actions

$$\begin{aligned}(\ldots,w_i,\ldots,w_j,\ldots)\,t_{ij}&=(\ldots,w_j,\ldots,w_i,\ldots),\\ (\ldots,w_i,\ldots,w_j,\ldots)\,\bar{t}_{ij}&=(\ldots,\overline{w}_j,\ldots,\overline{w}_i,\ldots),\quad\text{and}\\ (\ldots,w_i,\ldots)\,\bar{t}_{ii}&=(\ldots,\overline{w}_i,\ldots),\end{aligned}$$

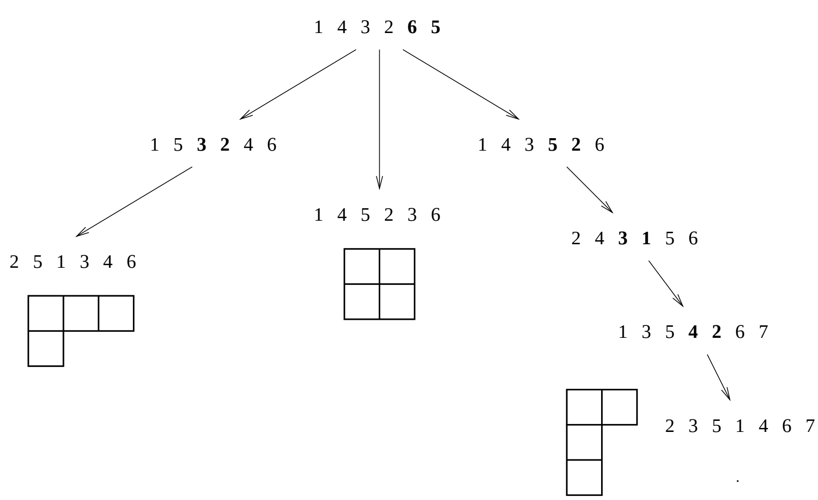
and let $\bar{t}_{ji}:=\bar{t}_{ij}$.

Following Lascoux and Schützenberger [LS3], for any permutation $\varpi\in S_\infty$, we construct a rooted tree $T(\varpi)$ with nodes given by permutations of the same length $\ell(\varpi)$, and root ϖ, as follows. If $\varpi=1$ or ϖ is Grassmannian, then set $T(\varpi):=\{\varpi\}$. Otherwise, let r be the largest descent of ϖ, and set $s:=\max(i>r\mid\varpi_i<\varpi_r)$. The definitions of r and s imply that we have $\ell(\varpi t_{rs})=\ell(\varpi)-1$. Let

$$I(\varpi):=\{i\mid 1\leq i<r\ \text{ and }\ \ell(\varpi t_{rs}t_{ir})=\ell(\varpi)\}.$$

If $I(\varpi)\neq\emptyset$, then let $\Psi(\varpi):=\{\varpi t_{rs}t_{ir}\mid i\in I(\varpi)\}$; otherwise, let $\Psi(\varpi):=\Psi(1\times\varpi)$. To recursively define $T(\varpi)$, we join ϖ by an edge to each $v\in\Psi(\varpi)$, and attach to each $v\in\Psi(\varpi)$ its tree $T(v)$. One can show that $T(\varpi)$ is a finite tree whose leaves are all Grassmannian permutations. The tree $T(\varpi)$ is the *Lascoux-Schützenberger transition tree* of ϖ.[1] We define the *Stanley coefficient* c_λ^ϖ to be the number of leaves of shape λ in the transition tree $T(\varpi)$ associated to ϖ.

Example 8. The transition tree of $\varpi=143265$ is displayed below. At all nodes where the branching rule is applied, the positions r and s are shown in boldface.



There exist alternative combinatorial formulas for the coefficients c_λ^ϖ in terms of Young tableaux [EG, FG, RS]; see also [Lit]. We note here the result of Fomin and Greene [FG] that c_λ^ϖ is equal to the number of semistandard tableaux T of shape $\widetilde{\lambda}$ such that the *column word* of T, obtained by reading the entries of T from bottom to top and left to right, is a reduced word for w. We display below the three tableaux associated to the leaves of the tree for $143265=s_3s_2s_3s_5$ in Example 8.

[1]The trees in [LS3] differ slightly from the ones here; their leaves are vexillary permutations.

2	3	5
3		

2	3
3	5

2	3
3	
5	

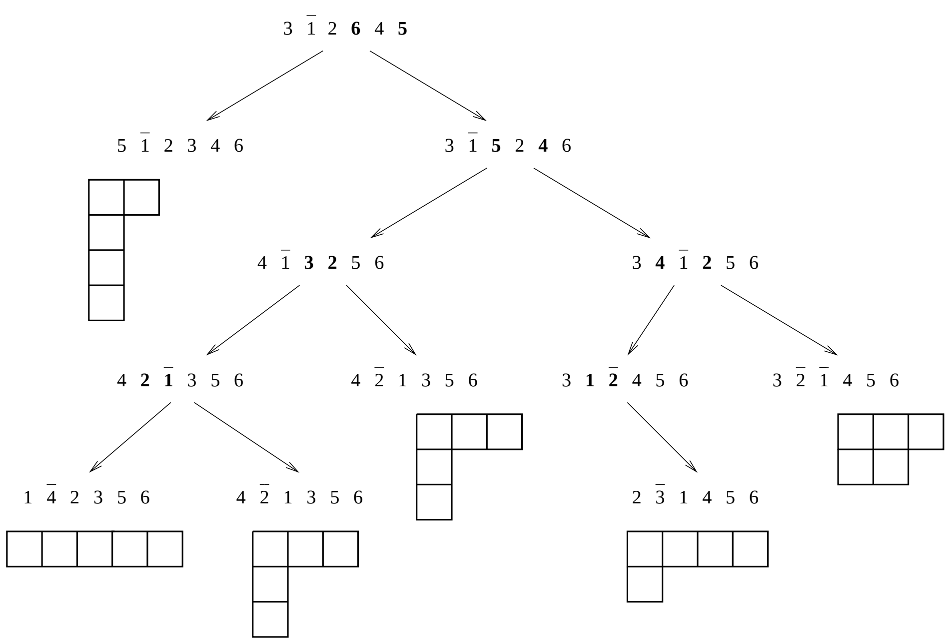
A signed permutation $w \in W_\infty$ is said to be *increasing up to k* if it has no descents less than k. This condition is automatically satisfied if $k = 0$, and for positive k it means that $0 < w_1 < w_2 < \cdots < w_k$. If $k \geq 2$, we say that an element $w \in \widetilde{W}_\infty$ is *increasing up to k* if it has no descents less than k; this means that $|w_1| < w_2 < \cdots < w_k$. By convention we agree that every element of $\widetilde{W}_\infty$ is increasing up to $\Box$ and also increasing up to 1.

For any $w \in W_\infty$ which is increasing up to k, we construct a rooted tree $T^k(w)$ with nodes given by elements of W_∞ and root w as follows. Let r be the largest descent of w. If $w = 1$ or $r = k$, then set $T^k(w) := \{w\}$. Otherwise, let $s := \max(i > r \mid w_i < w_r)$ and $\Phi(w) := \Phi_1(w) \cup \Phi_2(w)$, where

$$\Phi_1(w) := \{wt_{rs}t_{ir} \mid 1 \leq i < r \text{ and } \ell(wt_{rs}t_{ir}) = \ell(w)\},$$
$$\Phi_2(w) := \{wt_{rs}\overline{t}_{ir} \mid i \geq 1 \text{ and } \ell(wt_{rs}\overline{t}_{ir}) = \ell(w)\}.$$

To recursively define $T^k(w)$, we join w by an edge to each $v \in \Phi(w)$, and attach to each $v \in \Phi(w)$ its tree $T^k(v)$. We call $T^k(w)$ the *k-transition tree* of w.

Example 9. The 1-transition tree of $w = 3\overline{1}2645$ is displayed below, with positions r and s shown in boldface.



For any $w \in \widetilde{W}_\infty$ which is increasing up to k, we construct the k-transition tree $\widetilde{T}^k(w)$ with nodes given by elements of $\widetilde{W}_\infty$ and root w in a manner parallel to the above. Let r be the largest descent of w. If $w = 1$, or $k \neq 1$ and $r = k$, or $k = 1$ and $r \in \{\Box, 1\}$, then set $\widetilde{T}^k(w) := \{w\}$. Otherwise, let $s := \max(i > r \mid w_i < w_r)$

and $\widetilde{\Phi}(w) := \widetilde{\Phi}_1(w) \cup \widetilde{\Phi}_2(w)$, where

$$\widetilde{\Phi}_1(w) := \{wt_{rs}t_{ir} \mid 1 \leq i < r \ \text{ and } \ \ell(wt_{rs}t_{ir}) = \ell(w)\},$$
$$\widetilde{\Phi}_2(w) := \{wt_{rs}\bar{t}_{ir} \mid i \neq r \ \text{ and } \ \ell(wt_{rs}\bar{t}_{ir}) = \ell(w)\}.$$

To define $\widetilde{T}^k(w)$, we join w by an edge to each $v \in \widetilde{\Phi}(w)$, and attach to each $v \in \widetilde{\Phi}(w)$ its tree $\widetilde{T}^k(v)$.

According to [T7, Lemma 3 and §6], the k-transition trees $T^k(w)$ and $\widetilde{T}^k(w)$ are finite trees, all of whose nodes are increasing up to k. Moreover, the leaves of these trees are k-Grassmannian elements. For any k-strict (respectively, typed k-strict) partition λ, the *mixed Stanley coefficient* e^w_λ (respectively, d^w_λ) is defined to be the number of leaves of $T^k(w)$ (respectively, $\widetilde{T}^k(w)$) of shape λ. For instance, from Example 9 we deduce that

$$e^{3\bar{1}2645}_{(4,1)} = e^{3\bar{1}2645}_{(3,2)} = 1 \quad \text{and} \quad e^{3\bar{1}2645}_{(3,1,1)} = 2.$$

Example 10. When $k = 0$, there exist alternative combinatorial formulas for the integers e^w_λ and d^w_λ involving certain tableaux [H, Kr, La1, La2]. For instance, choose any w in W_∞ and let λ be a Young diagram with r rows such that $|\lambda| = \ell(w)$. We call a sequence $b = (b_1, \ldots, b_m)$ *unimodal* if for some index j we have

$$b_1 > b_2 > \cdots > b_{j-1} > b_j < b_{j+1} < \cdots < b_m.$$

A *subsequence* of b is any sequence $(b_{i_1}, \ldots, b_{i_p})$ with $1 \leq i_1 < \cdots < i_p \leq m$. A *Kraśkiewicz tableau* (also known as a *standard decomposition tableau*) for w of shape λ is a filling T of the boxes of λ with nonnegative integers in such a way that (i) if t_i is the sequence of entries in the i-th row of T, reading from left to right, then the row word $t_r \ldots t_1$ is a reduced word for w, and (ii) for each i, t_i is a unimodal subsequence of maximum length in $t_r \ldots t_{i+1}t_i$. Lam [La2] proved that e^w_λ is equal to the number of Kraśkiewicz tableaux for w of shape λ.

5. Schubert polynomials and symmetric functions

5.1. **The nilCoxeter algebra and Schubert polynomials.** Bernstein-Gelfand-Gelfand [BGG] and Demazure [D1, D2] used *divided difference operators* to construct an algorithm that produces Giambelli polynomials which represent the Schubert classes on the flag variety G/B in the Borel presentation (27) of the cohomology ring, starting from the choice of a representative for the class of a point. For the general linear group, Lascoux and Schützenberger [LS1, LS3] applied this algorithm with a very natural choice of top degree polynomial to define *Schubert polynomials*, a theory which included both single and double versions.

In a series of papers [FS, FK2, FK3, La1], Fomin, Stanley, Kirillov, and Lam developed an alternative approach to the theory of Schubert polynomials, which extended to all the classical types. The main idea was to use the nilCoxeter algebra of the Weyl group, an abstract algebra which is isomorphic to the algebra of divided difference operators. The work was complicated by the fact that in Lie types B, C, and D, there are many candidate theories of Schubert polynomials (see [BH, F2, LP1, LP2, T3, T4] for examples), and it was not clear why any one of them should be preferred from the others. However, the theory that was best understood from a combinatorial point of view was that of Billey and Haiman [BH], which incorporated both the Lascoux-Schützenberger type A Schubert polynomials and the classical Schur Q-functions. Double versions of the Billey-Haiman polynomials were recently introduced and studied by Ikeda, Mihalcea, and Naruse [IMN]. Their type B, C, and D double Schubert polynomials will be key ingredients in our story, and we discuss their construction using the nilCoxeter algebra below.

The *nilCoxeter algebra* $\mathcal{W}_n$ of the hyperoctahedral group W_n is the free associative algebra with unity generated by the elements $u_0, u_1, \ldots, u_{n-1}$ modulo the relations

$$\begin{array}{rcll} u_i^2 & = & 0, & i \geq 0\ ; \\ u_i u_j & = & u_j u_i, & |i-j| \geq 2\ ; \\ u_i u_{i+1} u_i & = & u_{i+1} u_i u_{i+1}, & i > 0\ ; \\ u_0 u_1 u_0 u_1 & = & u_1 u_0 u_1 u_0. & \end{array}$$

For any $w \in W_n$, choose a reduced word $a_1 \cdots a_\ell$ for w and define $u_w := u_{a_1} \ldots u_{a_\ell}$. Since the last three relations listed are the Coxeter relations for the Weyl group W_n, it follows that u_w is well defined. Moreover, the u_w for $w \in W_n$ form a free $\mathbb{Z}$-basis of $\mathcal{W}_n$. We denote the coefficient of $u_w \in \mathcal{W}_n$ in the expansion of the element $h \in \mathcal{W}_n$ by $\langle h, w \rangle$; thus $h = \sum_{w \in W_n} \langle h, w \rangle\, u_w$ for all $h \in \mathcal{W}_n$.

Let t be an indeterminate and define

$$\begin{aligned} A_i(t) &:= (1 + t u_{n-1})(1 + t u_{n-2}) \cdots (1 + t u_i)\ ; \\ \tilde{A}_i(t) &:= (1 - t u_i)(1 - t u_{i+1}) \cdots (1 - t u_{n-1})\ ; \\ C(t) &:= (1 + t u_{n-1}) \cdots (1 + t u_1)(1 + t u_0)(1 + t u_0)(1 + t u_1) \cdots (1 + t u_{n-1}). \end{aligned}$$

Suppose that $X = (x_1, x_2, \ldots)$, $Y = (y_1, y_2, \ldots)$, and $Z = (z_1, z_2, \ldots)$ are three infinite sequences of commuting independent variables. Let

$$C(X) := C(x_1) C(x_2) \cdots$$

and for $w \in W_n$, define

$$\mathfrak{C}_w(X\,; Y, Z) := \left\langle \tilde{A}_{n-1}(z_{n-1}) \cdots \tilde{A}_1(z_1) C(X) A_1(y_1) \cdots A_{n-1}(y_{n-1}), w \right\rangle. \tag{34}$$

Set $\mathfrak{C}_w(X\,; Y) := \mathfrak{C}_w(X\,; Y, 0)$. The polynomials $\mathfrak{C}_w(X\,; Y)$ are the type C Schubert polynomials of Billey and Haiman and the $\mathfrak{C}_w(X\,; Y, Z)$ are their double versions due to Ikeda, Mihalcea, and Naruse. Note that $\mathfrak{C}_w$ is really a polynomial in the Y and Z variables, with coefficients which are formal power series in X. In fact, these formal power series are symmetric in the X variables; this follows immediately from Fomin and Kirillov's result [FK3, Prop. 4.2] that $C(s)C(t) = C(t)C(s)$, for any two commuting variables s and t. We set

$$F_w(X) := \mathfrak{C}_w(X\,; 0, 0) = \langle C(X), w \rangle$$

and call F_w the *type C Stanley symmetric function* indexed by $w \in W_n$. For any $\varpi \in S_n$, the double Schubert polynomial $\mathfrak{S}_\varpi(Y, Z)$ of Lascoux and Schützenberger is given by

$$\mathfrak{S}_\varpi(Y, Z) := \mathfrak{C}_\varpi(0\,; Y, Z) = \left\langle \tilde{A}_{n-1}(z_{n-1}) \cdots \tilde{A}_1(z_1) A_1(y_1) \cdots A_{n-1}(y_{n-1}), \varpi \right\rangle.$$

The polynomial $\mathfrak{S}_\varpi(Y) := \mathfrak{S}_\varpi(Y, 0)$ is the single Schubert polynomial.

It is transparent from the definition that $\mathfrak{C}_w$ is stable under the natural inclusion of W_n in W_{n+1}; it follows that $\mathfrak{S}_\varpi$ and $\mathfrak{C}_w$ are well defined for $\varpi \in S_\infty$ and $w \in W_\infty$, respectively. This *stability property* of Schubert polynomials is special to the classical groups, and will be used in §7.3 in order to characterize them via their compatibility with divided difference operators (Theorem 11).

Given any Weyl group elements $u_1, \ldots, u_p, w$, we will write $u_1 \cdots u_p = w$ if $\ell(u_1) + \cdots + \ell(u_p) = \ell(w)$ and the product of $u_1, \ldots, u_p$ is equal to w. In this case we say that $u_1 \cdots u_p$ is a *reduced factorization* of w. Equation (34) is equivalent to the relations

$$\mathfrak{C}_w(X\,;Y,Z) = \sum_{uv=w} \mathfrak{S}_{u^{-1}}(-Z)\mathfrak{C}_v(X\,;Y) = \sum_{uvu'=w} \mathfrak{S}_{u^{-1}}(-Z)F_v(X)\mathfrak{S}_{u'}(Y) \tag{35}$$

summed over all reduced factorizations $uv = w$ and $uvu' = w$ (respectively) with $u, u' \in S_\infty$ (compare with [IMN, Cor. 8.10]). In particular, we have

$$\mathfrak{S}_\varpi(Y,Z) = \sum_{uu'=\varpi} \mathfrak{S}_{u^{-1}}(-Z)\mathfrak{S}_{u'}(Y) \tag{36}$$

summed over all reduced factorizations $uu' = \varpi$ in S_∞.

For the orthogonal types B and D we will work with coefficients in the ring $\mathbb{Z}[\frac{1}{2}]$. For $w \in W_\infty$, let $s(w)$ denote the number of i such that $w_i < 0$. The type B double Schubert polynomials $\mathfrak{B}_w$ are related to the type C polynomials by the equation $\mathfrak{B}_w = 2^{-s(w)}\mathfrak{C}_w$. We will use the nilCoxeter algebra $\widetilde{\mathcal{W}}_n$ of $\widetilde{W}_n$ to define type D Schubert polynomials. $\widetilde{\mathcal{W}}_n$ is the free associative algebra with unity generated by the elements $u_\Box, u_1, \ldots, u_{n-1}$ modulo the relations

$$\begin{array}{rcll}
u_i^2 & = & 0, & i \in \mathbb{N}_\Box\ ; \\
u_\Box u_1 & = & u_1 u_\Box, & \\
u_\Box u_2 u_\Box & = & u_2 u_\Box u_2, & \\
u_i u_{i+1} u_i & = & u_{i+1} u_i u_{i+1}, & i > 0\ ; \\
u_i u_j & = & u_j u_i, & j > i+1, \text{ and } (i,j) \neq (\Box, 2).
\end{array}$$

For any $w \in \widetilde{W}_n$, choose a reduced word $a_1 \cdots a_\ell$ for w and define $u_w := u_{a_1} \ldots u_{a_\ell}$. We denote the coefficient of $u_w \in \widetilde{\mathcal{W}}_n$ in the expansion of the element $h \in \widetilde{\mathcal{W}}_n$ by $\langle h, w\rangle$. Let t be an indeterminate and, following Lam [La1], define

$$D(t) := (1 + tu_{n-1}) \cdots (1 + tu_2)(1 + tu_1)(1 + tu_\Box)(1 + tu_2) \cdots (1 + tu_{n-1}).$$

Let $D(X) = D(x_1)D(x_2)\cdots$, and define

$$\mathfrak{D}_w(X\,;Y,Z) := \left\langle \tilde{A}_{n-1}(z_{n-1}) \cdots \tilde{A}_1(z_1)D(X)A_1(y_1) \cdots A_{n-1}(y_{n-1}), w \right\rangle.$$

The polynomials $\mathfrak{D}_w(X\,;Y) := \mathfrak{D}_w(X\,;Y,0)$ are the type D Billey-Haiman Schubert polynomials, and the $\mathfrak{D}_w(X\,;Y,Z)$ are their double versions studied in [IMN].

Example 11. For $\varpi = 321 \in S_3$ we have that

$$\begin{aligned}
\mathfrak{S}_{321}(Y,Z) &= \langle (1 - z_2u_2)(1 - z_1u_1)(1 - z_1u_2)(1 + y_1u_2)(1 + y_1u_1)(1 + y_2u_2), 321\rangle \\
&= -z_2z_1^2 + z_2z_1y_1 + z_2z_1y_2 - z_2y_1y_2 + z_1^2y_1 - z_1y_1^2 - z_1y_1y_2 + y_1^2y_2 \\
&= (y_1 - z_1)(y_1 - z_2)(y_2 - z_1).
\end{aligned}$$

Let $\varpi_0 = (n, n-1, \ldots, 1)$ be the longest element of S_n. Then one can show (see [FS, Cor. 4.4]) that

$$\mathfrak{S}_{\varpi_0}(Y,Z) = \prod_{i+j \leq n} (y_i - z_j).$$

There are more complicated formulas (see [IMN, Thm. 1.2]) for the double Schubert polynomials $\mathfrak{C}_{w_0}$ and $\mathfrak{D}_{w_0}$, where w_0 denotes the longest element in the respective Weyl group. We do not require these formulas in this article.

5.2. **Schur, theta, and eta polynomials.** In this section, we use raising operators to construct the polynomials from [BKT2, BKT4] which will appear later in splitting formulas for the double Schubert polynomials.

For $m,n\geq 1$, set $Y_{(m)}=(y_1,\ldots,y_m)$ and $Z_{(n)}=(z_1,\ldots,z_n)$. Define the complete supersymmetric polynomials $h_r=h_r(Y_{(m)}/Z_{(n)})$ by their generating function

$$\prod_{i=1}^m(1-y_it)^{-1}\prod_{j=1}^n(1-z_jt)=\sum_{r=0}^\infty h_rt^r.$$

The *supersymmetric Schur polynomial* $s_\lambda(Y_{(m)}/Z_{(n)})$ is obtained by setting

$$s_\lambda(Y_{(m)}/Z_{(n)}):=R^0\,h_\lambda=\det\left(h_{\lambda_i+j-i}(Y_{(m)}/Z_{(n)})\right)_{i,j}$$

for any partition λ, where $h_\lambda:=\prod_i h_{\lambda_i}$. The usual Schur polynomials satisfy the identities

$$s_\lambda(Y_{(m)})=s_\lambda(Y_{(m)}/Z_{(n)})|_{Z_{(n)}=0}$$

and

$$s_\lambda(0/Z_{(n)})=s_\lambda(Y_{(m)}/Z_{(n)})|_{Y_{(m)}=0}=(-1)^{|\lambda|}s_{\widetilde{\lambda}}(Z_{(n)}).$$

Fix an integer $k\geq 0$, and define formal power series $\vartheta_r=\vartheta_r(X\,;Y_{(k)})$ for $r\in\mathbb{Z}$ by the equation

$$\prod_{i=1}^\infty\frac{1+x_it}{1-x_it}\prod_{j=1}^k(1+y_jt)=\sum_{r=0}^\infty\vartheta_rt^r.$$

Set $\vartheta_\lambda:=\prod_i\vartheta_{\lambda_i}$ and

$$\Theta_\lambda(X\,;Y_{(k)}):=R^\lambda\,\vartheta_\lambda \tag{37}$$

for any k-strict partition λ, where R^λ is the raising operator in (12). Following [BKT2], we call $\Theta_\lambda(X\,;Y_{(k)})$ a *theta polynomial*, although this is a slight abuse of language (compare with Remark 1). Define

$$\eta_k:=\frac{1}{2}\vartheta_k+\frac{1}{2}e_k(Y_{(k)})\ \text{ and }\ \eta'_k:=\frac{1}{2}\vartheta_k-\frac{1}{2}e_k(Y_{(k)})=\frac{1}{2}\sum_{i=0}^{k-1}q_{k-i}(X)e_i(Y_{(k)}). \tag{38}$$

For any typed k-strict partition λ, consider the *eta polynomial*

$$H_\lambda(X\,;Y_{(k)}):=2^{-\ell_k(\lambda)}\,R^\lambda\star\vartheta_\lambda. \tag{39}$$

The raising operator expression in (39) is defined in the same way as the analogous one in equation (25), but using ϑ_r and η_k,η'_k in place of c_r and τ_k,τ'_k, respectively.

When $k=0$, the indexing partitions λ are strict, we have that $\vartheta_r(X)=q_r(X)$ and $\Theta_\lambda(X\,;Y_{(0)})=Q_\lambda(X)$ are the classical Schur Q-functions discussed in §2.1, and $H_\lambda(X\,;Y_{(0)})=P_\lambda(X)$ is a Schur P-function. The ring $\Gamma:=\mathbb{Z}[q_1,q_2,\ldots]$ is the ring of Schur Q-functions and $\Gamma':=\mathbb{Z}[P_1,P_2,\ldots]$ is the ring of Schur P-functions (see [HH] and [M, III.8]). For our purposes here we need to know that $\mathfrak{C}_w(X\,;Y,Z)$ lies in $\Gamma[Y,Z]$ and $\mathfrak{D}_w(X\,;Y,Z)$ lies in $\Gamma'[Y,Z]$. These assertions follow from Example 13 below; see also equation (65) for a quick proof of the first one.

Theorem 4 ([LS1, BKT2, BKT4])**.** (a) *If $\varpi\in S_\infty$ is a Grassmannian permutation with a unique descent at m and λ is the corresponding partition of length at most m, then we have*

$$\mathfrak{S}_\varpi(Y)=s_\lambda(Y_{(m)}). \tag{40}$$

(b) *If $w_\lambda \in W_\infty$ is the k-Grassmannian element corresponding to the k-strict partition λ, then we have*

$$\mathfrak{C}_{w_\lambda}(X\,;Y) = \Theta_\lambda(X\,;Y_{(k)}). \tag{41}$$

(c) *If $w_\lambda \in \widetilde{W}_\infty$ is the k-Grassmannian element corresponding to the typed k-strict partition λ, then we have*

$$\mathfrak{D}_{w_\lambda}(X\,;Y) = H_\lambda(X\,;Y_{(k)}). \tag{42}$$

Although the equality of polynomials (40) in $\mathbb{Z}[Y]$ is directly analogous to (41) and (42), the latter two equations are much harder to prove. One reason for this is that e.g. (41) is an equality in the ring

$$\Gamma[Y] = \mathbb{Z}[q_1(X), q_2(X), \ldots\,;\, y_1, y_2, \ldots],$$

where there are relations among the q_r, and these relations play a crucial role in the proof. In fact, the proofs of (41) and (42) in [BKT2, BKT4] rely on Theorems 2 and 3 (see §7.2). We will generalize Theorem 4 below to a result (Theorem 5) which gives analogous formulas for arbitrary double Schubert polynomials.

Remark 3. Each of the Schur, theta, and eta polynomials in Theorem 4 may be expressed as a sum of monomials indexed by certain tableaux on λ. The latter are fillings of the boxes of the Young diagram of λ with (marked) positive integers satisfying natural conditions. These *tableau formulas* have the form

$$s_\lambda(Y_{(m)}) = \sum_T y^T,\ \ \Theta_\lambda(X\,;Y_{(k)}) = \sum_U 2^{n(U)}(xy)^U,\ \ H_\lambda(X\,;Y_{(k)}) = \sum_V 2^{n(V)}(xy)^V$$

summed over all semistandard tableaux T with entries at most m, all k-bitableaux U, and all typed k'-bitableaux V, respectively, on the diagram of λ. The precise definitions of these objects are given in [F4, T5, T8].

Example 12. (a) Let ϖ be a Grassmannian permutation with unique descent at m and $\lambda = (\varpi_m - m, \ldots, \varpi_1 - 1)$ be the corresponding partition. The double Schubert polynomial $\mathfrak{S}_\varpi(Y,Z)$ is a *double Schur polynomial* $s_\lambda(Y_{(m)}, Z)$, also known as a factorial Schur function. In this case, formula (36) can be made more explicit, by working as follows. In any reduced factorization $\varpi = uu'$, the right factor u' is also Grassmannian, and corresponds to a partition μ whose diagram is contained in the diagram of λ; moreover, we have $\mathfrak{S}_{u'}(Y) = s_\mu(Y_{(m)})$. The left factor u is a fully commutative element of S_∞ (in the sense of [St2]), and it follows from [BJS, §2] that the Schubert polynomial $\mathfrak{S}_{u^{-1}}(Z)$ in (36) is a flagged skew Schur polynomial. Setting $Z_{(r)} = (z_1, \ldots, z_r)$ for each r, we conclude (see also [Kre, Prop. 4.1]) that

$$s_\lambda(Y_{(m)}, Z) = \sum_{\mu \subset \lambda} s_\mu(Y_{(m)}) \det\left(e_{\lambda_i - \mu_j - i + j}(-Z_{(\lambda_i + m - i)})\right)_{1 \leq i,j \leq m}.$$

(b) Let λ be a k-strict partition. In any reduced factorization $w_\lambda = uv$ of the k-Grassmannian element w_λ in W_∞, the right factor v is also k-Grassmannian, and hence $v = w_\mu$ for some k-strict partition μ. Assuming that $u \in S_\infty$, we note that u may not be fully commutative (for example, let $k = 1$ and consider the factorization $2\overline{3}1 = uv$ with $u = 321$ and $v = 2\overline{1}3$), although it is a *skew* element of W_∞, in the sense of [T5, Def. 4]. Equations (35) and (41) give

$$\mathfrak{C}_{w_\lambda}(X\,;Y,Z) = \sum_{uw_\mu = w_\lambda} \Theta_\mu(X\,;Y_{(k)})\mathfrak{S}_{u^{-1}}(-Z) \tag{43}$$

summed over all reduced factorizations $uw_\mu = w_\lambda$ with $u \in S_\infty$. It follows from [T5, Prop. 4] that the k-strict partitions μ in (43) satisfy $\mu \subset \lambda$ and $\ell_k(\mu) = \ell_k(\lambda)$. Suppose next that $k = 0$ and λ is a strict partition of length ℓ. Then w_λ is a fully commutative element and $\mathfrak{C}_{w_\lambda}(X\,;Y,Z) = Q_\lambda(X\,;Z)$ is a double analogue of Schur's Q-function introduced by Ivanov [Iv]. Equation (43) becomes

$$Q_\lambda(X\,;Z) = \sum_{\mu\subset\lambda} Q_\mu(X) \det\left(e_{\lambda_i-\mu_j}(-Z_{(\lambda_i-1)})\right)_{1\leq i,j\leq\ell}$$

summed over all strict partitions $\mu \subset \lambda$ with $\ell(\mu) = \ell(\lambda) = \ell$.

5.3. **Splitting formulas for Schubert polynomials.** Following [BKTY1, T7], we say that an element $w \in W_\infty$ is *compatible* with the sequence $\mathfrak{a} : a_1 < \cdots < a_p$ of elements of $\mathbb{N}_0$ if all descent positions of w are contained in $\mathfrak{a}$. We say that an element $w \in \widetilde{W}_\infty$ is *compatible* with the sequence $\mathfrak{a} : a_1 < \cdots < a_p$ of elements of $\mathbb{N}_\Box$ if all descent positions of w are listed among $\Box, a_1, \ldots, a_p$, if $a_1 = 1$, or contained in $\mathfrak{a}$, otherwise. Such sequences $\mathfrak{a}$ parametrize parabolic subgroups of the respective Weyl groups, and w is compatible with $\mathfrak{a}$ if and only if it indexes a Schubert variety in the associated homogeneous space, or a corresponding degeneracy locus (see §3 and §6).

Let w be an element of W_∞ (respectively $\widetilde{W}_\infty$) compatible with $\mathfrak{a}$ as above and let $\mathfrak{b} : b_1 < \cdots < b_q$ be a second sequence of elements of $\mathbb{N}_0$ (respectively $\mathbb{N}_\Box$) such that w^{-1} is compatible with $\mathfrak{b}$. Suppose that $w = \varpi \in S_\infty$ and $a_1 > 0$. Given any sequence of partitions $\underline{\lambda} = (\lambda^1, \ldots, \lambda^{p+q-1})$, we define

$$c_{\underline{\lambda}}^{\varpi} := \sum_{u_1\cdots u_{p+q-1}=\varpi} c_{\lambda^1}^{u_1} \cdots c_{\lambda^{p+q-1}}^{u_{p+q-1}}. \tag{44}$$

Next suppose that $w \in W_\infty$ and $b_1 = 0$. Given any sequence of partitions $\underline{\lambda} = (\lambda^1, \ldots, \lambda^{p+q-1})$ with λ^q a_1-strict, we define

$$f_{\underline{\lambda}}^{w} := \sum_{u_1\cdots u_{p+q-1}=w} c_{\lambda^1}^{u_1} \cdots c_{\lambda^{q-1}}^{u_{q-1}} e_{\lambda^q}^{u_q} c_{\lambda^{q+1}}^{u_{q+1}} \cdots c_{\lambda^{p+q-1}}^{u_{p+q-1}}. \tag{45}$$

Finally, suppose that $w \in \widetilde{W}_\infty$ and $b_1 = \Box$. Given any sequence of partitions $\underline{\lambda} = (\lambda^1, \ldots, \lambda^{p+q-1})$ with λ^q a_1-strict and typed, define

$$g_{\underline{\lambda}}^{w} := \sum_{u_1\cdots u_{p+q-1}=w} c_{\lambda^1}^{u_1} \cdots c_{\lambda^{q-1}}^{u_{q-1}} d_{\lambda^q}^{u_q} c_{\lambda^{q+1}}^{u_{q+1}} \cdots c_{\lambda^{p+q-1}}^{u_{p+q-1}}. \tag{46}$$

The sums in equations (44), (45), and (46) are over all reduced factorizations $u_1 \cdots u_{p+q-1} = w$ such that $u_i \in S_\infty$ for all $i \neq q$, $u_j(i) = i$ whenever $j < q$ and $i \leq b_{q-j}$ or whenever $j > q$ and $i \leq a_{j-q}$ (and where we set $u_j(0) = 0$). Moreover, the nonnegative integers c_λ^u, e_λ^u, and d_λ^u in these formulas are the (mixed) Stanley coefficients which were defined in §4.3.

It is shown in [BKTY1, §4] that the $c_{\underline{\lambda}}^{\varpi}$ in (44) may be interpreted as special cases of type A *quiver coefficients*. These are certain integers which appear in a formula of Buch and Fulton [BF] for the cohomology class of a quiver variety (a degeneracy locus associated to a sequence of vector bundles and vector bundle morphisms over a fixed base, in the form of an oriented quiver of type A). Equations (45) and (46) therefore provide analogues $f_{\underline{\lambda}}^w$ and $g_{\underline{\lambda}}^w$ of these coefficients in the other classical Lie types, for the degeneracy loci considered in §6.

Set $Y_i := \{y_{a_{i-1}+1}, \ldots, y_{a_i}\}$ for each $i \geq 1$ and $Z_j := \{z_{b_{j-1}+1}, \ldots, z_{b_j}\}$ for each $j \geq 1$. Notice in particular that $Y_1 = \emptyset$ if $a_1 = 0$ or $a_1 = \Box$.

Theorem 5 (Splitting Schubert polynomials, [BKTY1, T7])**.** (a) *Suppose that $a_1 > 0$ and that ϖ and ϖ^{-1} in S_∞ are compatible with the sequences $\mathfrak{a}$ and $\mathfrak{b}$, respectively. Then the Schubert polynomial $\mathfrak{S}_\varpi(Y, Z)$ satisfies*

$$(47) \quad \mathfrak{S}_\varpi = \sum_{\underline{\lambda}} c_{\underline{\lambda}}^{\varpi}\, s_{\lambda^1}(0/Z_q) \cdots s_{\lambda^{q-1}}(0/Z_2) s_{\lambda^q}(Y_1/Z_1) s_{\lambda^{q+1}}(Y_2) \cdots s_{\lambda^{p+q-1}}(Y_p)$$

summed over all sequences of partitions $\underline{\lambda} = (\lambda^1, \ldots, \lambda^{p+q-1})$.

(b) *Suppose that $b_1 = 0$ and that w and w^{-1} in W_∞ are compatible with the sequences $\mathfrak{a}$ and $\mathfrak{b}$, respectively. Then the Schubert polynomial $\mathfrak{C}_w(X\,; Y, Z)$ satisfies*

$$(48) \quad \mathfrak{C}_w = \sum_{\underline{\lambda}} f_{\underline{\lambda}}^{w}\, s_{\lambda^1}(0/Z_q) \cdots s_{\lambda^{q-1}}(0/Z_2) \Theta_{\lambda^q}(X\,; Y_1) s_{\lambda^{q+1}}(Y_2) \cdots s_{\lambda^{p+q-1}}(Y_p)$$

summed over all sequences of partitions $\underline{\lambda} = (\lambda^1, \ldots, \lambda^{p+q-1})$ with λ^q a_1-strict.

(c) *Suppose that $b_1 = \Box$ and that w and w^{-1} in $\widetilde{W}_\infty$ are compatible with the sequences $\mathfrak{a}$ and $\mathfrak{b}$, respectively. Then the Schubert polynomial $\mathfrak{D}_w(X\,; Y, Z)$ satisfies*

$$(49) \quad \mathfrak{D}_w = \sum_{\underline{\lambda}} g_{\underline{\lambda}}^{w}\, s_{\lambda^1}(0/Z_q) \cdots s_{\lambda^{q-1}}(0/Z_2) H_{\lambda^q}(X\,; Y_1) s_{\lambda^{q+1}}(Y_2) \cdots s_{\lambda^{p+q-1}}(Y_p)$$

summed over all sequences of partitions $\underline{\lambda} = (\lambda^1, \ldots, \lambda^{p+q-1})$ with λ^q a_1-strict and typed.

Theorem 5 includes multiple formulas for the double Schubert polynomials in each Lie type, depending on the choice of the two sequences $\mathfrak{a}$ and $\mathfrak{b}$. For each fixed Weyl group element w, there is a basic formula which involves the descents of w and w^{-1}, defined by choosing $\mathfrak{a}$ and $\mathfrak{b}$ to be minimal, and all the other formulas are obtained from it by splitting the groups of variables further.

On the other hand, suppose that the sequences $\mathfrak{a}$ and $\mathfrak{b}$ are fixed; then the expressions for the Schubert polynomials displayed in equations (47), (48), and (49) are *uniquely determined*. This follows from the fact that the Schur, theta, and eta polynomials in the respective groups of variables appearing in Theorem 5 are linearly independent over the ring of integers (see also Remark 4). It would be desirable to eliminate the hypothesis on b_1 from parts (b) and (c) of the theorem; for some obstacles in the way of achieving this, see [T7, Example 3].

Example 13. Suppose that $a_1 = 0$ and $a_1 = \Box$ in parts (b) and (c) of Theorem 5. Then we obtain the splitting formulas

$$(50) \qquad \mathfrak{C}_w = \sum_{\underline{\lambda}} f_{\underline{\lambda}}^{w}\, s_{\lambda^1}(0/Z_q) \cdots s_{\lambda^{q-1}}(0/Z_2) Q_{\lambda^q}(X) s_{\lambda^{q+1}}(Y_2) \cdots s_{\lambda^{p+q-1}}(Y_p)$$

and

$$(51) \qquad \mathfrak{D}_w = \sum_{\underline{\lambda}} g_{\underline{\lambda}}^{w}\, s_{\lambda^1}(0/Z_q) \cdots s_{\lambda^{q-1}}(0/Z_2) P_{\lambda^q}(X) s_{\lambda^{q+1}}(Y_2) \cdots s_{\lambda^{p+q-1}}(Y_p),$$

respectively, where the sums are over all sequences of partitions $\underline{\lambda} = (\lambda^1, \ldots, \lambda^{p+q-1})$ with λ^q strict. Thus we see that in all cases, the Schubert polynomials can be expressed as a *positive* linear combination of products of Jacobi-Trudi determinants times (at most) a single Schur Pfaffian. Recall also that in this situation, there exist tableau-based combinatorial expressions for the coefficients $f^w_{\underline{\lambda}}$ and $g^w_{\underline{\lambda}}$. Single Schubert polynomial versions of equations (50) and (51) were given in [Yo].

Example 14. To simplify the notation, write $\mathfrak{C}_w$ for $\mathfrak{C}_w(X\,;Y,Z)$, Q_λ for $Q_\lambda(X)$, and $\Theta^{(k)}_\lambda$ for $\Theta_\lambda(X\,;Y_{(k)})$.

(a) Let $w = 3\overline{1}\overline{2} = s_0s_1s_0s_1s_2s_1 \in W_3$ and set $\mathfrak{b} = (0 < 1)$. The following equalities correspond to $\mathfrak{a} = (1 < 2)$ and $\mathfrak{a} = (0 < 1 < 2)$, respectively, in Theorem 5(b).

$$\begin{aligned}
\mathfrak{C}_{3\overline{1}\overline{2}} &= \Theta^{(1)}_{(4,2)} + \Theta^{(1)}_{(3,2)}\, y_2 - z_1\, \Theta^{(1)}_{(3,2)} \\
&= Q_{(4,2)} + Q_{(4,1)}\, y_1 + Q_{(3,2)}\, y_1 + Q_{(3,1)}\, y_1^2 + \left(Q_{(3,2)} + Q_{(3,1)}\, y_1 + Q_{(2,1)}\, y_1^2\right) y_2 \\
&\quad - z_1 \left(Q_{(3,2)} + Q_{(3,1)}\, y_1 + Q_{(2,1)}\, y_1^2\right).
\end{aligned}$$

(b) Let $w = 12\overline{3} = s_2s_1s_0s_1s_2 \in W_3$, set $\mathfrak{b} = (0 < 2)$, $e^y_i = e_i(y_1, y_2)$ and $e^z_i = e_i(z_1, z_2)$ for $i = 1, 2$. The following equalities correspond to taking $\mathfrak{a}$ to be (2), $(1 < 2)$, and $(0 < 2)$, respectively, in Theorem 5(b).

$$\begin{aligned}
\mathfrak{C}_{12\overline{3}} &= \Theta^{(2)}_5 - e^z_1\, \Theta^{(2)}_4 + e^z_2\, \Theta^{(2)}_3 \\
&= \Theta^{(1)}_5 + \Theta^{(1)}_4\, y_2 - e^z_1 \left(\Theta^{(1)}_4 + \Theta^{(1)}_3\, y_2\right) + e^z_2 \left(\Theta^{(1)}_3 + \Theta^{(1)}_2\, y_2\right) \\
&= Q_5 + Q_4\, e^y_1 + Q_3\, e^y_2 - e^z_1 \left(Q_4 + Q_3\, e^y_1 + Q_2\, e^y_2\right) + e^z_2 \left(Q_3 + Q_2\, e^y_1 + Q_1\, e^y_2\right).
\end{aligned}$$

6. Degeneracy loci

In this section, we show how the splitting formulas of Theorem 5 translate directly into Chern class formulas for degeneracy loci in the sense of [F3] and §3.2, with the symmetries native to the corresponding G/P space.

Let $c = (c_0, c_1, c_2, \ldots)$ and $d = (d_0, d_1, d_2, \ldots)$ be two families of commuting variables, with $c_0 = d_0 = 1$ as usual. We first extend the definitions of the polynomials $s_\lambda(c)$, $\Theta_\lambda(c)$, and $H_\lambda(c)$ from §1 and §2 to obtain polynomials in the formal difference $c - d$ (this notation stems from the theory of λ-rings). Define elements g_r and h_r for $r \in \mathbb{Z}$ by the identities of formal power series

$$\sum_{r=-\infty}^{+\infty} g_r t^r := \left(\sum_{i=0}^{\infty} c_i t^i\right)\left(\sum_{i=0}^{\infty} d_i t^i\right)^{-1} ;$$

$$\sum_{r=-\infty}^{+\infty} h_r t^r := \left(\sum_{i=0}^{\infty} (-1)^i c_i t^i\right)^{-1}\left(\sum_{i=0}^{\infty} (-1)^i d_i t^i\right).$$

For any partition λ, define the *Schur polynomial* $s_\lambda(c-d)$ by

$$s_\lambda(c-d) := R^0\, h_\lambda = \det(h_{\lambda_i+j-i})_{i,j}.$$

For any k-strict partition λ, define the theta polynomial $\Theta_\lambda(c-d)$ by

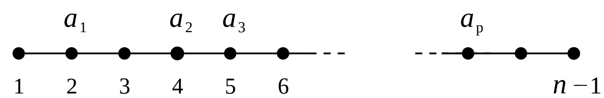
$$\Theta_\lambda(c-d) := R^\lambda\, g_\lambda,$$

where R^λ denotes the raising operator in (12). Finally, for any typed k-strict partition λ, define the eta polynomial $H_\lambda(c-d)$ by

$$H_\lambda(c-d) := 2^{-\ell_k(\lambda)} R^\lambda \star g_\lambda,$$

where R^λ denotes the raising operator in (24), and the action $\star$ is defined as in §2.2, replacing c_λ by g_λ throughout. If V and V' are two vector bundles on an algebraic variety M with total Chern classes $c(V)$ and $c(V')$, respectively, we denote the class $s_\lambda(c(V)-c(V'))$ by $s_\lambda(V-V')$, the class $\Theta_\lambda(c(V)-c(V'))$ by $\Theta_\lambda(V-V')$, and the class $H_\lambda(c(V)-c(V'))$ by $H_\lambda(V-V')$.

6.1. **Type A degeneracy loci.** Fix a sequence $\mathfrak{a} : a_1 < \cdots < a_p$ of positive integers with $a_p < n$, which gives a subset of the nodes of the Dynkin diagram for the root system of type A_{n-1}:



Let $S^{\mathfrak{a}}$ be the set of permutations $\varpi \in S_n$ whose descent positions are listed among the integers $a_1, \ldots, a_p$, i.e., that are compatible with $\mathfrak{a}$. These elements are the minimal length coset representatives in $S_n/S_{\mathfrak{a}}$, where $S_{\mathfrak{a}}$ denotes the parabolic subgroup of S_n generated by the simple transpositions s_i for $i \notin \{a_1, \ldots, a_p\}$.

Let $E \to M$ be a vector bundle of rank n on an algebraic variety M, assumed to be smooth for simplicity. Consider a partial flag of subbundles of E

$$0 \subset E_1 \subset \cdots \subset E_p \subset E$$

with $\operatorname{rank} E_r = a_r$ for each r, and a complete flag

$$0 \subset F_1 \subset \cdots \subset F_n = E$$

of subbundles of E (with $\operatorname{rank} F_s = s$ for each s). For every $\varpi \in S^{\mathfrak{a}}$, we define the *degeneracy locus* $\mathfrak{X}_\varpi \subset M$ as the locus of $b \in M$ such that

$$\dim(E_r(b) \cap F_s(b)) \geq \#\,\{\, i \leq a_r \mid \varpi_i > n-s \,\} \ \ \forall\, r, s.$$

A precise definition of $\mathfrak{X}_\varpi$ as a subscheme of $\mathfrak{X}$ can be obtained by pulling back from the universal case, which occurs on the partial flag bundle $\mathrm{F}^{\mathfrak{a}}(E)$ (see [F3] and [FP, §6.2 and App. A] for more details). Assume further that $\mathfrak{X}_\varpi$ has pure codimension $\ell(\varpi)$ in M, which is the case when the vector bundles are in general position. The next result will be a formula for the cohomology class $[\mathfrak{X}_\varpi]$ in $\mathrm{H}^{2\ell(\varpi)}(M)$ in terms of the Chern classes of the bundles E_r and F_s.

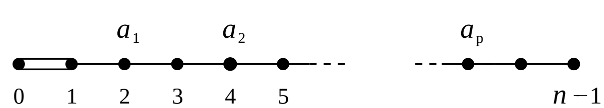
Consider a second sequence $\mathfrak{b} : 0 \leq b_1 < \cdots < b_q$ with $b_q < n$.

Theorem 6 ([BKTY1])**.** *Suppose that $\varpi \in S^{\mathfrak{a}}$ and that ϖ^{-1} is compatible with $\mathfrak{b}$. Then we have*

$$[\mathfrak{X}_\varpi] = \sum_{\underline{\lambda}} c^{\varpi}_{\underline{\lambda}}\, s_{\widetilde{\lambda}^1}(F_{n-b_{q-1}} - F_{n-b_q}) \cdots s_{\widetilde{\lambda}^q}(E - E_1 - F_{n-b_1}) \cdots s_{\widetilde{\lambda}^{p+q-1}}(E_{p-1} - E_p)$$

in $\mathrm{H}^*(M)$, where the sum is over all sequences of partitions $\underline{\lambda} = (\lambda^1, \ldots, \lambda^{p+q-1})$ and the coefficients $c^w_{\underline{\lambda}}$ are given by (44).

6.2. **Symplectic degeneracy loci.** Fix a sequence $\mathfrak{a} : a_1 < \cdots < a_p$ of nonnegative integers with $a_p < n$, which is a subset of the nodes of the Dynkin diagram for the root system of type C_n:



Denote by $W^{\mathfrak{a}}$ the set of signed permutations $w \in W_n$ whose descent positions are listed among the integers $a_1, \ldots, a_p$.

Let $E \to M$ be a vector bundle of rank $2n$ on a smooth algebraic variety M. Assume that E is a *symplectic* bundle, so that E is equipped with an everywhere nondegenerate skew-symmetric form $E \otimes E \to \mathbb{C}$. Consider a partial flag of subbundles of E

$$0 \subset E_p \subset \cdots \subset E_1 \subset E$$

with $\operatorname{rank} E_r = n - a_r$ and E_1 isotropic, and a (complete) isotropic flag

$$0 \subset F_1 \subset \cdots \subset F_{2n} = E$$

of subbundles of E with $\operatorname{rank} F_s = s$ for each s (and $F_{n+s} = F_{n-s}^{\perp}$ for $0 \leq s < n$). Fix a second sequence $\mathfrak{b} : 0 = b_1 < \cdots < b_q$ with $b_q < n$, and define quotient bundles

$$\begin{gathered} Q_1 := E/E_1,\ Q_2 := E_1/E_2,\ \ldots,\ Q_p := E_{p-1}/E_p \\ \widehat{Q}_2 := F_n/F_{n-b_2},\ \ldots,\ \widehat{Q}_q := F_{n-b_{q-1}}/F_{n-b_q}. \end{gathered} \tag{52}$$

There is a group monomorphism $\phi : W_n \hookrightarrow S_{2n}$ with image

$$\phi(W_n) = \{\, \varpi \in S_{2n} \mid \varpi_i + \varpi_{2n+1-i} = 2n+1, \quad \text{for all } i \,\}.$$

The map ϕ is determined by setting, for each $w = (w_1, \ldots, w_n) \in W_n$ and $1 \leq i \leq n$,

$$\phi(w)_i := \begin{cases} n + 1 - w_{n+1-i} & \text{if } w_{n+1-i} \text{ is unbarred,} \\ n + \overline{w}_{n+1-i} & \text{otherwise.} \end{cases}$$

For every $w \in W^{\mathfrak{a}}$ we have the *degeneracy locus* $\mathfrak{X}_w \subset M$, which is the locus of $b \in M$ such that

$$\dim(E_r(b) \cap F_s(b)) \geq \#\{\, i \leq n - a_r \mid \phi(w)_i > 2n - s \,\} \ \ \forall\, r, s.$$

As in the type A case, assuming that $\mathfrak{X}_w$ has pure codimension $\ell(w)$ in M, we give a formula for the class $[\mathfrak{X}_w]$ in $\mathrm{H}^{2\ell(w)}(M)$.

Theorem 7 ([T7])**.** *Suppose that $w \in W^{\mathfrak{a}}$ and that w^{-1} is compatible with $\mathfrak{b}$. Then we have*

$$\begin{aligned} [\mathfrak{X}_w] &= \sum_{\underline{\lambda}} f^w_{\underline{\lambda}}\, s_{\widetilde{\lambda}^1}(\widehat{Q}_q) \cdots s_{\widetilde{\lambda}^{q-1}}(\widehat{Q}_2) \Theta_{\lambda^q}(Q_1 - F_n) s_{\lambda^{q+1}}(Q_2) \cdots s_{\lambda^{p+q-1}}(Q_p) \\ &= \sum_{\underline{\lambda}} f^w_{\underline{\lambda}}\, s_{\lambda^1}(F_{n+b_{q-1}} - F_{n+b_q}) \cdots \Theta_{\lambda^q}(E - E_1 - F_n) \cdots s_{\lambda^{p+q-1}}(E_{p-1} - E_p) \end{aligned}$$

in $\mathrm{H}^*(M)$, where the sum is over all sequences of partitions $\underline{\lambda} = (\lambda^1, \ldots, \lambda^{p+q-1})$ with λ^q a_1-strict, and the coefficients $f^w_{\underline{\lambda}}$ are given by (45).

6.3. **Orthogonal degeneracy loci.**

6.3.1. *The odd orthogonal case.* Let $E \to M$ be a vector bundle of rank $2n+1$ on a smooth algebraic variety M. Assume that E is an *orthogonal* bundle, i.e. E is equipped with an everywhere nondegenerate symmetric form $E \otimes E \to \mathbb{C}$. Fix a sequence $\mathfrak{a} : a_1 < \cdots < a_p$ of nonnegative integers with $a_p < n$, as in the symplectic case. Consider a partial flag of subbundles of E

$$0 \subset E_p \subset \cdots \subset E_1 \subset E$$

with $\operatorname{rank} E_r = n - a_r$ and E_1 isotropic, and a complete isotropic flag

$$0 \subset F_1 \subset \cdots \subset F_{2n+1} = E$$

of subbundles of E. Let $\mathfrak{b}\,:\,0=b_1<\cdots<b_q$ be a sequence of integers with $b_q<n$ and define the quotient bundles Q_r and $\widehat{Q}_s$ using the same equations (52) as in type C.

There is a group monomorphism $\phi' : W_n \hookrightarrow S_{2n+1}$ with image

$$\phi'(W_n)=\{\,\varpi\in S_{2n+1}\mid \varpi_i+\varpi_{2n+2-i}=2n+2,\quad \text{for all } i\,\}.$$

The map ϕ is determined by setting, for each $w=(w_1,\ldots,w_n)\in W_n$ and $1\leq i\leq n$,

$$\phi'(w)_i := \begin{cases} n+1-w_{n+1-i} & \text{if } w_{n+1-i} \text{ is unbarred,}\\ n+1+\overline{w}_{n+1-i} & \text{otherwise.}\end{cases}$$

Fix an element w in $W^{\mathfrak{a}}$. The *degeneracy locus* $\mathfrak{X}_w\subset M$ is the locus of $b\in M$ such that

$$\dim(E_r(b)\cap F_s(b))\geq \#\{\,i\leq n-a_r \mid \phi'(w)_i>2n+1-s\,\}\ \ \forall\, r,s,$$

and we assume as before that $\mathfrak{X}_w$ has pure codimension $\ell(w)$ in M. For any a_1-strict partition λ, let $\Theta'_\lambda=2^{-s(w)}\Theta_\lambda$, where $s(w)=\#\{i\mid w_i<0\}$. We then have the following analogue of Theorem 7.

Theorem 8 ([T7]). *Suppose that $w\in W^{\mathfrak{a}}$ and that w^{-1} is compatible with $\mathfrak{b}$. Then we have*

$$[\mathfrak{X}_w]=\sum_{\underline{\lambda}} f^w_{\underline{\lambda}}\, s_{\widetilde{\lambda}^1}(\widehat{Q}_q)\cdots s_{\widetilde{\lambda}^{q-1}}(\widehat{Q}_2)\Theta'_{\lambda^q}(Q_1-F_{n+1})s_{\lambda^{q+1}}(Q_2)\cdots s_{\lambda^{p+q-1}}(Q_p)$$

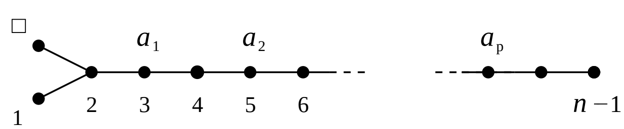
$$=\sum_{\underline{\lambda}} f^w_{\underline{\lambda}}\, s_{\lambda^1}(F_{n+1+b_{q-1}}-F_{n+1+b_q})\cdots\Theta'_{\lambda^q}(E-E_1-F_{n+1})\cdots s_{\lambda^{p+q-1}}(E_{p-1}-E_p)$$

in $\mathrm{H}^*(M)$, where the sum is over all sequences of partitions $\underline{\lambda}=(\lambda^1,\ldots,\lambda^{p+q-1})$ with λ^q a_1-strict, and the coefficients $f^w_{\underline{\lambda}}$ are given by (45).

6.3.2. *The even orthogonal case.* Let $E\to M$ be an orthogonal vector bundle of rank $2n$ on a smooth algebraic variety M. Fix a complete isotropic flag

$$0\subset F_1\subset\cdots\subset F_{2n}=E$$

of subbundles of E, and a sequence $\mathfrak{a}\,:\,a_1<\cdots<a_p$ of elements of $\mathbb{N}_\Box$ with $a_p<n$, which gives a subset of the vertices of the Dynkin diagram for the root system of type D_n:



Consider a partial flag of subbundles of E

$$0\subset E_p\subset\cdots\subset E_1\subset E$$

with $\operatorname{rank} E_r=n-a_r$ and E_1 isotropic and in the same family as F_n if $a_1=\Box$. Fix a sequence $\mathfrak{b}\,:\,\Box=b_1<\cdots<b_q$ with $b_q<n$, and define the quotient bundles Q_r and $\widehat{Q}_s$ as in type C, using the equations (52).

Recall that $\widetilde{W}_n$ is a subgroup of W_n, and hence we have a group monomorphism $\phi:\widetilde{W}_n\hookrightarrow S_{2n}$, defined by restricting the map ϕ of §6.2 to $\widetilde{W}_n$. Let $\widetilde{W}^{\mathfrak{a}}$ be the set of signed permutations $w\in\widetilde{W}_n$ which are compatible with $\mathfrak{a}$. For every $w\in\widetilde{W}^{\mathfrak{a}}$, we define the *degeneracy locus* $\mathfrak{X}_w \subset M$ as the closure of the locus of $b \in M$ such that

$$\dim(E_r(b) \cap F_s(b)) = \#\{\, i \leq n - a_r \mid \phi(w_0ww_0)_i > 2n - s \,\} \ \forall\, r, s,$$

with the reduced scheme structure. Assume further that $\mathfrak{X}_w$ has pure codimension $\ell(w)$ in M, and consider its cohomology class $[\mathfrak{X}_w]$ in $\mathrm{H}^{2\ell(w)}(M)$.

Theorem 9 ([T7]). *Suppose that $w \in \widetilde{W}^{\mathfrak{a}}$ and that w^{-1} is compatible with $\mathfrak{b}$. Then we have*

$$\begin{aligned}[\mathfrak{X}_w] &= \sum_{\underline{\lambda}} g^w_{\underline{\lambda}}\, s_{\widetilde{\lambda}^1}(\widehat{Q}_q) \cdots s_{\widetilde{\lambda}^{q-1}}(\widehat{Q}_2) H_{\lambda^q}(Q_1 - F_n) s_{\lambda^{q+1}}(Q_2) \cdots s_{\lambda^{p+q-1}}(Q_p) \\ &= \sum_{\underline{\lambda}} g^w_{\underline{\lambda}}\, s_{\lambda^1}(F_{n+b_{q-1}} - F_{n+b_q}) \cdots H_{\lambda^q}(E - E_1 - F_n) \cdots s_{\lambda^{p+q-1}}(E_{p-1} - E_p),\end{aligned}$$

in $\mathrm{H}^*(M)$, where the sum is over all sequences of partitions $\underline{\lambda} = (\lambda^1, \ldots, \lambda^{p+q-1})$ with λ^q a_1-strict and typed, and the coefficients $g^w_{\underline{\lambda}}$ are given by (46).

Example 15. Suppose that $E = \mathbb{C}^N$ is a complex vector space, which in types B, C, and D is equipped with a nondegenerate symmetric or skew-symmetric bilinear form. The homogeneous space $M = F^{\mathfrak{a}}(E)$ which parametrizes all partial (isotropic) flags $E_\bullet$ in E is a G/P space for the appropriate classical group G and parabolic subgroup P corresponding to the sequence $\mathfrak{a}$. In this case $E_\bullet$ is also used to denote the tautological filtration of the trivial rank N vector bundle E over M, the vector bundles F_s are all trivial, and the formulas of this section solve the Giambelli problem for $\mathrm{H}^*(G/P)$ (see [T7, §5]). If we furthermore assume that $p = 1$, so that $\mathfrak{a} = \{a_1\}$ and M is an (isotropic) Grassmannian, then Theorems 6, 7, and 9 specialize to Theorems 1, 2, and 3, respectively.

Using the identification of degeneracy locus formulas with Giambelli polynomials in equivariant cohomology (see [Gr] and 3.2 and §7.3 of this paper), the theorems in this section translate easily into corresponding results for the T-equivariant cohomology ring of G/P. The Chern roots of the vector bundle F_n (up to a sign) are identified with the weights of the linear T-action on the standard representation V of G; specifically, the class $-c_1(F_{n+1-i}/F_{n-i})$ is mapped onto the variable t_i in the notation of [IMN, §10] (and the variable y_i in §7.3 below). The results may also be used to deduce formulas for the restriction of an equivariant Schubert class in $\mathrm{H}^*_T(G/P)$ to a torus fixed point; see op. cit. for more details.

7. Proofs of the main theorems

We discuss here what is involved in proving the aforementioned theorems. We pay particular attention to the proofs of Theorems 6–9, and establish the connection between the Schubert polynomials of [IMN] and the geometry of degeneracy loci.

7.1. **Proofs of** IG **and** OG **Giambelli.** Although briefly mentioned in §1 and §2, so far there has been little discussion of *Pieri rules* for isotropic Grassmannians. The first such rules were proved by Hiller and Boe [HB] for the maximal isotropic Grassmannians, and applied by Pragacz [P2] to prove his Giambelli formulas for these spaces. Pragacz and Ratajski [PR1, PR3] later obtained Pieri type rules for general isotropic Grassmannians, but used a different notion of ‘special Schubert classes' than ours (their special classes were the Chern classes of the universal subbundles E' in §2).

The proofs of Theorems 2 and 3 from [BKT2, BKT4] require the classical Pieri rules from [BKT1], which hold for the special Schubert classes c_r and τ_r defined in §2. For the symplectic Grassmannian $\mathrm{IG}(n-k, 2n)$, the Pieri rule has the form

$$c_p \cdot [X_\lambda] = \sum_{\lambda \xrightarrow{p} \mu} 2^{N(\lambda,\mu)}\, [X_\mu]\,. \tag{53}$$

When $k > 0$, the Pieri relation $\lambda \xrightarrow{p} \mu$ and the definition of the exponents $N(\lambda, \mu)$ is more complicated than in the $k = 0$ case of the Lagrangian Grassmannian. However, the Pieri rule (53) can still be used recursively to show that a general Schubert class $[X_\lambda]$ may be written as a polynomial in the special classes. Therefore, in order to prove Theorem 2, it suffices to show that the expression $R^\lambda\, c_\lambda$ satisfies the same Pieri rule (53), and this is the approach taken in [BKT2]. The complex argument that proves that $R^\lambda\, c_\lambda$ obeys (53) uses subtle alternating properties of $R^\lambda\, c_\lambda$ which depend on the order ideal $\mathcal{C}(\lambda)$, and a substitution algorithm which is a mathematical model of controlled evolution. The reader will find an exposition of the main ideas in [T5, §3]. The proof of Theorem 3 is similar to that of Theorem 2. It exploits the weight space decomposition of $\mathrm{H}^*(\mathrm{OG}(m, 2n), \mathbb{Q})$ induced by the natural involution of the Dynkin diagram of type D_n, and a surprising relation between the Schubert calculus on even and odd orthogonal Grassmannians.

Currently, there is a second, simpler proof of Theorem 2, using the *double theta polynomials* $\Theta_\lambda(c\,|\,t)$ defined by Wilson [TW, W]. These polynomials are analogues of the usual double (or factorial) Schur polynomials, and give equivariant Giambelli expressions for the Schubert classes in $\mathrm{H}_T^*(\mathrm{IG}(m, 2n))$. This was proved by Ikeda and Matsumura [IM] and by the author and Wilson [TW] in two different ways. Both proofs rely on a key technical result from [IM], which is a certain compatibility of the $\Theta_\lambda(c\,|\,t)$ with the action of t-divided difference operators.

7.2. **Proofs of Theorems 4 and 5.** Following [LS1, BH, FK3] or by specializing the arguments of §7.3, one shows that the single Schubert polynomials represent the Schubert classes in $\mathrm{H}^*(G/B)$, and that the algebra that they span is isomorphic to the stable cohomology ring $\mathbb{H}(G/B)$ of the complete flag manifold as $n \to \infty$. Therefore, to prove Theorem 4 it suffices to show that (a) the Schur, theta, and eta polynomials indexed by single row partitions agree with the corresponding single Schubert polynomials, and (b) the Giambelli formulas of §1 and §2 hold in $\mathbb{H}(G/B)$. Assertion (a) is easy to check directly from the definition of the Schubert polynomials, and (b) is an immediate consequence of Theorems 1, 2, and 3.

The proof of Theorem 5 is based on the transition equations for Schubert polynomials mentioned in §4.3, and some important actors that have remained largely in the background until now, the *Stanley symmetric functions* of [Sta, BH, FK3] and the *mixed Stanley functions* of [T7]. We will state the key result here for the type C Billey-Haiman Schubert polynomials $\mathfrak{C}_w(X\,;Y)$. Let $\mathfrak{C}_w(X\,;Y_{(k)})$ denote the power series obtained from $\mathfrak{C}_w(X\,;Y)$ by setting $y_j = 0$ for all $j > k$.

Theorem 10 ([T7]). *Suppose that $w \in W_\infty$ is increasing up to k. Then we have*

$$\mathfrak{C}_w(X\,;Y_{(k)}) = \sum_\lambda e^w_\lambda\, \Theta_\lambda(X\,;Y_{(k)}), \tag{54}$$

where the sum is over all k-strict partitions λ with $|\lambda| = \ell(w)$.

For instance, the $k=1$ transition tree in Example 9 corresponds to the equation

$$\mathfrak{C}_{3\bar{1}2645}(X\,;Y_{(1)}) = \Theta_{(2,1,1,1)} + \Theta_5 + 2\,\Theta_{(3,1,1)} + \Theta_{(4,1)} + \Theta_{(3,2)}.$$

Taking $k=0$ in Theorem 10 gives the expansion of the type C Stanley symmetric function F_w as a sum of Schur Q-functions, which was first proved by Billey [Bi1]:

$$F_w(X) = \sum_{\lambda\,:\,|\lambda|=\ell(w)} e^w_\lambda\, Q_\lambda(X) \tag{55}$$

with the sum over strict partitions λ of $\ell(w)$.

When w is increasing up to k, we can identify $\mathfrak{C}_w(X\,;Y_{(k)})$ in (54) with the restriction $J_w(X\,;Y_{(k)})$ of a 'mixed Stanley function' $J_w(X\,;Y)$ (see [T7, §2] for the precise definitions). Furthermore, one shows directly from their definition in §5.1 that the Schubert polynomials satisfy splitting formulas which express them as sums of products of (mixed) Stanley functions in mutually disjoint groups of variables. Theorem 5 follows immediately by combining these two ingredients.

7.3. **Divided differences and geometrization maps.** We define an action of W_∞ on $\Gamma[Y,Z]$ by ring automorphisms as follows. The simple reflections s_i for $i>0$ act by interchanging y_i and y_{i+1} and leaving all the remaining variables fixed. The reflection s_0 maps y_1 to $-y_1$, fixes the y_j for $j\geq 2$ and all the z_j, and satisfies

$$s_0(q_r(X)) = q_r(y_1,x_1,x_2,\ldots) = q_r(X) + 2\sum_{j=1}^r y_1^j q_{r-j}(X). \tag{56}$$

For each $i\geq 0$, define the *divided difference operator* ∂_i^y on $\Gamma[Y,Z]$ by

$$\partial_0^y f := \frac{f-s_0f}{-2y_1}, \qquad \partial_i^y f := \frac{f-s_if}{y_i-y_{i+1}} \quad \text{for } i>0.$$

Consider the ring involution $\omega:\Gamma[Y,Z]\to\Gamma[Y,Z]$ determined by

$$\omega(y_j) = -z_j, \qquad \omega(z_j) = -y_j, \qquad \omega(q_r(X)) = q_r(X)$$

and set $\partial_i^z := \omega\partial_i^y\omega$ for each $i\geq 0$.

Theorem 11 ([IMN])**.** *The polynomials $\mathfrak{C}_w(X\,;Y,Z)$ for $w\in W_\infty$ are the unique family of elements of $\Gamma[Y,Z]$ satisfying the equations*

$$\partial_i^y\mathfrak{C}_w = \begin{cases} \mathfrak{C}_{ws_i} & \text{if } \ell(ws_i)<\ell(w), \\ 0 & \text{otherwise,}\end{cases} \qquad \partial_i^z\mathfrak{C}_w = \begin{cases} \mathfrak{C}_{s_iw} & \text{if } \ell(s_iw)<\ell(w), \\ 0 & \text{otherwise,}\end{cases} \tag{57}$$

for all $i\geq 0$, together with the condition that the constant term of $\mathfrak{C}_w$ is 1 if $w=1$, and 0 otherwise.

Proof. Define $\mathfrak{C}^{(n)} := C(X)A_1(y_1)\cdots A_{n-1}(y_{n-1})$, so that we have $\mathfrak{C}_w(X\,;Y) = \left\langle\mathfrak{C}^{(n)},w\right\rangle$, for any $w\in W_n$. In [FS, Lemma 3.5] and [FK3, Theorem 7.1] Fomin, Stanley, and Kirillov provide simple proofs (using the nilCoxeter relations) that the single Schubert polynomials $\mathfrak{C}_w(X\,;Y)$ satisfy the equations in (57) which involve the ∂_i^y operators. We give the proof for ∂_0^y below for the reader's convenience, since our setup differs from that in [FK3]. It follows from (55) and (56) that $s_0C(X) = C(X)C(y_1)$, and we clearly have

$$C(y_1)A_1(-y_1) = A_1(y_1)(1+y_1u_0)(1+y_1u_0) = A_1(y_1)(1+2y_1u_0).$$

We deduce that

$$\begin{aligned}\partial_0^y\,\mathfrak{C}^{(n)} &= \frac{1}{-2y_1}\left(C(X)(A_1(y_1)-C(y_1)A_1(-y_1))A_2(y_2)\cdots A_{n-1}(y_{n-1})\right)\\ &= \frac{1}{-2y_1}\,C(X)A_1(y_1)(-2y_1u_0)A_2(y_2)\cdots A_{n-1}(y_{n-1}) = \mathfrak{C}^{(n)}u_0,\end{aligned}$$

which is equivalent the desired conditions $\partial_0^y\,\mathfrak{C}_w(X\,;Y) = \mathfrak{C}_{ws_0}(X\,;Y)$, if $\ell(ws_0) < \ell(w)$, and $\partial_0^y\,\mathfrak{C}_w(X\,;Y) = 0$, otherwise. Moreover, in view of equation (35), the arguments here and in loc. cit. extend easily to show that the double Schubert polynomials $\mathfrak{C}_w(X\,;Y,Z)$ fulfill the entire list of conditions in the theorem.

Following [IMN, §7.4], the uniqueness is shown as follows. If $\{\mathfrak{C}'_w\}$ for $w \in W_\infty$ is a second family of elements of $\Gamma[Y,Z]$ satisfying the displayed conditions, then by inducting on the length of w one sees that $\partial_i^y(\mathfrak{C}'_w - \mathfrak{C}_w) = \partial_i^z(\mathfrak{C}'_w - \mathfrak{C}_w) = 0$ for all $i \geq 0$. We deduce that the difference $\mathfrak{C}'_w - \mathfrak{C}_w$ is invariant under the action of s_i and $\omega s_i \omega$ for every i, and hence must be a constant. Finally, the condition on the constant term implies that $\mathfrak{C}'_w = \mathfrak{C}_w$, for all $w \in W_\infty$. □

The connection between Theorem 5 and Theorems 6, 7, 8, and 9 depends on an important ring homomorphism derived from [T3, T4], [IMN, §10], and [Gr], which we call the *geometrization map*. We will discuss this homomorphism in detail in the Lie types A, C, and D (leaving type B as an exercise for the reader), and use it to give a complete proof of Theorem 9.

Let G denote the group $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{Sp}_{2n}(\mathbb{C})$, or $\mathrm{SO}_{2n}(\mathbb{C})$ with its standard representation $V = \mathbb{C}^n$ for GL_n or $V = \mathbb{C}^{2n}$ in the latter two cases. In type A, equip V with the zero form, and in types C and D equip V with an antidiagonal symplectic or orthogonal form $(\ ,\ )$, so that the standard basis $\{e_1,\ldots,e_{2n}\}$ of V satisfies $(e_i,e_j) = 0$ for $i+j \neq 2n+1$ and $(e_i, e_{2n+1-i}) = 1$, for $1 \leq i \leq n$. We obtain an induced vector bundle $E = EG \times^G V$ over BG and bilinear form $E \otimes E \to \mathbb{C}$. Let $V_\bullet$ be the isotropic flag in V with $V_i = \langle e_1,\ldots,e_i\rangle$ for each $i \in [1,n]$, and B denote the stabilizer of $V_\bullet$. The pullback of the bundle E to BB has an isotropic flag $\mathcal{V}'_\bullet = \{EG \times^B V_i\}_i$ of subbundles of E. If pr_1 and pr_2 are the two projection maps $BB \times_{BG} BB \to BB$, then we obtain the two isotropic flags of subbundles $\mathcal{V}_\bullet := \mathrm{pr}_1^*\,\mathcal{V}'_\bullet$ and $F_\bullet := \mathrm{pr}_2^*\,\mathcal{V}'_\bullet$ of the pullback of E to $BB \times_{BG} BB$. This is the universal case of the degeneracy locus problems considered in §6.1, §6.2, and §6.3.2, for the parabolic subgroup $P = B$.

Type A. Introduce two new sets of commuting variables $\mathrm{X} = (\mathrm{x}_1,\mathrm{x}_2,\ldots)$, $\mathrm{Y} = (\mathrm{y}_1,\mathrm{y}_2,\ldots)$ and let $\mathrm{X}_n = (\mathrm{x}_1,\ldots,\mathrm{x}_n)$ and $\mathrm{Y}_n = (\mathrm{y}_1,\ldots,\mathrm{y}_n)$. In this case $G = \mathrm{GL}_n$ and it follows from the above discussion and (31) that there is a natural presentation

$$\mathrm{H}^*(BB \times_{BG} BB, \mathbb{Z}) \cong \mathbb{Z}[\mathrm{X}_n, \mathrm{Y}_n]/\mathrm{I}_n, \tag{58}$$

where I_n is the ideal generated by the differences $e_i(\mathrm{x}_1,\ldots,\mathrm{x}_n) - e_i(\mathrm{y}_1,\ldots,\mathrm{y}_n)$ for $1 \leq i \leq n$. The inverse of the isomorphism (58) sends the class of x_i to $-c_1(\mathcal{V}_i/\mathcal{V}_{i-1})$ and of y_i to $-c_1(F_{n+1-i}/F_{n-i})$ for each i with $1 \leq i \leq n$.

The *geometrization map* is the ring homomorphism

$$\rho_n : \mathbb{Z}[Y,Z] \to \mathbb{Z}[\mathrm{X}_n, \mathrm{Y}_n]/\mathrm{I}_n$$

defined by setting $\rho_n(y_i) := \mathrm{x}_i$ and $\rho_n(z_i) := \mathrm{y}_i$ for $1 \leq i \leq n$ and $\rho_n(y_i) = \rho_n(z_i) = 0$ for $i > n$. Fulton [F1, F3] showed that for $\varpi \in S_n$, the homomorphism ρ_n sends $\mathfrak{S}_\varpi(Y,Z)$ to a polynomial which represents the universal Schubert class $[\mathfrak{X}_\varpi]$ in the presentation (58). A different way to establish this is obtained by arguing as in the proof of Theorem 12 below.

Type C. Here $G = \mathrm{Sp}_{2n}$ and we have a natural presentation

$$\mathrm{H}^*(BB \times_{BG} BB, \mathbb{Z}) \cong \mathbb{Z}[\mathrm{X}_n, \mathrm{Y}_n]/\mathrm{J}_n, \tag{59}$$

where J_n is the ideal generated by the differences $e_i(\mathrm{x}_1^2, \ldots, \mathrm{x}_n^2) - e_i(\mathrm{y}_1^2, \ldots, \mathrm{y}_n^2)$ for $1 \leq i \leq n$. The inverse of the isomorphism (59) sends the class of x_i to $-c_1(\mathcal{V}_{n+1-i}/\mathcal{V}_{n-i})$ and of y_i to $-c_1(F_{n+1-i}/F_{n-i})$ for each i with $1 \leq i \leq n$.

Recall that $h_j(\mathrm{Y}_n) = s_j(\mathrm{Y}_n)$ denotes the j-th complete symmetric polynomial, which is the sum of all monomials of total degree j in the variables Y_n. The *geometrization map* is the ring homomorphism

$$\pi_n : \Gamma[Y, Z] \to \mathbb{Z}[\mathrm{X}_n, \mathrm{Y}_n]/\mathrm{J}_n$$

determined by setting

$$\pi_n(q_r(X)) := \sum_{i=0}^{r} e_i(\mathrm{X}_n) h_{r-i}(\mathrm{Y}_n) \ \text{ for all } r,$$

$$\pi_n(y_i) := \begin{cases} -\mathrm{x}_i & \text{if } 1 \leq i \leq n, \\ 0 & \text{if } i > n, \end{cases} \quad \text{and} \quad \pi_n(z_j) := \begin{cases} \mathrm{y}_j & \text{if } 1 \leq j \leq n, \\ 0 & \text{if } j > n. \end{cases}$$

To show that π_n is well defined, set $\xi_r := \sum_{i=0}^{r} e_i(\mathrm{X}_n) h_{r-i}(\mathrm{Y}_n)$ and observe that the generating function $\Xi(t)$ for the ξ_r satisfies

$$\Xi(t) := \sum_{r=0}^{\infty} \xi_r t^r = \left(\sum_{i=0}^{\infty} e_i(\mathrm{X}_n) t^i\right)\left(\sum_{j=0}^{\infty} h_j(\mathrm{Y}_n) t^j\right) = \prod_{i=1}^{n} \frac{1 + \mathrm{x}_i t}{1 - \mathrm{y}_i t}.$$

Since the difference $\prod_i (1 - \mathrm{x}_i^2 t^2) - \prod_i (1 - \mathrm{y}_i^2 t^2)$ lies in the ideal $\mathrm{J}_n[t^2]$, we have

$$\Xi(t)\Xi(-t) = \prod_{i=1}^{n} \frac{1 - \mathrm{x}_i^2 t^2}{1 - \mathrm{y}_i^2 t^2} \in 1 + \mathrm{J}_n[[t^2]]. \tag{60}$$

Extracting the coefficient of t^{2r} from both sides of equation (60) gives

$$\xi_r^2 + 2\sum_{i=1}^{r} (-1)^i \xi_{r+i} \xi_{r-i} \in \mathrm{J}_n, \quad \text{for all } \ r > 0. \tag{61}$$

The relations (61), which agree with the $k = 0$ case of (15), generate the ideal of relations among the $q_r(X)$. Therefore, π_n is a well defined ring homomorphism.

Theorem 12 (Geometrization, [IMN])**.** *For any $w \in W_n$, the geometrization map π_n sends $\mathfrak{C}_w(X\,; Y, Z)$ to a polynomial which represents the universal Schubert class $[\mathfrak{X}_w] \in \mathrm{H}^*(BB \times_{BG} BB, \mathbb{Z})$ in the presentation* (59).

Proof. The simple reflections s_i in W_∞ act on $\mathbb{Z}[\mathrm{X}, \mathrm{Y}]$ as follows. The reflection s_i interchanges x_i and x_{i+1} for $i > 0$, while s_0 replaces x_1 by $-\mathrm{x}_1$; all other variables remain fixed. We have corresponding divided difference operators $\partial_i^{\mathrm{x}} : \mathbb{Z}[\mathrm{X}, \mathrm{Y}] \to \mathbb{Z}[\mathrm{X}, \mathrm{Y}]$. For each $i \geq 0$ and $f \in \mathbb{Z}[\mathrm{X}, \mathrm{Y}]$, they are defined by

$$\partial_0^{\mathrm{x}} f := \frac{f - s_0 f}{2\mathrm{x}_1}, \qquad \partial_i^{\mathrm{x}} f := \frac{f - s_i f}{\mathrm{x}_{i+1} - \mathrm{x}_i} \quad \text{for } i > 0.$$

Let $\tilde{\omega}$ denote the involution of $\mathbb{Z}[\mathrm{X}, \mathrm{Y}]$ obtained by interchanging the variable x_j with y_j for all $j \geq 1$. Geometrically, this corresponds to the automorphism of

$BB \times_{BG} BB$ given by switching the two factors. Define the y-divided difference operators ∂_i^{y} on $\mathbb{Z}[\mathrm{X},\mathrm{Y}]$ by $\partial_i^{\mathrm{y}} = \tilde{\omega}\partial_i^{\mathrm{x}}\tilde{\omega}$ for each $i \geq 0$.

Restricting the above to W_n and $0 \leq i \leq n-1$, we obtain divided differences ∂_i^{x}, ∂_i^{y} acting on $\mathbb{Z}[\mathrm{X}_n,\mathrm{Y}_n]/\mathrm{J}_n$ and hence on $\mathrm{H}^*(BB \times_{BG} BB, \mathbb{Z})$. In the mid 1990s, building on the work of Bernstein-Gelfand-Gelfand [BGG] and Demazure [D1, D2], Fulton [F2, F3] studied the action of the operators ∂_i^{x} on the universal Schubert classes $[\mathfrak{X}_w]$, constructing them geometrically using correspondences by $\mathbb{P}^1$-bundles. It follows from this work that we have

(62)
$$\partial_i^{\mathrm{x}}[\mathfrak{X}_w] = \begin{cases} [\mathfrak{X}_{ws_i}] & \text{if } \ell(ws_i) < \ell(w), \\ 0 & \text{otherwise,} \end{cases} \qquad \partial_i^{\mathrm{y}}[\mathfrak{X}_w] = \begin{cases} [\mathfrak{X}_{s_iw}] & \text{if } \ell(s_iw) < \ell(w), \\ 0 & \text{otherwise,} \end{cases}$$

for all i.

Write BB_n and BG_n for BB and BG, respectively, to emphasize the dependence on the rank n, and denote by BM_n the Borel mixing space $BB_n \times_{BG_n} BB_n$. The natural embedding of W_n into W_{n+1} induces a morphism $\phi_n : BM_n \to BM_{n+1}$ and hence a map of cohomology rings

$$\phi_n^* : \mathrm{H}^*(BM_{n+1}, \mathbb{Z}) \to \mathrm{H}^*(BM_n, \mathbb{Z}) \tag{63}$$

which in terms of the presentation (59) is given by sending x_{n+1} and y_{n+1} to zero. Let

$$\mathbb{H}(BM_\infty) := \varprojlim \mathrm{H}^*(BM_n, \mathbb{Z})$$

be the stable cohomology ring of BM_n, which is the inverse limit in the category of graded rings of the system of maps $\{\phi_n^*\}_{n\geq 1}$ in (63). For each $w \in W_\infty$, we have a stable Schubert class C_w in $\mathbb{H}(BM_\infty)$, defined as the element $\varprojlim[\mathfrak{X}_w]$. It follows from the equations (62) that

$$\partial_i^{\mathrm{x}} C_w = \begin{cases} C_{ws_i} & \text{if } \ell(ws_i) < \ell(w), \\ 0 & \text{otherwise,} \end{cases} \qquad \partial_i^{\mathrm{y}} C_w = \begin{cases} C_{s_iw} & \text{if } \ell(s_iw) < \ell(w), \\ 0 & \text{otherwise,} \end{cases} \tag{64}$$

for all $i \geq 0$, while the degree zero component of C_w is 1 if $w = 1$, and 0 otherwise. Moreover, using the presentation (59) and arguing as Theorem 11, it is easy to check that the family $\{C_w\}$ for $w \in W_\infty$ is uniquely determined by these conditions.

The composite homomorphism

$$\Gamma[Y,Z] \xrightarrow{\pi_n} \mathbb{Z}[\mathrm{X}_n,\mathrm{Y}_n]/\mathrm{J}_n \xrightarrow{\simeq} \mathrm{H}^*(BM_n,\mathbb{Z})$$

is compatible with the maps ϕ_n^*, therefore there is an induced ring homomorphism

$$\pi_\infty : \Gamma[Y,Z] \to \mathbb{H}(BM_\infty).$$

One verifies that π_∞ respects the actions of the divided differences on its domain and codomain (compare with [BH]). We deduce the theorem, since both $\mathfrak{C}_w$ and C_w are characterized by the equations (57) and (64), respectively, and the same degree zero condition. One can show that, in fact, the map π_∞ is a canonical isomorphism of graded rings. □

For any $w \in W_n$, the image of $\mathfrak{C}_w$ under the geometrization map π_n may be computed as follows. Use (35) and (55) to write

$$\mathfrak{C}_w(X\,;Y,Z) = \sum_{u,v,u',\lambda} e_\lambda^v\, \mathfrak{S}_{u^{-1}}(-Z) Q_\lambda(X) \mathfrak{S}_{u'}(Y), \tag{65}$$

where the sum is over all reduced factorizations $uvu' = w$ and strict partitions λ with $u, u' \in S_n$ and $|\lambda| = \ell(v)$. We then have

$$\pi_n(\mathfrak{C}_w(X\,;Y,Z)) = \sum_{u,v,u',\lambda} e^v_\lambda\, \mathfrak{S}_{u^{-1}}(-\mathrm{Y}_n)\widetilde{Q}_\lambda(\mathrm{X}_n/\mathrm{Y}_n)\mathfrak{S}_{u'}(-\mathrm{X}_n). \tag{66}$$

We call the polynomial $\widetilde{Q}_\lambda(\mathrm{X}_n/\mathrm{Y}_n)$ a *supersymmetric $\widetilde{Q}$-polynomial*; it is obtained from the Q-polynomial $Q_\lambda(c)$ in (16) by specializing c_r to $\sum_i e_i(\mathrm{X}_n)h_{r-i}(\mathrm{Y}_n)$ for each r. The supersymmetric $\widetilde{Q}$-polynomials have properties directly analogous to those of the $\widetilde{Q}$-polynomials $\widetilde{Q}_\lambda(\mathrm{X}_n)$ of Pragacz and Ratajski [PR2], which satisfy the identity

$$\widetilde{Q}_\lambda(\mathrm{X}_n) = \widetilde{Q}_\lambda(\mathrm{X}_n/\mathrm{Y}_n)|_{\mathrm{Y}_n=0}.$$

Moreover, setting $Z = 0$ and $\mathrm{Y}_n = 0$ in (66), one recovers the *symplectic Schubert polynomials* $\mathfrak{C}_w(\mathrm{X}_n)$ of [T3].

Remark 4. Let $\mathfrak{a}$ be a sequence of integers as in §6.2 and let $P \subset G$ be the parabolic subgroup corresponding to $\mathfrak{a}$. Suppose further that we are concerned only with degeneracy loci having the symmetries of P, or equivalently, with Giambelli formulas in $\mathrm{H}^*_T(G/P)$. Let $k = a_1$ and $\Gamma^{(k)} := \mathbb{Z}[\vartheta_1, \vartheta_2, \ldots]$ denote the associated ring of theta polynomials. Then it suffices to work with polynomials in the ring

$$\Gamma^{(k)}[y_{k+1}, y_{k+2}, \ldots, z_1, z_2, \ldots],$$

as these elements map under π_n to the cohomology classes that lie in $\mathrm{H}^*_T(G/P)$. This follows from equation (35) and [T7, Prop. 5].

Type D. Here $G = \mathrm{SO}_{2n}$ and according to (31) the ring $\mathrm{H}^*(BB \times_{BG} BB, \mathbb{Q})$ is presented as a quotient

$$\mathrm{H}^*(BB \times_{BG} BB, \mathbb{Q}) \cong \mathbb{Q}[\mathrm{X}_n, \mathrm{Y}_n]/\mathrm{J}'_n, \tag{67}$$

where J'_n is the ideal generated by the differences $e_i(\mathrm{x}_1^2, \ldots, \mathrm{x}_n^2) - e_i(\mathrm{y}_1^2, \ldots, \mathrm{y}_n^2)$ for $1 \leq i \leq n-1$ and $\mathrm{x}_1 \cdots \mathrm{x}_n - \mathrm{y}_1 \cdots \mathrm{y}_n$. The inverse of the isomorphism (67) sends the class of x_i to $-c_1(\mathcal{V}_{n+1-i}/\mathcal{V}_{n-i})$ and of y_i to $-c_1(F_{n+1-i}/F_{n-i})$ for each i with $1 \leq i \leq n$. The *geometrization map* is the ring homomorphism

$$\pi'_n : \Gamma'[Y, Z] \to \mathbb{Q}[\mathrm{X}_n, \mathrm{Y}_n]/\mathrm{J}'_n$$

defined by setting

$$\pi'_n(P_r(X)) := \frac{1}{2}\sum_{i=0}^r e_i(\mathrm{X}_n)h_{r-i}(\mathrm{Y}_n) \quad \text{for all}\, r,$$

$$\pi'_n(y_i) := \begin{cases} -\mathrm{x}_i & \text{if } 1 \leq i \leq n, \\ 0 & \text{if } i > n, \end{cases} \qquad \text{and} \qquad \pi'_n(z_j) := \begin{cases} \mathrm{y}_j & \text{if } 1 \leq j \leq n, \\ 0 & \text{if } j > n. \end{cases}$$

One can now prove a type D version of Theorem 12: for any $w \in \widetilde{W}_n$, the map π'_n sends $\mathfrak{D}_w(X\,;Y,Z)$ to a polynomial which represents the universal Schubert class $[\mathfrak{X}_w]$ in the presentation (67). Moreover, the system of maps π'_n induces a graded ring isomorphism π'_∞ between $\Gamma'[Y,Z]$ and the stable cohomology ring of the Borel mixing space associated to $\mathrm{SO}_{2n}(\mathbb{C})$. In the single (non-equivariant) case when $Z = 0$ and $\mathrm{Y}_n = 0$, we observe that $\pi'_n(\mathfrak{D}_w(X\,;Y))$ is equal to the *orthogonal Schubert polynomial* $\mathfrak{D}_w(\mathrm{X}_n)$ of [T4].

7.4. **Proof of Theorems 6–9.** We will give our proof of Theorem 9 below; the arguments in types A–C are simpler, and the reader may also consult the references [BKTY1, §4.1] (for Theorem 6) and [T7, Thm. 3] (for Theorems 7 and 8). All of the proofs rely on the *splitting principle*, which allows one to express the Chern classes of the relevant vector bundles as elementary symmetric polynomials in their Chern roots; for more details on this, see for example [F5, §3.2] and [Ma, §3.5].

The variables x_i and y_i for $1 \leq i \leq n$ give the Chern roots of the various vector bundles over $BB \times_{BG} BB$, which pull back to give the roots of the corresponding vector bundles over M. In particular the Chern roots of Q_1 are (the pullbacks of) $\mathrm{x}_1, \ldots, \mathrm{x}_n, -\mathrm{x}_1, \ldots, -\mathrm{x}_{a_1}$, while those of Q_r for $r \geq 2$ are $-\mathrm{x}_{a_{r-1}+1}, \ldots, -\mathrm{x}_{a_r}$. Similarly the Chern roots of F_{n+1-r} are represented by $-\mathrm{y}_r, \ldots, -\mathrm{y}_n$ for each r. With $k = a_1$, we have

$$\vartheta_r(X\,;Y_1) = \sum_{i=0}^{r} q_{r-i}(X) e_i(y_1, \ldots, y_{a_1})$$

for each $r \geq 0$. It follows that

$$\begin{aligned}
\pi'_n(\vartheta_r(X\,;Y_1)) &= \sum_{i,j\geq 0} e_{r-i-j}(\mathrm{X}_n) h_j(\mathrm{Y}_n) e_i(-\mathrm{x}_1, \ldots, -\mathrm{x}_{a_1}) \\
&= \sum_{j\geq 0} e_{r-j}(\mathrm{x}_1, \ldots, \mathrm{x}_n, -\mathrm{x}_1, \ldots, -\mathrm{x}_{a_1}) h_j(\mathrm{Y}_n) \\
&= c_r(Q_1 - F_n) = c_r(E - E_1 - F_n).
\end{aligned}$$

Moreover, we have

$$\pi'_n(e_{a_1}(Y_1)) = (-1)^{a_1} \mathrm{x}_1 \cdots \mathrm{x}_{a_1} = c_{a_1}(E_0 - E_1),$$

where E_0 denotes the maximal isotropic subbundle of E. We deduce from the equations (38) that

$$\begin{aligned}
\pi'_n(\eta_{a_1}(X\,;Y_1)) &= \frac{1}{2}(c_{a_1}(E - E_1 - F_n) + c_{a_1}(E_0 - E_1)) \quad \text{and} \\
\pi'_n(\eta'_{a_1}(X\,;Y_1)) &= \frac{1}{2}(c_{a_1}(E - E_1 - F_n) - c_{a_1}(E_0 - E_1)).
\end{aligned}$$

Furthermore, for any partition μ and $r \geq 2$, we have

$$\pi'_n(s_\mu(Y_r)) = s_\mu(-\mathrm{x}_{a_{r-1}+1}, \ldots, -\mathrm{x}_{a_r}) = s_\mu(Q_r) = s_\mu(E_{r-1} - E_r),$$

while

$$\pi'_n(s_\mu(0/Z_r)) = s_{\widetilde{\mu}}(-\mathrm{y}_{b_{r-1}+1}, \ldots, -\mathrm{y}_{b_r}) = s_{\widetilde{\mu}}(\widehat{Q}_r) = s_\mu(F_{n+b_{r-1}} - F_{n+b_r}).$$

The equality in Theorem 9 is therefore obtained by applying π'_n to formula (49).

8. Suggestions for future research.

In this section we propose some natural directions to follow in future work.

8.1. **Quantum cohomology of G/P.** The last two decades have seen much exploration of the Gromov-Witten theory and quantum cohomology rings of homogeneous spaces. In particular, one seeks to extend the classical understanding of Schubert calculus to the quantum setting, with analogues of the theorems of Pieri, Giambelli, and computations of Schubert structure constants for $\mathrm{QH}^*(G/P)$. One of the motivations for [T7] was the fact that the known classical Giambelli formulas expressed in terms of the special Schubert classes in this article have straightforward extensions to the small quantum cohomology ring of G/P; see [Be, FGP, CF, KT2, KT3, BKT3, BKT4]. We expect that there should be quantum (double) Schubert polynomials and (equivariant) Giambelli formulas in the Lie types B, C, and D which restrict to the results found in the present paper when the quantum parameters are set equal to zero.

8.2. **K-theory of G/P.** The classes of the structure sheaves $\mathcal{O}_{X_w}$ of the Schubert varieties X_w, $w \in W^P$ provide a natural $\mathbb{Z}$-basis for the Grothendieck group $K(G/P)$ of vector bundles on G/P. When G is the general linear group GL_n, one has the Lascoux-Schützenberger theory of Grothendieck polynomials [LS2, FLa], and many of the type A results for cohomology can be generalized to K-theory [L3, Bu, BKTY2]. Even for the Grassmannian $\mathrm{G}(m,n)$, however, one does not yet have a Giambelli formula for $[\mathcal{O}_{X_\lambda}]$ in terms of special Schubert classes that clearly extends the raising operator expression (5) in §1 (see [Bu, Thm. 1] for a Jacobi-Trudi recursion). There should be versions of the double Grothendieck polynomials for the other classical Lie types, defined using the degenerate Hecke algebra in place of the nilCoxeter algebra; the type A theory is worked out in [FK1]. One could then seek analogues of Lascoux's transition equations [L3] and the splitting formula of [BKTY2, Thm. 4] for these objects. Some recent related work on equivariant K-theory in the other types may be found in [IN2, GK]; see also [KK, GR].

8.3. **Combinatorial questions.** There is a large body of research on the combinatorial aspects of the Schubert calculus, most all of it in the situation where the underlying Weyl group elements are *fully commutative* in the sense of [St2]. By contrast, the combinatorial theory developed in [BKT1, BKT2, BKT4, T5, T7, T8] is in its infancy, with many questions worth exploring further. In particular, we would like a deeper understanding of the connections between the formulas that appear in special cases, such as in the works cited in the introduction, which differ from those given here (see [TW] for recent progress on this).

We mentioned in §4.3 that the type A Stanley coefficients c_λ^{ϖ} enumerate Young tableaux as well as leaves in transition trees. However the mixed Stanley coefficients e_λ^w and d_λ^w are only known to be positive through transition when $k > 0$. Are there alternative combinatorial formulas for them, and is there a geometric proof of their positivity? In type A, Little [Lit] has studied the combinatorics of the Lascoux-Schützenberger tree; is there an analogue of his bijection for the c_λ^{ϖ} which involves the 0-transition trees of [Bi1] and Kraśkiewicz-Lam tableaux [Kr, La1]? [2] Do the mixed Stanley coefficients include a rule which computes the Schubert structure constants on isotropic Grassmannians? Some partial results on this last question are obtained in [T7, §2.4].

8.4. **A theory for the exceptional types.** It natural to ask whether the uniform choice of special Schubert classes on the Grassmannians for the classical groups from [BKT1] and §1-§2 extends to the exceptional Lie types, and to look for canonical Giambelli expressions native to G/P for arbitrary reductive groups G and P. Anderson [A] has obtained degeneracy locus formulas for vector bundles with structure group G_2, and there are parallels between the combinatorics of Schubert calculus

[2] Such a bijection in Lie type B has recently been defined in [BHRY].

on the Grassmannians $\mathrm{G}(m,n)$ and the hermitian symmetric (or cominuscule) quotients of E_6 and E_7 [TY]. More remains to be understood however to obtain a generalization of the results discussed in this article to any G/P space.

8.5. **Connections with representation theory.** Do the raising operator formulas of this paper appear in other areas, and in particular in representation theory? The Schur S- and Q-polynomials were studied by Schur (and his advisor Frobenius) in order to compute the characters and projective characters of the symmetric group and the polynomial characters of the general linear group. The discovery of formulas such as (7) and (18), (19) in the algebra of symmetric functions and associated group representations long preceded their realizations in the Schubert calculus. A representation-theoretic understanding of the raising operator expressions (37) and (39), parallel to the extensive theory of Schur polynomials, would certainly be desirable. Young's raising operators are well known in algebraic combinatorics, but we suspect that their full potential has not yet been exploited. For a non-exhaustive list of references which apply them in various settings, see [BKT2, BKT3, BKT4, DLT, Ga, HH, LaSc, Le, Li, M, Mo, Ro, T5, T6, T8, TW, To, Y].

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