QUANTUM COHOMOLOGY OF ISOTROPIC GRASSMANNIANS

HARRY TAMVAKIS

ABSTRACT. Let G be a classical Lie group and P a maximal parabolic subgroup. We describe a quantum Pieri rule which holds in the small quantum cohomology ring of G/P. We also give a presentation of this ring in terms of special Schubert class generators and relations. This is a survey paper which reports on joint work with Anders S. Buch and Andrew Kresch.

1. INTRODUCTION

Let G be a classical Lie group and P any maximal parabolic subgroup of G. Our aim in this paper is to discuss what is known about two questions regarding the small quantum cohomology ring of the homogeneous space $X = G/P$. The first problem is to formulate and prove a 'quantum Pieri rule' in the ring $QH^*(X)$, that is, a combinatorial rule which describes the quantum product of a general Schubert class with a 'special' Schubert class. The special classes should generate $QH^*(X)$, and we also seek a presentation for this ring in terms of these generators and relations.

In Lie type A, the answers to both of the above questions are well known, as X is an ordinary Grassmannian; see e.g. [HP] for the classical story and [ST, Be] for the extension to quantum cohomology. In the symplectic and orthogonal Lie types, the homogeneous space X is a Grassmannian which parametrizes isotropic subspaces in a vector space equipped with a nondegenerate skew-symmetric or symmetric form. When the subspaces in question have the maximum possible dimension, the corresponding classical analysis is contained in [Eh, Bo, HB], while [KT1, KT2] deal with the quantum case.

The above results about $QH^*(X)$ depend upon a study of the three point, genus zero Gromov-Witten invariants on X. The analysis in [Be, KT1, KT2] required intersection theory on certain Quot scheme compactifications of the moduli space of degree d rational maps to X. More recently, using his idea of the kernel and span of a rational map to a Grassmannian, Buch [Bu] gave new proofs of the main structure theorems for the quantum cohomology ring. These arguments require only basic algebraic geometry, assuming the associativity of the quantum product. The paper [BKT1] used similar techniques to deal with the case of maximal isotropic Grassmannians. It emerged in each of these examples that the relevant Gromov-Witten invariants on X were equal (or related) to classical triple intersection numbers on other homogeneous spaces.

The latter ideas and methods have been extended in [BKT2] to study the quantum cohomology ring of non-maximal isotropic Grassmannians X. In contrast, even

---

Date: August 18, 2004.
2000 *Mathematics Subject Classification.* Primary 14N35; Secondary 14M15, 14N15.
The author was supported in part by NSF Grant DMS-0296023.

the classical cohomology of these spaces is still rather unexplored; [Bo], [Se], and [PR1, PR2] are some previous works in this direction. The present paper includes the main results of [BKT2], which give a presentation over $\mathbb{Z}$ for the quantum cohomology ring $QH^*(X)$ and a quantum Pieri rule as well. An important feature of the analysis in loc. cit. is that it does not lead to a quantum extension of the classical Pieri rules of Pragacz and Ratajski from [PR1, PR2]. Instead, we need to use a different set of special Schubert classes, which are equal (or related) to the Chern classes of the universal *quotient* bundle over X. The resulting classical Pieri rules are simpler than the previously known ones, admit straightforward quantum extensions, and are parallel to the aforementioned examples (both type A and maximal isotropic).

This paper is organized as follows. Section 2 discusses the type A Grassmannian. The next section considers both the Lagrangian Grassmannian (in type C) and the maximal isotropic orthogonal Grassmannian (in types B and D). The last three sections study the non-maximal isotropic Grassmannians in types C, B, and D, respectively, and present the main results of [BKT2].

Some of the results included here were announced at the Miami Winter School on 'Geometric Methods in Algebra and Number Theory' in December of 2003. It is a pleasure to thank the organizers Fedor Bogomolov, Bruno de Oliveira, Yuri Tschinkel, and Alan Zame for making this stimulating event possible. I also thank my collaborators Anders Buch and Andrew Kresch, without whom this survey paper could not have been written.

2. The type A Grassmannian

We first discuss the relevant facts about the classical and small quantum cohomology of type A Grassmannians, namely the Pieri rule and the presentation of the ring in terms of generators and relations. Let $V = \mathbb{C}^N$ and X be the Grassmannian which parametrizes m-dimensional complex linear subspaces Σ of V. We will use $G(m, N)$ or $G(m, V)$ to denote X, depending on the context. X is a smooth complex manifold of dimension mn, where $n = N - m$.

Let $\mathcal{R}(m, n)$ denote the set of integer partitions $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_m \geqslant 0)$ with $\lambda_1 \leqslant n$, so that the Young diagram of λ fits inside an $m \times n$ rectangle (see Figure 1). For every $\lambda \in \mathcal{R}(m, n)$, we have a Schubert variety $X_\lambda(F_\bullet)$ in X, which

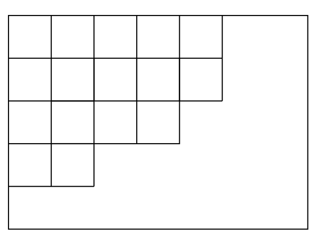

Figure 1. The partition $(5, 5, 4, 2)$ in $\mathcal{R}(5, 7)$

also depends the choice of a complete flag of subspaces

$$F_\bullet \ : \ 0 = F_0 \subset F_1 \subset \cdots \subset F_N = V$$

in V. The variety $X_\lambda(F_\bullet)$ is defined as the locus of $\Sigma \in X$ such that

$$\dim(\Sigma \cap F_{n+i-\lambda_i}) \geqslant i$$

for all $i = 1, \ldots, m$, and has codimension $|\lambda| = \sum \lambda_i$ in X. We let $\sigma_\lambda = [X_\lambda(F_\bullet)]$ denote the corresponding Schubert class in $H^{2|\lambda|}(X, \mathbb{Z})$; the set of all Schubert classes σ_λ for $\lambda \in \mathcal{R}(m,n)$ forms a $\mathbb{Z}$-basis for the cohomology of X. As all cohomology classes occur in even degrees, we will adopt the convention that the degree of a class $\alpha \in H^{2k}(X, \mathbb{Z})$ is equal to k throughout this paper.

The varieties $X_p(F_\bullet)$ for $p = 1, \ldots, n$ are called *special Schubert varieties*, and the corresponding cohomology classes are the *special Schubert classes* $\sigma_1, \ldots, \sigma_n$. The Schubert variety $X_p(F_\bullet)$ may be defined by a single Schubert condition; in fact, it only depends on the subspace F_{n+1-p}:

$$X_p(F_\bullet) = \{\, \Sigma \in X \mid \Sigma \cap F_{n+1-p} \neq 0 \,\}.$$

Consider the universal short exact sequence of vector bundles over X

$$0 \to S \to V_X \to Q \to 0 \tag{1}$$

where V_X denotes the trivial vector bundle of rank N over X, S is the tautological subbundle of rank m, and Q is the quotient bundle. The special Schubert class σ_p is equal to the p-th Chern class $c_p(Q)$, essentially by the definition of Chern classes. Indeed, let $\pi : P(S) \to X$ be the projection map and $\eta : P(S) \to P(V) = \mathbb{P}^{N-1}$ be the natural morphism induced by the inclusion $S \hookrightarrow V_X$. We then have that

$$\sigma_p = [X_p(F_\bullet)] = \pi_* \eta^* [P(F_{n+1-p})] = \pi_* \eta^* c_1(\mathcal{O}_{P(V)}(1))^{m-1+p}.$$

On the other hand, $\eta^* c_1(\mathcal{O}_{P(V)}(1)) = c_1(\eta^* \mathcal{O}_{P(V)}(1)) = c_1(\mathcal{O}_{P(S)}(1))$. Therefore

$$\pi_* \eta^* c_1(\mathcal{O}_{P(V)}(1))^{m-1+p} = \pi_* c_1(\mathcal{O}_{P(S)}(1))^{m-1+p} = s_p(S),$$

where $s_p(S)$ denotes the p-th Segre class of S (see e.g. [Fu, Sect. 3.1] for the definition of Segre classes). Now the Whitney sum formula applied to the sequence (1) states that $c(S)c(Q) = 1$ in $H^*(X, \mathbb{Z})$; it follows that $s_p(S) = c_p(Q)$.

The ring structure of $H^*(X, \mathbb{Z})$ is determined by the classical *Pieri rule* [Pi], which gives the product of a general Schubert class with a special one. For $1 \leqslant p \leqslant n$ we have

$$\sigma_\lambda \, \sigma_p = \sum \sigma_\mu,$$

where the sum is over all $\mu \in \mathcal{R}(m,n)$ obtained from the Young diagram of λ by adding p boxes, with no two in the same column. A skew diagram μ/λ which does not contain two boxes in the same column is called a *horizontal strip*.

One can use the Pieri rule to show that the special Schubert classes generate the ring $H^*(X, \mathbb{Z})$. Moreover, the cohomology of X may be presented as a quotient of the polynomial ring $\mathbb{Z}[\sigma_1, \ldots, \sigma_n]$ by the relations

$$\det(\sigma_{1+j-i})_{1 \leqslant i,j \leqslant r} = 0, \qquad m+1 \leqslant r \leqslant N.$$

The Whitney sum formula $c(S)c(Q) = 1$ can be used to show that these relations hold in $H^*(X, \mathbb{Z})$, since the coefficients of the inverse power series

$$c_t(Q)^{-1} = (1 + \sigma_1 t + \cdots + \sigma_n t^n)^{-1} = c_t(S)$$

must vanish in degrees higher than $m = \operatorname{rank}(S)$.

To extend the above picture to the quantum cohomology of X, we need to recall the enumerative definition of three point, genus zero Gromov-Witten invariants. We agree that a rational map of degree d to X is a morphism $f \colon \mathbb{P}^1 \to X$ such that

$$\int_X f_*[\mathbb{P}^1] \cdot \sigma_1 = d.$$

Given a degree $d \geqslant 0$ and partitions λ, μ, and ν such that $|\lambda|+|\mu|+|\nu| = mn+dN$, the Gromov-Witten invariant $\langle \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle_d$ is defined as the number of rational maps $f \colon \mathbb{P}^1 \to X$ of degree d such that $f(0) \in X_\lambda(F_\bullet)$, $f(1) \in X_\mu(G_\bullet)$, and $f(\infty) \in X_\nu(H_\bullet)$, for given flags $F_\bullet$, $G_\bullet$, and $H_\bullet$ in general position.

The (small) *quantum cohomology ring* $QH^*(X)$ is a $\mathbb{Z}[q]$-algebra which is isomorphic to $H^*(X,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$ as a module over $\mathbb{Z}[q]$, where q is a formal variable. This will be the case for all the varieties considered in this paper; for the type A Grassmannian $G(m,N)$, the degree of the variable q is equal to N. The multiplicative structure of $QH^*(X)$ is defined using the relation

$$\sigma_\lambda \cdot \sigma_\mu = \sum \langle \sigma_\lambda, \sigma_\mu, \sigma_{\widehat{\nu}} \rangle_d \, \sigma_\nu \, q^d, \tag{2}$$

the sum over $d \geqslant 0$ and partitions ν with $|\nu| = |\lambda| + |\mu| - dN$. Here $\widehat{\nu}$ denotes the dual partition of ν, defined so that

$$\int_X \sigma_\nu \sigma_{\widehat{\nu}} = 1.$$

One observes that the $d = 0$ terms of the sum in (2) just give the classical cup product in the cohomology ring of X.

The ring structure of $QH^*(X)$ (and hence all three point, genus zero Gromov-Witten invariants on X) is determined by Bertram's quantum Pieri rule. To state it, we let $\mathcal{R}'(m+1,n)$ denote the set of partitions $\mu \in \mathcal{R}(m+1,n)$ with $\mu_1 = n$ and $\mu_{m+1} \geqslant 1$. For any $\mu \in \mathcal{R}'(m+1,n)$, define a partition $\widetilde{\mu} \in \mathcal{R}(m,n)$ by removing a hook of length N from μ; in other words,

$$\widetilde{\mu} = (\mu_2 - 1, \ldots, \mu_{m+1} - 1).$$

Theorem 1 ([Be])**.** *For $1 \leqslant p \leqslant n$, we have*

$$\sigma_\lambda \cdot \sigma_p = \sum_{\mu \in \mathcal{R}(m,n)} \sigma_\mu + \sum_{\mu \in \mathcal{R}'(m+1,n)} \sigma_{\widetilde{\mu}} \, q, \tag{3}$$

where both sums are over diagrams μ obtained from λ by adding p boxes, with no two in the same column.

We remark that the partitions $\widetilde{\mu}$ which appear in the second sum in (3) are exactly those $\nu \in \mathcal{R}(m,n)$ such that $|\nu| = |\lambda| + p - N$ and

$$\lambda_1 - 1 \geqslant \nu_1 \geqslant \lambda_2 - 1 \geqslant \nu_2 \geqslant \cdots \geqslant \lambda_m - 1 \geqslant \nu_m \geqslant 0.$$

Example 1. For the Grassmannian $G(4,8)$, we have

$$\sigma_{4,3,1,1} \cdot \sigma_2 = \sigma_{4,4,2,1} + \sigma_{4,3,3,1} + \sigma_3 \, q + \sigma_{2,1} \, q$$

in the quantum cohomology ring $QH^*(G(4,8))$.

In particular, it is easy to deduce Siebert and Tian's presentation of the quantum cohomology ring of X from the quantum Pieri rule; see [Bu] for details.

Theorem 2 ([ST])**.** *The ring $QH^*(X)$ is presented as a quotient of the polynomial ring $\mathbb{Z}[\sigma_1, \ldots, \sigma_n, q]$ by the relations*

$$\det(\sigma_{1+j-i})_{1 \leqslant i,j \leqslant r} = 0, \qquad m+1 \leqslant r \leqslant N-1$$

and

$$\det(\sigma_{1+j-i})_{1 \leqslant i,j \leqslant N} = (-1)^{n-1} q.$$

We will now give an approach to proving Theorem 1 using results of [BKT1] together with the projection formula. Although one does not need all of these constructions for the proof (see the Introduction and [Bu]), they play an important role in the following sections. The main result of [BKT1] equates all of the above degree d Gromov-Witten invariants on X with classical triple intersection numbers on a two-step flag variety $Y_d = F(m-d, m+d; N)$. The variety Y_d parametrizes pairs of subspaces (A, B) with $A \subset B \subset V$, $\dim A = m-d$ and $\dim B = m+d$; we agree that Y_d is empty if $d > \min(m, n)$. To each Schubert variety $X_\lambda(F_\bullet)$ in X, we associate a Schubert variety $X_\lambda^{(d)}(F_\bullet)$ in Y_d via the prescription

$$X_\lambda^{(d)}(F_\bullet) = \{\, (A,B) \in Y_d \mid \exists\ \Sigma \in X_\lambda(F_\bullet) : A \subset \Sigma \subset B \,\}\,. \tag{4}$$

We let $\sigma_\lambda^{(d)}$ denote the class of $X_\lambda^{(d)}(F_\bullet)$ in $H^*(Y_d, \mathbb{Z})$. Consider three partitions λ, μ, and ν such that $|\lambda| + |\mu| + |\nu| = mn + dN$. We then have

$$\langle \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle_d = \int_{F(m-d,m+d;N)} \sigma_\lambda^{(d)} \cdot \sigma_\mu^{(d)} \cdot \sigma_\nu^{(d)}. \tag{5}$$

It is easy to justify the $d = 1$ case of (5), since a straightforward argument shows that the flag variety $Y_1 = F(m-1, m+1; N)$ is exactly the parameter space of lines on the Grassmannian $G(m, N)$.

The proof of (5) in general proceeds by considering, for any morphism $f : \mathbb{P}^1 \to X$, the pair $(\mathrm{Ker}(f), \mathrm{Span}(f))$ consisting of the *kernel* and *span* of f. Here, the kernel (respectively, the span) of f is defined as the intersection (respectively, the linear span) of all the subspaces $\Sigma \subset V$ corresponding to image points of f. Setting $a = \dim \mathrm{Ker}(f)$ and $b = \dim \mathrm{Span}(f)$, a dimension count on the flag variety $F(a, b; N)$ establishes that $d \leqslant \min(m, n)$ whenever $\langle \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle_d \neq 0$. For such d, one then shows that the map $f \mapsto (\mathrm{Ker}(f), \mathrm{Span}(f))$ is a bijection between the set of morphisms f counted by the Gromov-Witten invariant on the left hand side of (5) and the points (A, B) in the triple intersection $X_\lambda^{(d)}(F_\bullet) \cap X_\mu^{(d)}(G_\bullet) \cap X_\nu^{(d)}(H_\bullet)$ in Y_d, assuming the flags $F_\bullet$, $G_\bullet$, and $H_\bullet$ are in general position.

For any Young diagram λ, let $\overline{\lambda}$ denote the diagram obtained by deleting the leftmost column of λ. In terms of partitions, we have $\overline{\lambda}_i = \max\{\lambda_i - 1, 0\}$. Given any Schubert variety $X_\lambda(F_\bullet)$ in $G(m, V)$, we will consider an associated Schubert variety $X_{\overline{\lambda}}(F_\bullet)$ in $G(m+1, V)$, with cohomology class $\sigma_{\overline{\lambda}}$. It is straightforward to check that quantum Pieri rule follows from the two relations

$$\langle \sigma_\lambda, \sigma_\mu, \sigma_p \rangle_d = 0 \quad \text{for all } d \geqslant 2 \tag{6}$$

and

$$\langle \sigma_\lambda, \sigma_\mu, \sigma_p \rangle_1 = \int_{G(m+1,N)} \sigma_{\overline{\lambda}} \cdot \sigma_{\overline{\mu}} \cdot \sigma_{\overline{p}}. \tag{7}$$

Given (5), the vanishing assertion (6) is proved by a dimension count, which shows that the sum of the codimensions of the three Schubert varieties $X_\lambda^{(d)}(F_\bullet)$, $X_\mu^{(d)}(G_\bullet)$, and $X_p^{(d)}(H_\bullet)$ is strictly greater than the dimension of Y_d.

To establish (7), one may work as follows. Consider the three-step flag variety $Z = F(m-1, m, m+1; N)$, with its natural projections $\pi_1 : Z \to X$ and $\pi_2 : Z \to Y_1$. Note that for every $\lambda \in \mathcal{R}(m, n)$, we have $X_\lambda^{(1)}(F_\bullet) = \pi_2(\pi_1^{-1}(X_\lambda(F_\bullet)))$. The morphism π_2 lies on the left hand side of a commutative diagram

$$\begin{tikzcd} Z \arrow[r, "\varphi_2"] \arrow[d, "\pi_2"] & F(m,m+1;N) \arrow[r, "\varphi_1"] \arrow[d, "\pi"] & G(m,N) \\ Y_1 \arrow[r, "\psi"] & G(m+1,N) & \end{tikzcd}$$

where every arrow is a natural smooth projection map and $\pi_1 = \varphi_1\varphi_2$. We now apply (5) when $d = 1$, $\nu = (p)$ and use the projection formula repeatedly to obtain

$$\begin{aligned}
\langle \sigma_\lambda, \sigma_\mu, \sigma_p \rangle_1 &= \int_{Y_1} \sigma_\lambda^{(1)} \cdot \sigma_\mu^{(1)} \cdot \sigma_p^{(1)} \\
&= \int_{Y_1} \pi_{2*}\pi_1^*\sigma_\lambda \cdot \pi_{2*}\pi_1^*\sigma_\mu \cdot \psi^*\sigma_{p-1} \\
&= \int_Z \pi_2^*\pi_{2*}\pi_1^*\sigma_\lambda \cdot \pi_1^*\sigma_\mu \cdot \varphi_2^*\pi^*\sigma_{p-1} \\
&= \int_{F(m,m+1;N)} \varphi_{2*}\pi_2^*\pi_{2*}\pi_1^*\sigma_\lambda \cdot \varphi_1^*\sigma_\mu \cdot \pi^*\sigma_{p-1} \\
&= \int_{F(m,m+1;N)} \pi^*\pi_*\varphi_1^*\sigma_\lambda \cdot \varphi_1^*\sigma_\mu \cdot \pi^*\sigma_{p-1} \\
&= \int_{G(m+1;N)} \pi_*\varphi_1^*\sigma_\lambda \cdot \pi_*\varphi_1^*\sigma_\mu \cdot \sigma_{p-1} \\
&= \int_{G(m+1,N)} \sigma_{\overline{\lambda}} \cdot \sigma_{\overline{\mu}} \cdot \sigma_{\overline{p}}.
\end{aligned}$$

3. Maximal isotropic Grassmannians

In this section we will consider the Grassmannians of maximal isotropic subspaces in a vector space equipped with a nondegenerate symmetric or skew-symmetric bilinear form.

3.1. **The Lagrangian Grassmannian $LG(n, 2n)$.** We begin in type C with the Lagrangian Grassmannian $LG = LG(n, 2n)$ parametrizing Lagrangian subspaces in $V = \mathbb{C}^{2n}$. The variety LG has complex dimension $n(n+1)/2$. Let $\mathcal{D}_n$ denote the set of all strict partitions $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_\ell > 0)$ with $\lambda_1 \leqslant n$, and fix a complete isotropic flag of subspaces of V

$$F_\bullet \ : \ 0 = F_0 \subset F_1 \subset \cdots \subset F_n \subset V$$

where $\dim(F_i) = i$ for each i, and F_n is Lagrangian. For each $\lambda \in \mathcal{D}_n$, the codimension $|\lambda|$ Schubert variety $X_\lambda(F_\bullet) \subset LG$ is defined as the locus of $\Sigma \in LG$ such that

$$\dim(\Sigma \cap F_{n+1-\lambda_i}) \geqslant i, \quad \text{for} \quad i = 1, \ldots, \ell(\lambda). \tag{8}$$

Here $\ell(\lambda)$ denotes the length of λ, that is, the number of non-zero parts in λ. Let σ_λ be the class of $X_\lambda(F_\bullet)$ in the cohomology group $H^{2|\lambda|}(LG, \mathbb{Z})$; the classes σ_λ for $\lambda \in \mathcal{D}_n$ then form a $\mathbb{Z}$-basis of the cohomology of LG.

The classes $\sigma_1, \ldots, \sigma_n$ are again called special; each σ_p is the class of a special Schubert variety $X_p(F_\bullet)$, which is defined by a single Schubert condition, as in type A:

$$X_p(F_\bullet) = \{\, \Sigma \in LG \mid \Sigma \cap F_{n+1-p} \neq 0 \,\}.$$

Furthermore, if

$$0 \to S \to V_X \to Q \to 0$$

denotes the tautological short exact sequence of vector bundles over LG, then Q can be canonically identified with S^*, and we have $\sigma_p = c_p(S^*)$, for $0 \leqslant p \leqslant n$, as in Section 2.

The classical Pieri rule for LG is due to Hiller and Boe [HB]. It states that for any $\lambda \in \mathcal{D}_n$ and $p = 1, \ldots, n$ we have

$$\sigma_\lambda\, \sigma_p = \sum_\mu 2^{N(\lambda,\mu)} \sigma_\mu \tag{9}$$

in $H^*(LG, \mathbb{Z})$, where the sum is over all strict partitions μ obtained from λ by adding p boxes, with no two in the same column, and $N(\lambda, \mu)$ is the number of connected components of the skew diagram μ/λ which do not meet the first column (the connected components of a skew diagram α are defined by letting two boxes in α be connected if they share a vertex or an edge).

The ring $H^*(LG, \mathbb{Z})$ is presented as a quotient of the polynomial ring $\mathbb{Z}[\sigma_1, \ldots, \sigma_n]$ modulo the relations

$$c_t(S)c_t(S^*) = (1 - \sigma_1 t + \cdots + (-1)^n \sigma_n t^n)(1 + \sigma_1 t + \cdots + \sigma_n t^n) = 1. \tag{10}$$

By equating the coefficients of like powers of t in (10), we see that these relations are given by

$$\sigma_r^2 + 2\sum_{i=1}^{n-r} (-1)^i \sigma_{r+i}\sigma_{r-i} = 0$$

for $1 \leqslant r \leqslant n$, where we define $\sigma_0 = 1$ and $\sigma_j = 0$ for $j < 0$.

A rational map to LG is a morphism $f \colon \mathbb{P}^1 \to LG$, and its degree is the degree of $f_*[\mathbb{P}^1] \cdot \sigma_1$. The Gromov-Witten invariant $\langle \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle_d$ is defined for $|\lambda| + |\mu| + |\nu| = \dim(LG) + d(n+1)$ and counts the number of rational maps $f \colon \mathbb{P}^1 \to LG(n, 2n)$ of degree d such that $f(0) \in X_\lambda(F_\bullet)$, $f(1) \in X_\mu(G_\bullet)$, and $f(\infty) \in X_\nu(H_\bullet)$, for given isotropic flags $F_\bullet$, $G_\bullet$, and $H_\bullet$ in general position. The quantum cohomology ring of LG is a $\mathbb{Z}[q]$-algebra isomorphic to $H^*(LG, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$ as a module over $\mathbb{Z}[q]$, but here q is a formal variable of degree $n + 1$. The product in $QH^*(LG)$ is defined by the same equation (2) as before. We can now state the quantum Pieri rule for LG, which extends the classical rule of Hiller and Boe.

Theorem 3 ([KT1]). *For any $\lambda \in \mathcal{D}_n$ and $p \geqslant 1$ we have*

$$\sigma_\lambda \cdot \sigma_p = \sum_\mu 2^{N(\lambda,\mu)} \sigma_\mu + \sum_\nu 2^{N'(\nu,\lambda)} \sigma_\nu\, q$$

in $QH^*(LG(n, 2n))$, where the first sum is classical, as in (9), while the second is over all strict ν obtained from λ by subtracting $n+1-p$ boxes, no two in the same column, and $N'(\nu, \lambda)$ is one less than the number of connected components of λ/ν.

Example 2. For the Grassmannian $LG(4, 8)$, the relations

$$\sigma_{3,2} \cdot \sigma_3 = 2\, \sigma_{4,3,1} + \sigma_3\, q + \sigma_{2,1}\, q \quad \text{and} \quad \sigma_{4,2} \cdot \sigma_3 = \sigma_{4,3,2} + \sigma_4\, q + 2\, \sigma_{3,1}\, q$$

hold in the quantum cohomology ring $QH^*(LG)$.

The proof of Theorem 3 from [BKT1] uses a result similar to (5) which holds for LG. In this setting, the role of the two step flag variety Y_d is played by a non-maximal isotropic Grassmannian $IG(n-d, 2n)$. This is because the span of a rational map $\mathbb{P}^1 \to X$ is the orthogonal complement of its kernel, and hence is redundant. We have that for any λ, μ, $\nu \in \mathcal{D}_n$ such that $|\lambda| + |\mu| + |\nu| = n(n+1)/2 + d(n+1)$,

$$\langle \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle_d = \int_{IG(n-d,2n)} \sigma_\lambda^{(d)} \cdot \sigma_\mu^{(d)} \cdot \sigma_\nu^{(d)}.$$

Here, for each $\lambda \in D_n$, $\sigma_\lambda^{(d)}$ denotes the cohomology class of a Schubert variety $X_\lambda^{(d)}(F_\bullet)$ in $IG(n-d, 2n)$, defined by an equation directly analogous to (4). To compute the line numbers $\langle \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle_1$, one shows that up to a factor of 2, they are equal to classical intersection numbers on the Lagrangian Grassmannian $LG(n+1, 2n+2)$. More precisely, we have that

$$\int_{IG(n-1,2n)} \sigma_\lambda^{(1)} \cdot \sigma_\mu^{(1)} \cdot \sigma_\nu^{(1)} = \frac{1}{2} \int_{LG(n+1,2n+2)} \sigma_\lambda^+ \cdot \sigma_\mu^+ \cdot \sigma_\nu^+, \tag{11}$$

where σ_λ^+, σ_μ^+, σ_ν^+ denote Schubert classes in $H^*(LG(n+1, 2n+2), \mathbb{Z})$.

We give a brief discussion of the geometric proof of (11), since an analogous argument works to prove the quantum Pieri rule on any isotropic Grassmannian in type C. Let H be a 2-dimensional symplectic vector space and let V^+ be the orthogonal direct sum of V and H. We then consider the correspondence between $LG(n+1, 2n+2)$ and $IG(n-1, 2n)$ consisting of pairs (Σ^+, Σ') with Σ^+ a Lagrangian subspace of V^+ and Σ' an isotropic $(n-1)$-dimensional subspace of V, given by the condition $\Sigma' \subset \Sigma^+$. This is the correspondence induced by the rational map which sends Σ^+ to $\Sigma^+ \cap V$.

Choose general isotropic flags $E_\bullet$, $F_\bullet$, and $G_\bullet$ in V, so that the corresponding varieties $X_\lambda(E_\bullet)$, $X_\mu(F_\bullet)$, and $X_\nu(G_\bullet)$ meet transversely. We then extend the three flags in V to a flag of subspaces in V^+ by adjoining, in each case, a general element of H. One then checks that to every point in $X_\lambda^+(E_\bullet^+) \cap X_\mu^+(F_\bullet^+) \cap X_\nu^+(G_\bullet^+)$ (intersection in $LG(n+1, 2n+2)$) there corresponds a point in $X_\lambda^{(1)}(E_\bullet) \cap X_\mu^{(1)}(F_\bullet) \cap X_\nu^{(1)}(G_\bullet)$, and conversely, each point in the latter intersection corresponds to exactly two points in $X_\lambda^+(E_\bullet^+) \cap X_\mu^+(F_\bullet^+) \cap X_\nu^+(G_\bullet^+)$, with the intersection transverse at both of these points. This is enough to prove (11).

We also have the following presentation of the quantum cohomology ring of LG.

Theorem 4 ([KT1]). *The ring $QH^*(LG)$ is presented as a quotient of the polynomial ring $\mathbb{Z}[\sigma_1, \ldots, \sigma_n, q]$ by the relations*

$$\sigma_r^2 + 2\sum_{i=1}^{n-r} (-1)^i \sigma_{r+i}\sigma_{r-i} = (-1)^{n-r} \sigma_{2r-n-1}\, q$$

for $1 \leqslant r \leqslant n$.

3.2. **The orthogonal Grassmannian $OG(n+1, 2n+2)$.** We now turn to the even orthogonal Grassmannian $OG = OG(n+1, 2n+2)$. Here $V = \mathbb{C}^{2n+2}$ is equipped with a nondegenerate symmetric form and OG parametrizes one component of the locus of maximal isotropic subspaces in V. The variety OG is projectively equivalent to the odd orthogonal Grassmannian $OG(n, 2n+1)$, and hence the following analysis (for the maximal isotropic case) will include both of the orthogonal Lie types B and D. The dimension of OG equals $n(n+1)/2$, the same as the dimension of LG.

A good part of the classical story for OG is similar to that for $LG(n,2n)$. The Schubert varieties $X_\lambda(F_\bullet)$ in OG are parametrized by partitions $\lambda \in \mathcal{D}_n$ and are defined by the same equations (8) as before, with respect to an isotropic flag $F_\bullet$ in V. The same is true for the special Schubert varieties $X_p(F_\bullet)$, for $p = 1, \ldots, n$. Let τ_λ be the cohomology class of $X_\lambda(F_\bullet)$; then the τ_λ for $\lambda \in \mathcal{D}_n$ form a Z-basis for $H^*(OG, \mathbb{Z})$. Furthermore, let S (respectively, Q) denote the tautological subbundle (respectively, quotient bundle) over OG.

One important difference between the orthogonal and symplectic Grassmannians is that the natural embedding of $OG(n+1, 2n+2)$ into the type A Grassmannian $G(n+1, 2n+2)$ multiplies all degrees by a factor of two. This occurs because the subspaces corresponding to points of OG all lie on the quadric of isotropic vectors in V. We therefore have a natural map $\theta : P(S) \to \mathcal{Q}$, where $\mathcal{Q} \hookrightarrow P(V)$ is a $2n$-dimensional quadric hypersurface.

Let us recall the structure of the cohomology ring $H^*(\mathcal{Q}, \mathbb{Z})$. If E and F are maximal isotropic subspaces in V, then $P(E)$ and $P(F)$ are subvarieties of $\mathcal{Q}$ called rulings. The two rulings represent the same class in $H^{2n}(\mathcal{Q}, \mathbb{Z})$ if and only if E and F lie in the same SO_{2n+2}-orbit. There are two families of rulings giving rise to two cohomology classes e and f, and, if $h = c_1(\mathcal{O}_{P(V)}(1)|_{\mathcal{Q}})$ denotes the hyperplane class, we have $h^n = e + f$. A $\mathbb{Z}$-basis for $H^*(\mathcal{Q}, \mathbb{Z})$ is given by $1, h, \ldots, h^n, e, eh = fh, eh^2, \ldots, eh^n$. Finally, one has that $e^2 = f^2 = 0$ and $ef = eh^n$, if n is even, while $e^2 = f^2 = eh^n$ and $ef = 0$, if n is odd.

We deduce that the special Schubert classes τ_p satisfy

$$\tau_p = \pi_*\theta^*[P(F_{n+1-p})] = \frac{1}{2}\,\pi_*\theta^* c_1(\mathcal{O}_{\mathcal{Q}}(1))^{n+p} = \frac{1}{2}\,\pi_* c_1(\mathcal{O}_{P(S)}(1))^{n+p},$$

where $\pi : P(S) \to OG$ is the projection map, and hence that $c_p(Q) = 2\tau_p$, for $p = 1, \ldots, n$. This explains the form of the Pieri rule for OG, as compared to that for $LG(n,2n)$. We have

$$\tau_\lambda\,\tau_p = \sum_\mu 2^{N'(\lambda,\mu)}\tau_\mu$$

in $H^*(OG, \mathbb{Z})$, where the sum is over all strict partitions μ obtained from λ by adding p boxes, with no two in the same column, and $N'(\lambda,\mu)$ is one less than the number of connected components of μ/λ. The cohomology ring of OG is presented as a quotient of $\mathbb{Z}[\tau_1, \ldots, \tau_n]$ modulo the relations

$$\tau_r^2 + 2\sum_{i=1}^{r-1}(-1)^i\tau_{r+i}\tau_{r-i} + (-1)^r\tau_{2r} = 0$$

for $1 \leqslant r \leqslant n$, where as usual $\tau_0 = 1$ and $\tau_j = 0$ for $j < 0$ or $j > n$.

The quantum cohomology of OG is isomorphic to $H^*(OG, \mathbb{Z}) \otimes \mathbb{Z}[q]$ as a module over $\mathbb{Z}[q]$, but this time the variable q has degree $2n$.

Theorem 5 ([KT2]). *For any $\lambda \in \mathcal{D}_n$ and $p \geqslant 1$ we have*

$$\tau_\lambda \cdot \tau_p = \sum_\mu 2^{N'(\lambda,\mu)}\tau_\mu + \sum_\nu 2^{N'(\lambda,\nu)}\tau_{\nu\smallsetminus(n,n)}\,q,$$

where the first sum is over strict μ and the second over partitions $\nu = (n, n, \overline{\nu})$ with $\overline{\nu}$ strict, such that both μ and ν are obtained from λ by adding p boxes, with no two in the same column.

Example 3. For the Grassmannian $OG(5, 10)$, we have

$$\tau_{3,2} \cdot \tau_3 = 2\,\tau_{4,3,1} \quad \text{and} \quad \tau_{4,2} \cdot \tau_3 = \tau_{4,3,2} + 2\,\sigma_1\, q$$

in the quantum cohomology ring $QH^*(OG)$. Compare this with Example 2.

The degree doubling phenomenon discussed previously allows us to conclude that for every degree d map $f : \mathbb{P}^1 \to OG$, the pullback of the quotient bundle Q has degree $2d$. It follows that the relevant parameter space of kernels of the maps counted by a Gromov-Witten invariant is the non-maximal isotropic Grassmannian $OG(n+1-2d, 2n+2)$. A dimension counting argument now implies that $\langle \tau_\lambda, \tau_\mu, \tau_p \rangle_d = 0$ for $d > 1$, as before.

One observes that the relation $\tau_n^2 = q$ holds in $QH^*(OG)$; this follows from the easy enumerative fact that there is a unique line in OG through a given point and incident to $X_n(E_\bullet)$ and $X_n(F_\bullet)$, for general complete flags $E_\bullet$ and $F_\bullet$ in V. Since the degree of q equals $2n$, this is enough to deduce the presentation which follows.

Theorem 6 ([KT2])**.** *The ring $QH^*(OG)$ is presented as a quotient of the polynomial ring $\mathbb{Z}[\tau_1, \ldots, \tau_n, q]$ modulo the relations*

$$\tau_r^2 + 2\sum_{i=1}^{r-1} (-1)^i \tau_{r+i}\tau_{r-i} + (-1)^r \tau_{2r} = 0$$

for all $r < n$, together with the quantum relation

$$\tau_n^2 = q.$$

The remainder of the proof of the quantum Pieri rule for OG differs from those discussed in previous sections. One proves by geometric considerations that the product $\tau_\lambda \cdot \tau_p$ in $QH^*(OG)$ is classical whenever $\lambda_1 < n$. In other words, if the first row of λ is not full, then multiplying τ_λ by a special Schubert class carries no quantum correction. This implies that if λ is such that $\lambda_1 = n$, and $\lambda \smallsetminus n = (\lambda_2, \lambda_3, \ldots)$, then the equation

$$\tau_{\lambda \smallsetminus n} \cdot \tau_n = \tau_\lambda$$

holds in $QH^*(OG)$. On the other hand,

$$\tau_\lambda \cdot \tau_n = \tau_{\lambda \smallsetminus n} \cdot \tau_n^2 = \tau_{\lambda \smallsetminus n}\, q.$$

We thus have established the quantum Pieri rule for multiplication by the special Schubert class τ_n. The general case of the rule follows easily from this and the aforementioned properties.

We refer the reader to the lecture notes [T] for a more detailed discussion of the quantum cohomology of type A and maximal isotropic Grassmannians. This includes some aspects of the story which we have not touched on here, such as classical and quantum Giambelli formulas and Littlewood-Richardson rules.

4. The isotropic Grassmannian $IG(n-k, 2n)$

4.1. **The classical theory.** The next three sections report on joint work Anders Buch and Andrew Kresch [BKT2]. We return here to the symplectic vector space $V = \mathbb{C}^{2n}$ as in Section 3.1, and consider the Grassmannian $IG = IG(n-k, 2n)$ which parametrizes $(n-k)$-dimensional isotropic subspaces of V. The variety IG has dimension $(n-k)(n+3k+1)/2$. The Schubert varieties in IG are parametrized by a certain subset of the hyperoctahedral group, which is described below.

The elements of Weyl group $W_n = S_n \ltimes \mathbb{Z}_2^n$ for the root system C_n are permutations with a sign attached to each entry; we will write these elements as barred permutations. The hyperoctahedral group W_n is an extension of the symmetric group S_n by an element s_0 which acts on the right by

$$(u_1, u_2, \ldots, u_n)s_0 = (\overline{u}_1, u_2, \ldots, u_n),$$

and is generated by the simple reflections $s_0, s_1, \ldots, s_{n-1}$, where each s_i for $i > 0$ is a simple transposition in S_n. If W_k is the parabolic subgroup of W_n generated by $\{s_i \mid i \neq k\}$, then the set $W^{(k)} \subset W_n$ of minimal length coset representatives of W_k parametrizes the Schubert varieties in $IG(n-k, 2n)$. This indexing set $W^{(k)}$ consists of barred permutations of the form

$$w = w_{u,\lambda} = (u_k, \ldots, u_1, \overline{\lambda}_1, \ldots, \overline{\lambda}_\ell, v_{n-k-\ell}, \ldots, v_1)$$

where $\lambda \in \mathcal{D}_n$ with $\ell = \ell(\lambda) \leqslant n-k$, $u_k < \cdots < u_1$, and $v_{n-k-\ell} < \cdots < v_1$.

We can define the Schubert varieties X_w in IG geometrically using a group monomorphism $\phi \colon W_n \hookrightarrow S_{2n}$ with image

$$\phi(W_n) = \{\, \sigma \in S_{2n} \mid \sigma(i) + \sigma(2n+1-i) = 2n+1, \quad \text{for all} \quad i \,\}.$$

The map ϕ is determined by setting, for each $w = (w_1, \ldots, w_n) \in W_n$ and $1 \leqslant i \leqslant n$,

$$\phi(w)(i) = \begin{cases} n+1-w_{n+1-i} & \text{if } w_{n+1-i} \text{ is unbarred,} \\ n + \overline{w}_{n+1-i} & \text{otherwise.} \end{cases}$$

Consider a complete isotropic flag

$$F_\bullet \ : \ 0 \subset F_1 \subset F_2 \subset \cdots \subset F_n \subset V$$

and extend it to a complete flag in V by letting $F_{n+p} = F_{n-p}^\perp$ for $1 \leqslant p \leqslant n$. For $w = w_{u,\lambda}$, the Schubert variety $X_w = X_w(F_\bullet) \subset IG$ is defined as the locus of isotropic Σ such that

$$\dim(\Sigma \cap F_i) \geqslant \#\{\, p \leqslant n-k \mid \phi(w)(p) > 2n-i \,\} \quad \text{for} \quad 1 \leqslant i \leqslant 2n.$$

Following Pragacz and Ratajski [PR1], each Weyl group element $w_{u,\lambda}$ corresponds to a pair of partitions $\Lambda = (\alpha \,|\, \lambda)$, where the 'top' partition $\alpha = \alpha(u, \lambda)$ is defined by

$$\alpha_i = u_i + i - k - 1 + \#\{j \mid \lambda_j > u_i\}$$

for $1 \leqslant i \leqslant k$; the 'bottom' partition is λ. The Schubert varieties X_Λ in IG are thus parametrized by the set $\mathcal{P}(k, n)$ of pairs $\Lambda = (\alpha \,|\, \lambda)$ with $\alpha \in \mathcal{R}(k, n-k)$, $\lambda \in \mathcal{D}_n$, and such that $\alpha_k \geqslant \ell(\lambda)$. Figure 2 illustrates a partition pair indexing a Schubert variety in $IG(4, 14)$, with $\alpha = (4, 3, 3)$ and $\lambda = (5, 4, 1)$.
The codimension of the variety $X_\Lambda(F_\bullet)$ in IG is given by the weight $|\Lambda| = |\alpha| + |\lambda|$; we let σ_Λ denote the corresponding cohomology class in $H^{2|\Lambda|}(IG, \mathbb{Z})$.

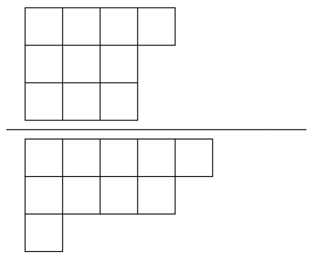

FIGURE 2. The partition pair $(4,3,3\,|\,5,4,1)$

The *special* Schubert varieties used in our analysis of IG are the varieties $X_p(F_\bullet)$ for $1 \leqslant p \leqslant n+k$ defined by a single Schubert condition, as follows:

$$X_p(F_\bullet) = \{\, \Sigma \in IG \mid \Sigma \cap F_{n+k+1-p} \neq 0 \,\}.$$

Our main reason for this choice of special classes is the application to the classical and quantum Pieri rules which follow. We note that $X_p = X_{(1^{\min(p,k)}|\max(p-k,0))}$. If σ_p denotes the cohomology class of X_p in $H^{2p}(IG,\mathbb{Z})$, and

$$0 \to S \to V \to Q \to 0$$

is the tautological exact sequence of vector bundles over IG, then σ_p is equal to the Chern class $c_p(Q)$.

We proceed to describe the classical Pieri rule for multiplying a general Schubert class σ_Λ with a special class. In contrast, Pragacz and Ratajski [PR1] obtain a Pieri rule for multiplication with the Chern classes of S^*.

Recall that the number of components of skew diagram is by definition the number of connected components of its vertical projection. We adopt the following shifting conventions for the two diagrams α, λ in a pair $\Lambda = (\alpha\,|\,\lambda)$. For each i, the ith part of α is shifted to the right $k-i+1$ units, and the ith part of λ is shifted to the right $i-1$ units. Upon performing this operation, we obtain the *shifted diagram* $\mathcal{S}(\Lambda)$ of Λ. We say that a box of $(\alpha\,|\,\lambda)$ lies *shift-under* some given reference box(es) if, in $\mathcal{S}(\Lambda)$, the box lies under (at least one of) the reference box(es). In Figure 3, the boxes in $(4,3,3\,|\,5,4,1)$ which lie shift-under the box marked with an 'x' are marked with an 'o'.

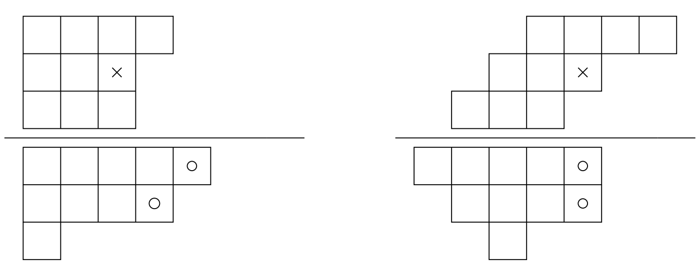

FIGURE 3. $(4,3,3\,|\,5,4,1)$ and $\mathcal{S}(4,3,3\,|\,5,4,1)$

Theorem 7. *For any $\Lambda = (\alpha\,|\,\lambda) \in \mathcal{P}(k,n)$ and $p \geqslant 1$ we have*

$$\sigma_\Lambda \sigma_p = \sum_M 2^{N(\Lambda,M)} \sigma_M,$$

where the sum is over all $M = (\beta\,|\,\mu)$ with $|M| = |\Lambda| + p$ such that

(i) *$\alpha_{i+1} \leqslant \beta_i \leqslant \alpha_i + 1$ for each i,*
(ii) *$\mu \supset \lambda$, with μ/λ a horizontal strip,*
(iii) *for each i with $\beta_i = \alpha_i$, there is at most one box of μ/λ shift-under the rightmost box of β_i,*
(iv) *for each i with $\beta_i < \alpha_i$, there are exactly $\alpha_i - \beta_i + 1$ boxes of μ/λ shift-under the boxes between the rightmost box of β_i and the rightmost box of α_i, inclusive; furthermore, these $\alpha_i - \beta_i + 1$ boxes of the bottom part are contained in a single row.*

Let R_i denote the collection of box(es) of μ/λ indicated in (iii) and (iv) for given i, with $R_i = \emptyset$ when $\beta_i > \alpha_i$, and set $R = \bigcup_{i=1}^k R_i$. Then the exponent $N(\Lambda, M)$ equals the number of components of $(\mu/\lambda) \smallsetminus R$ not meeting the first column.

If the partition pair M with $|M| = |\Lambda|+p$ satisfies conditions (i)-(iv) of Theorem 7, then we will write $\Lambda \xrightarrow{p} M$. The argument used to prove Theorem 7 in [BKT2] is geometric, along the lines of Hodge and Pedoe's proof of the classical Pieri rule for type A Grassmannians, via triple intersections [HP, §XIV.4].

Theorem 8. *The cohomology ring $H^*(IG, \mathbb{Z})$ is presented as a quotient of the polynomial ring $\mathbb{Z}[\sigma_1, \ldots, \sigma_{n+k}]$ modulo the relations*

$$\det(\sigma_{1+j-i})_{1\leqslant i,j\leqslant r} = 0, \qquad n-k+1 \leqslant r \leqslant n+k \tag{12}$$

and

$$\sigma_r^2 + 2\sum_{i=1}^{n+k-r} (-1)^i \sigma_{r+i}\sigma_{r-i} = 0, \qquad k+1 \leqslant r \leqslant n. \tag{13}$$

The first set of relations (12) in Theorem 8 follow from the Whitney sum formula $c(S)c(Q) = 1$, as in Section 2. To see the relations (13), note that the symplectic form on V gives rise to a pairing $S \otimes Q \to \mathcal{O}$, and hence an injection $S \to Q^*$. The Chern classes $c_j(Q^*/S)$ vanish for $j > 2k$; multiplying with the previous relation, we deduce that the cohomology class $c(Q)c(Q^*)$ vanishes in degrees larger than $2k$. We have thus shown that the relations (12) and (13) hold in $H^*(IG, \mathbb{Z})$; more work is required to deduce that these two sets of relations suffice to obtain the presentation in the theorem.

4.2. **The quantum theory.** The quantum cohomology ring $QH^*(IG)$ is a $\mathbb{Z}[q]$-algebra as before, where the degree of the formal variable q is given by $\deg(q) = n + k + 1$. The ring structure on $QH^*(IG)$ is determined by a relation analogous to (2)

$$\sigma_\Lambda \cdot \sigma_M = \sum \langle \sigma_\Lambda, \sigma_M, \sigma_{\widehat{N}} \rangle_d\, \sigma_N\, q^d,$$

the sum over $d \geqslant 0$ and $N \in \mathcal{P}(k, n)$ with $|N| = |\Lambda| + |M| - (n + k + 1)d$.

For each partition pair $M = (\beta\,|\,\mu)$ with $\mu_1 > 0$, define a new pair $M^* = (\beta^*\,|\,\mu^*)$ by setting $\beta_i^* = \beta_i - 1$ and $\mu_i^* = \mu_{i+1}$ for each i. In other words, M^* is obtained from M by removing one box from each row of β and the entire first row of μ. Let $\mathcal{Q}(k, n)$ denote the set of $M = (\beta\,|\,\mu) \in \mathcal{P}(k, n)$ such that $\mu_1 = n$.

Theorem 9. *For any $\Lambda \in \mathcal{P}(k, n)$ and p with $1 \leqslant p \leqslant n + k$, we have*

$$\sigma_\Lambda \cdot \sigma_p = \sum_{M\in\mathcal{P}(k,n)} 2^{N(\Lambda,M)}\, \sigma_M + \sum_{M\in\mathcal{Q}(k,n+1)} 2^{N(\Lambda,M)-1}\, \sigma_{M^*}\, q$$

in the quantum cohomology ring of $IG(n-k,2n)$, where both sums involve partition pairs M such that $\Lambda \xrightarrow{p} M$.

Example 4. In the quantum cohomology ring of $IG(4,12)$, we have

$$\sigma_5 \cdot \sigma_{(4,3|3,1)} = 2\,\sigma_{(4,4|5,3)} + 4\,\sigma_{(4,4|6,2)} + 2\,\sigma_{(4,4|5,2,1)} + \sigma_{(4,4|4,3,1)} + 2\,\sigma_{(4,3|6,2,1)}$$

$$+\sigma_{(4,2|1)}\,q + 2\,\sigma_{(3,3|1)}\,q + 4\,\sigma_{(3,2|2)}\,q + \sigma_{(2,2|2,1)}\,q.$$

The analysis used in the proof of Theorem 9 is more involved, but analogous, to the corresponding one in Section 3.1. To study the Gromov-Witten invariants $\langle\sigma_\Lambda,\sigma_M,\sigma_N\rangle_d$, we use the auxilliary variety Z_d parametrizing pairs (A,B) of subspaces of V with $A\in IG(n-k-d,2n)$, $\dim B = n-k+d$, and $A\subset B\subset A^\perp$, for each $d\leqslant n-k$. For every $\Lambda\in\mathcal{P}(k,n)$, define a subvariety $Y_\Lambda\subset Z_d$ by

$$Y_\Lambda(F_\bullet) = \{\,(A,B)\in Z_d \mid A\subset\Sigma\subset B \text{ for some } \Sigma \text{ with } \Sigma\in X_\Lambda(F_\bullet)\,\}.$$

One then shows that if $\Lambda,M,N\in\mathcal{P}(k,n)$ are such that $|\Lambda|+|M|+|N| = \dim(IG)+n+k+1$, then

$$\langle\sigma_\Lambda,\sigma_M,\sigma_N\rangle_1 = \int_{Z_1}[Y_\Lambda]\cdot[Y_M]\cdot[Y_N].$$

Moreover, if $N = (1^{\min(p,k)}\,|\max(p-k,0))$ indexes a special Schubert class, then the Gromov-Witten invariant $\langle\sigma_\Lambda,\sigma_M,\sigma_p\rangle_d$ vanishes whenever $d\geqslant 2$. The key to proving these results is to associate to any rational map $f\colon\mathbb{P}^1\to IG$ of degree d counted by $\langle\sigma_\Lambda,\sigma_M,\sigma_N\rangle_d$ the pair consisting of the kernel and the span of f.

Finally, one uses a correspondence between lines on $IG = IG(n-k,2n)$ and points on $IG(n+1-k,2n+2)$ to obtain an analogue of (11) for IG. For Λ, M, $N\in\mathcal{P}(k,n)$ such that the first parts of the top partitions of Λ, M, N sum to at most $2(n-k)+1$, we obtain that

$$\int_{Z_1}[Y_\Lambda]\cdot[Y_M]\cdot[Y_N] = \frac{1}{2}\int_{IG(n+1-k,2n+2)}[X^+_\Lambda]\cdot[X^+_M]\cdot[X^+_N],$$

where X^+_Λ, X^+_M, X^+_N denote Schubert varieties in $IG(n+1-k,2n+2)$. It follows that when $\sigma_N=\sigma_p$ is a special Schubert class, then

$$\langle\sigma_\Lambda,\sigma_M,\sigma_p\rangle_1 = \frac{1}{2}\int_{IG(n+1-k,2n+2)}\sigma^+_\Lambda\cdot\sigma^+_M\cdot\sigma^+_p.$$

We deduce Theorem 9 from this, Theorem 7, and an analysis of the Poincaré duality involution on $\mathcal{P}(k,n)$.

Theorem 10. *The quantum cohomology ring $QH^*(IG,\mathbb{Z})$ is presented as a quotient of the polynomial ring $\mathbb{Z}[\sigma_1,\ldots,\sigma_{n+k},q]$ modulo the relations*

$$\det(\sigma_{1+j-i})_{1\leqslant i,j\leqslant r} = 0, \qquad n-k+1\leqslant r\leqslant n+k$$

and

$$\sigma_r^2 + 2\sum_{i=1}^{n+k-r}(-1)^i\sigma_{r+i}\sigma_{r-i} = (-1)^{n+k-r}\sigma_{2r-n-k-1}\,q, \qquad k+1\leqslant r\leqslant n.$$

5. The odd orthogonal Grassmannian $OG(n-k, 2n+1)$

5.1. **The classical theory.** In this section, we consider a vector space $V = \mathbb{C}^{2n+1}$ equipped with a nondegenerate symmetric bilinear form. The odd orthogonal Grassmannian $OG = OG(n-k, 2n+1)$ parametrizes the $(n-k)$-dimensional isotropic subspaces in V; it has the same dimension as the isotropic Grassmannian $IG(n-k, 2n)$. The Weyl group for the root system B_n is the same as that for C_n, hence most of the analysis in §4.1 applies here as well. To define the Schubert variety in OG indexed by $w \in W_n$, we use a monomorphism $\psi \colon W_n \hookrightarrow S_{2n+1}$ with image

$$\psi(W_n) = \{\, \sigma \in S_{2n+1} \mid \sigma(i) + \sigma(2n+2-i) = 2n+2, \quad \text{for all} \quad i \,\},$$

determined by the equalities

$$\psi(w)(i) = \begin{cases} n+1-w_{n+1-i} & \text{if } w_{n+1-i} \text{ is unbarred,} \\ n+1+\overline{w}_{n+1-i} & \text{otherwise,} \end{cases}$$

for each $w = (w_1, \dots, w_n) \in W_n$ and $1 \leqslant i \leqslant n$. Given an isotropic flag $F_\bullet$ in V, we extend it to a complete flag by setting $F_{n+p} = F^{\perp}_{n+1-p}$ for $1 \leqslant p \leqslant n+1$. For $w = w_{u,\lambda}$, the Schubert variety $X_w(F_\bullet) \subset OG$ is defined as the locus of isotropic subspaces Σ in V such that

$$\dim(\Sigma \cap F_i) \geqslant \#\{\, p \leqslant n-k \mid \psi(w)(p) > 2n+1-i \,\} \quad \text{for} \quad 1 \leqslant i \leqslant 2n.$$

To each Weyl group element $w_{u,\lambda}$ we associate a pair of partitions $\Lambda = (\alpha \mid \lambda)$ as in §4.1. The Schubert varieties in OG are indexed by the set $\mathcal{P}(k,n)$ of partition pairs; we let $\tau_\Lambda \in H^{2|\Lambda|}(OG, \mathbb{Z})$ be the cohomology class determined by the Schubert variety $X_\Lambda(F_\bullet)$.

The *special* Schubert varieties for OG are the varieties

$$X_p = X_{(1^{\min(p,k)} | \max(p-k,0))}, \qquad 1 \leqslant p \leqslant n+k.$$

These are defined by a single Schubert condition as follows: let

$$\varepsilon(p) = n+k-p+\begin{cases} 1 & \text{if } p > k, \\ 2 & \text{if } p \leqslant k. \end{cases} \tag{14}$$

Then

$$X_p(F_\bullet) = \{\, \Sigma \in OG \mid \Sigma \cap F_{\varepsilon(p)} \neq 0 \,\}.$$

We let τ_p denote the cohomology class of X_p, and S (respectively Q) be the tautological subbundle (respectively, quotient bundle) over $OG(n-k, 2n+1)$. As in Section 3.2, by considering the natural map $\theta : P(S) \to \mathcal{Q} \subset P(V)$, where $\mathcal{Q}$ is the $(2n-1)$-dimensional quadric of isotropic vectors, one shows that

$$c_p(Q) = \begin{cases} \tau_p & \text{if } p \leqslant k, \\ 2\tau_p & \text{if } p > k. \end{cases}$$

The classical Pieri rule for OG involves the same conditions (i)-(iv) and set $R \subset \mu/\lambda$ that appeared in Theorem 7.

Theorem 11. *For any $\Lambda = (\alpha \mid \lambda) \in \mathcal{P}(k,n)$ and $p \geqslant 1$ we have*

$$\tau_\Lambda \, \tau_p = \sum_M 2^{N'(\Lambda,M)} \tau_M, \tag{15}$$

where the sum is over all $M=(\beta\,|\,\mu)$ *with* $\Lambda \xrightarrow{p} M$*. Moreover,* $N'(\Lambda,M)$ *equals the number (respectively, one less than the number) of components of* $(\mu/\lambda)\smallsetminus R$*, if* $p\leqslant k$ *(respectively, if* $p>k$*).*

Let $c_p=c_p(Q)$. Using *rational* coefficients, one has a presentation of the cohomology ring of OG in terms of the c variables directly analogous to that in Theorem 8. The ring $H^*(OG,\mathbb{Q})$ is presented as a quotient of the polynomial ring $\mathbb{Q}[c_1,\ldots,c_{n+k}]$ modulo the relations

$$\det(c_{1+j-i})_{1\leqslant i,j\leqslant r}=0, \qquad n-k+1\leqslant r\leqslant n+k$$

and

$$c_r^2+2\sum_{i=1}^{r}(-1)^i c_{r+i}c_{r-i}=0, \qquad k+1\leqslant r\leqslant n.$$

However, a presentation with integer coefficients using the special Schubert classes τ_p is more difficult to obtain. Let $\delta_p=1$, if $p\leqslant k$, and $\delta_p=2$, otherwise.

Theorem 12. *The cohomology ring* $H^*(OG(n-k,2n+1),\mathbb{Z})$ *is presented as a quotient of the polynomial ring* $\mathbb{Z}[\tau_1,\ldots,\tau_{n+k}]$ *modulo the relations*

$$\det(\delta_{1+j-i}\tau_{1+j-i})_{1\leqslant i,j\leqslant r}=0, \qquad n-k+1\leqslant r\leqslant n,$$

$$\sum_{p=k+1}^{r}(-1)^p\tau_p\det(\delta_{1+j-i}\tau_{1+j-i})_{1\leqslant i,j\leqslant r-p}=0, \qquad n+1\leqslant r\leqslant n+k,$$

and

$$\tau_r^2+\sum_{i=1}^{r}(-1)^i\delta_{r-i}\tau_{r+i}\tau_{r-i}=0, \qquad k+1\leqslant r\leqslant n.$$

5.2. **The quantum theory.** Given a degree $d\geqslant 0$ and partition pairs $\Lambda,M,N\in\mathcal{P}(k,n)$ such that $|\Lambda|+|M|+|N|=\dim(OG)+d(n+k)$, we define the Gromov-Witten invariant $\langle\tau_\Lambda,\tau_M,\tau_N\rangle_d$ as in Section 3.1. The degree of q in the quantum cohomology ring $QH^*(OG)$ is equal to $n+k$, and the quantum product is defined as usual. A notable feature of the quantum Pieri rule for OG is that it involves q^2 terms as well as q terms. To formulate it, we require some additional notation.

Let $\mathcal{P}'(k,n+1)$ be the set of $(\beta\,|\,\mu)\in\mathcal{P}(k,n+1)$ such that $\beta_1=n+1-k$ and $\max(\beta_2+k-1,1)\leqslant\mu_1\leqslant n$. For any $M=(\beta\,|\,\mu)\in\mathcal{P}'(k,n+1)$, define a partition pair $\widetilde{M}\in\mathcal{P}(k,n)$ by

$$\widetilde{(\beta\,|\,\mu)}=(\mu_1-k,\beta_2-1,\ldots,\beta_k-1\,|\,\mu_2,\mu_3,\ldots).$$

In other words, $\widetilde{M}$ is obtained from $(\beta\,|\,\mu)$ by removing a hook of length n from β and the entire first row of μ, then adding the part μ_1-k to what remains in the top partition. Recall also the operation $M\mapsto M^*$ from Section 4.2.

Theorem 13. *For any* $\Lambda=(\alpha\,|\,\lambda)\in\mathcal{P}(k,n)$ *and* p *with* $1\leqslant p\leqslant n+k$*, the quantum product* $\tau_\Lambda\cdot\tau_p\in QH^*(OG(n-k,2n+1))$ *is equal to*

$$\sum_{M\in\mathcal{P}(k,n)}2^{N'(\Lambda,M)}\,\tau_M+\sum_{M\in\mathcal{P}'(k,n+1)}2^{N'(\Lambda,M)}\,\tau_{\widetilde{M}}\,q+\sum_{M\in\mathcal{Q}(k,n)}2^{N'(\Lambda^*,M)}\,\tau_{M^*}\,q^2,$$

where (i) the first sum is classical, as in (15), (ii) the second sum is over $M\in\mathcal{P}'(k,n+1)$ *with* $\Lambda\xrightarrow{p}M$*, and (iii) the third sum is empty unless* $\lambda_1=n$*, and over* $M\in\mathcal{Q}(k,n)$ *such that* $\Lambda^*\xrightarrow{p}M$*.*

The q terms in Theorem 13 are explained by a phenomenon similar to what happens in type A (Section 2), while the q^2 terms are in analogy with the maximal orthogonal Grassmannian (Section 3.2); a dimension count shows that there are no higher degree contributions. For the line numbers, we observe that the parameter space of lines on OG is the orthogonal two-step flag variety $Y_1 = OF(n-k-1, n-k+1; 2n+1)$. It follows that

$$\langle \tau_\Lambda, \tau_M, \tau_p \rangle_1 = \int_{Y_1} \tau_\Lambda^{(1)} \cdot \tau_M^{(1)} \cdot \tau_p^{(1)}$$

where $\tau_\Lambda^{(1)}$, $\tau_M^{(1)}$, and $\tau_p^{(1)}$ are the associated Schubert classes in $H^*(Y_1, \mathbb{Z})$. The relevant diagram of smooth projections here is

$$\begin{array}{ccc} OF(n-k, n-k+1; 2n+1) & \xrightarrow{\varphi_1} & OG(n-k, 2n+1) \\ \big\downarrow \pi & & \\ OG(n-k+1, 2n+1). & & \end{array}$$

We associate to any partition pair $\Lambda = (\alpha \,|\, \lambda)$ in $\mathcal{P}(k,n)$ the pair $\overline{\Lambda} \in \mathcal{P}(k-1,n)$ obtained by removing the first part α_1 of the top partition:

$$\overline{(\alpha \,|\, \lambda)} = (\alpha_2, \dots, \alpha_k \,|\, \lambda).$$

One checks that $\pi(\varphi_1^{-1}(X_\Lambda(F_\bullet))) = X_{\overline{\Lambda}}(F_\bullet)$, for any $\Lambda \in \mathcal{P}(k,n)$. By arguing as in Section 2, we deduce that for $\Lambda, M \in \mathcal{P}(k,n)$ and $1 \leqslant p \leqslant n+k$, if $|\Lambda| + |M| + p = \dim OG + n + k$, then

$$\langle \tau_\Lambda, \tau_M, \tau_p \rangle_1 = \int_{OG(n-k+1, 2n+1)} \tau_{\overline{\Lambda}} \cdot \tau_{\overline{M}} \cdot \tau_{p-1}. \tag{16}$$

This result characterizes the degree 1 quantum correction terms in Theorem 13.

To understand the q^2 terms that appear in the quantum Pieri rule, one uses the relation $\tau_{n+k}^2 = q^2$ and follows the model of $OG(n, 2n+1)$, which is isomorphic to $OG(n+1, 2n+2)$. Note that the correct analogy is that the variable $q \in QH^*(OG(n, 2n+1))$ corresponds to $q^2 \in QH^*(OG(n-k, 2n+1))$, when $k \geqslant 1$. This is explained by degree considerations: on non-maximal orthogonal Grassmannians OG, we have $\deg(q) = n+k$, which specializes under $k=0$ to n, or half of the degree of q on $OG(n, 2n+1)$. It is also related to the degree doubling phenomenon, whereby lines on the maximal orthogonal Grassmannian are mapped to conics in projective space under the Plücker embedding.

Example 5. In the ring $QH^*(OG(2,9))$, we have

$$\tau_1 \cdot \tau_{(2,2|4,3)} = \tau_{(2,1|3)}\, q + 2\, q^2.$$

The term $2\,q^2$ is explained by the enumerative fact that there is a unique conic on $OG(2,9)$ passing through two general points. This conic meets a general hyperplane section (encoded by the class τ_1) in 2 points, and each pair, consisting of the conic together with a point of intersection with the given hyperplane section, contributes to the coefficient of q^2.

Theorem 14. *The quantum cohomology ring $QH^*(OG,\mathbb{Z})$ is presented as a quotient of the polynomial ring $\mathbb{Z}[\tau_1,\ldots,\tau_{n+k},q]$ modulo the relations*

$$\det(\delta_{1+j-i}\tau_{1+j-i})_{1\leqslant i,j\leqslant r}=0,\qquad n-k+1\leqslant r\leqslant n,$$

$$\sum_{p=k+1}^{r}(-1)^p\tau_p\det(\delta_{1+j-i}\tau_{1+j-i})_{1\leqslant i,j\leqslant r-p}=0,\qquad n+1\leqslant r<n+k,$$

$$\sum_{p=k+1}^{n+k}(-1)^p\tau_p\det(\delta_{1+j-i}\tau_{1+j-i})_{1\leqslant i,j\leqslant n+k-p}=q$$

and

$$\tau_r^2+\sum_{i=1}^{r}(-1)^i\delta_{r-i}\tau_{r+i}\tau_{r-i}=0,\qquad k+1\leqslant r\leqslant n.$$

6. The even orthogonal Grassmannian $OG(n+1-k,2n+2)$

6.1. **The classical theory.** In this section, we take the vector space $V=\mathbb{C}^{2n+2}$ equipped with a nondegenerate symmetric form, and consider the Grassmannian $OG'=OG(n+1-k,2n+2)$ which parametrizes the $(n+1-k)$-dimensional isotropic subspaces in V. The dimension of OG' is equal to $(n+1-k)(n+3k)/2$.

To define the Schubert varieties in OG', we require the Weyl group $\widetilde{W}_{n+1}$ of type D_{n+1}. This group is an extension of S_{n+1} by an element $s_\Box$ which acts by

$$(u_1,u_2,\ldots,u_{n+1})s_\Box=(\overline{u}_2,\overline{u}_1,u_3,\ldots,u_{n+1}).$$

If $\widetilde{W}_k$ is the subgroup of $\widetilde{W}_{n+1}$ generated by $\{s_i\mid i\neq k\}$, then the set $\widetilde{W}^{(k)}\subset\widetilde{W}_{n+1}$ of minimal length coset representatives of $\widetilde{W}_k$ parametrizes the Schubert varieties in $OG(n+1-k,2n+2)$. The set $\widetilde{W}^{(k)}$ consists of barred permutations of the form

$$w=w_{u,\lambda}=(\widehat{u}_k,\ldots,u_1,\overline{z}_1,\ldots,\overline{z}_r,v_{n+1-k-r},\ldots,v_1)\tag{17}$$

where $z_1>\cdots>z_r$ with $r\leqslant n-k$, $u_k<\cdots<u_1$, and $v_{n+1-k-r}<\cdots<v_1$. The convention here is that $\widehat{u}_k$ is equal to u_k or $\overline{u}_k$, according to the parity of r.

The elements in $\widetilde{W}^{(k)}$ correspond to a set $\widetilde{\mathcal{P}}(k,n)$ of partition pairs, which involve a partition $\alpha\in\mathcal{R}(k,n+1-k)$ and a $\lambda\in\mathcal{D}_n$ such that $\alpha_k\geqslant\ell(\lambda)$. More precisely, $\widetilde{\mathcal{P}}(k,n)$ consists of (i) pairs $(\alpha\,|\,\lambda)$ such that $\alpha_k=\ell(\lambda)$; these correspond to elements

$$w_{(\alpha|\lambda)}=(\widehat{u}_k,\ldots,u_1,\overline{\lambda_1+1},\ldots,\overline{\lambda_\ell+1},v_{n+1-k-\ell},\ldots,v_1)$$

in $\widetilde{W}_{n+1}$; (ii) two types of pairs $(\alpha\,|\,\lambda)$ and $(\alpha\,|\,\lambda]$ such that $\alpha_k>\ell(\lambda)$; these correspond to the Weyl group elements

$$w_{(\alpha|\lambda)}=(\widehat{u}_k,\ldots,u_1,\overline{\lambda_1+1},\ldots,\overline{\lambda_\ell+1},v_{n+1-k-\ell},\ldots,v_1)$$

and

$$w_{(\alpha|\lambda]}=(\widehat{u}_k,\ldots,u_1,\overline{\lambda_1+1},\ldots,\overline{\lambda_\ell+1},\overline{1},v_{n-k-\ell},\ldots,v_1),$$

respectively. We refer to the objects in this case as partition pairs of *type 1* and of *type 2*, respectively, and let $\mathrm{type}(\Lambda)\in\{1,2\}$ denote the type of a partition pair Λ. If Λ falls under case (i), then we set $\mathrm{type}(\Lambda)=0$. In both cases (i) and (ii), the equalities

$$\alpha_i=u_i+i-k-1+\#\{j\mid\lambda_j+1>u_i\}$$

for $1\leqslant i\leqslant k$ determine the numbers u_i, while $v_{n+1-k-\ell}<\cdots<v_1$. The weight of a partition pair $(\alpha\,|\,\lambda)$ or $(\alpha\,|\,\lambda]$ equals $|\alpha|+|\lambda|$, as before.

The Schubert varieties in OG' are defined using a monomorphism $\varphi : \widetilde{W}_{n+1} \hookrightarrow S_{2n+2}$ whose image consists of those permutations $\sigma \in S_{2n+2}$ such that $\sigma(i) + \sigma(2n+3-i) = 2n+3$ for all i and the number of $i \leqslant n+1$ such that $\sigma(i) > n+1$ is even. The map φ is defined by the equation

$$\varphi(w)(i) = \begin{cases} n+2-w_{n+2-i} & \text{if } w_{n+2-i} \text{ is unbarred,} \\ n+1+\overline{w}_{n+2-i} & \text{otherwise,} \end{cases}$$

for $w = (w_1, \ldots, w_{n+1}) \in \widetilde{W}_{n+1}$ and $1 \leqslant i \leqslant n+1$. Choose also an isotropic flag $F_\bullet$ in V, and extend it to a complete flag of V by setting $F_{n+1+p} = F_{n+1-p}^{\perp}$ for $1 \leqslant p \leqslant n+1$.

Set $\iota(w) = 0$ if $\overline{1}$ is a part of w, and $\iota(w) = 1$ otherwise, and define $\widetilde{\varphi} : \widetilde{W}_{n+1} \hookrightarrow S_{2n+2}$ by

$$\widetilde{\varphi}(w) = s_{n+1}^{\iota(w)} \varphi(w).$$

The map $\widetilde{\varphi}$ is a modification of φ so that in the sequence of values of $\widetilde{\varphi}(w)$, $n+2$ always comes before $n+1$. We need also the alternate complete flag $\widetilde{F}_\bullet$, with $\widetilde{F}_i = F_i$ for $i \leqslant n$ but completed with a maximal isotropic subspace $\widetilde{F}_{n+1}$ in the opposite family from F_{n+1} (thus we have $\widetilde{F}_{n+1} \cap F_{n+1} = F_n$). Define

$$F_\bullet^\iota = \begin{cases} F_\bullet & \text{if } n \neq \iota \pmod 2, \\ \widetilde{F}_\bullet & \text{if } n = \iota \pmod 2. \end{cases}$$

For $w \in \widetilde{W}^{(k)}$, the Schubert variety $X_w(F_\bullet)$ is defined as the locus of isotropic Σ in OG' such that

$$\dim(\Sigma \cap F_i^{\iota(w)}) \geqslant \#\{\, p \leqslant n+1-k \mid \widetilde{\varphi}(w)(p) > 2n+2-i \,\}$$

for $1 \leqslant i \leqslant 2n+2$. We let $\tau_\Lambda \in H^{2|\Lambda|}(OG', \mathbb{Z})$ denote the cohomology class determined by the Schubert variety indexed by w_Λ for an element $\Lambda \in \widetilde{\mathcal{P}}(k, n)$.

The *special* Schubert varieties for $OG(n+1-k, 2n+2)$ are the varieties

$$X_p = X_{(1^{\min(p,k)} | \max(p-k, 0))}, \quad 1 \leqslant p \leqslant n+k, \quad \text{and} \quad X'_k = X_{(1^k|0]}.$$

These are defined by a single Schubert condition as follows. For $p \neq k$, we have

$$X_p(F_\bullet) = \{\, \Sigma \in OG' \mid \Sigma \cap F_{\varepsilon(p)} \neq 0 \,\}$$

where $\varepsilon(p)$ is given by (14). If n is even, then

$$X_k(F_\bullet) = \{\, \Sigma \in OG' \mid \Sigma \cap F_{n+1} \neq 0 \,\}$$

and

$$X'_k(F_\bullet) = \{\, \Sigma \in OG' \mid \Sigma \cap \widetilde{F}_{n+1} \neq 0 \,\},$$

while the roles of F_{n+1} and $\widetilde{F}_{n+1}$ are switched if n is odd. We let τ_p denote the cohomology class of $X_p(F_\bullet)$ and τ'_k denote the cohomology class of $X'_k(F_\bullet)$.

The Pieri rule for OG' requires a slightly different shifting convention than that used for IG and OG. The shifted diagram of a partition pair $(\alpha \mid \lambda)$ or $(\alpha \mid \lambda]$ is obtained by shifting the ith part of α to the right by $k-i+1$ units, and the ith part of λ to the right by i units. Using this convention, the notation $\Lambda \xrightarrow{p} M$ and the multiplicity $N'(\Lambda, M)$ are defined in exactly the same way as in Theorem 11. Given two partition pairs Λ and M which both fall in case (ii), above, we set

$$\epsilon_{\Lambda M} = \begin{cases} 1 & \text{if } \mathrm{type}(\Lambda) = \mathrm{type}(M) \text{ and } \ell(\lambda) = \ell(\mu) \\ 1 & \text{if } \mathrm{type}(\Lambda) \neq \mathrm{type}(M) \text{ and } \ell(\lambda) \neq \ell(\mu) \\ 0 & \text{otherwise.} \end{cases}$$

In addition, let

$$h(\Lambda, M) = \#\{i \mid \beta_i \leqslant \alpha_i\} + \ell(\lambda) + \begin{cases} 1 & \text{if } \mathrm{type}(\Lambda) = 2 \text{ or } \mathrm{type}(M) = 2 \\ 0 & \text{otherwise.} \end{cases}$$

If $p \neq k$, then set $\delta_{\Lambda M} = 1$. If $p = k$ and $N'(\Lambda, M) > 0$, then set

$$\delta_{\Lambda M} = \delta'_{\Lambda M} = 1/2,$$

while if $N'(\Lambda, M) = 0$, define

$$\delta_{\Lambda M} = \begin{cases} 1 & \text{if } h(\Lambda, M) \text{ is even} \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \delta'_{\Lambda M} = \begin{cases} 1 & \text{if } h(\Lambda, M) \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 15. *Let $\Lambda = (\alpha \,|\, \lambda)$ or $\Lambda = (\alpha \,|\, \lambda]$ be an element of $\widetilde{\mathcal{P}}(k, n)$ and $p \geqslant 1$. If $\ell(\lambda) = \alpha_k$, then*

$$\tau_\Lambda \, \tau_p = \sum_M 2^{N'(\Lambda,M)} \delta_{\Lambda M} \tau_M,$$

while if $\ell(\lambda) < \alpha_k$, then

$$\tau_\Lambda \, \tau_p = \sum_{\ell(\mu) = \beta_k} 2^{N'(\Lambda,M)} \delta_{\Lambda M} \tau_M + \sum_{\ell(\mu) < \beta_k} 2^{N'(\Lambda,M)} \delta_{\Lambda M} \epsilon_{\Lambda M} \tau_M,$$

where the sums are over all $M = (\beta \,|\, \mu)$ and $M = (\beta \,|\, \mu]$ in $\widetilde{\mathcal{P}}(k, n)$ with $\Lambda \xrightarrow{p} M$ and satisfying the indicated conditions. Furthermore, the product $\tau_\Lambda \, \tau'_k$ is obtained by replacing $\delta_{\Lambda M}$ with $\delta'_{\Lambda M}$ throughout.

Let

$$0 \to S \to V_X \to Q \to 0$$

denote the tautological exact sequence of vector bundles over OG'. One checks as in Section 3.2 that

$$c_p(Q) = \begin{cases} \tau_p & \text{if } p < k, \\ \tau_k + \tau'_k & \text{if } p = k, \\ 2\tau_p & \text{if } p > k. \end{cases} \tag{18}$$

For each $r > 0$, let Δ_r denote the $r \times r$ Schur determinant

$$\Delta_r = \det(c_{1+j-i})_{1 \leqslant i,j \leqslant r},$$

where each variable c_p represents the Chern class $c_p(Q)$. Using rational coefficients, the ring $H^*(OG', \mathbb{Q})$ is presented as a quotient of the polynomial ring $\mathbb{Q}[c_1, \ldots, c_{n+k}, \xi]$ modulo the relations

$$\Delta_r = 0, \qquad n - k + 1 < r \leqslant n + k, \tag{19}$$

$$\xi \, \Delta_{n+1-k} = 0, \tag{20}$$

and

$$c_k^2 + 2\sum_{i=1}^{k}(-1)^i c_{k+i}c_{k-i} = \xi^2, \tag{21}$$

$$c_r^2 + 2\sum_{i=1}^{r}(-1)^i c_{r+i}c_{r-i} = 0, \qquad k+1 \leqslant r \leqslant n. \tag{22}$$

In the above relations, the variable ξ represents the difference $\tau_k - \tau'_k$ of the two special Schubert classes in codimension k.

The new relations (20) and (21) both come from the cohomology of the quadric $\mathcal{Q} \subset P(V)$. As in Section 3.2, let $\pi : P(S) \to OG'$ and $\theta : P(S) \to \mathcal{Q}$ denote the natural morphisms. We then have

$$\xi = \tau_k - \tau'_k = \pi_*(\theta^*(e-f)),$$

where e and f are the two ruling classes in $H^{2n}(\mathcal{Q},\mathbb{Z})$. If $\zeta = c_1(\mathcal{O}_{P(S)}(1))$, the projective bundle formula dictates

$$\zeta^{n+1-k} + c_1(\pi^*S)\zeta^{n-k} + \cdots + c_{n+1-k}(\pi^*S) = 0. \tag{23}$$

The relations in the cohomology ring of $\mathcal{Q}$ imply that $\theta^*(e-f)\zeta = \theta^*(eh-fh) = 0$, and hence $\theta^*(e-f)c_{n+1-k}(\pi^*S) = 0$ in $H^*(P(S),\mathbb{Z})$, using (23). Applying π_*, we see that $(\tau_k - \tau'_k)c_{n+1-k}(S) = 0$ in $H^*(OG',\mathbb{Z})$, which is exactly relation (20). We leave the proof of (21) as an exercise for the reader.

As in Section 5.1, the analogous presentation of $H^*(OG',\mathbb{Z})$ with integer coefficients is rather more involved.

Theorem 16. *Define polynomials c_p using the equations (18). Then the cohomology ring $H^*(OG(n+1-k,2n+2),\mathbb{Z})$ is presented as a quotient of the polynomial ring $\mathbb{Z}[\tau_1,\ldots,\tau_k,\tau'_k,\tau_{k+1},\ldots,\tau_{n+k}]$ modulo the relations*

$$\Delta_r = 0, \qquad n-k+1 < r \leqslant n,$$

$$\tau_k\Delta_{n+1-k} = \tau'_k\Delta_{n+1-k} = \sum_{p=k+1}^{n+1}(-1)^{p+k+1}\tau_p\Delta_{n+1-p},$$

$$\sum_{p=k+1}^{r}(-1)^p\tau_p\Delta_{r-p} = 0, \qquad n+1 < r \leqslant n+k,$$

and

$$\tau_k\tau'_k + \sum_{i=1}^{k}(-1)^i\tau_{k+i}\tau_{k-i} = 0,$$

$$\tau_r^2 + \sum_{i=1}^{r}(-1)^i\tau_{r+i}c_{r-i} = 0, \qquad k+1 \leqslant r \leqslant n.$$

6.2. **The quantum theory.** The story here is rather similar to that for the odd orthogonal Grassmannian of Section 5. The degree of q in the ring $QH(OG')$ is equal to $n+k$, and the quantum Pieri rule will involve both q and q^2 terms. Since $OG(n,2n+2)$ is not a quotient of SO_{2n+2} by a *maximal* parabolic subgroup, we will assume that $k \geqslant 2$ in this subsection.

Suppose that the classical Pieri rule for $OG(n+1-k,2n+2)$ is given by

$$\tau_\Lambda \tau_p = \sum_M f(\Lambda, M)\tau_M, \qquad \tau_\Lambda \tau'_k = \sum_M f'(\Lambda, M)\tau_M, \tag{24}$$

where the sums are over all $M \in \widetilde{\mathcal{P}}(k,n)$, $1 \leqslant p \leqslant n+k$, and the coefficients $f(\Lambda, M)$ and $f'(\Lambda, M)$ are given as in Theorem 15.

Let $\widetilde{\mathcal{Q}}(k,n)$ denote the set of $M = (\beta \,|\, \mu)$ (or $M = (\beta \,|\, \mu]$) in $\widetilde{\mathcal{P}}(k,n)$ such that $\mu_1 = n$. For a partition pair $M \in \widetilde{\mathcal{P}}(k,n)$, the pair M^* is defined using the same prescription as in Section 4.2. Let $\widetilde{\mathcal{P}}'(k,n+1)$ be the set of pairs $(\beta \,|\, \mu)$ or $(\beta \,|\, \mu]$ in $\widetilde{\mathcal{P}}(k,n+1)$ such that $\beta_1 = n+2-k$ and $\max(\beta_2+k-2,1) \leqslant \mu_1 \leqslant n$. For any $M \in \widetilde{\mathcal{P}}'(k,n+1)$, define a partition pair $\widetilde{M} \in \widetilde{\mathcal{P}}(k,n)$ by the equations

$$\widetilde{(\beta \,|\, \mu)} = (\mu_1 - k + 1, \beta_2 - 1, \ldots, \beta_k - 1 \,|\, \mu_2, \mu_3, \ldots);$$
$$\widetilde{(\beta \,|\, \mu]} = (\mu_1 - k + 1, \beta_2 - 1, \ldots, \beta_k - 1 \,|\, \mu_2, \mu_3, \ldots].$$

Theorem 17. *For any $\Lambda = (\alpha \,|\, \lambda)$ or $\Lambda = (\alpha \,|\, \lambda]$ in $\widetilde{\mathcal{P}}(k,n)$ and p with $1 \leqslant p \leqslant n+k$, the quantum product $\tau_\Lambda \cdot \tau_p \in QH^*(OG(n+1-k,2n+2))$ is equal to*

$$\sum_{M \in \widetilde{\mathcal{P}}(k,n)} f(\Lambda, M)\, \tau_M + \sum_{M \in \widetilde{\mathcal{P}}'(k,n+1)} f(\Lambda, M)\, \tau_{\widetilde{M}}\, q + \sum_{M \in \widetilde{\mathcal{Q}}(k,n)} f(\Lambda^*, M)\, \tau_{M^*}\, q^2,$$

where (i) the first sum is classical, as in (24), (ii) the second sum is over $M \in \widetilde{\mathcal{P}}'(k,n+1)$ with $\Lambda \xrightarrow{p} M$, and (iii) the third sum is empty unless $\lambda_1 = n$, and over $M \in \widetilde{\mathcal{Q}}(k,n)$ such that $\Lambda^* \xrightarrow{p} M$. Furthermore, the product $\tau_\Lambda \cdot \tau'_k$ is obtained by replacing f with f' throughout.

Theorem 17 is proved using the same set of arguments used to obtain Theorem 13. For the terms which are linear in q, we consider the diagram

$$\begin{array}{ccc} OF(n+1-k,n+2-k;2n+2) & \longrightarrow & OG(n+1-k,2n+2) \\ \downarrow & & \\ OG(n+2-k,2n+2) & & \end{array}$$

and apply the projection formula as in Sections 2 and 5.2. The q^2 terms in the quantum Pieri rule for OG' come from the basic relation $\tau_{n+k}^2 = q^2$.

Example 6. The quantum Pieri relations

$$\tau_2 \cdot \tau_{(2,2|1)} = \tau_2\, q \quad \text{and} \quad \tau'_2 \cdot \tau_{(2,2|1)} = \tau_{(2,2|3)}$$

hold in $QH^*(OG(2,8))$. Also, in $QH^*(OG(3,10))$, we have

$$\tau_2 \cdot \tau_{(3,2|4,2)} = \tau_{(3,3|4,3]} + \tau_{(3,3|4,2,1)} + \tau_{(3,2|2)}\, q + \tau_{(3,1|3)}\, q + \tau_1\, q^2;$$
$$\tau'_2 \cdot \tau_{(3,2|4,2)} = \tau_{(3,3|4,3)} + \tau_{(3,3|4,2,1)} + \tau_{(3,2|2]}\, q + \tau_{(3,1|3)}\, q + \tau_1\, q^2.$$

We conclude with a presentation of the ring $QH^*(OG(n+1-k,2n+2))$.

Theorem 18. *The quantum cohomology ring $QH^*(OG')$ is presented as a quotient of the polynomial ring $\mathbb{Z}[\tau_1, \ldots, \tau_k, \tau'_k, \tau_{k+1}, \ldots, \tau_{n+k}, q]$ modulo the relations*

$$\Delta_r = 0, \qquad n-k+1 < r \leqslant n,$$

$$\tau_k \Delta_{n+1-k} = \tau'_k \Delta_{n+1-k} = \sum_{p=k+1}^{n+1} (-1)^{p+k+1} \tau_p \Delta_{n+1-p},$$

$$\sum_{p=k+1}^{r} (-1)^p \tau_p \Delta_{r-p} = 0, \qquad n+1 < r < n+k,$$

$$\sum_{p=k+1}^{n+k} (-1)^p \tau_p \Delta_{n+k-p} = -q,$$

and

$$\tau_k \tau'_k + \sum_{i=1}^{k} (-1)^i \tau_{k+i} \tau_{k-i} = 0,$$

$$\tau_r^2 + \sum_{i=1}^{r} (-1)^i \tau_{r+i} c_{r-i} = 0, \qquad k+1 \leqslant r \leqslant n,$$

where the polynomials c_p are defined by (18).

References

[Be] A. Bertram : *Quantum Schubert calculus*, Adv. in Math. **128** (1997), 289–305.

[Bo] A. Borel : *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math. **57** (1953), 115–207.

[Bu] A. Buch : *Quantum cohomology of Grassmannians*, Compositio Math. **137** (2003), 227–235.

[BKT1] A. Buch, A. Kresch and H. Tamvakis : *Gromov-Witten invariants on Grassmannians*, J. Amer. Math. Soc. **16** (2003), 901–915.

[BKT2] A. Buch, A. Kresch and H. Tamvakis : *Quantum Pieri rules for isotropic Grassmannians*, in preparation.

[Eh] C. Ehresmann : *Sur la topologie de certains espaces homogènes*, Ann. of Math. (2) **35** (1934), 396–443.

[Fu] W. Fulton : *Intersection Theory*, Second edition, Ergebnisse der Math. **2**, Springer-Verlag, Berlin, 1998.

[HB] H. Hiller and B. Boe : *Pieri formula for SO_{2n+1}/U_n and Sp_n/U_n*, Adv. in Math. **62** (1986), 49–67.

[HP] W.V.D. Hodge and D. Pedoe : *Methods of Algebraic Geometry*, Cambridge Univ. Press, Cambridge, 1952.

[KT1] A. Kresch and H. Tamvakis : *Quantum cohomology of the Lagrangian Grassmannian*, J. Algebraic Geom. **12** (2003), 777–810.

[KT2] A. Kresch and H. Tamvakis : *Quantum cohomology of orthogonal Grassmannians*, Compositio Math. **140** (2004), 482–500.

[Pi] M. Pieri : *Sul problema degli spazi secanti. Nota 1^a*, Rend. Ist. Lombardo (2) **26** (1893), 534–546.

[PR1] P. Pragacz and J. Ratajski : *A Pieri-type theorem for Lagrangian and odd orthogonal Grassmannians*, J. Reine Angew. Math. **476** (1996), 143–189.

[PR2] P. Pragacz and J. Ratajski : *A Pieri-type formula for even orthogonal Grassmannians*, Fund. Math. **178** (2003), 49–96.

[Se] S. Sertöz : *A triple intersection theorem for the varieties* $\mathrm{SO}(n)/P_d$, Fund. Math. **142** (1993), 201–220.

[ST] B. Siebert and G. Tian : *On quantum cohomology rings of Fano manifolds and a formula of Vafa and Intriligator*, Asian J. Math. **1** (1997), 679–695.

[T] H. Tamvakis : *Gromov-Witten invariants and quantum cohomology of Grassmannians*, preprint (2003), available at math.AG/0306415.

Department of Mathematics, Brandeis University - MS 050, P. O. Box 9110, Waltham, MA 02454, USA

E-mail address: harryt@brandeis.edu