A GIAMBELLI FORMULA FOR EVEN ORTHOGONAL GRASSMANNIANS

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Abstract. Let X be an orthogonal Grassmannian parametrizing isotropic subspaces in an even dimensional vector space equipped with a nondegenerate symmetric form. We prove a Giambelli formula which expresses an arbitrary Schubert class in the classical and quantum cohomology rings of X as a polynomial in certain special Schubert classes. Our analysis reveals a surprising relation between the Schubert calculus on even and odd orthogonal Grassmannians. We also study *eta polynomials*, a family of polynomials defined using raising operators whose algebra agrees with the Schubert calculus on X.

0. Introduction

Consider a complex vector space V of dimension N equipped with a nondegenerate symmetric form. Choose an integer $m < N/2$ and consider the Grassmannian $\mathrm{OG} = \mathrm{OG}(m, N)$ parametrizing isotropic m-dimensional subspaces of V. Our aim in this paper is to prove a *Giambelli formula* that expresses the Schubert classes on OG as polynomials in certain special Schubert classes that generate the cohomology ring $\mathrm{H}^*(\mathrm{OG}, \mathbb{Z})$. When $N = 2n + 1$ is odd, this was the main result of [BKT2]; what is new here concerns the even case $N = 2n + 2$.

The proof of our main theorem (Theorem 2) exploits the weight space decomposition of $\mathrm{H}^*(\mathrm{OG}(m, 2n + 2), \mathbb{Q})$ induced by the natural involution of the Dynkin diagram of type D_{n+1}. We require the Giambelli formula for odd orthogonal Grassmannians from [BKT2] and a similar result for the (+1)-eigenspace of $\mathrm{H}^*(\mathrm{OG}(m, 2n + 2), \mathbb{Q})$, which is the subring generated by the Chern classes of the tautological vector bundles over OG. These ingredients combine to establish Theorem 2 thanks to a surprising new relation between the cohomology of even and odd orthogonal Grassmannians (Proposition 2).

Define nonnegative integers K and k by the equations

$$K = N - 2m = \begin{cases} 2k + 1 & \text{if } N \text{ is odd,} \\ 2k & \text{if } N \text{ is even.} \end{cases}$$

Observe that $n + k = N - m - 1$. An integer partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ is *k-strict* if no part λ_i greater than k is repeated. Let λ be a k-strict partition whose Young diagram is contained in an $m \times (n + k)$ rectangle. For $1 \leq j \leq m$, let

$$(1) \qquad \overline{p}_j(\lambda) = N - m + j - \lambda_j - \#\{i \leq j \mid \lambda_i + \lambda_j \geq K + j - i \text{ and } \lambda_i > k\},$$

and notice that $\overline{p}_j(\lambda) \neq n + 1$ for every j and λ.

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An isotropic flag is a complete flag $0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_N = V$ of subspaces of V such that $F_i = F_j^\perp$ whenever $i + j = N$. For any fixed isotropic flag $F_\bullet$ and any k-strict partition λ whose Young diagram is contained in an $m \times (n+k)$ rectangle, we define a closed subset $Y_\lambda = Y_\lambda(F_\bullet) \subset \mathrm{OG}$ by setting

$$Y_\lambda(F_\bullet) = \{\Sigma \in \mathrm{OG} \mid \dim(\Sigma \cap F_{\overline{p}_j}) \geq j \text{ for } 1 \leq j \leq m\}\,. \tag{2}$$

If N is odd, the varieties Y_λ are exactly the Schubert varieties in OG. If N is even, and k is not a part of λ, then Y_λ is again a Schubert variety in OG. Otherwise, Y_λ is a union of two Schubert varieties X_λ and X'_λ, which will be defined below. The algebraic set Y_λ has pure codimension $|\lambda| = \sum \lambda_i$ and determines a class $[Y_\lambda]$ in $\mathrm{H}^{2|\lambda|}(\mathrm{OG}, \mathbb{Z})$.

Consider the exact sequence of vector bundles over $X = \mathrm{OG}$

$$0 \to \mathcal{S} \to V_X \to \mathcal{Q} \to 0,$$

where V_X denotes the trivial bundle of rank N and $\mathcal{S}$ is the tautological subbundle of rank m. The Chern classes $c_p = c_p(\mathcal{Q})$ of $\mathcal{Q}$ satisfy

$$c_p = \begin{cases} [Y_p] & \text{if } p \leq k, \\ 2[Y_p] & \text{if } p > k. \end{cases} \tag{3}$$

As in [BKT2], we will express our Giambelli formulas using Young's raising operators [Y]. For any integer sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ with finite support and $i < j$, we define

$$R_{ij}(\alpha) = (\alpha_1, \ldots, \alpha_i + 1, \ldots, \alpha_j - 1, \ldots).$$

We also set $c_\alpha = \prod_i c_{\alpha_i}$. If R is any finite monomial in the operators R_{ij}, then set $R\, c_\alpha = c_{R\,\alpha}$; we stress that the operator R acts on the subscript α and not on the monomial c_α itself. Given a k-strict partition λ we define the operator

$$R^\lambda = \prod (1 - R_{ij}) \prod_{\lambda_i + \lambda_j \geq K + j - i} (1 + R_{ij})^{-1} \tag{4}$$

where the first product is over all pairs $i < j$ and the second product is over pairs $i < j$ such that $\lambda_i + \lambda_j \geq K + j - i$. Let $\ell_k(\lambda)$ denote the number of parts λ_i which are strictly greater than k.

Theorem 1. *For any k-strict partition λ contained in an $m \times (n+k)$ rectangle, we have $[Y_\lambda] = 2^{-\ell_k(\lambda)} R^\lambda\, c_\lambda$ in the cohomology ring of $\mathrm{OG}(m, N)$.*

When N is odd, Theorem 1 is the Giambelli formula for the Schubert classes on odd orthogonal Grassmannians from [BKT2, §2]; the result for even N is proved along the same lines. We next will refine Theorem 1 to obtain a Giambelli polynomial representing any Schubert class in the even orthogonal case.

For the rest of this section, we assume that $N = 2n + 2$ is even, so that $m = n + 1 - k$ and $K = 2k > 0$. Fix a maximal isotropic subspace L of V, i.e. with $\dim(L) = n+1$. Two maximal isotropic subspaces E and F of the space V are said to be in the same family if $\dim(E \cap F) \equiv n + 1 \pmod 2$. The Schubert varieties in OG are defined relative to an isotropic flag $F_\bullet$, and their classes are independent of this flag as long as F_{n+1} is in the same family as L.

A *typed k-strict partition* λ consists of a k-strict partition $(\lambda_1, \ldots, \lambda_\ell)$ together with an integer $\mathrm{type}(\lambda) \in \{0, 1, 2\}$, such that $\mathrm{type}(\lambda) > 0$ if and only if $\lambda_j = k$ for some index j. Let $\widetilde{\mathcal{P}}(k, n)$ denote the set of all typed k-strict partitions whose Young diagrams are contained in an $m \times (n+k)$ rectangle. Notice that $\lambda_j = k < \lambda_{j-1}$

if and only if $\overline{p}_j(\lambda) = n+2$. For every $\lambda \in \widetilde{\mathcal{P}}(k,n)$, define the index function $p_j = p_j(\lambda)$ by

$$p_j(\lambda) = \begin{cases} \overline{p}_j(\lambda) - 1 & \text{if } \lambda_j = k < \lambda_{j-1} \text{ and } n + j + \text{type}(\lambda) \text{ is even,} \\ \overline{p}_j(\lambda) & \text{otherwise.} \end{cases}$$

According to [BKT1], for every isotropic flag $F_\bullet$ we have a Schubert cell $X_\lambda^\circ = X_\lambda^\circ(F_\bullet)$ in OG, defined as the locus of $\Sigma \in \mathrm{OG}$ such that $\dim(\Sigma \cap F_i) = \#\{j \mid p_j \leq i\}$ for each i. The Schubert variety X_λ is the Zariski closure of the Schubert cell X_λ°. We let $\tau_\lambda = [X_\lambda]$ denote the corresponding Schubert class in $\mathrm{H}^{2|\lambda|}(\mathrm{OG}, \mathbb{Z})$; this class has a type which agrees with the type of λ.

We say that a k-strict partition λ has *positive type* if $\lambda_i = k$ for some index i. If the (untyped) k-strict partition λ has positive type, we agree that $\tau_\lambda = [X_\lambda]$ and $\tau'_\lambda = [X'_\lambda]$ denote the Schubert classes in $\mathrm{H}^{2|\lambda|}(\mathrm{OG}(m, 2n+2))$ of type 1 and 2, respectively, associated to λ. If λ does not have positive type, then $\tau_\lambda = [X_\lambda]$ denotes the associated Schubert class of type zero. We then have that $[Y_\lambda] = \tau_\lambda + \tau'_\lambda$, if λ has positive type, while $[Y_\lambda] = \tau_\lambda$, otherwise.

The non-trivial automorphism of the Dynkin diagram of type D_{n+1} gives rise to an involution ι of OG, which interchanges X_λ and X'_λ. This in turn results in a weight space decomposition

$$\mathrm{H}^*(\mathrm{OG}, \mathbb{Q}) = \mathrm{H}^*(\mathrm{OG}, \mathbb{Q})_1 \oplus \mathrm{H}^*(\mathrm{OG}, \mathbb{Q})_{-1}. \tag{5}$$

Proposition 1. *The ι-invariant subring of $\mathrm{H}^*(\mathrm{OG}, \mathbb{Q})$ is generated by the Chern classes of $\mathcal{Q}$ and is spanned by the classes $[Y_\lambda]$, i.e.,*

$$\mathrm{H}^*(\mathrm{OG}, \mathbb{Q})_1 = \mathbb{Q}[c_1(Q), \ldots, c_{n+k}(Q)] = \bigoplus_\lambda \mathbb{Q} \cdot [Y_\lambda],$$

where the sum is over all k-strict partitions λ contained in an $m \times (n+k)$ rectangle.

There are certain special Schubert varieties in $\mathrm{OG}(m, 2n+2)$, defined by a single Schubert condition, as the locus of $\Sigma \in \mathrm{OG}$ which non-trivially intersect a given isotropic subspace or its orthogonal complement. The corresponding *special Schubert classes*

$$\tau_1, \ldots, \tau_{k-1}, \tau_k, \tau'_k, \tau_{k+1}, \ldots, \tau_{n+k} \tag{6}$$

are indexed by the typed k-strict partitions with a single non-zero part, and generate the cohomology ring $\mathrm{H}^*(\mathrm{OG}, \mathbb{Z})$. We have $\text{type}(\tau_k) = 1$, $\text{type}(\tau'_k) = 2$, and in this case equation (3) becomes

$$c_p(\mathcal{Q}) = \begin{cases} \tau_p & \text{if } p < k, \\ \tau_k + \tau'_k & \text{if } p = k, \\ 2\tau_p & \text{if } p > k. \end{cases} \tag{7}$$

Set $\overline{\mathrm{OG}} = \mathrm{OG}(n-k, 2n+1) = \mathrm{OG}(m-1, N-1)$. If the k-strict partition λ is contained in an $(n-k) \times (n+k)$ rectangle, let σ_λ denote the corresponding Schubert class in $\mathrm{H}^*(\overline{\mathrm{OG}}, \mathbb{Z})$. Both $\mathrm{H}^*(\overline{\mathrm{OG}}, \mathbb{Q})$ and $\mathrm{H}^*(\mathrm{OG}, \mathbb{Q})$ are modules over the ring $\mathbb{Q}[c] := \mathbb{Q}[c_1, \ldots, c_{n+k}]$, where the variables c_p act as multiplication with the Chern classes of the respective quotient bundles on $\overline{\mathrm{OG}}$ and OG. For any k-strict partition λ we let $\lambda + k$ denote the partition obtained by adding one copy of k to λ (and arranging the parts in decreasing order).

Proposition 2. *The linear map* $\mathrm{H}^*(\overline{\mathrm{OG}},\mathbb{Q}) \to \mathrm{H}^*(\mathrm{OG},\mathbb{Q})_{-1}$ *defined by*

$$\sigma_\lambda \mapsto \tau_{\lambda+k} - \tau'_{\lambda+k}$$

is an isomorphism of $\mathbb{Q}[c_1,\ldots,c_{n+k}]$*-modules. Moreover, we have*

$$\mathrm{H}^*(\mathrm{OG},\mathbb{Q})_{-1} = \mathbb{Q}[c_1,\ldots,c_{n+k}]\cdot(\tau_k - \tau'_k) = \bigoplus_\lambda \mathbb{Q}\cdot(\tau_{\lambda+k} - \tau'_{\lambda+k})\,,$$

where the sum is over all k*-strict partitions* λ *contained in an* $(m-1)\times(n+k)$ *rectangle, and*

$$\tau_{\lambda+k} - \tau'_{\lambda+k} = 2^{-\ell_k(\lambda)}(\tau_k - \tau'_k)\,\widetilde{R}^\lambda\, c_\lambda,$$

where $\widetilde{R}^\lambda$ *is defined by equation* (4) *with* $K = 2k+1$*.*

The statement in Proposition 2 that the isomorphism of vector spaces

$$\sigma_\lambda \mapsto \tau_{\lambda+k} - \tau'_{\lambda+k} \tag{8}$$

is also an isomorphism of $\mathbb{Q}[c]$-modules is an apparently new relation between the cohomology of even and odd orthogonal Grassmannians. This result depends crucially on our convention for assigning types to Schubert classes on OG, which was introduced in [BKT1]. The convention was chosen in loc. cit. because it results in a relatively simple Pieri formula for products with the special Schubert classes. Our proof that (8) gives a $\mathbb{Q}[c]$-module homomorphism is based on this Pieri rule; it would be interesting to expose a direct geometric argument.

Theorem 1 and Proposition 2 imply our main result, a Giambelli formula which expresses any Schubert class τ_λ in terms of the above special classes. Let R be any finite monomial in the operators R_{ij} which appears in the expansion of the power series R^λ in (4). If $\mathrm{type}(\lambda) = 0$, then set $R\star c_\lambda = c_{R\,\lambda}$. Suppose that $\mathrm{type}(\lambda) > 0$, let $d = \ell_k(\lambda)+1$ be the index such that $\lambda_d = k < \lambda_{d-1}$, and set $\widehat{\alpha} = (\alpha_1,\ldots,\alpha_{d-1},\alpha_{d+1},\ldots,\alpha_\ell)$ for any integer sequence α of length ℓ. If R involves any factors R_{ij} with $i=d$ or $j=d$, then let $R\star c_\lambda = \frac{1}{2}\,c_{R\,\lambda}$. If R has no such factors, then let

$$R\star c_\lambda = \begin{cases} \tau_k\, c_{\widehat{R\,\lambda}} & \text{if } \mathrm{type}(\lambda)=1,\\ \tau'_k\, c_{\widehat{R\,\lambda}} & \text{if } \mathrm{type}(\lambda)=2.\end{cases}$$

Theorem 2 (Classical Giambelli for OG)**.** *For every* $\lambda \in \widetilde{\mathcal{P}}(k,n)$*, we have*

$$\tau_\lambda = 2^{-\ell_k(\lambda)} R^\lambda \star c_\lambda$$

in the cohomology ring of $\mathrm{OG}(n+1-k,2n+2)$*.*

For example, consider the ring $\mathrm{H}^*(\mathrm{OG}(4,12))$ (where $k=2$) and the partition $\lambda = (3,2,2)$. Then the Schubert class for λ of type 2 is given by

$$\begin{aligned}\tau'_\lambda &= \frac{1}{2}\frac{1-R_{12}}{1+R_{12}}(1-R_{13})(1-R_{23})\star c_{322}\\ &= \frac{1}{2}(1-2R_{12}+2R_{12}^2-2R_{12}^3)(1-R_{13}-R_{23}+R_{13}R_{23})\star c_{322}\\ &= \tau_3\tau'_2(\tau_2+\tau'_2) - \tau_4\tau'_2\tau_1 + \tau_6\tau_1 - \tau_3^2\tau_1 + \tau_4\tau_3 - \tau_7.\end{aligned}$$

We remark that in general, the Giambelli formula expresses the Schubert class τ_λ as a polynomial in the special Schubert classes (6) with *integer* coefficients.

The small quantum cohomology ring $\mathrm{QH}(\mathrm{OG})$ is a q-deformation of the cohomology ring $\mathrm{H}^*(\mathrm{OG},\mathbb{Z})$ whose structure constants are defined by the three point, genus zero Gromov-Witten invariants of OG. Here q denotes a formal variable if $k \geq 2$, or a pair (q_1, q_2) of formal variables if $k = 1$. The quantum Pieri rule of [BKT1] implies that the ring QH(OG) is generated by the special Schubert classes. The following quantum Giambelli formula is the analogue of [BKT3, Thm. 2] in the even orthogonal case.

Theorem 3 (Quantum Giambelli for OG)**.** *For every $\lambda \in \widetilde{\mathcal{P}}(k,n)$, we have*

$$\tau_\lambda = 2^{-\ell_k(\lambda)} R^\lambda \star c_\lambda$$

in the quantum cohomology ring $\mathrm{QH}(\mathrm{OG}(n+1-k, 2n+2))$*. In other words, the quantum Giambelli formula for* OG *is the same as the classical Giambelli formula.*

As in [BKT2], we will use raising operators to define a family of polynomials $\{H_\lambda\}$ indexed by typed k-strict partitions whose algebra agrees with the Schubert calculus in the stable cohomology ring of OG. Let $x = (x_1, x_2, \ldots)$ be an infinite sequence of variables and $y = (y_1, \ldots, y_k)$ be a finite set of k variables. We define the functions $q_r(x)$ and $e_r(y)$ by the equations

$$\prod_{i=1}^{\infty} \frac{1+x_i t}{1-x_i t} = \sum_{r=0}^{\infty} q_r(x) t^r \quad \text{and} \quad \prod_{j=1}^{k} (1+y_j t) = \sum_{r=0}^{k} e_r(y) t^r$$

and set

$$\vartheta_r = \vartheta_r(x\,; y) = \sum_{i=0}^{r} q_{r-i}(x) e_i(y)$$

for each $r \geq 0$. (The ϑ_r will play the role of the Chern classes $c_r(\mathcal{Q})$.) Define $\eta_r = \vartheta_r$ for $r < k$, $\eta_r = \frac{1}{2}\vartheta_r$ for $r > k$, and set

$$\eta_k = \frac{1}{2}\vartheta_k + \frac{1}{2}e_k(y) \quad \text{and} \quad \eta'_k = \frac{1}{2}\vartheta_k - \frac{1}{2}e_k(y) = \frac{1}{2}\sum_{i=0}^{k-1} q_{k-i}(x) e_i(y).$$

We call $B^{(k)} = \mathbb{Z}[\eta_1, \ldots, \eta_{k-1}, \eta_k, \eta'_k, \eta_{k+1}, \ldots]$ the ring of *eta polynomials.* For any typed k-strict partition λ, define the eta polynomial

$$H_\lambda = 2^{-\ell_k(\lambda)}\, R^\lambda \star \vartheta_\lambda. \tag{9}$$

The raising operator expression in (9) is defined in the same way as the analogous one in Theorem 2, but using ϑ_r and η_k, η'_k in place of c_r and τ_k, τ'_k, respectively.

Theorem 4. *The H_λ, for λ a typed k-strict partition, form a $\mathbb{Z}$-basis of $B^{(k)}$. There is a surjective ring homomorphism $B^{(k)} \to \mathrm{H}^*(\mathrm{OG}(n+1-k, 2n+2), \mathbb{Z})$ such that H_λ is mapped to τ_λ, if λ fits inside an $(n+1-k) \times (n+k)$ rectangle, and to zero, otherwise.*

Thus far, we have excluded k equal to 0 from consideration. In fact, if we set $k = 0$ in equation (9), then we see that the eta polynomial $H_\lambda(x\,; y)$ is equal to a Schur P-function $P_\lambda(x)$, indexed by a strict partition λ. The connection between the algebra of Schur P-functions and the Schubert calculus on maximal orthogonal Grassmannians was established by Pragacz [P]. The eta polynomials studied here therefore serve as an analogue of the Schur P-functions for non-maximal even orthogonal Grassmannians.

Billey and Haiman [BH] have introduced a theory of Schubert polynomials $\mathfrak{D}_w(x, z)$ indexed by elements w of the Weyl group of type D. To any typed k-strict partition λ we associate a k-Grassmannian element w_λ, and prove (Proposition 6.3) that

$$H_\lambda(x\,;z_1,\ldots,z_k) = \mathfrak{D}_{w_\lambda}(x,z).$$

What is new in this equality is the expression of $\mathfrak{D}_{w_\lambda}$ via raising operators, as in (9). Note also that the equality takes place in the full ring $B^{(0)}[z_1,z_2,\ldots]$ of type D Billey-Haiman polynomials, where there are relations among the generators of $B^{(0)}$. A general theorem of op. cit. shows that $\mathfrak{D}_{w_\lambda}$ can be written as a sum of products of type D Stanley symmetric functions and type A Schubert polynomials. Lam [La] has shown that the type D Stanley symmetric functions are positive integer linear combinations of Schur P-functions, where the coefficients count Kraśkiewicz-Lam tableaux. Using these results, we can express $H_\lambda(x\,;y)$ as an explicit positive linear combination of products of Schur P-functions and S-polynomials (Theorem 6). A different expression for $H_\lambda(x\,;y)$, which writes it as a sum of monomials $2^{n(U)}(xy)^U$ over all 'typed k'-bitableaux' U of shape λ, is obtained in [T5].

Following [BH], the polynomials $\mathfrak{D}_{w_\lambda}(x,z)$ represent the pullbacks of the Schubert classes τ_λ in the stable cohomology ring of the complete flag variety SO_{2n}/B. We emphasize here that this theory does not imply our raising operator Giambelli formula, which is an expression for τ_λ in terms of the special Schubert classes τ_r. On the other hand, the eta polynomials defined here are used in [T4] to obtain combinatorially explicit Giambelli and degeneracy locus formulas which generalize Theorem 2. Similar formulas related to type A geometry were obtained earlier in [BKTY]. See also [Fu, G, IMN, T2, T3] for related results and [T6] for an exposition.

This paper is organized as follows. Sections 1 and 2 are concerned with the proof of Theorem 1. Propositions 1 and 2 and Theorem 2 are proved in section 3. The quantum Giambelli formula (Theorem 3) is established in section 4, where we also give a quantum version of Proposition 2. Section 5 develops the theory of eta polynomials and contains the proof of Theorem 4. In section 6 we show that the eta polynomials are equal to certain Billey-Haiman Schubert polynomials of type D, and give some applications. Finally, the appendix contains a detailed study of the Schubert varieties in orthogonal Grassmannians which justifies our claims about the spaces Y_λ, and corrects a related error in [BKT1].

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1. The Pieri rule for Chern classes

In this section we let the integer N have arbitrary parity and work in the cohomology ring $\mathrm{H}^*(\mathrm{OG}(m,N),\mathbb{Z})$. Recall that the integer $K = N - 2m$ is equal to $2k$ or $2k+1$. Let $c_p = c_p(\mathcal{Q})$ be the p-th Chern class of the universal quotient bundle $\mathcal{Q}$ over OG and define $Y_\lambda \subset \mathrm{OG}$ as in the introduction. We will formulate a Pieri rule for the cup products $c_p \cdot [Y_\lambda]$ in $\mathrm{H}^*(\mathrm{OG},\mathbb{Z})$.

We identify each partition λ with its Young diagram of boxes. Given two Young diagrams μ and ν with $\mu \subset \nu$, the skew diagram ν/μ is called a horizontal (resp. vertical) strip if it does not contain two boxes in the same column (resp. row). We say that the boxes $[r,c]$ and $[r',c']$ in row r (resp. r') and column c (resp. c') of λ

are *K-related* if $c \leq k < c'$ and $c + c' = K + 1 + r - r'$. This notion also makes sense for boxes outside the Young diagram of λ.

For any two k-strict partitions λ and μ, we write $\lambda \to \mu$ if μ may be obtained by removing a vertical strip from the first k columns of λ and adding a horizontal strip to the result, so that

(1) if one of the first k columns of μ has the same number of boxes as the same column of λ, then the bottom box of this column is K-related to at most one box of $\mu \smallsetminus \lambda$; and

(2) if a column of μ has fewer boxes than the same column of λ, then the removed boxes and the bottom box of μ in this column must each be K-related to exactly one box of $\mu \smallsetminus \lambda$, and these boxes of $\mu \smallsetminus \lambda$ must all lie in the same row.

Let $\mathbb{A}$ be the set of boxes of $\mu \smallsetminus \lambda$ in columns $k+1$ and higher which are not mentioned in (1) or (2), and define $N(\lambda, \mu)$ to be the number of connected components of $\mathbb{A}$. Here two boxes are connected if they share at least a vertex.

We say that a k-strict partition λ has *positive type* if $\lambda_i = K/2$ for some index i (note that this can only happen if K is even, so equal to $2k$). Given two k-strict partitions λ and μ with $\lambda \to \mu$, define

$$\widehat{N}(\lambda, \mu) = \begin{cases} N(\lambda, \mu) + 1 & \text{if } \lambda \text{ has positive type and } \mu \text{ does not,} \\ N(\lambda, \mu) & \text{otherwise.} \end{cases}$$

For any k-strict partition λ and any integer $p \geq 1$, the multiplication rule

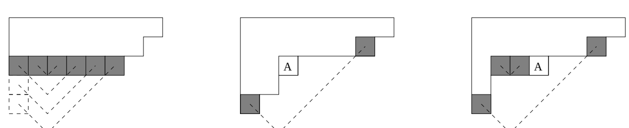
$$c_p \cdot [Y_\lambda] = \sum_{\lambda \to \mu,\, |\mu| = |\lambda| + p} 2^{\widehat{N}(\lambda, \mu)} \, [Y_\mu] \tag{10}$$

holds in $\mathrm{H}^*(\mathrm{OG}, \mathbb{Z})$. Indeed, when N is odd, (10) is equivalent to the Pieri rule for odd orthogonal Grassmannians from [BKT1, Thm. 2.1]. When N is even, the result follows easily from the Pieri rule for even orthogonal Grassmannians [BKT1, Thm. 3.1], which is recalled in §3.

Example 1.1. For the Grassmannian $\mathrm{OG}(5, 14)$ with $k = 2$, $n = 6$, and $K = 4$, we have

$$c_2 \cdot [Y_{87211}] = 2\,[Y_{876}] + 2\,[Y_{87321}] + 4\,[Y_{87411}].$$

Here $\lambda = (8, 7, 2, 1, 1)$ gives rise to 3 partitions μ with $\lambda \to \mu$ and $|\mu| = |\lambda| + 2$ as shown in the following pictures.



For any k-strict partition λ contained in an $m \times (n + k)$ rectangle, define a class W_λ in the cohomology ring of $\mathrm{OG}(m, N)$ by

$$W_\lambda = 2^{-\ell_k(\lambda)} R^\lambda \, c_\lambda.$$

The next result extends [BKT2, Eqn. (14)] to include the even values of N.

Theorem 5. *We have*

$$c_p \cdot W_\lambda = \sum_{\lambda \to \mu,\, |\mu| = |\lambda| + p} 2^{\widehat{N}(\lambda, \mu)} \, W_\mu \tag{11}$$

in $\mathrm{H}^*(\mathrm{OG},\mathbb{Z})$. *In other words, the cohomology classes* $[Y_\lambda]$ *and* W_λ *satisfy the same Pieri rule for products with the Chern classes of* $\mathcal{Q}$.

Observe that Theorem 1 follows easily from Theorem 5. To see this, write $\mu \succ \lambda$ if μ strictly dominates λ, i.e., $\mu \neq \lambda$ and $\mu_1 + \cdots + \mu_i \geq \lambda_1 + \cdots + \lambda_i$ for each $i \geq 1$. It follows from (10) and (11) that

$$2^{\ell_k(\lambda)} W_\lambda + \sum_{\mu \succ \lambda} a_{\lambda\mu}\, W_\mu = c_{\lambda_1} \cdots c_{\lambda_\ell} = 2^{\ell_k(\lambda)} [Y_\lambda] + \sum_{\mu \succ \lambda} a_{\lambda\mu}\, [Y_\mu]$$

for some constants $a_{\lambda\mu} \in \mathbb{Z}$. Using this and induction on λ, we deduce that $W_\lambda = [Y_\lambda]$, for each k-strict partition λ. We will prove Theorem 5 (and hence also Theorem 1) in the following section.

2. Proof of Theorem 5

Our proof of Theorem 5 is almost identical to the proof of [BKT2, Eqn. (14)]. We will not repeat the arguments of loc. cit. here, but will give an overview of the proof, pointing out where it needs to be modified to include the even case $K = 2k$.

2.1. If λ is any sequence of (possibly negative) integers, we say that λ has length ℓ if $\lambda_i = 0$ for all $i > \ell$ and $\ell \geq 0$ is the smallest number with this property. All integer sequences in this paper have finite length. In analogy with Young diagrams of partitions, we will say that a pair $[i,j]$ is a *box* of the integer sequence λ if $i \geq 1$ and $1 \leq j \leq \lambda_i$. A *composition* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r, \ldots)$ is a sequence of integers from the set $\mathbb{N} = \{0, 1, 2, \ldots\}$; we let $|\alpha| = \sum \alpha_i$.

Let $\Delta = \{(i,j) \in \mathbb{N} \times \mathbb{N} \mid 1 \leq i \leq j\}$ and define a partial order on Δ by agreeing that $(i',j') \leq (i,j)$ if $i' \leq i$ and $j' \leq j$. We call a finite subset D of Δ a *valid set of pairs* if $(i,j) \in D$ implies $(i',j') \in D$ for all $(i',j') \in \Delta$ with $(i',j') \leq (i,j)$. An *outer corner* of a valid set of pairs D is a pair $(i,j) \in \Delta \smallsetminus D$ such that $D \cup (i,j)$ is also a valid set of pairs.

Definition 2.1. For any valid set of pairs D, we define the raising operator

$$R^D = \prod_{i<j} (1 - R_{ij}) \prod_{i<j\,:\,(i,j) \in D} (1 + R_{ij})^{-1}.$$

For any finite monomial R in the operators R_{ij} which appears in the expansion of R^D, and any integer sequence λ, we let $R\, c_\lambda = c_{R\lambda}$. We define the element $T(D,\lambda)$ in $\mathrm{H}^*(\mathrm{OG}(m,N),\mathbb{Z})$ by the formula

$$T(D,\lambda) = 2^{-\#\{i \,|\, (i,i) \in D\}}\, R^D c_\lambda.$$

It follows from [BKT1, Thms. 2.2 and 3.2] that the Chern classes c_r satisfy the relations

$$\frac{1 - R_{12}}{1 + R_{12}}\, c_{(r,r)} = c_r^2 + 2 \sum_{i=1}^{r} (-1)^i c_{r+i} c_{r-i} = 0 \quad \text{for } r > k. \tag{12}$$

The next three lemmas are proved using the relations (12) in the same way as their counterparts in [BKT2, Lemmas 1.2–1.4].

Lemma 2.2. *Let* $\lambda = (\lambda_1, \ldots, \lambda_{j-1})$ *and* $\mu = (\mu_{j+2}, \ldots, \mu_\ell)$ *be integer vectors. Assume that* $(j, j+1) \notin D$ *and that for each* $h < j$, $(h,j) \in D$ *if and only if* $(h, j+1) \in D$. *Then for any integers* r *and* s *we have*

$$T(D, (\lambda, r, s, \mu)) = -T(D, (\lambda, s-1, r+1, \mu))\,.$$

In particular, $T(D,(\lambda,r,r+1,\mu))=0$.

Lemma 2.3. *Let* $\lambda=(\lambda_1,\ldots,\lambda_{j-1})$ *and* $\mu=(\mu_{j+2},\ldots,\mu_\ell)$ *be integer vectors, assume* $(j,j+1)\in D$*, and that for each* $h>j+1$*,* $(j,h)\in D$ *if and only if* $(j+1,h)\in D$*. If* $r,s\in\mathbb{Z}$ *are such that* $r+s>2k$*, then we have*

$$T(D,(\lambda,r,s,\mu))=-T(D,(\lambda,s,r,\mu))\,.$$

In particular, $T(D,(\lambda,r,r,\mu))=0$ *for any* $r>k$.

Lemma 2.4. *If* $(i,j)\notin D$ *and* $D\cup(i,j)$ *is a valid set of pairs, then*

$$T(D,\lambda)=T(D\cup(i,j),\lambda)+T(D\cup(i,j),R_{ij}\lambda).$$

2.2. Throughout the rest of this section we fix K, $p>0$, and the k-strict partition λ of length ℓ. We will work in the ring $\mathrm{H}^*(\mathrm{OG},\mathbb{Z})$, where $\mathrm{OG}=\mathrm{OG}(m,2m+K)$ for a sufficiently large integer m. Define a valid set of pairs $\mathcal{C}=\mathcal{C}(\lambda)$ by

$$\mathcal{C}(\lambda)=\{(i,j)\in\Delta\mid\lambda_i+\lambda_j\geq K+j-i,\ \lambda_i>k,\ \text{and}\ j\leq\ell\}.$$

Notice that $W_\lambda=T(\mathcal{C},\lambda)$. For any $d\geq\ell$ define the raising operator R_d^λ by

$$R_d^\lambda=\prod_{1\leq i<j\leq d}(1-R_{ij})\prod_{i<j\,:\,(i,j)\in\mathcal{C}}(1+R_{ij})^{-1}.$$

We compute that

$$c_p\cdot W_\lambda=c_p\cdot 2^{-\ell_k(\lambda)}R_\ell^\lambda\,c_\lambda=2^{-\ell_k(\lambda)}R_{\ell+1}^\lambda\cdot\prod_{i=1}^{\ell}(1-R_{i,\ell+1})^{-1}\,c_{\lambda,p}$$

$$=2^{-\ell_k(\lambda)}R_{\ell+1}^\lambda\cdot\prod_{i=1}^{\ell}(1+R_{i,\ell+1}+R_{i,\ell+1}^2+\cdots)\,c_{\lambda,p}$$

and therefore

$$c_p\cdot T(\mathcal{C},\lambda)=\sum_{\nu\in\mathcal{N}}T(\mathcal{C},\nu),\tag{13}$$

where $\mathcal{N}=\mathcal{N}(\lambda,p)$ is the set of all compositions $\nu\geq\lambda$ such that $|\nu|=|\lambda|+p$ and $\nu_j=0$ for $j>\ell+1$.

2.3. We will prove that the right hand side of equation (13) is equal to the right hand side of the Pieri rule (11), thus proving Theorem 5. For the rest of this section we set $m=\ell_k(\lambda)+1$, i.e. m is minimal such that $\lambda_m\leq k$. We call m the *middle row* of λ.

Definition 2.5. A *valid 4-tuple* of *level* h is a 4-tuple $\psi=(D,\mu,S,h)$, such that h is an integer with $0\leq h\leq\ell+1$, D is a valid set of pairs containing $\mathcal{C}$, all pairs (i,j) in D satisfy $i\leq m$ and $j\leq\ell+1$, S is a subset of $D\smallsetminus\mathcal{C}$, and μ is an integer sequence of length at most $\ell+1$. The evaluation of ψ is defined by $\mathrm{ev}(\psi)=T(D,\mu)\in\mathrm{H}^*(\mathrm{OG},\mathbb{Z})$.

In the following we set $\mu_0=\infty$ whenever μ is an integer sequence.

Definition 2.6. For any $y\in\mathbb{Z}$ we let $r(y)$ denote the largest integer such that $r(y)\leq\ell+1$ and $\lambda_{r(y)-1}\geq K+r(y)-y$.

We have the relation $\mathcal{C} = \{(i,j) \in \Delta \mid j < r(i+\lambda_i+1) \text{ and } \lambda_i > k\}$. Notice also that $r(m+k) = m$ while

$$r(m+k+1) = \begin{cases} m+1 & \text{if } \lambda_m = K/2, \\ m & \text{otherwise.} \end{cases} \tag{14}$$

Definition 2.7. Let $h \in \mathbb{N}$ satisfy $1 \leq h \leq m$ and let μ be an integer sequence.

(a) We define $b_h = r(h+\lambda_h+1)$ and $g_h = b_{h-1}$. By convention we set $g_1 = \ell+1$.

(b) Set $R(\mu) = \{[i,c] \in \mu \smallsetminus \lambda \mid c > k \text{ and } \mu_{r(i+c)} < K + r(i+c) - i - c\}$.

(c) Assume that $h \geq 2$ and $\mu_h \geq \lambda_{h-1}$. If $[h, \lambda_{h-1}] \in R(\mu)$ then set $e_h(\mu) = \lambda_{h-1}$. Otherwise, if $h < m$ (respectively, if $h = m$) choose $e_h(\mu) > \lambda_h$ (respectively, $e_h(\mu) \geq K/2$) minimal such that $[h,c] \notin R(\mu)$ for $e_h(\mu) \leq c \leq \lambda_{h-1}$. Finally, set $f_h(\mu) = r(h+e_h(\mu))$.

If μ is a k-strict partition such that $\lambda \to \mu$, then the set $\mathbb{A}$ from §1 consists of the boxes of $\mu \smallsetminus \lambda$ in columns $k+1$ and higher which are not in $R(\mu)$. For a general sequence μ, when $[h, \lambda_{h-1}] \notin R(\mu)$, the integer $e_h(\mu)$ is the least such that $e_h(\mu) \geq K/2$, $(e_h(\mu), \lambda_h)$ is a k-strict partition, and $[h,c] \notin R(\mu)$ for $e_h(\mu) \leq c \leq \lambda_{h-1}$.

If we are given a fixed valid 4-tuple (D, μ, S, h) with $1 \leq h \leq m$, we will use the shorthand notation $b = b_h$, $g = g_h$, $R = R(\mu)$, $e = e_h(\mu)$, and $f = f_h(\mu)$.

Definition 2.8. Let $(i,j) \in \Delta$ be arbitrary. We define two conditions $\mathrm{W}(i,j)$ and X on a valid 4-tuple (D, μ, S, h) as follows.

$$\mathrm{W}(i,j) \;:\; \mu_i + \mu_j \geq K + j - i \text{ and } \mu_i > k.$$

Condition X is true if and only if $(h,h) \in D$ and

$$\mu_h \geq \mu_{h-1} \text{ or } \mu_h > \lambda_{h-1} \text{ or } (\mu_h = \lambda_{h-1} \text{ and } (h,f) \notin S)\,.$$

2.4. The following *substitution rule* will be applied iteratively to rewrite the right hand side of (13). Both this rule and the algorithm which follows it are identical to the one in [BKT2, §3.3], but we recall them here for the sake of exposition.

<u>Substitution Rule</u>

Let (D, μ, S, h) be a valid 4-tuple of level $h \geq 1$. Assume first that $(h,h) \notin D$. If

(i) there is an outer corner (i,h) of D with $i \leq m$ such that $\mathrm{W}(i,h)$ holds

then REPLACE (D, μ, S, h) with

$$(D \cup (i,h), \mu, S, h) \text{ and } (D \cup (i,h), R_{ih}\mu, S \cup (i,h), h).$$

Otherwise, if

(ii) D has no outer corner in column h and $\mu_h > \lambda_{h-1}$,

then STOP.

Assume now that $(h,h) \in D$. If

(iii) there is an outer corner (h,j) of D with $j \leq \ell+1$ such that $\mathrm{W}(h,j)$ holds, then REPLACE (D,μ,S,h) with

$$\begin{cases}(D\cup(h,j),\mu,S,h) \ \text{ and } \ (D\cup(h,j),R_{hj}\mu,S\cup(h,j),h) & \text{if } \mu_j\leq\mu_{j-1},\\ (D\cup(h,j),R_{hj}\mu,S\cup(h,j),h) & \text{if } \mu_j>\mu_{j-1}.\end{cases}$$

Otherwise, if

(iv) $\mathrm{W}(h,g)$ or X holds, and D has an outer corner (i,g) with $i\leq h$,

then REPLACE (D,μ,S,h) with

$$(D\cup(i,g),\mu,S,h) \ \text{ and } \ (D\cup(i,g),R_{ig}\mu,S\cup(i,g),h).$$

Otherwise, if

(v) X holds,

then STOP.

If none of the above conditions hold, REPLACE (D,μ,S,h) with $(D,\mu,S,h-1)$.

2.5. Define the set $\Psi=\{(\mathcal{C},\nu,\emptyset,\ell+1)\mid\nu\in\mathcal{N}(\lambda,p)\}$, so that $\sum_{\psi\in\Psi}\mathrm{ev}(\psi)$ agrees with the right hand side of (13). Consider the following algorithm which will change Ψ by replacing some 4-tuples with one or two new valid 4-tuples. The algorithm applies the Substitution Rule to each element (D,μ,S,h) of level $h\geq 1$. If the substitution rule results in a REPLACE statement, then the set is changed by replacing (D,μ,S,h) by one or two new 4-tuples accordingly; otherwise the substitution rule results in a STOP statement, and the 4-tuple (D,μ,S,h) is left untouched. These substitutions are iterated until no further elements can be REPLACED.

Suppose that the 4-tuple $\psi=(D,\mu,S,h)$ occurs in the algorithm. If ψ is replaced by two 4-tuples ψ_1 and ψ_2, then it follows from Lemma 2.4 that $\mathrm{ev}(\psi)=\mathrm{ev}(\psi_1)+\mathrm{ev}(\psi_2)$. Moreover, if ψ meets **(iii)** and is replaced by the single 4-tuple $\psi'=(D\cup(h,j),R_{hj}\mu,S\cup(h,j),h)$, then one can show that $\mu_j=\mu_{j-1}+1$ and $D\cup(h,j)$ has no outer corner in column j, so Lemmas 2.2 and 2.4 imply that $\mathrm{ev}(\psi)=\mathrm{ev}(\psi')$.

When the algorithm terminates, let Ψ_0 (respectively Ψ_1) denote the collection of all 4-tuples (D,μ,S,h) in the final set such that $h=0$ (respectively $h>0$). We deduce from the above analysis that

$$\sum_{\nu\in\mathcal{N}}T(\mathcal{C},\nu)=\sum_{\psi\in\Psi_0}\mathrm{ev}(\psi)+\sum_{\psi\in\Psi_1}\mathrm{ev}(\psi).$$

Claim 1. For each 4-tuple $\psi=(D,\mu,S,0)$ in Ψ_0 with $\mu_{\ell+1}\geq 0$, μ is a k-strict partition with $\lambda\to\mu$,

$$D=\mathcal{C}_{\ell+1}(\mu):=\{(i,j)\in\Delta\mid\mu_i+\mu_j\geq K+j-i,\ \ \mu_i>k,\ \text{ and } \ j\leq\ell+1\}$$

is uniquely determined by μ, and

$$\mathrm{ev}(\psi)=T(\mathcal{C}(\mu),\mu).$$

Furthermore, for each such partition μ, there are exactly $2^{\widehat{N}(\lambda,\mu)}$ such 4-tuples ψ, in accordance with the Pieri rule (11).

Given a valid 4-tuple $\psi=(D,\mu,S,h)$ with $h\geq 2$ and $\mu_h\geq\lambda_{h-1}$, we define a new 4-tuple $\iota\psi$ as follows. If $(h,h)\notin D$, then set $\iota\psi=(D,\widetilde{\mu},S,h)$, where the composition $\widetilde{\mu}$ is defined by $\widetilde{\mu}_{h-1}=\mu_h-1$, $\widetilde{\mu}_h=\mu_{h-1}+1$, and $\widetilde{\mu}_t=\mu_t$ for $t\notin\{h-1,h\}$. If $(h,h)\in D$ and $\mu_{h-1}=\mu_h$, then set $\iota\psi=\psi$. Assume that $(h,h)\in D$ and $\mu_{h-1}\neq\mu_h$. Let ϖ be the involution of Δ that exchanges $(h-1,g)$

with (h, f), and fixes all other pairs. Then set $\iota\psi = (D, \widetilde{\mu}, \widetilde{S}, h)$, where $\widetilde{S} = \varpi(S)$, and $\widetilde{\mu}$ is the composition obtained from μ by switching the parts μ_{h-1} and μ_h.

Claim 2. The above map ι restricts to an involution $\Psi_1 \to \Psi_1$ such that

$$\mathrm{ev}(\psi) + \mathrm{ev}(\iota(\psi)) = 0,$$

for every $\psi \in \Psi_1$.

We remark that the 4-tuples $\psi \in \Psi_0$ with $\mu_{\ell+1} < 0$ evaluate to zero trivially, by Definition 2.1; the two claims therefore suffice to prove Theorem 5. The proofs of these claims are nearly identical to the ones in [BKT2, §4], replacing the value $2k+1$ by K throughout. The following explicit construction of the sets S in the 4-tuples which appear in Ψ_0 accounts for the multiplicities $\widehat{N}(\lambda, \mu)$ in Claim 1.

Fix an arbitrary k-strict partition μ such that $\lambda \to \mu$ and $|\mu| = |\lambda| + p$. Recall the set $\mathbb{A}$ of §1, and define a new set $\widehat{\mathbb{A}}$ by

$$\widehat{\mathbb{A}} = \begin{cases} \mathbb{A} \cup \{[m, k]\} & \text{if } \lambda_m = K/2 < \mu_m, \\ \mathbb{A} & \text{otherwise.} \end{cases}$$

A *component* means an (edge or vertex) connected component of the set $\widehat{\mathbb{A}}$. We say that a box B of $\widehat{\mathbb{A}}$ is *distinguished* if the box directly to the left of B does not lie in $\widehat{\mathbb{A}}$. We say that B is *optional* if it is the rightmost distinguished box in its component. Using (14), we deduce that $\widehat{N}(\lambda, \mu)$ is equal to the number of optional distinguished boxes in $\widehat{\mathbb{A}}$.

To each distinguished box $B = [i, c]$ we associate the pair $(i, j) = (i, r(i + c))$. The inequality $\lambda_{i-1} \geq K + i - (i + c)$ implies that $i \leq j$, so $(i, j) \in \Delta$. Let E (respectively F) be the set of pairs associated to optional (respectively non-optional) distinguished boxes. We furthermore let G be the set of all pairs $(i, j) \in \Delta$ for which some box in row i of $\mu \smallsetminus \lambda$ is K-related to a box in row j of $\lambda \smallsetminus \mu$.

Suppose that $(\mathcal{C}_{\ell+1}(\mu), \mu, S, 0) \in \Psi_0$ and $(i, j) \in S \smallsetminus G$. Then one can show that (i, j) is the pair associated to a distinguished box of $\widehat{\mathbb{A}}$. For the last statement, observe that if $i = j = m$, then $\lambda_m \leq K/2 < \mu_m$ and (m, m) is associated to the distinguished box $[m, k{+}1] \in \widehat{\mathbb{A}}$ (respectively, $[m, k] \in \widehat{\mathbb{A}}$) if $\lambda_m < K/2$ (respectively, $\lambda_m = K/2$). To every subset E' of E we associate the set of pairs

$$S(E') := E' \cup F \cup G.$$

This is a disjoint union, and there are exactly $2^{\widehat{N}(\lambda, \mu)}$ sets of this form. Let $S \subset \Delta$ be any subset. Then one may prove as in [BKT2, §4] that $(\mathcal{C}_{\ell+1}(\mu), \mu, S, 0) \in \Psi_0$ if and only if $S = S(E')$ for some subset $E' \subset E$.

3. The classical Giambelli formula

In this section we prove Propositions 1 and 2 and Theorem 2 of the introduction. Throughout the section we will be in the even orthogonal case where $K = 2k$. We assume that $k > 0$, however all of our results about the cohomology of $\mathrm{OG}(n+1-k, 2n+2)$ also hold when $k = 0$, provided that in the latter case OG parametrizes both families of maximal isotropic subspaces, and hence is a disjoint union of two irreducible components.

3.1. The Schubert classes τ_λ for all typed k-strict partitions λ whose diagrams are contained in an $m \times (n+k)$ rectangle form a $\mathbb{Z}$-basis of $\mathrm{H}^*(\mathrm{OG}(m,N),\mathbb{Z})$. For typed k-strict partitions λ and μ, we write $\lambda \to \mu$ if the underlying k-strict partitions satisfy $\lambda \to \mu$ (with $K = 2k$) and furthermore $\mathrm{type}(\lambda) + \mathrm{type}(\mu) \neq 3$. According to [BKT1, Thm. 3.1], the following Pieri rule holds in $\mathrm{H}^*(\mathrm{OG},\mathbb{Z})$. For any typed k-strict partition λ and any integer $p \geq 1$, we have

$$c_p \cdot \tau_\lambda = \sum_{\substack{\lambda \to \mu \\ |\mu| = |\lambda| + p}} 2^{N(\lambda,\mu)}\, \tau_\mu. \tag{15}$$

3.2. In the sequel we will have to work both with k-strict and with typed k-strict partitions. We will therefore adopt the following conventions for use with Schubert classes and (later) representing polynomials indexed by such objects. If the k-strict partition λ has positive type, we agree that τ_λ and τ'_λ denote the Schubert classes in $\mathrm{H}^*(\mathrm{OG},\mathbb{Z})$ of type 1 and 2, respectively, associated to λ. If λ does not have positive type, then τ_λ denotes the associated Schubert class of type zero.

For any k-strict partition λ, we define a cohomology class $\widehat{\tau}_\lambda \in \mathrm{H}^{2|\lambda|}(\mathrm{OG},\mathbb{Z})$ by the equations

$$\widehat{\tau}_\lambda = \begin{cases} \tau_\lambda + \tau'_\lambda & \text{if } \lambda \text{ has positive type,} \\ \tau_\lambda & \text{otherwise.} \end{cases} \tag{16}$$

Let $\mathbb{Q}[c]$ denote the $\mathbb{Q}$-subalgebra of $\mathrm{H}^*(\mathrm{OG}(m,N),\mathbb{Q})$ generated by the Chern classes c_p for all $p \geq 1$.

Recall from the introduction that for each k-strict partition λ contained in an $m \times (n+k)$ rectangle, we have a Zariski closed subset Y_λ of the even orthogonal Grassmannian $\mathrm{OG}(n+1-k, 2n+2)$, defined by the equations (2). See the appendix for a proof that if λ has type zero, then Y_λ is the Schubert variety X_λ in OG, and otherwise Y_λ has two irreducible components, which are the Schubert varieties X_λ and X'_λ. We deduce that the cohomology class $[Y_\lambda]$ in $\mathrm{H}^{2|\lambda|}(\mathrm{OG},\mathbb{Z})$ is equal to $\widehat{\tau}_\lambda$, for each such λ. The Pieri rule (15) therefore implies that the classes $[Y_\lambda]$ satisfy the rule (10). Proposition 1 follows directly from these observations.

We next prove Proposition 2. For any k-strict partition λ, define a class $\widetilde{\tau}_\lambda$ in $\mathrm{H}^{2|\lambda|}(\mathrm{OG},\mathbb{Z})$ by the equations

$$\widetilde{\tau}_\lambda = \begin{cases} \tau_\lambda - \tau'_\lambda & \text{if } \lambda \text{ has positive type,} \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

The Pieri rule (15) implies that for any $p \geq 1$, we have

$$c_p \cdot \widetilde{\tau}_\lambda = \begin{cases} \sum_{\lambda \to \mu,\, |\mu| = |\lambda| + p} 2^{N(\lambda,\mu)}\, \widetilde{\tau}_\mu & \text{if } \lambda \text{ has positive type,} \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

The positivity of the coefficients in this formula is a consequence of the type convention for Schubert classes introduced in [BKT1].

Consider the $\mathbb{Q}$-linear map

$$\psi : \mathrm{H}^*(\mathrm{OG}(m-1, N-1),\mathbb{Q}) \to \mathrm{H}^*(\mathrm{OG}(m,N),\mathbb{Q})_{-1}$$

defined by $\psi(\sigma_\lambda) = \widetilde{\tau}_{\lambda+k}$ for each k-strict partition λ contained in an $(m-1)\times(n+k)$ rectangle. By comparing (18) with (10) for $K = 2k+1$, we obtain

$$\psi(c_p \cdot \sigma_\lambda) = c_p \cdot \psi(\sigma_\lambda).$$

We deduce that ψ is an isomorphism of $\mathbb{Q}[c]$-modules, and that

$$\mathrm{H}^*(\mathrm{OG},\mathbb{Q})_{-1} = \bigoplus_{\lambda} \mathbb{Q}\cdot(\tau_{\lambda+k}-\tau'_{\lambda+k}) = \mathbb{Q}[c_1,\ldots,c_{n+k}]\cdot(\tau_k-\tau'_k).$$

Next, use Theorem 1 to expand $\sigma_\lambda \in \mathrm{H}^*(\mathrm{OG}(m-1,N-1))$ as a Giambelli polynomial in the Chern classes of $\mathcal{Q}$. By mapping this equation to $\mathrm{H}^*(\mathrm{OG}(m,N),\mathbb{Q})_{-1}$ via ψ and using the fact that $\psi(1)=\tau_k-\tau'_k$, it follows that

$$\tau_{\lambda+k}-\tau'_{\lambda+k} = 2^{-\ell_k(\lambda)}(\tau_k-\tau'_k)\,\widetilde{R}^\lambda\,c_\lambda, \tag{19}$$

where $\widetilde{R}^\lambda$ denotes the operator defined by formula (4) with $K=2k+1$. This completes the proof of Proposition 2.

Let λ be a k-strict partition contained in an $m\times(n+k)$ rectangle, and let R be a finite monomial in the operators R_{ij} that occurs in the expansion of R^λ. If λ does not have positive type, then set $R\diamond c_\lambda=0$. If λ has positive type, then set $d=\ell_k(\lambda)+1$, so that $\lambda_d=k<\lambda_{d-1}$. If R contains any operator R_{ij} for which i or j is equal to d, then set $R\diamond c_\lambda=0$. Otherwise define $R\diamond c_\lambda=(\tau_k-\tau'_k)\,c_{\widehat{R\lambda}}$, where $\widehat{\alpha}=(\alpha_1,\ldots,\alpha_{d-1},\alpha_{d+1},\ldots)$. Equation (19) is then equivalent to the identity

$$\widetilde{\tau}_\lambda = 2^{-\ell_k(\lambda)}R^\lambda\diamond c_\lambda\,. \tag{20}$$

Now observe that for any typed k-strict partition $\lambda\in\widetilde{\mathcal{P}}(k,n)$ and monomial R we have

$$R\star c_\lambda = \begin{cases} R\,c_\lambda & \text{if } \mathrm{type}(\lambda)=0,\\ \frac{1}{2}R\,c_\lambda+\frac{1}{2}R\diamond c_\lambda & \text{if } \mathrm{type}(\lambda)=1,\\ \frac{1}{2}R\,c_\lambda-\frac{1}{2}R\diamond c_\lambda & \text{if } \mathrm{type}(\lambda)=2.\end{cases} \tag{21}$$

Theorem 2 follows by combining Theorem 1, equations (20) and (21), and the identity

$$\tau_\lambda = \begin{cases} \widehat{\tau}_\lambda & \text{if } \mathrm{type}(\lambda)=0,\\ \frac{1}{2}(\widehat{\tau}_\lambda+\widetilde{\tau}_\lambda) & \text{if } \mathrm{type}(\lambda)=1,\\ \frac{1}{2}(\widehat{\tau}_\lambda-\widetilde{\tau}_\lambda) & \text{if } \mathrm{type}(\lambda)=2.\end{cases} \tag{22}$$

Note that we have ignored the type of λ in the expressions $R\,c_\lambda$, $R\diamond c_\lambda$, $\widehat{\tau}_\lambda$, $\widetilde{\tau}_\lambda$ which appear on the right hand sides of equations (21) and (22).

4. The quantum Giambelli formula

4.1. Quantum cohomology of orthogonal Grassmannians. As in the introduction, we consider the orthogonal Grassmannian $\mathrm{OG}=\mathrm{OG}(m,N)$ with $N=2m+K$, where $K\geq 2$. When $K\neq 2$, the quantum cohomology ring $\mathrm{QH}(\mathrm{OG})$ is a $\mathbb{Z}[q]$-algebra which is isomorphic to $\mathrm{H}^*(\mathrm{OG})\otimes_{\mathbb{Z}}\mathbb{Z}[q]$ as a module over $\mathbb{Z}[q]$. The degree of the formal variable q is $n+k$. When $K=2$, $\mathrm{QH}(\mathrm{OG})$ is a $\mathbb{Z}[q_1,q_2]$-algebra, we have

$$\mathrm{QH}(\mathrm{OG})\cong \mathrm{H}^*(\mathrm{OG},\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{Z}[q_1,q_2]$$

as a $\mathbb{Z}[q_1,q_2]$-module, and $\deg(q_1)=\deg(q_2)=n+1$. In both cases, we set

$$\mathrm{QH}(\mathrm{OG},\mathbb{Q}) := \mathrm{QH}(\mathrm{OG})\otimes_{\mathbb{Z}}\mathbb{Q}.$$

The precise definition of the ring structure on $\mathrm{QH}(\mathrm{OG})$ is given in [FPa].

We first correct an error in the definition of $\widetilde{\nu}$ given in [BKT1, §3.5], which appears in the quantum Pieri rule [BKT1, Thm. 3.4]. In the first paragraph of

[BKT1, §3.5], $\widetilde{\nu}$ should be obtained from ν by removing the first row of ν as well as $n+k-\nu_1$ boxes in the first column, i.e. $\widetilde{\nu}=(\nu_2,\nu_3,\dots,\nu_r)$, where $r=\nu_1-2k+2$.

We next recall some further definitions from [BKT1], using a more uniform notation. Let $P(m,N)$ denote the set of all k-strict partitions contained in an $m\times(n+k)$ rectangle. For $\lambda\in P(m,N)$ we let $\lambda^*=(\lambda_2,\lambda_3,\dots)$ be the partition obtained by removing the first row of λ. Let $P'(m,N)$ be the set of k-strict partitions ν contained in an $(m+1)\times(n+k)$ rectangle for which $\ell(\nu)=m+1$, $\nu_1\geq K-1$, and the number of boxes in the second column of ν is at most ν_1-K+2. For each element $\nu\in P'(m,N)$, we let $\widetilde{\nu}\in P(m,N)$ denote the partition obtained by removing the first row of ν as well as $n+k-\nu_1$ boxes from the first column. That is,

$$\widetilde{\nu}=(\nu_2,\nu_3,\dots,\nu_r)\,,\quad\text{where } r=\nu_1-K+2.$$

For each k-strict partition $\lambda\in P(m,N)$, we define $Y_\lambda\subset\mathrm{OG}$ as in the introduction and let $[Y_\lambda]$ denote the corresponding class in $\mathrm{QH}(\mathrm{OG})$. For odd integers N, we set $\sigma_\lambda=[Y_\lambda]$, while for even N, we let $\widehat{\tau}_\lambda$ and $\widetilde{\tau}_\lambda$ denote the classes in $\mathrm{QH}(\mathrm{OG})$ defined by the equations (16) and (17). We proceed to describe the products of these classes with the Chern classes $c_p=c_p(\mathcal{Q})$.

Assume first that $K\geq 3$. The following quantum Pieri rule for Chern classes is a direct consequence of [BKT1, Thms. 2.4 and 3.4]. For any $\lambda\in P(m,N)$ and integer $p\in[1,n+k]$, we have

$$c_p\cdot[Y_\lambda]=\sum_{\lambda\to\mu}2^{\widehat{N}(\lambda,\mu)}\,[Y_\mu]+\sum_{\lambda\to\nu}2^{\widehat{N}(\lambda,\nu)}\,[Y_{\widetilde{\nu}}]\,q+\sum_{\lambda^*\to\rho}2^{\widehat{N}(\lambda^*,\rho)}\,[Y_{\rho^*}]\,q^2\,. \tag{23}$$

Here the first sum is classical, the second sum is over $\nu\in P'(m,N)$ with $\lambda\to\nu$ and $|\nu|=|\lambda|+p$, and the third sum is empty unless $\lambda_1=n+k$, and over $\rho\in P(m,N)$ such that $\rho_1=n+k$, $\lambda^*\to\rho$, and $|\rho|=|\lambda^*|+p$.

Suppose now that $K\geq 4$ is even. Then the quantum cohomology ring $\mathrm{QH}(\mathrm{OG},\mathbb{Q})$ has a decomposition

$$\mathrm{QH}(\mathrm{OG},\mathbb{Q})=\mathrm{QH}(\mathrm{OG},\mathbb{Q})_1\oplus\mathrm{QH}(\mathrm{OG},\mathbb{Q})_{-1}$$

where $\mathrm{QH}(\mathrm{OG},\mathbb{Q})_1$ (respectively $\mathrm{QH}(\mathrm{OG},\mathbb{Q})_{-1}$) is the $\mathbb{Q}[q]$-submodule spanned by the classes $\widehat{\tau}_\lambda$ (respectively $\widetilde{\tau}_\lambda$). This is the eigenspace decomposition for the action of the involution $\iota:\mathrm{OG}\to\mathrm{OG}$ defined in the introduction. Set $\overline{\mathrm{OG}}=\mathrm{OG}(m-1,N-1)$. Proposition 2 has the following generalization.

Proposition 4.1. *The $\mathbb{Q}$-linear map $\mathrm{QH}(\overline{\mathrm{OG}},\mathbb{Q})\to\mathrm{QH}(\mathrm{OG},\mathbb{Q})_{-1}$ defined by*

$$\sigma_\lambda\,q^d\mapsto\widetilde{\tau}_{\lambda+k}\,(-q)^d$$

is an isomorphism of $\mathbb{Q}[c_1,\dots,c_{n+k}]$-modules.

Proof. We argue as in the proof of Proposition 2 given in §3.2, using (23) and the quantum Pieri rule for the products $c_p\cdot\tau_\lambda$ obtained from [BKT1, Thm. 3.4]. □

Observe that (23) and Proposition 4.1 determine the products $c_p\cdot\widetilde{\tau}_\lambda$ in $\mathrm{QH}(\mathrm{OG})$ for all $p\geq 1$ and even $K\geq 4$.

Assume next that $K=2$, so that $k=1$ and $m=n$. Set $\widehat{q}=q_1+q_2$ and $\widetilde{q}=q_1-q_2$. The following results are direct consequences of [BKT1, Thm. A.1]. For $\lambda\in P(n,2n+2)$ and $p\in[1,n+1]$ such that $p\neq 1$ or $\lambda\neq(1^n)$, we have

$$c_p\cdot\widehat{\tau}_\lambda=\sum_{\lambda\to\mu}2^{\widehat{N}(\lambda,\mu)}\,\widehat{\tau}_\mu+\sum_{\lambda\to\nu}2^{\widehat{N}(\lambda,\nu)-1}\,(\widehat{\tau}_{\widetilde{\nu}}\,\widehat{q}-\widetilde{\tau}_{\widetilde{\nu}}\,\widetilde{q}\,)+\sum_{\lambda^*\to\rho}2^{\widehat{N}(\lambda^*,\rho)}\,\widehat{\tau}_{\rho^*}\,q_1q_2\,, \tag{24}$$

where the first sum is classical, the second sum is over $\nu \in P'(n, 2n+2)$ with $\lambda \to \nu$ and $|\nu| = |\lambda| + p$, and the third sum is empty unless $\lambda_1 = n+1$ and over $\rho \in P(n, 2n+2)$ such that $\rho_1 = n+1$, $\lambda^* \to \rho$, and $|\rho| = |\lambda^*| + p$.

If $\lambda \in P(n-1, 2n+1)$ and $p \in [1, n+1]$ with $p \neq 1$ or $\lambda \neq (1^{n-1})$, we have

$$\begin{aligned} c_p \cdot \widetilde{\tau}_{(\lambda,1)} = \sum_{\lambda \to \mu} 2^{N(\lambda,\mu)}\, \widetilde{\tau}_{(\mu,1)} + \sum_{\lambda \to \nu} 2^{N(\lambda,\nu)-1}\, (\widehat{\tau}_{(\widetilde{\nu},1)}\, \widetilde{q} - \widetilde{\tau}_{(\widetilde{\nu},1)}\, \widehat{q}\,) \\ + \sum_{\lambda^* \to \rho} 2^{N(\lambda^*,\rho)}\, \widetilde{\tau}_{(\rho^*,1)}\, q_1 q_2\,, \end{aligned} \tag{25}$$

where the first sum is classical, the second sum is over $\nu \in P'(n-1, 2n+1)$ with $\lambda \to \nu$ (for $K = 3$) and $|\nu| = |\lambda| + p$, and the third sum is empty unless $\lambda_1 = n+1$ and over $\rho \in P(n-1, 2n+1)$ such that $\rho_1 = n+1$, $\lambda^* \to \rho$ (for $K = 3$), and $|\rho| = |\lambda^*| + p$. Moreover, the integers $N(\lambda, \mu)$, $N(\lambda, \nu)$, and $N(\lambda^*, \rho)$ are computed for $K = 3$.

Finally, we have

$$c_1 \cdot \widehat{\tau}_{(1^n)} = \begin{cases} 2\,\widehat{\tau}_{n+1} + 2\,\widehat{\tau}_{(2,1^{n-1})} + \widehat{q} & \text{if } n > 1 \\ 2\,\widehat{\tau}_{n+1} + \widehat{q} & \text{if } n = 1 \end{cases} \tag{26}$$

and

$$c_1 \cdot \widetilde{\tau}_{(1^n)} = 2\,\widetilde{\tau}_{(2,1^{n-1})} + \widetilde{q}\,. \tag{27}$$

Remark 4.2. Observe that Proposition 4.1 is valid for $K = 2$ and $\mathrm{OG} = \mathrm{OG}(n, 2n+2)$ if the quantum cohomology ring of the latter space is replaced with $\mathrm{QH}(\mathrm{OG}, \mathbb{Q})/(q_1 - q_2)$, with $q := q_1 = q_2$. Similarly, equation (23) is valid in this quotient ring, except for the product $c_1 \cdot \widehat{\tau}_{(1^n)}$. This product is special because it is the only one that can produce a partition $\nu \in P'(n, 2n+2)$ such that ν has positive type but $\widetilde{\nu}$ does not have positive type.

4.2. The stable cohomology ring of OG. In this section, we recall the definition of the stable cohomology ring of the orthogonal Grassmannian $\mathrm{OG}(m, N)$; this depends only on $K = N - 2m$, and is denoted $\mathbb{H}(\mathrm{OG}_K)$. The ring $\mathbb{H}(\mathrm{OG}_K)$ is defined as the inverse limit in the category of graded rings of the system

$$\cdots \leftarrow \mathrm{H}^*(\mathrm{OG}(m, N), \mathbb{Z}) \leftarrow \mathrm{H}^*(\mathrm{OG}(m+1, N+2), \mathbb{Z}) \leftarrow \cdots$$

We set

$$\mathbb{H}(\mathrm{OG}_k(\mathrm{odd})) := \mathbb{H}(\mathrm{OG}_{2k+1}) \quad \text{and} \quad \mathbb{H}(\mathrm{OG}_k(\mathrm{even})) := \mathbb{H}(\mathrm{OG}_{2k}).$$

The ring $\mathbb{H}(\mathrm{OG}_k(\mathrm{odd}))$ was studied in [BKT2, BKT3].

It follows from [BKT1, Thm. 3.2] that $\mathbb{H}(\mathrm{OG}_k(\mathrm{even}))$ may be presented as a quotient of the polynomial ring $\mathbb{Z}[\tau_1, \ldots, \tau_{k-1}, \tau_k, \tau'_k, \tau_{k+1}, \ldots]$ modulo the relations

$$\tau_r^2 + \sum_{i=1}^{r} (-1)^i \tau_{r+i} c_{r-i} = 0 \quad \text{for } r > k, \tag{28}$$

$$\tau_k \tau'_k + \sum_{i=1}^{k} (-1)^i \tau_{k+i} \tau_{k-i} = 0, \tag{29}$$

where the c_i obey the equations (7).

The stable cohomology ring $\mathbb{H}(\mathrm{OG}_k(\mathrm{even}))$ has a free $\mathbb{Z}$-basis of Schubert classes τ_λ, one for each typed k-strict partition λ. There is a natural surjective ring homomorphism

$$\mathbb{H}(\mathrm{OG}_k(\mathrm{even})) \to \mathrm{H}^*(\mathrm{OG}(n+1-k, 2n+2), \mathbb{Z})$$

that maps τ_λ to τ_λ, when $\lambda \in \widetilde{\mathcal{P}}(k,n)$, and to zero, otherwise. All the conclusions of §3 remain true for the ring $\mathbb{H}(\mathrm{OG}_k(\mathrm{even}))$, with no restrictions on the size of the (typed) k-strict partitions involved.

4.3. Proof of Theorem 3. We first generalize two results from [BKT3].

Lemma 4.3. *Let λ be a k-strict partition contained in an $m \times (n+k)$ rectangle. Then the stable Giambelli polynomial $R^\lambda c_\lambda$ for $[Y_\lambda]$ in $\mathbb{H}(\mathrm{OG}_K)$ involves only Chern classes c_p with $p \leq 2n+2k-1$.*

Proof. The proof is the same as that of [BKT3, Cor. 1]. $\square$

For any abelian group A, let $A_\mathbb{Q} = A \otimes_\mathbb{Z} \mathbb{Q}$.

Proposition 4.4. *Let λ be a k-strict partition. Then there exist unique coefficients $a_{p,\mu} \in \mathbb{Q}$ for $p \geq \lambda_1$ and (p,μ) a k-strict partition, such that the recursive identity*

$$[Y_\lambda] = \sum_{p \geq \lambda_1} \sum_{\mu\,:\,(p,\mu)\ k\text{-strict}} a_{p,\mu}\, c_p\, [Y_\mu]$$

holds in the stable cohomology ring $\mathbb{H}(\mathrm{OG}_K)_\mathbb{Q}$. Furthermore, $a_{p,\mu} = 0$ whenever $\mu \not\subset \lambda^*$, or when λ is contained in an $m \times (n+k)$ rectangle and $p \geq 2n+2k$.

Proof. The proof is the same as that of [BKT3, Prop. 3]. $\square$

Next, we give the even orthogonal analogue of [BKT3, Prop. 5]. If λ is a k-strict partition of positive type, we let $\lambda - k$ denote the partition obtained by removing one part equal to k from λ.

Proposition 4.5. *There exists a unique ring homomorphism*

$$\pi : \mathbb{H}(\mathrm{OG}_k(\mathrm{even})) \to \mathrm{QH}(\mathrm{OG}(n+1-k, 2n+2))$$

such that the following relations are satisfied:

$$\pi(\tau_i) = \begin{cases} \tau_i & \text{if } 1 \leq i \leq n+k, \\ 0 & \text{if } n+k < i < 2n+2k, \\ 0 & \text{if } i \text{ is odd and } i > 2n+2k, \end{cases}$$
$$\pi(\tau'_k) = \tau'_k.$$

Furthermore, we have $\pi(\tau_\lambda) = \tau_\lambda$ for each $\lambda \in \widetilde{\mathcal{P}}(k,n)$.

Proof. The relations (28)–(29) for $r \geq n+k$ uniquely specify the values $\pi(\tau_i)$ for even integers $i \geq 2n+2k$. We must show that the remaining relations for $k < r < n+k$ are mapped to zero by π. When $k < n-1$ the individual terms in these relations carry no q correction. It remains only to consider the case $k = n-1$, and $\mathrm{OG} = \mathrm{OG}(1, 2n+2)$ is a quadric. The relation in degree $2n$ is treated in [BKT1, Thms. 3.5, A.2], and the other relations are handled similarly, with the expression in degree $2r$ ($n < r < 2n-1$) yielding a coefficient of $qc_{2(r-n)+1}$ of $1 - 2 + 2 - \cdots \pm 2 \mp 1 = 0$.

Let λ be any k-strict partition contained in an $(n+1-k)\times(n+k)$ rectangle. It follows from Proposition 4.4 that

$$\widehat{\tau}_\lambda = \sum_{\lambda_1\leq p<2n+2k,\ \mu\subset\lambda^*} a_{p,\mu}\, c_p\, \widehat{\tau}_\mu \tag{30}$$

holds in $\mathbb{H}(\mathrm{OG}_k(\mathrm{even}))_{\mathbb{Q}}$. If λ has positive type, then Proposition 4.4 applied to $\mathrm{OG}(n-k,2n+1)$ also gives

$$\sigma_{\lambda-k} = \sum_{p,\mu} a'_{p,\mu}\, c_p\, \sigma_\mu \tag{31}$$

in $\mathbb{H}(\mathrm{OG}_k(\mathrm{odd}))_{\mathbb{Q}}$. Using Proposition 2, equation (31) implies that

$$\widetilde{\tau}_\lambda = \sum_{p,\mu} a'_{p,\mu}\, c_p\, \widetilde{\tau}_{\mu+k} \tag{32}$$

holds in $\mathbb{H}(\mathrm{OG}_k(\mathrm{even}))_{\mathbb{Q}}$. Observe that we have $\mu\subset(\lambda-k)^*$ for each partition μ appearing in the sum (32), and hence also $\mu+k\subset\lambda^*$.

To prove that $\pi(\tau_\lambda)=\tau_\lambda$, it is enough to show that $\pi(\widehat{\tau}_\lambda)=\widehat{\tau}_\lambda$ and $\pi(\widetilde{\tau}_\lambda)=\widetilde{\tau}_\lambda$. We argue by induction on $\ell(\lambda)$, the case $\ell(\lambda)=1$ being clear. When λ has more than one part, we apply the ring homomorphism π to both sides of (30) and (32) and use the inductive hypothesis to show that

$$\pi(\widehat{\tau}_\lambda) = \sum_{p,\mu} a_{p,\mu}\, c_p\, \widehat{\tau}_\mu \quad \text{and} \quad \pi(\widetilde{\tau}_\lambda) = \sum_{p,\mu} a'_{p,\mu}\, c_p\, \widetilde{\tau}_{\mu+k}$$

hold in $\mathrm{QH}(\mathrm{OG}(n+1-k,2n+2),\mathbb{Q})$. The quantum Pieri rules of §4.1 imply that none of the products appearing in these sums involve q correction terms. Specifically, when $K\geq 3$ this follows from (23) and Proposition 4.1, while the case $K=2$ uses (24), (25), (26), and (27). This completes the proof. $\square$

Theorem 3 follows immediately from Proposition 4.5 and Lemma 4.3.

5. Eta Polynomials

5.1. Given any power series $\sum_{i\geq 0} c_i t^i$ in the variable t and an integer sequence $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_\ell)$, we write $c_\alpha = c_{\alpha_1}c_{\alpha_2}\cdots c_{\alpha_\ell}$ and set $R\,c_\alpha = c_{R\alpha}$ for any raising operator R. We will always work with power series with constant term 1, so that $c_0=1$ and $c_i=0$ for $i<0$.

Let $x=(x_1,x_2,\ldots)$ be a list of commuting independent variables and let $\Lambda=\Lambda(x)$ be the ring of symmetric functions in x. Consider the generating series

$$\prod_{i=1}^{\infty}(1+x_it) = \sum_{r=0}^{\infty} e_r(x)t^r$$

for the elementary symmetric functions e_r. If λ is any partition, let λ' denote the partition conjugate to λ, and define the Schur S-function $s_{\lambda'}(x)$ by the equation

$$s_{\lambda'} = \prod_{i<j}(1-R_{ij})\, e_\lambda.$$

Moreover, define the functions $q_r(x)$ by the generating series

$$\prod_{i=1}^{\infty}\frac{1+x_it}{1-x_it} = \sum_{r=0}^{\infty} q_r(x)t^r$$

and let $\Gamma = \mathbb{Z}[q_1, q_2, \ldots]$. Given any strict partition λ of length $\ell(\lambda)$, the Schur Q-function $Q_\lambda(x)$ is defined by the equation

$$Q_\lambda = \prod_{i<j} \frac{1 - R_{ij}}{1 + R_{ij}}\, q_\lambda$$

and the P-function is given by

$$P_\lambda(x) = 2^{-\ell(\lambda)}\, Q_\lambda(x).$$

Fix an integer $k \geq 1$, let $y = (y_1, \ldots, y_k)$ and $\Lambda_y = \mathbb{Z}[y_1, \ldots, y_k]^{S_k}$. For each integer r, define $\vartheta_r = \vartheta_r(x\,;y)$ by

$$\vartheta_r = \sum_{i\geq 0} q_{r-i}(x)e_i(y).$$

We let $\Gamma_y = \Gamma \otimes_{\mathbb{Z}} \Lambda_y$ and $\Gamma^{(k)}$ be the subring of Γ_y generated by the ϑ_r:

$$\Gamma^{(k)} = \mathbb{Z}[\vartheta_1, \vartheta_2, \vartheta_3, \ldots].$$

According to [BKT2, (19)], we have

$$\vartheta_r^2 + 2\sum_{i=1}^{r}(-1)^i \vartheta_{r+i}\vartheta_{r-i} = e_r(y^2) \tag{33}$$

for any integer r, where y^2 denotes $(y_1^2, \ldots, y_k^2)$.

Proposition 5.1. *The ϑ_λ for λ a strict partition form a free Λ_y-basis of Γ_y.*

Proof. It is known e.g. from [M, III.(8.6)] that the $q_\lambda(x)$ for λ strict form a $\mathbb{Z}$-basis of Γ, and therefore also a Λ_y-basis of Γ_y. Since

$$\vartheta_\lambda(x\,;y) = q_\lambda(x) + \sum_{\alpha\neq 0} q_{\lambda-\alpha}(x)e_\alpha(y)$$

with the sum over nonzero compositions α, we deduce that the ϑ_λ for λ strict also form a Λ_y-basis of Γ_y. □

Following [BKT2, Prop. 5.2], the ϑ_λ for λ k-strict form a $\mathbb{Z}$-basis of $\Gamma^{(k)}$. For any k-strict partition λ, the theta polynomial $\Theta_\lambda(x\,;y)$ is defined by $\Theta_\lambda = \widetilde{R}^\lambda \vartheta_\lambda$, where, as before, $\widetilde{R}^\lambda$ denotes the operator in formula (4) with $K = 2k+1$. The polynomials Θ_λ for all k-strict partitions λ form another $\mathbb{Z}$-basis of $\Gamma^{(k)}$ (Theorem 2 of op. cit.).

5.2. Recall that $P_0 = 1$ and for each integer $r \geq 1$, we have $P_r = q_r/2$. Set

$$\eta_r(x\,;y) = \begin{cases} e_r(y) + 2\sum_{i=0}^{r-1} P_{r-i}(x)e_i(y) & \text{if } r < k, \\ \sum_{i=0}^{r} P_{r-i}(x)e_i(y) & \text{if } r \geq k \end{cases}$$

and

$$\eta'_k(x\,;y) = \sum_{i=0}^{k-1} P_{k-i}(x)e_i(y).$$

Observe that we have, for any $r \geq 0$,

$$\vartheta_r = \begin{cases} \eta_r & \text{if } r < k, \\ \eta_k + \eta'_k & \text{if } r = k, \\ 2\eta_r & \text{if } r > k, \end{cases} \tag{34}$$

while $\eta_k - \eta'_k = e_k(y)$. Define the ring of eta polynomials

$$B^{(k)} = \mathbb{Z}[\eta_1, \ldots, \eta_{k-1}, \eta_k, \eta'_k, \eta_{k+1} \ldots].$$

Proposition 5.2. *The $\mathbb{Q}$-algebra $B^{(k)}_{\mathbb{Q}}$ is a free $\Gamma^{(k)}_{\mathbb{Q}}$-module with basis 1, $e_k(y)$.*

Proof. The definition implies that $B^{(k)}_{\mathbb{Q}} = \Gamma^{(k)}_{\mathbb{Q}}[e_k(y)]$. We claim that

$$\Gamma^{(k)}_{\mathbb{Q}} \cap e_k(y)\Gamma^{(k)}_{\mathbb{Q}} = 0.$$

Indeed, we know that the ϑ_λ for k-strict partitions λ form a $\mathbb{Q}$-basis of $\Gamma^{(k)}_{\mathbb{Q}}$. We deduce from (33) and Proposition 5.1 that for each k-strict partition λ, there is a unique expression

$$\vartheta_\lambda(x\,;y) = \sum_{\mu,\nu} a_{\mu\nu}\, \vartheta_\mu(x\,;y)e_\nu(y^2) \tag{35}$$

where the $a_{\mu\nu}$ are integers and the sum is over strict partitions μ and partitions ν with $\nu_1 \leq k$ such that $|\mu|+2|\nu| = |\lambda|$. The claim follows from this and the fact that the $\mathbb{Z}$-linear submodules of Λ_y spanned by the $e_\nu(y^2)$ and $e_k(y)e_\nu(y^2)$, respectively, have trivial intersection. We deduce that $B^{(k)}_{\mathbb{Q}}$ is the $\Gamma^{(k)}_{\mathbb{Q}}$-algebra generated by $e_k(y)$ modulo the quadratic relation (33). The proposition now follows by elementary algebra. □

Let λ be a typed k-strict partition and let R be any finite monomial in the operators R_{ij} which appears in the expansion of the power series R^λ in (4). If $\mathrm{type}(\lambda) = 0$, then set $R \star \vartheta_\lambda = \vartheta_{R\,\lambda}$. Suppose that $\mathrm{type}(\lambda) > 0$, let $d = \ell_k(\lambda) + 1$ be the index such that $\lambda_d = k < \lambda_{d-1}$, and set $\widehat{\alpha} = (\alpha_1, \ldots, \alpha_{d-1}, \alpha_{d+1}, \ldots, \alpha_\ell)$ for any integer sequence α of length ℓ. If R involves any factors R_{ij} with $i = d$ or $j = d$, then let $R \star \vartheta_\lambda = \frac{1}{2}\,\vartheta_{R\,\lambda}$. If R has no such factors, then let

$$R \star \vartheta_\lambda = \begin{cases} \eta_k\, \vartheta_{\widehat{R\,\lambda}} & \text{if } \mathrm{type}(\lambda) = 1, \\ \eta'_k\, \vartheta_{\widehat{R\,\lambda}} & \text{if } \mathrm{type}(\lambda) = 2. \end{cases}$$

Definition 5.3. For any typed k-strict partition λ, the *eta polynomial* $H_\lambda = H_\lambda(x\,;y)$ is the element of $B^{(k)}$ defined by the raising operator formula

$$H_\lambda = 2^{-\ell_k(\lambda)} R^\lambda \star \vartheta_\lambda.$$

The *type* of the polynomial H_λ is the same as the type of λ.

If λ is a k-strict partition, we define the polynomials H_λ, H'_λ, $\widehat{H}_\lambda$, and $\widetilde{H}_\lambda$ using the same conventions as in §3.2 in the case of Schubert classes. Note that

$$\widehat{H}_\lambda = 2^{-\ell_k(\lambda)} R^\lambda\, \vartheta_\lambda = \begin{cases} H_\lambda + H'_\lambda & \text{if } \lambda \text{ has positive type,} \\ \quad H_\lambda & \text{otherwise,} \end{cases}$$

while if λ has positive type then

$$\widetilde{H}_\lambda = H_\lambda - H'_\lambda = 2^{-\ell_k(\lambda)} e_k(y)\, \widetilde{R}^{\lambda - k}\, \vartheta_{\lambda-k} = 2^{-\ell_k(\lambda)} e_k(y)\, \Theta_{\lambda-k}\,.$$

The raising operator expression for $\widehat{H}_\lambda$ implies that $\Gamma^{(k)}_{\mathbb{Q}} = \bigoplus_\lambda \mathbb{Q}\, \widehat{H}_\lambda$, summed over all k-strict partitions λ. It follows from Proposition 5.2 that there is a direct sum decomposition

$$B^{(k)}_{\mathbb{Q}} = \Gamma^{(k)}_{\mathbb{Q}} \oplus e_k(y)\, \Gamma^{(k)}_{\mathbb{Q}} = \bigoplus_{\lambda\ k\text{-strict}} \mathbb{Q}\, \widehat{H}_\lambda \oplus \bigoplus_{\substack{\lambda\ k\text{-strict} \\ \text{of positive type}}} \mathbb{Q}\, \widetilde{H}_\lambda.$$

For any typed k-strict partition λ, we have

$$H_\lambda = \begin{cases} \widehat{H}_\lambda & \text{if } \mathrm{type}(\lambda) = 0, \\ \frac{1}{2}(\widehat{H}_\lambda + \widetilde{H}_\lambda) & \text{if } \mathrm{type}(\lambda) = 1, \\ \frac{1}{2}(\widehat{H}_\lambda - \widetilde{H}_\lambda) & \text{if } \mathrm{type}(\lambda) = 2 \end{cases} \tag{36}$$

in analogy with (22). We deduce that $B^{(k)}$ is isomorphic to the stable cohomology ring $\mathbb{H}(\mathrm{OG}_k(\text{even}))$ via the map which sends H_λ to τ_λ, and that the polynomials H_λ indexed by typed k-strict partitions λ form a $\mathbb{Z}$-basis of $B^{(k)}$. Indeed, the relations (28) and (29) are satisfied by the η_r, $r \geq 1$ and η'_k, as follows immediately from the equations (33) for the values $r \geq k$. This completes the proof of Theorem 4.

5.3. Define the elements $S_\lambda(x\,;y)$ and $Q_\lambda(x\,;y)$ of $\Gamma^{(k)}$ by the equations

$$S_\lambda(x\,;y) = \prod_{i<j}(1 - R_{ij})\,\vartheta_\lambda = \det(\vartheta_{\lambda_i+j-i})_{i,j}$$

and

$$Q_\lambda(x\,;y) = \prod_{i<j}\frac{1 - R_{ij}}{1 + R_{ij}}\,\vartheta_\lambda.$$

These polynomials were studied in [BKT2, Thm. 3 and Prop. 5.9]. Using equation (36), we easily obtain the following result.

Proposition 5.4. *Let λ be a typed k-strict partition of length ℓ.*

(a) *If $\lambda_i + \lambda_j < 2k + j - i$ for all $i < j$, then we have*

$$H_\lambda(x\,;y) = \begin{cases} S_\lambda(x\,;y) & \text{if } \lambda_1 < k, \\ \frac{1}{2}\,S_\lambda(x\,;y) + \frac{1}{2}\,e_k(y)\,S_{\lambda-k}(x\,;y) & \text{if } \mathrm{type}(\lambda) = 1, \\ \frac{1}{2}\,S_\lambda(x\,;y) - \frac{1}{2}\,e_k(y)\,S_{\lambda-k}(x\,;y) & \text{if } \mathrm{type}(\lambda) = 2, \\ \frac{1}{2}\,S_\lambda(x\,;y) & \text{if } \lambda_1 > k. \end{cases}$$

(b) *If $\lambda_i + \lambda_j \geq 2k + j - i$ for all $i < j \leq \ell$, then we have*

$$H_\lambda(x\,;y) = \begin{cases} 2^{-\ell}\,Q_\lambda(x\,;y) & \text{if } \lambda_\ell > k, \\ 2^{-\ell}\,Q_\lambda(x\,;y) + 2^{-\ell}\,e_k(y)\,Q_{\lambda-k}(x\,;y) & \text{if } \mathrm{type}(\lambda) = 1, \\ 2^{-\ell}\,Q_\lambda(x\,;y) - 2^{-\ell}\,e_k(y)\,Q_{\lambda-k}(x\,;y) & \text{if } \mathrm{type}(\lambda) = 2, \\ 2^{1-\ell}\,Q_\lambda(x\,;y) & \text{if } \lambda_\ell < k. \end{cases}$$

6. Schubert polynomials for even orthogonal Grassmannians

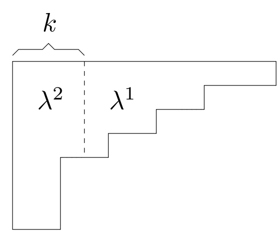
6.1. In this section, we prove that the polynomials $H_\lambda(x\,;y)$ are special cases of Billey-Haiman type D Schubert polynomials $\mathfrak{D}_w(x,z)$. Let $\widetilde{W}_{n+1}$ be the Weyl group for the root system of type D_{n+1}. The elements of $\widetilde{W}_{n+1}$ may be represented as signed permutations of the set $\{1,\ldots,n+1\}$; we will denote a sign change by a bar over the corresponding entry. The group $\widetilde{W}_{n+1}$ is generated by the simple transpositions $s_i = (i,i+1)$ for $1 \leq i \leq n$, and an element s_0 which acts on the right by

$$(u_1,u_2,\ldots,u_{n+1})s_0 = (\overline{u}_2,\overline{u}_1,u_3,\ldots,u_{n+1}).$$

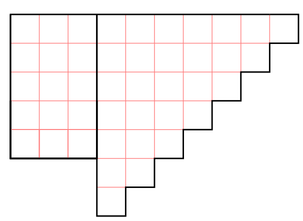
Set $\widetilde{W}_\infty = \bigcup_n \widetilde{W}_n$ and let $w \in \widetilde{W}_\infty$. A *reduced factorization* of w is a product $w = uv$ in $\widetilde{W}_\infty$ such that $\ell(w) = \ell(u) + \ell(v)$. A *reduced word* of $w \in \widetilde{W}_\infty$ is a sequence $a_1 \cdots a_\ell$ of elements in $\mathbb{N}$ such that $w = s_{a_1} \cdots s_{a_\ell}$ and $\ell = \ell(w)$. If we convert all the 0's which appear in the reduced word $a_1 \ldots a_r$ to 1's, we obtain a *flattened word* of w. For example, 23012 is a reduced word of $\overline{1}4\overline{3}2$, and 23112 is the corresponding flattened word (note that the flattened word need not be reduced). We say that w has a *descent* at position $r \geq 0$ if $\ell(ws_r) < \ell(w)$, where s_r is the simple reflection indexed by r.

For $k \neq 1$, an element $w \in \widetilde{W}_\infty$ is k-Grassmannian if $\ell(ws_i) = \ell(w) + 1$ for all $i \neq k$. We say that w is 1-Grassmannian if $\ell(ws_i) = \ell(w) + 1$ for all $i \geq 2$. The elements of $\widetilde{W}_{n+1}$ index the Schubert classes in the cohomology ring of the flag variety SO_{2n+2}/B, which contains $\mathrm{H}^*(\mathrm{OG}(n+1-k, 2n+2), \mathbb{Z})$ as the subring spanned by Schubert classes given by k-Grassmannian elements. In particular, each typed k-strict partition λ in $\widetilde{\mathcal{P}}(k,n)$ corresponds to a k-Grassmannian element $w_\lambda \in \widetilde{W}_{n+1}$ which we proceed to describe; more details and relations to other indexing conventions can be found in [T1, §6].

Given any typed k-strict partition λ, we let λ^1 be the strict partition obtained by removing the first k columns of λ, and let λ^2 be the partition of boxes contained in the first k columns of λ.



A typed k-strict partition λ belongs to $\widetilde{\mathcal{P}}(k,n)$ if and only if its Young diagram fits inside the shape Π obtained by attaching an $(n+1-k) \times k$ rectangle to the left side of a staircase partition with n rows. When $n = 7$ and $k = 3$, this shape looks as follows.

$\Pi =$ 

The boxes of the staircase partition that are outside λ form south-west to north-east diagonals. Such a diagonal is called *related* if it is K-related to one of the bottom boxes in the first k columns of λ, or to any box $[0, i]$ for which $\lambda_1 < i \leq k$; the remaining diagonals are *non-related*. Let $r_1 < r_2 < \cdots < r_k$ denote the lengths of the related diagonals, let $u_1 < u_2 < \cdots < u_t$ be the lengths of the non-related diagonals, and set $p = \ell(\lambda^1) = \ell_k(\lambda)$. If $\mathrm{type}(\lambda)$ is non-zero, then $t = n - k - p$. If $\mathrm{type}(\lambda) = 1$, then the k-Grassmannian element corresponding to λ is given by

$$w_\lambda = (r_1 + 1, \ldots, r_k + 1, \overline{(\lambda^1)_1 + 1}, \ldots, \overline{(\lambda^1)_p + 1}, \widehat{1}, u_1 + 1, \ldots, u_{n-k-p} + 1)\,,$$

while if $\mathrm{type}(\lambda) = 2$, then

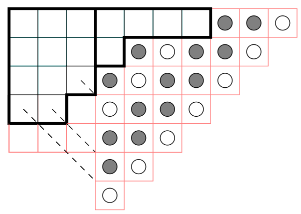
$$w_\lambda = (\overline{r_1 + 1}, \ldots, r_k + 1, \overline{(\lambda^1)_1 + 1}, \ldots, \overline{(\lambda^1)_p + 1}, \widehat{1}, u_1 + 1, \ldots, u_{n-k-p} + 1)\,.$$

Here we use the convention that $\widehat{1}$ is equal to either 1 or $\overline{1}$, determined so that w_λ contains an even number of barred integers. Finally, if $\mathrm{type}(\lambda) = 0$, then $r_1 = 0$, i.e., one of the related diagonals has length zero. In this case we have $t = n-k-p+1$ and

$$w_\lambda = (\widehat{1}, r_2+1, \ldots, r_k+1, \overline{(\lambda^1)_1+1}, \ldots, \overline{(\lambda^1)_p+1}, u_1+1, \ldots, u_{n+1-k-p}+1)\,.$$

The element $w_\lambda \in \widetilde{W}_\infty$ depends on λ and k, but is independent of n.

Example 6.1. The element $\lambda = (7,4,3,2) \in \widetilde{\mathcal{P}}(3,7)$ of type 2 corresponds to $w_\lambda = (\overline{3}, 6, 7, \overline{5}, \overline{2}, \overline{1}, 4, 8)$.

$\lambda \ =$ 

6.2. We say that a sequence $a = (a_1, \ldots, a_m)$ is *unimodal* if for some $r \leq m$, we have

$$a_1 > a_2 > \cdots > a_{r-1} \geq a_r < a_{r+1} < \cdots < a_m,$$

and if $a_{r-1} = a_r$ then $a_r = 1$.

Let $w \in \widetilde{W}_n$ and λ be a Young diagram with r rows and $\ell(w)$ boxes. Following [Kr, La], a *Kraśkiewicz-Lam tableau* for w of shape λ is a filling T of the boxes of λ with positive integers in such a way that

a) If t_i is the sequence of entries in the i-th row of T, reading from left to right, then the row word $t_r \ldots t_1$ is a flattened word for w.

b) For each i, t_i is a unimodal subsequence of maximum length in $t_r \ldots t_{i+1} t_i$.

If T is a Kraśkiewicz-Lam tableau of shape λ with row word $a_1 \ldots a_\ell$, set

$$m(T) = \ell(\lambda) + 1 - p,$$

where p denotes the number of distinct values of $s_{a_1} \cdots s_{a_j}(1)$ for $0 \leq j \leq \ell$. It follows from [La, Thm. 4.35] that $m(T) \geq 0$.

For each $w \in \widetilde{W}_\infty$ one has a *type D Stanley symmetric function* $E_w(x)$, which is a positive linear combination of Schur P-functions [BH, FK, La]. In particular, Lam [La] has shown that

$$E_w(x) = \sum_\lambda d_w^\lambda \, P_\lambda(x) \tag{37}$$

where $d_w^\lambda = \sum_T 2^{m(T)}$, summed over all Kraśkiewicz-Lam tableaux T for w of shape λ.

Example 6.2. Let λ be a strict partition of length ℓ with $\lambda_1 \leq n$, and let μ be the strict partition whose parts are the numbers from 1 to $n+1$ which do not belong to the set $\{1, \lambda_\ell+1, \ldots, \lambda_1+1\}$. Then the signed permutation

$$w_\lambda = (\overline{\lambda_1+1}, \ldots, \overline{\lambda_\ell+1}, \widehat{1}, \mu_{n-\ell}, \ldots, \mu_1)$$

is the 0-Grassmannian element of $\widetilde{W}_{n+1}$ corresponding to the partition λ. There exists a unique Kraśkiewicz-Lam tableau T_λ for w_λ. This tableau has shape λ and its ith row contains the integers between 1 and λ_i in decreasing order; moreover, we have $m(T_\lambda) = 0$. For example,

$$T_{(6,5,2)} = \begin{array}{|c|c|c|c|c|c|}\hline 6 & 5 & 4 & 3 & 2 & 1 \\ \hline 5 & 4 & 3 & 2 & 1 & \multicolumn{1}{c}{} \\ \cline{1-5} 2 & 1 & \multicolumn{4}{c}{} \\ \cline{1-2}\end{array} .$$

We deduce that $E_{w_\lambda}(x) = P_\lambda(x)$.

6.3. Following Billey and Haiman [BH], each $w \in \widetilde{W}_\infty$ defines a type D Schubert polynomial $\mathfrak{D}_w(x,z)$. Here $z = (z_1, z_2, \ldots)$ is another infinite set of variables and each $\mathfrak{D}_w$ is a polynomial in the ring $A = \mathbb{Z}[P_1(x), P_2(x), \ldots ; z_1, z_2, \ldots]$. The polynomials $\mathfrak{D}_w$ for $w \in \widetilde{W}_\infty$ form a $\mathbb{Z}$-basis of A, and their algebra agrees with the Schubert calculus on orthogonal flag varieties SO_{2n}/B, when n is sufficiently large. According to [BH, Thm. 4], for any $w \in \widetilde{W}_n$ we have

$$\mathfrak{D}_w(x,z) = \sum_{uv=w} E_u(x)\mathfrak{S}_v(z)\,, \tag{38}$$

summed over all reduced factorizations $w = uv$ in $\widetilde{W}_n$ for which $v \in S_n$. Here $\mathfrak{S}_v(z)$ denotes the type A Schubert polynomial of Lascoux and Schützenberger [LS].

Proposition 6.3. *The ring $B^{(k)}$ of eta polynomials is, by the identification of y_i with z_i for $i = 1, \ldots, k$, a subring of the ring of Billey-Haiman Schubert polynomials of type D. For every typed k-strict partition λ we have $H_\lambda(x\,; z_1, \ldots, z_k) = \mathfrak{D}_{w_\lambda}(x,z)$.*

Proof. We first show that the eta polynomial η_r, $r \geq 1$ (respectively η'_k) agrees with the Billey-Haiman Schubert polynomial indexed by the k-Grassmannian element $w_{(r)} \in \widetilde{W}_\infty$ corresponding to $\lambda = (r)$ (respectively, by $w'_{(k)}$ corresponding to $\lambda = (k)$ with $\mathrm{type}(\lambda) = 2$). One sees that $w_{(r)}$ has a reduced word given by $(k - r+1, k-r+2, \ldots, k)$ when $1 \leq r \leq k$, by $(r-k, r-k-1, \ldots, 1, 0, 2, 3, \ldots, k)$ when $r > k$, and that $w'_{(k)}$ has the reduced word $(0, 2, \ldots, k)$. It follows that if $w_{(r)} = uv$ is any reduced factorization of $w_{(r)}$ with $v \in S_\infty$, then $v = w_{(i)}$ for some integer i with $0 \leq i \leq k$. The type A Schubert polynomial for $w_{(i)}$ is given by $\mathfrak{S}_{w_{(i)}}(z) = e_i(z_1, \ldots, z_k)$, and (37) implies that the type D Stanley symmetric function for $u = w_{(r)}w_{(i)}^{-1}$ is

$$E_u(x) = \begin{cases} 2P_{r-i}(x) & \text{if } r < k, \\ P_{r-i}(x) & \text{if } r \geq k \end{cases} .$$

We conclude from (38) that

$$\mathfrak{D}_{w_{(r)}}(x,z) = \eta_r(x\,; z) \quad \text{and} \quad \mathfrak{D}_{w'_{(k)}}(x,z) = \eta'_k(x\,; z),$$

as required. Since the Schubert polynomials $\mathfrak{D}_w$ multiply like the Schubert classes on even orthogonal flag varieties, the proposition now follows from Theorem 2. $\square$

Remark 6.4. It is important to note that the equality in Proposition 6.3 is taking place in the ring A, where there are relations among the P_r. These relations are used crucially in its proof, which requires Theorem 2.

Theorem 6. *For any typed k-strict partition λ, the polynomial H_λ is a linear combination of products of Schur P-functions and S-polynomials:*

$$H_\lambda(x\,;y) = \sum_{\mu,\nu} d_{\mu\nu}^\lambda P_\mu(x) s_{\nu'}(y) \tag{39}$$

where the sum is over partitions μ and ν such that μ is strict and $\nu \subset \lambda^2$. Moreover, the coefficients $d_{\mu\nu}^\lambda$ are nonnegative integers, equal to the number of Kraśkiewicz-Lam tableaux for $w_\lambda w_\nu^{-1}$ of shape μ.

Proof. Proposition 6.3 and (38) imply that for every typed k-strict partition λ we have

$$H_\lambda(x\,;y) = \sum_{uv=w_\lambda} E_u(x)\mathfrak{S}_v(y)\,, \tag{40}$$

where the sum is over all reduced factorizations $w_\lambda = uv$ in W_∞ with $v \in S_\infty$. The right factor v in any such factorization must be the identity or a Grassmannian permutation with descent at position k. In fact, it is not hard to check that the right reduced factors of w_λ that belong to S_∞ are permutations w_ν given by partitions $\nu \subset \lambda^2$ (and are exactly these partitions whenever $\mathrm{type}(\lambda) \neq 2$). We now use (37) and the fact that the Schubert polynomial $\mathfrak{S}_{w_\nu}(y)$ is equal to the Schur polynomial $s_{\nu'}(y)$. ☐

If λ is a k-strict partition of type 2, let

$$\lambda^3 = \lambda^1 + r_1 = ((\lambda^1)_1, (\lambda^1)_2, \ldots, r_1, r_1 - 1, \ldots, 2, 1)$$

be λ^1 with a part r_1 added, and let λ^4 be λ^2 with r_1 boxes subtracted from the k-th column, so that

$$(\lambda^4)' = ((\lambda^2)'_1, \ldots, (\lambda^2)'_{k-1}, (\lambda^2)'_k - r_1).$$

Corollary 6.5. *Let λ be a typed k-strict partition.*

(a) *The homogeneous summand of $H_\lambda(x\,;y)$ of highest x-degree is the type D Stanley symmetric function $E_{w_\lambda}(x)$, and satisfies $E_{w_\lambda}(x) = 2^{-\ell_k(\lambda)}\, R^\lambda\, q_\lambda(x)$.*

(b) *The homogeneous summand of $H_\lambda(x\,;y)$ of lowest x-degree is $P_{\lambda^1}(x)\, s_{(\lambda^2)'}(y)$, if $\mathrm{type}(\lambda) \neq 2$, and $P_{\lambda^3}(x)\, s_{(\lambda^4)'}(y)$, if $\mathrm{type}(\lambda) = 2$.*

Proof. Part (a) follows by setting $y = 0$ in (40) and also in the raising operator expression $H_\lambda(x\,;y) = 2^{-\ell_k(\lambda)}\, R^\lambda \star \vartheta_\lambda(x\,;y)$. Part (b) is deduced from (39), Example 6.2, and the observation that $w_\lambda w_{\lambda^2}^{-1}$ (respectively, $w_\lambda w_{\lambda^4}^{-1}$) is the 0-Grassmannian Weyl group element corresponding to the strict partition λ^1, if $\mathrm{type}(\lambda) \neq 2$ (respectively, to the strict partition λ^3, if $\mathrm{type}(\lambda) = 2$). ☐

Appendix A. Schubert varieties in orthogonal Grassmannians

Our goal in this section is to give a geometric description of the Schubert varieties in the orthogonal Grassmannians $\mathrm{OG}(m, N)$, and to establish the assertions about the subsets Y_λ claimed in the introduction and Section 3.2. We also correct errors in the definition of the type D Schubert varieties and degeneracy loci which appeared in the earlier papers [KT, §3.2, p. 1713], [T1, §6.1, p. 333], and [BKT1, §3.1 and §4.3, p. 377 and 389]. In each of these references, the type D Schubert variety is claimed to be the locus of isotropic subspaces (or flags) Σ which satisfies a system of dimension inequalities, obtained by intersecting Σ with the subspaces in a fixed isotropic flag (or its alternate). The Schubert variety should be defined as the closure of the corresponding Schubert cell, which consists of the locus of Σ satisfying a system of dimension equalities. See e.g. [T3, §2.1] for a precise definition of the Schubert varieties in even orthogonal flag varieties in these terms, and the discussion in [Fu] and [FPr, §6.1] for more on this phenomenon.

We thank Vijay Ravikumar for pointing out that the description of the type D Schubert varieties and their Bruhat order in [BKT1, §4.3] is wrong, which led us to the above errors. Fortunately, the mistaken description of the Schubert varieties in [KT, T1, BKT1] does not affect the correctness of the results of op. cit. outside of [BKT1, Prop. 4.5], as this description was never used. We fix the errors in [BKT1, §4] below.

Let $V \cong \mathbb{C}^N$ be a complex vector space equipped with a non-degenerate symmetric bilinear form $(-,-)$. Fix $m < N/2$ and let $\mathrm{OG} = \mathrm{OG}(m,N)$ be the orthogonal Grassmannian of m-dimensional isotropic subspaces of V. This variety has a transitive action of the group $\mathrm{SO}(V)$ of linear automorphisms that preserve the form on V. For any subset $A \subset V$ we let $\langle A \rangle \subset V$ denote the $\mathbb{C}$-linear span of A. Fix an isotropic flag $F_\bullet$ in V and let $B \subset \mathrm{SO}(V)$ be the Borel subgroup stabilizing $F_\bullet$. The *Schubert varieties* in OG are the orbit closures of the action of B on OG. We also fix a basis $e_1, \ldots, e_N$ of V such that $(e_i, e_j) = \delta_{i+j,N+1}$ and $F_p = \langle e_1, \ldots, e_p \rangle$ for each p.

We call a subset $P \subset [1,N]$ of cardinality m an *index set* if for all $i, j \in P$ we have $i + j \neq N + 1$. Equivalently, the subspace $\langle e_p : p \in P \rangle \subset V$ is a point in OG. Let

$$X_P^\circ = B.\langle e_p : p \in P \rangle \subset \mathrm{OG}$$

be the orbit of this point, and let $X_P = \overline{X_P^\circ}$ be the corresponding Schubert variety. Any point $\Sigma \in \mathrm{OG}$ defines an index set

$$P(\Sigma) = \{p \in [1,N] \mid \Sigma \cap F_p \supsetneq \Sigma \cap F_{p-1}\}\,,$$

since no vector in $F_p \smallsetminus F_{p-1}$ is orthogonal to a vector in $F_{N+1-p} \smallsetminus F_{N-p}$. It follows from [BKT1, Lemma 4.1] that we have $X_P^\circ = \{\Sigma \in \mathrm{OG} \mid P(\Sigma) = P\}$. In particular, the Schubert varieties in OG are in one-to-one correspondence with the index sets. Given any two index sets $P = \{p_1 < \cdots < p_m\}$ and $Q = \{q_1 < \cdots < q_m\}$, we write $Q \leq P$ if $q_j \leq p_j$ for each j.

Recall that the integer n is defined so that $N = 2n+1$ if N is odd, $N = 2n+2$ if N is even, and k satisfies $n + k = N - m - 1$. Following [BKT1, §4] and the introduction, equation (1) establishes a bijection between the index sets Q for which $n+1 \notin Q$ and the set of k-strict partitions λ whose Young diagram is contained in an $m \times (n+k)$ rectangle. Notice that the condition on Q is always true if N is odd. If $P = \{p_j\}$ is an arbitrary index set we let $\overline{P} = \{\overline{p}_j\}$ denote the index set with $\overline{p}_j = n+2$, if $p_j = n+1$, and $\overline{p}_j = p_j$, otherwise. Define a Zariski closed subset $Y_P \subset \mathrm{OG}$ by

$$Y_P = \{\Sigma \in \mathrm{OG} \mid \dim(\Sigma \cap F_{\overline{p}_j}) \geq j \text{ for } 1 \leq j \leq m\}\,.$$

If λ is the k-strict partition corresponding to $\overline{P}$, then Y_P agrees with the set Y_λ defined in the introduction. Observe that $Y_P = \bigcup_{Q \leq \overline{P}} X_Q^\circ$. In particular, $X_P \subset Y_P$ for any index set P.

Type B. We assume in this section that $N = 2n+1$ is odd.

Proposition A.1. *For any index set* $P = \{p_1 < \cdots < p_m\} \subset [1, 2n+1]$*, we have*

$$X_P = Y_P = \{\Sigma \in \mathrm{OG} \mid \dim(\Sigma \cap F_{p_j}) \geq j, \quad \forall\, 1 \leq j \leq m\}\,.$$

For any two index sets P *and* Q*, we have* $X_Q \subset X_P$ *if and only if* $Q \leq P$*.*

Proof. It suffices to prove that $Q \leq P$ implies $X_Q \subset X_P$. Assuming $Q < P$, it is enough to construct an index set P' such that $Q \leq P' < P$ and $X_{P'} \subset X_P$.

Choose j minimal such that $q_j < p_j$, and notice that $[q_j, p_j - 1] \cap P = \emptyset$. If some integer $x \in [q_j, p_j - 1]$ satisfies that $x \neq n+1$ and $N + 1 - x \notin P$, then set

$$P' = \{p_1, \ldots, p_{j-1}, x, p_{j+1}, \ldots, p_m\},$$

and observe that $Q \leq P' < P$. Define a morphism of varieties $\mathbb{P}^1 \to \mathrm{OG}$ by

$$[s:t] \mapsto \Sigma_{[s:t]} = \langle e_{p_1}, \ldots, e_{p_{j-1}}, s\, e_x + t\, e_{p_j}, e_{p_{j+1}}, \ldots, e_{p_m}\rangle\,. \tag{41}$$

Since $\Sigma_{[1:0]} \in X^\circ_{P'}$ and $\Sigma_{[s:t]} \in X^\circ_P$ for $t \neq 0$, it follows that $X^\circ_{P'} \subset X_P$.

Otherwise, we must have $N + 1 - x \in P$ for all $x \in [q_j, p_j - 1] \smallsetminus \{n+1\}$. We deduce that $q_j \leq n$ and $p_j \leq n+2$. If $p_j = n+2$, then we may use the set

$$P' = \{p_1, \ldots, p_{j-1}, n, p_{j+1}, \ldots, p_m\}$$

and the morphism

$$[s:t] \mapsto \Sigma_{[s:t]} = \langle e_{p_1}, \ldots, e_{p_{j-1}}, s^2\, e_n + 2st\, e_{n+1} - 2t^2\, e_{n+2}, e_{p_{j+1}}, \ldots, e_{p_m}\rangle$$

from $\mathbb{P}^1$ to OG to conclude as above that $X^\circ_{P'} \subset X_P$.

We may therefore assume that $q_j < p_j \leq n$. Set

$$P' = (P \smallsetminus \{p_j, N+1-q_j\}) \cup \{q_j, N+1-p_j\}.$$

We then use the morphism $\mathbb{P}^1 \to \mathrm{OG}$ given by

$$[s:t] \mapsto \Sigma_{[s:t]} = \langle e_p : p \in P \cap P'\rangle \oplus \langle s\, e_{q_j} + t\, e_{p_j}, s\, e_{N+1-p_j} - t\, e_{N+1-q_j}\rangle \tag{42}$$

to show that $X^\circ_{P'} \subset X_P$. We finally check that $Q \leq P' < P$. The relation $P' < P$ is true because $q_j < p_j$ and $N+1-p_j < N+1-q_j$. Set $u = p_j - q_j$ and choose $v > j$ such that $p_v = N + 1 - q_j$. Then we have $q_v < p_v$ and $p_{v-i} = p_v - i$ for $i \in [0, u-1]$. It follows that $q_{v-i} \leq q_v - i \leq p_v - i - 1 = p'_{v-i}$ for $i \in [0, u-1]$. Since $q_j = p'_j$ and $p'_r = p_r$ for $r \notin \{j\} \cup [v-u+1, v]$, this implies that $Q \leq P'$. $\square$

Type D. We assume in this section that $N = 2n+2$ is even. Following [BKT1, §4.3] we agree that every index set $P \subset [1, 2n+2]$ has a *type*, which is an integer $\mathrm{type}(P) \in \{0, 1, 2\}$. If $P \cap \{n+1, n+2\} = \emptyset$, then the type of P is zero. Otherwise, $\mathrm{type}(P)$ is equal to 1 plus the parity of the number of integers in $[1, n+1] \smallsetminus P$. As in loc. cit. and the introduction, there is a type preserving bijection between the index sets $P \subset [1, 2n+2]$ and the typed k-strict partitions λ in $\widetilde{\mathcal{P}}(k, n)$. If P corresponds to λ, then X_P is also denoted X_λ.

For any index set P, let $[P] = P \cup \{N+1-p \mid p \in P\}$. A pair (Q, P) of index sets is called *critical* if there exists an integer $c \leq n+1$ such that $[c, n+1] \subset [P] \cap [Q]$ and $\#Q \cap [1, c-1] = \#P \cap [1, c-1]$. Such an integer c is then called a *critical index.* We will write $Q \preceq P$ if (i) $Q \leq P$ and (ii) if (Q, P) is critical then $\mathrm{type}(Q) = \mathrm{type}(P)$.

Notice that if $q_j = n+1$ and $p_j = n+2$ for some j, then $n+1$ is a critical index for (Q, P) and $\mathrm{type}(Q) \neq \mathrm{type}(P)$, so $Q \not\preceq P$. Let ι be the involution on index sets that interchanges $n+1$ and $n+2$. We have $Q \preceq P$ if and only if $\iota(Q) \preceq \iota(P)$. We note that $Q \leq \overline{P}$ if and only if $Q \preceq P$ or $Q \preceq \iota(P)$.

Proposition A.2. *For any index set $P = \{p_1 < \cdots < p_m\} \subset [1, 2n+2]$, the Schubert variety X_P is equal to the set of all $\Sigma \in Y_P$ such that for all c with $[c, n+1] \subset [P]$, we have $\dim(\Sigma \cap F_{c-1}) > \#P \cap [1, c-1]$ or*

$$\dim((\Sigma + F_{c-1}) \cap F_{n+1}) \;\equiv\; c - 1 + \#P \cap [c, n+1] \pmod 2\,. \tag{43}$$

If $\operatorname{type}(P) = 0$*, then $X_P = Y_P$, while if* $\operatorname{type}(P) > 0$*, then $X_P \cup X_{\iota(P)} = Y_P$. For any two index sets P and Q, we have $X_Q \subset X_P$ if and only if $Q \preceq P$.*

Proof. Let Z_P be the subset of Y_P indicated in the proposition. Since Z_P is B-stable and contains X_P°, it suffices to show that Z_P is closed, that $X_Q^\circ \subset Z_P$ implies $Q \preceq P$, and that $Q \preceq P$ implies $X_Q \subset X_P$. The last of these assertions reduces to showing that, for $Q \prec P$, there exists an index set P' such that $Q \preceq P' < P$ and $X_{P'} \subset X_P$. Choose j minimal such that $q_j < p_j$. Then $[q_j, p_j - 1] \cap P = \emptyset$.

Assume first that we have $[q_j, p_j - 1] \not\subset [P]$, and set $x = \min([q_j, p_j - 1] \smallsetminus [P])$ and $P' = \{p_1, \dots, p_{j-1}, x, p_{j+1}, \dots, p_m\}$. Then $Q \leq P' < P$, and the morphism $\mathbb{P}^1 \to \mathrm{OG}$ defined by (41) shows that $X_{P'}^\circ \subset X_P$. If $Q \not\preceq P'$, then (Q, P') is critical and $\operatorname{type}(Q) \neq \operatorname{type}(P')$. This implies that $q_j \leq n$. Let c be a critical index for (Q, P'). Since $p_j \notin [P']$ we must have $p_j < c$ or $p_j \geq n+3$. If $p_j < c$, then c is also a critical index for (Q, P) and $\operatorname{type}(P') = \operatorname{type}(P)$, which contradicts $Q \preceq P$. On the other hand, if $p_j \geq n+3$, then $x = n+1 \in [P']$, $n \notin [P']$, and $P' \cap [1, n] \subsetneq Q \cap [1, n]$, which contradicts that (Q, P') is critical.

Otherwise we have $[q_j, p_j - 1] \subset [P] \smallsetminus P$. It follows that $q_j \leq n$ and $p_j \leq n+2$. Set $P' = (P \smallsetminus \{p_j, N+1-q_j\}) \cup \{q_j, N+1-p_j\}$. Then $P' < P$, and the morphism defined by (42) shows that $X_{P'}^\circ \subset X_P$. To see that $Q \preceq P'$, we first assume that $p_j \leq n+1$. Then the argument from the odd orthogonal case shows that $Q \leq P'$, so if $Q \not\preceq P'$, then (Q, P') is critical and $\operatorname{type}(Q) \neq \operatorname{type}(P')$. Since $Q \preceq P$ and $\operatorname{type}(P') = \operatorname{type}(P)$, we deduce that (Q, P) is not critical. Let c be a critical index for (Q, P'). Since c is not a critical index for (Q, P) we must have $c \leq p_j$. But then $[q_j, n+1] \subset [P]$, and since $Q \leq P$ we also have $[q_j, n+1] \subset [Q]$, so q_j is a critical index for (Q, P), a contradiction. Finally, if $p_j = n+2$, then the above argument applied to $\iota(Q)$ and $\iota(P)$ shows that $\iota(Q) \preceq \iota(P')$, and therefore that $Q \preceq P'$.

We next show that Z_P is closed. For each integer c with $[c, n+1] \subset [P]$, we set $j_c = \#P \cap [1, c-1]$ and $U_c = \{\Sigma \in Y_P \mid \dim(\Sigma \cap F_{c-1}) = j_c\}$, and we let $Z_c \subset U_c$ be the subset of points satisfying (43). Since all points $\Sigma \in Y_P$ satisfy $\dim(\Sigma \cap F_{c-1}) \geq j_c$, it follows that U_c is a Zariski open subset of Y_P. We claim that Z_c is closed in U_c. Let $\Sigma \in U_c$ be any point. The condition $[c, n+1] \subset [P]$ implies that

$$\#P \cap [1, N+1-c] = j_c + n + 2 - c,$$

so $\dim(\Sigma \cap F_{c-1}^\perp) \geq j_c + n + 2 - c$. It follows that $\dim((\Sigma \cap F_{c-1}^\perp) + F_{c-1}) \geq n+1$. But $(\Sigma \cap F_{c-1}^\perp) + F_{c-1} = (\Sigma + F_{c-1}) \cap F_{c-1}^\perp$ is an isotropic subspace of V, so its dimension is exactly $n+1$. Let $\mathrm{OG}_c = \mathrm{OG}_c(n-c+2, F_{c-1}^\perp / F_{c-1})$ denote the space of maximal isotropic subspaces of $F_{c-1}^\perp / F_{c-1}$, which has two connected components. The claim now follows because Z_c is the inverse image of one of these components under the morphism $U_c \to \mathrm{OG}_c$ defined by $\Sigma \mapsto ((\Sigma + F_{c-1}) \cap F_{c-1}^\perp)/F_{c-1}$. We conclude that Z_P is closed using the identity

$$Z_P = \bigcap_{[c,n+1] \subset [P]} ((Y_P \smallsetminus U_c) \cup Z_c)\,.$$

Finally, assume that $X_Q^\circ \subset Z_P$, and let $\Sigma \in X_Q^\circ$ be any point. Since $X_Q^\circ \subset Y_P$, we have $Q \leq \overline{P}$. It follows that $Q \leq P$, as otherwise $p_j = n+1$ and $q_j = n+2$ for some j, in which case $\Sigma \in U_{n+1} \smallsetminus Z_{n+1}$. Moreover, if c is any critical index for (Q,P), then $\Sigma \in U_c$, and (43) implies that $\mathrm{type}(Q) = \mathrm{type}(P)$. It follows that $Q \preceq P$, which completes the proof. $\square$

Example A.3. For the special Schubert varieties, indexed by typed k-strict partitions with a single non-zero part, Proposition A.2 gives

$$X_r = Y_r = \{\Sigma \in \mathrm{OG} \mid \Sigma \cap F_{\epsilon(r)} \neq 0\},$$

for $r \neq k$, where $\epsilon(r) = n+k+2-r$ if $r < k$, and $\epsilon(r) = n+k+1-r$ if $r > k$, while $X_k \cup X'_k = Y_k = \{\Sigma \in \mathrm{OG} \mid \Sigma \cap F_{n+2} \neq 0\}$. Let $\widetilde{F}_{n+1} \subset V$ be the unique maximal isotropic subspace such that $F_n \subset \widetilde{F}_{n+1} \neq F_{n+1}$. Proposition A.2 gives

$$X_k = \{\Sigma \in \mathrm{OG} \mid \Sigma \cap F_{n+1} \neq 0\} \quad \text{and} \quad X'_k = \{\Sigma \in \mathrm{OG} \mid \Sigma \cap \widetilde{F}_{n+1} \neq 0\},$$

if n is even, while the roles of F_{n+1} and $\widetilde{F}_{n+1}$ are exchanged if n is odd. This agrees with [BKT1, §3.2] and the assertions made in the introduction.

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