QUANTUM PIERI RULES FOR ISOTROPIC GRASSMANNIANS

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Abstract. We study the three point genus zero Gromov-Witten invariants on the Grassmannians which parametrize non-maximal isotropic subspaces in a vector space equipped with a nondegenerate symmetric or skew-symmetric form. We establish Pieri rules for the classical cohomology and the small quantum cohomology ring of these varieties, which give a combinatorial formula for the product of any Schubert class with certain special Schubert classes. We also give presentations of these rings, with integer coefficients, in terms of special Schubert class generators and relations.

0. Introduction

Let G be a classical Lie group of type B, C, or D, and P any maximal parabolic subgroup of G. The homogeneous space $X = G/P$ is a Grassmannian which parametrizes isotropic subspaces in a vector space equipped with a nondegenerate symmetric or skew-symmetric bilinear form. The small quantum cohomology ring $\mathrm{QH}(X)$ is a deformation of the cohomology ring $\mathrm{H}^*(X,\mathbb{Z})$, defined using structure constants given by the (three point, genus zero) Gromov-Witten invariants of X. The ring $\mathrm{QH}(X)$ is generated as an algebra over $\mathbb{Z}[q]$ by certain special Schubert classes, in a similar fashion to the ordinary cohomology ring of X, which is recovered when the formal variable q is set equal to zero. The main purpose of this paper is to formulate and prove a *quantum Pieri rule*, a combinatorial formula which describes the product of a special Schubert class with an arbitrary one in the quantum cohomology ring.

A quantum Pieri rule for the usual type A Grassmannian was proved by Bertram [Be], while corresponding rules for the Grassmannians of maximal isotropic subspaces were established by the second and third authors [KT1, KT2]. We deal here with the remaining cases, where X is a non-maximal isotropic symplectic or orthogonal Grassmannian. Our results are applied to obtain presentations of $\mathrm{QH}(X)$ with integer coefficients in terms of special Schubert class generators and relations.

One notable feature is that the quantum Pieri rule in the orthogonal cases involves quadratic q terms, which is a new phenomenon for Grassmannians. Algebraically, it is simplest to understand the transition between the maximal and non-maximal isotropic cases in type D, by considering the space $\mathrm{OG}(n-k,2n)$ of isotropic $(n-k)$-dimensional subspaces of $\mathbb{C}^{2n}$, for $k \geq 0$. When $k=1$, the Picard group of this orthogonal Grassmannian has rank two. Therefore the quantum cohomology ring of $\mathrm{OG}(n-1,2n)$ involves two deformation parameters q_1 and q_2,

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whereas $\mathrm{OG}(n,2n)$ and $\mathrm{OG}(n-k,2n)$ for $k>1$ have only one deformation parameter q. Formally, the single quantum parameter on $\mathrm{OG}(n,2n)$ is replaced with two square roots on $\mathrm{OG}(n-1,2n)$, which in turn are identified on $\mathrm{OG}(n-2,2n)$.

The proofs of the quantum Pieri rules in this article use the kernel–span and dimension counting ideas of the first author [Bu] in an essential way; the earlier papers [Be, KT1, KT2] used intersection theory on certain Quot scheme compactifications of the moduli space of maps from the projective line to X. Furthermore, we continue the program set out in [BKT1] of equating the Gromov-Witten invariants on X (of any degree) with classical triple intersection numbers on related varieties X'. However the analysis here is considerably more subtle, as the parameter space X' of kernel–span pairs will no longer be a homogeneous space for the group G, in general. For instance, the parameter space of lines on an orthogonal Grassmannian is a two step orthogonal flag variety, but the analogous statement is false in the symplectic case, where additional geometric arguments are needed to relate the degree one Gromov-Witten invariants to classical Schubert structure constants (Proposition 1.2).

On the other hand, in degrees greater than one, the symplectic Grassmannians are much better behaved than the orthogonal ones, from this point of view. We are able to show that the function ϕ which sends a rational map (or curve) counted by a Gromov-Witten invariant to the pair consisting of its kernel and span is a bijection in the symplectic case, but for orthogonal Grassmannians the map ϕ is N to 1, where N is an explicitly determined power of two. Additional complications stem from the fact that for any two points on the maximal orthogonal Grassmannian $\mathrm{OG}(d,2d)$, the corresponding subspaces of $\mathbb{C}^{2d}$ must intersect in at least a line, if the integer d is *odd*; this makes it impossible to place three points of $\mathrm{OG}(d,2d)$ in "pairwise general position" in the sense of [BKT1, Prop. 4].

Our Pieri rules and presentations are new even for the classical cohomology ring $\mathrm{H}^*(X,\mathbb{Z})$. Over each Grassmannian X, there is a universal exact sequence of vector bundles

$$0 \to \mathcal{S} \to \mathcal{V}_X \to \mathcal{Q} \to 0$$

with $\mathcal{S}$ the tautological subbundle of the trivial bundle $\mathcal{V}_X$. Pragacz and Ratajski [PR1, PR2] proved Pieri-type rules for the product of an arbitrary Schubert class with the Chern classes of $\mathcal{S}^*$. An important part of our work was the discovery of new classical Pieri rules for multiplying with the Chern classes of $\mathcal{Q}$. Aside from being much simpler than the formulas in [PR1, PR2], the new Pieri rules are also essential for our applications to quantum cohomology. To be specific, we prove that any quantum product of $c_p(\mathcal{Q})$ with another Schubert class on a symplectic Grassmannian can be obtained from the product of these classes in the ordinary cohomology ring of a larger Grassmannian. However, we do not know such a relation for all products involving the Chern classes of $\mathcal{S}^*$ (see Example 1.3).

Our method for proving the new classical Pieri rules is completely different (and much shorter) than that of loc. cit. We use a geometric approach, following Hodge's classical proof of the Pieri rule for the usual Grassmannians via triple intersections [H]; see also [Se, So]. In type D, the Chern classes of the tautological bundles $\mathcal{S}$ and $\mathcal{Q}$ do not generate the cohomology ring $\mathrm{H}^*(X,\mathbb{Q})$, and thus a direct approach by intersecting Schubert cells seems necessary to obtain the full picture. We also note that the classical Borel presentation [Bo] of the cohomology ring of orthogonal Grassmannians requires rational coefficients, and uses a different set of generators.

Another innovation of this article is our parametrization of the Schubert varieties by *k-strict partitions*, a convention which makes their codimension apparent. This notation is most convenient when working algebraically, in the classical or quantum cohomology ring, and extends the traditional parametrization for type A and maximal isotropic Grassmannians. An underlying reason which vindicates this choice is our work on Giambelli-type formulas for these varieties [BKT2]; in fact, the latter project has affected the exposition here in more ways than one. After the first three sections we introduce a different parametrization of the Schubert varieties by *index sets*, which is closer to their geometric definition as the closures of Schubert cells. Our proof of the classical Pieri rules is in this latter language, where they are somewhat easier to state.

The aforementioned motivation for the new classical Pieri rules is the reason behind the *in medias res* organization of this paper, where we begin by studying the quantum cohomology rings and follow this with a more detailed look at the Schubert varieties, culminating with our proofs of the classical Pieri rules. The various sections are ordered according to the three different Lie types considered. The quantum cohomology of isotropic Grassmannians of type C, B, and D is examined in Sections 1, 2, and 3, respectively. In Section 4 we begin our study of the Schubert varieties from scratch, parametrizing them by index sets and relating these to k-strict partitions. Section 5 contains short proofs of our classical Pieri rules, using the language of index sets, and then translates these rules to the initial statements in terms of k-strict partitions. The reader who is willing to grant the classical Pieri rules can omit Section 5 entirely. Finally, we provide an appendix which discusses the quantum cohomology of the orthogonal Grassmannian $\mathrm{OG}(n, 2n+2)$. Most of the results of this paper were announced in the survey [T], which employed the older notation of Weyl group elements and partition pairs.

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1. The Grassmannian $\mathrm{IG}(n-k, 2n)$

1.1. **Schubert classes.** Fix a vector space $V \cong \mathbb{C}^{2n}$ with a non-degenerate skew-symmetric bilinear form $(\ ,\ )$, and fix a non-negative integer $m \leq n$. Throughout this section we let the symbol IG, without parameters, denote the Grassmannian $\mathrm{IG}(m, 2n)$ which parametrizes m-dimensional isotropic subspaces of V. This algebraic variety has dimension $2m(n-m)+m(m+1)/2$. The Schubert varieties in IG are described below; our indexing conventions for these varieties appear to be new, and we refer the reader to Section 4 for elementary proofs of the relevant facts.

A partition is a weakly decreasing sequence of non-negative integers $\lambda = (\lambda_1 \geq \ldots \geq \lambda_m \geq 0)$. We will identify a partition λ with its Young diagram of boxes, which has λ_i boxes in row i. The weight of λ is the sum $|\lambda| = \sum \lambda_i$ of its parts, and the length $\ell(\lambda)$ is the number of non-zero parts of λ. We set $\lambda_i = 0$ for $i > \ell(\lambda)$.

Definition 1.1. Let k be a non-negative integer. We say that the partition λ is *k-strict* if no part greater than k is repeated, i.e., $\lambda_j > k \Rightarrow \lambda_{j+1} < \lambda_j$.

Set $k = n - m$. The Schubert varieties in IG are indexed by k-strict partitions λ which are contained in an $m \times (n+k)$ rectangle; we denote the set of all such partitions by $\mathcal{P}(k,n)$. An isotropic flag in V is a complete flag $0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_{2n} = V$ of subspaces such that $F_{n+i} = F_{n-i}^{\perp}$ for each $0 \leq i \leq n$. The Schubert variety for any $\lambda \in \mathcal{P}(k,n)$ relative to the isotropic flag $F_\bullet$ is defined by

$$X_\lambda(F_\bullet) = \{\Sigma \in \mathrm{IG} \mid \dim(\Sigma \cap F_{p_j(\lambda)}) \geq j, \quad \forall\, 1 \leq j \leq \ell(\lambda)\}\,,$$

where

$$p_j(\lambda) = n + k + 1 - \lambda_j + \#\{i < j : \lambda_i + \lambda_j \leq 2k + j - i\}.$$

The codimension of this variety is equal to $|\lambda|$. We let $\sigma_\lambda = [X_\lambda] \in \mathrm{H}^{2|\lambda|}(\mathrm{IG}) = \mathrm{H}^{2|\lambda|}(\mathrm{IG},\mathbb{Z})$ denote the cohomology class Poincaré dual to the cycle determined by $X_\lambda(F_\bullet)$. These Schubert classes form a $\mathbb{Z}$-basis for the cohomology ring of IG.

The k-strict partition λ has a unique dual partition $\lambda^\vee \in \mathcal{P}(k,n)$, for which $p_j(\lambda^\vee) = 2n + 1 - p_{m+1-j}(\lambda)$ for $1 \leq j \leq m$. With this notation we have

$$\int_{\mathrm{IG}} \sigma_\lambda \cdot \sigma_\mu = \delta_{\mu,\lambda^\vee}\,.$$

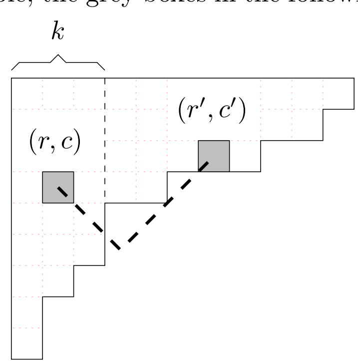
1.2. **Classical Pieri rule.** The Schubert varieties $X_p(F_\bullet) = X_{(p)}(F_\bullet)$ for $1 \leq p \leq n+k$ are called special, and are defined by the single Schubert condition $\Sigma \cap F_{n+k+1-p} \neq 0$. Consider the exact sequence of vector bundles over IG

$$0 \to \mathcal{S} \to \mathcal{V}_{\mathrm{IG}} \to \mathcal{Q} \to 0,$$

where $\mathcal{V}_{\mathrm{IG}}$ denotes the trivial bundle of rank $2n$ and $\mathcal{S}$ is the tautological subbundle of rank m. The special Schubert class $\sigma_p = [X_p(F_\bullet)]$ is equal to the Chern class $c_p(\mathcal{Q})$ of the tautological quotient bundle on IG. These classes generate the cohomology ring of IG. When $m < n$, one also has a set of special classes $\sigma_{(1^p)}$, for $1 \leq p \leq m$, which are the Chern classes $c_p(\mathcal{S}^*)$. These latter classes form a different set of generators of $\mathrm{H}^*(\mathrm{IG})$.

When $m = n$ and $\mathrm{IG} = \mathrm{LG}(n,2n)$ is the Lagrangian Grassmannian of maximal isotropic subspaces, Hiller and Boe proved a Pieri rule for any product involving a special class σ_p [HB]. For $m < n$, Pragacz and Ratajski have proved a more general formula for multiplying with the classes $\sigma_{(1^p)}$ [PR1]. Our first result is a new Pieri rule for products involving the special classes $\sigma_p = c_p(\mathcal{Q})$ for general isotropic Grassmannians. This rule is easier to state than the rule proved in [PR1]. Furthermore, our methods for obtaining quantum generalizations of the Pieri rules only work for the special classes σ_p (see Proposition 1.2 and Example 1.3).

Definition 1.2. Let λ be a k-strict partition. We will say that the box in row r and column c of λ is *k-related* to the box in row r' and column c' if $|c-k-1|+r = |c'-k-1|+r'$. For example, the grey boxes in the following partition are k-related.



The notion of k-related boxes is one of the ingredients of Pragacz and Ratajski's formula for products involving the special classes $\sigma_{(1^p)}$ [PR1].

Given two Young diagrams λ and μ with $\lambda \subset \mu$, the skew diagram μ/λ is called a horizontal (resp. vertical) strip if it does not contain two boxes in the same column (resp. row).

Definition 1.3. For any two k-strict partitions λ and μ, we have a relation $\lambda \to \mu$ if μ can be obtained by removing a vertical strip from the first k columns of λ and adding a horizontal strip to the result, so that

(1) if one of the first k columns of μ has the same number of boxes as the same column of λ, then the bottom box of this column is k-related to at most one box of $\mu \smallsetminus \lambda$; and

(2) if a column of μ has fewer boxes than the same column of λ, then the removed boxes and the bottom box of μ in this column must each be k-related to exactly one box of $\mu \smallsetminus \lambda$, and these boxes of $\mu \smallsetminus \lambda$ must all lie in the same row.

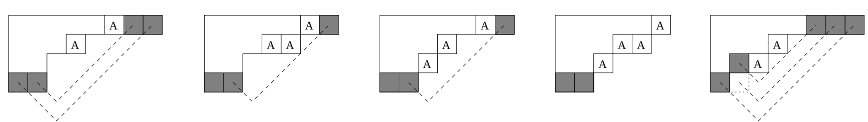
If $\lambda \to \mu$, we let $\mathbb{A}$ be the set of boxes of $\mu \smallsetminus \lambda$ in columns $k+1$ through $k+n$ which are *not* mentioned in (1) or (2). Then define $N(\lambda,\mu)$ to be the number of connected components of $\mathbb{A}$ which do not have a box in column $k+1$. Here two boxes are connected if they share at least a vertex.

Theorem 1.1 (Pieri rule for $\mathrm{IG}(m,2n)$)**.** *For any k-strict partition λ and integer $p \in [1, n+k]$, we have*

$$\sigma_p \cdot \sigma_\lambda = \sum_{\lambda \to \mu,\, |\mu| = |\lambda| + p} 2^{N(\lambda,\mu)}\, \sigma_\mu\,.$$

This theorem will be proved in Section 5.

Example 1.1. For the Grassmannian $\mathrm{IG}(4,12)$ we have $k=2$. For $\lambda = (5,3,2,2)$ we get the following shapes $\mu \in \mathcal{P}(2,6)$ such that $\lambda \to \mu$ and $|\mu| = |\lambda| + 4$:



By Theorem 1.1 we therefore obtain

$$\sigma_4 \cdot \sigma_\lambda = 4\sigma_{(8,4,2,2)} + 2\sigma_{(7,5,2,2)} + 2\sigma_{(7,4,3,2)} + \sigma_{(6,5,3,2)} + \sigma_{(8,4,3,1)}\,.$$

1.3. Presentation of $\mathrm{H}^*(\mathrm{IG})$.

For every partition λ, define a monomial σ^λ in the special Schubert classes by $\sigma^\lambda = \prod_i \sigma_{\lambda_i}$. We also use the convention that $\sigma_0 = 1$ and $\sigma_p = 0$ for $p < 0$ or $p > n+k$.

Theorem 1.2. a) *The cohomology ring $\mathrm{H}^*(\mathrm{IG},\mathbb{Z})$ is presented as a quotient of the polynomial ring $\mathbb{Z}[\sigma_1,\ldots,\sigma_{n+k}]$ modulo the relations*

$$\det(\sigma_{1+j-i})_{1\le i,j\le r} = 0\,, \qquad n-k+1 \le r \le n+k \tag{1}$$

and

$$\sigma_r^2 + 2\sum_{i=1}^{n+k-r} (-1)^i \sigma_{r+i}\sigma_{r-i} = 0\,, \qquad k+1 \le r \le n. \tag{2}$$

b) *The monomials σ^λ with $\lambda \in \mathcal{P}(k,n)$ form a $\mathbb{Z}$-basis for $\mathrm{H}^*(\mathrm{IG},\mathbb{Z})$.*

We prove Theorem 1.2 below based on the following lemma, which was shown to us by Miles Reid. It provides a method to obtain presentations of cohomology rings with integer coefficients which simplified our earlier proofs of Theorems 1.2, 2.2, and 3.2.

Lemma 1.1 (Reid). *Let $R = \mathbb{Z}[a_1, \ldots, a_d]$ be a free polynomial ring with homogeneous generators a_i, let $I \subset R$ be an ideal generated by homogeneous elements $b_1, \ldots, b_d \in R$, and let $\phi : R/I \to H$ be a surjective ring homomorphism. Assume that (i) H is a free $\mathbb{Z}$-module of rank $\prod_i (\deg b_i / \deg a_i)$, and (ii) for any field K, the K-vector space $(R/I) \otimes_{\mathbb{Z}} K$ has finite dimension. Then ϕ is an isomorphism.*

Proof. The second assumption implies that $(b_1, \ldots, b_d)$ is a regular sequence in $K[a_1, \ldots, a_d]$, for any field K. We deduce that R/I is flat as a $\mathbb{Z}$-module; this follows from [EGA, 0.15.1.16], which reduces the result to an application of the local criterion for flatness, as explained in [L, Lemma 4.3.16]. Since ϕ is a surjection from R/I onto a free $\mathbb{Z}$-module, the kernel of ϕ is also flat, so it is enough to show that $C = (R/I) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $H \otimes_{\mathbb{Z}} \mathbb{Q}$ have the same dimension as $\mathbb{Q}$-vector spaces. As the graded ring C is a complete intersection, its Hilbert series [Sta] is given by

$$\mathrm{H}(t) = \sum_{j=0}^{\infty} \dim(C_j)\, t^j = \prod_{i=1}^{d} \frac{1 - t^{\deg b_i}}{1 - t^{\deg a_i}}.$$

This is a polynomial which evaluates to $\prod_i (\deg b_i / \deg a_i)$ at $t = 1$. □

Lemma 1.2. *The quotient of the graded ring $\mathbb{Z}[a_1, \ldots, a_d]$ with $\deg a_i = i$ modulo the relations*

$$\det(a_{1+j-i})_{1 \le i,j \le r} = 0, \qquad m+1 \le r \le m+d$$

is a free $\mathbb{Z}$-module of rank $\binom{m+d}{d}$.

Proof. The displayed quotient ring is one of the standard presentations of the cohomology ring of the usual (type A) Grassmannian of m-dimensional subspaces of a vector space of dimension $m + d$. Moreover, the Euler characteristic of this Grassmannian equals $\binom{m+d}{d}$. □

Proof of Theorem 1.2. We write $\mu \succ \lambda$ if the partition μ strictly dominates λ, i.e. $\mu \neq \lambda$ and $\mu_1 + \cdots + \mu_i \geq \lambda_1 + \cdots + \lambda_i$ for each $i \geq 1$. It follows from the Pieri rule that $\sigma^\lambda = \sigma_\lambda + \sum_{\mu \succ \lambda} c_\mu \sigma_\mu$, where $c_\mu \in \mathbb{Z}$ and the sum is over partitions μ with $\mu \succ \lambda$. This implies (b). In particular, the special Schubert classes $\sigma_1, \ldots, \sigma_{n+k}$ generate the ring $\mathrm{H}^*(\mathrm{IG}, \mathbb{Z})$.

Set $R = \mathbb{Z}[a_1, \ldots, a_{n+k}]$, where a_i is a homogeneous variable of degree i, and let $\phi : R \to \mathrm{H}^*(\mathrm{IG}, \mathbb{Z})$ be the surjective ring homomorphism defined by $\phi(a_i) = \sigma_i$. We also write $a_0 = 1$ and $a_i = 0$ for $i < 0$ or $i > n + k$. For $r \geq 1$ we define

$$d_r = \det(a_{1+j-i})_{1 \le i,j \le r} \quad \text{and} \quad b_r = a_r^2 + 2 \sum_{i \geq 1} (-1)^i a_{r+i} a_{r-i}.$$

Let t be a formal variable. By expanding the determinant d_r along the top row, we obtain $d_r = a_1 d_{r-1} - a_2 d_{r-2} + \cdots + (-1)^{r-1} a_r$, which is equivalent to the identity of formal power series

$$\Big(\sum_{i=0}^{n+k} a_i t^i \Big) \Big(\sum_{i \geq 0} (-1)^i d_i t^i \Big) = 1. \tag{3}$$

The definition of b_r implies that

$$\sum_{i=0}^{n+k}(-1)^i b_i t^{2i} = \Big(\sum_{i=0}^{n+k} a_i t^i\Big)\Big(\sum_{i=0}^{n+k}(-1)^i a_i t^i\Big)\,. \tag{4}$$

Consider the ideal $I = (d_{m+1},\dots,d_{n+k},b_{k+1},\dots,b_n) \subset R$. We claim that $\phi(I) = 0$. Indeed, since $\sigma_i = c_i(\mathcal{Q})$ for each i, the Whitney sum formula $c_t(\mathcal{S})c_t(\mathcal{Q}) = 1$ and equation (3) imply that $\phi(d_r) = c_r(\mathcal{S}^*)$, so $\phi(d_r) = 0$ for $r > m$. Notice that the symplectic form gives a pairing $\mathcal{S}\otimes\mathcal{Q} \to \mathcal{O}$, which in turn produces an injection $\mathcal{S} \hookrightarrow \mathcal{Q}^*$. It therefore follows from (4) that

$$(-1)^r\phi(b_r) = c_{2r}(\mathcal{Q}\oplus\mathcal{Q}^*) = c_{2r}(\mathcal{V}_{\mathrm{IG}}/\mathcal{S}\oplus\mathcal{Q}^*) = c_{2r}(\mathcal{Q}^*/\mathcal{S}) = 0 \quad \text{for } r > k,$$

which proves the claim. Using the induced map $\phi : R/I \to \mathrm{H}^*(\mathrm{IG})$ we are left with checking (i) and (ii) of Lemma 1.1. Property (i) follows because $\deg(d_r) = r$, $\deg(b_r) = 2r$, and $\operatorname{rank}\mathrm{H}^*(\mathrm{IG}) = \#\mathcal{P}(k,n) = 2^m\binom{n}{k}$.

To prove (ii) it is enough to show that $d_r \in I$ for $n+k < r \leq 2n$, since this implies that R/I is a quotient of $R/(d_{m+1},\dots,d_{2n})$, which is a free $\mathbb{Z}$-module of finite rank by Lemma 1.2. It follows from (3) and (4) that

$$\sum_{i=0}^{n+k} a_i t^i = \Big(\sum_{i=0}^{n+k}(-1)^i b_i t^{2i}\Big)\Big(\sum_{i\geq 0} d_i t^i\Big),$$

which implies that

$$\sum_{i=0}^{n+k} a_i t^i \equiv \Big(\sum_{i=0}^{k}(-1)^i b_i t^{2i} + \sum_{i=n+1}^{n+k}(-1)^i b_i t^{2i}\Big)\Big(\sum_{i=0}^{m} d_i t^i + \sum_{i\geq n+k+1} d_i t^i\Big)$$

modulo the homogeneous ideal $I \subset R$. By equating terms of equal degrees in this congruence, we obtain $d_{n+k+1} \equiv d_{n+k+2} \equiv \cdots \equiv d_{2n} \equiv 0$ modulo I, as required. $\square$

1.4. **Gromov-Witten invariants.** A rational map of degree d to IG is a morphism of varieties $f : \mathbb{P}^1 \to \mathrm{IG}$ such that

$$\int_{\mathrm{IG}} f_*[\mathbb{P}^1]\cdot\sigma_1 = d\,,$$

i.e. d is the number of points in $f^{-1}(X_1(E_\bullet))$ when the isotropic flag $E_\bullet$ is in general position. All the Gromov-Witten invariants considered in this paper are three-point and genus zero. Given a degree $d \geq 0$ and k-strict partitions λ,μ,ν such that $|\lambda|+|\mu|+|\nu| = \dim(\mathrm{IG})+d(n+1+k)$, we define the Gromov-Witten invariant $\langle\sigma_\lambda,\sigma_\mu,\sigma_\nu\rangle_d$ to be the number of rational maps $f : \mathbb{P}^1 \to \mathrm{IG}$ of degree d such that $f(0) \in X_\lambda(E_\bullet)$, $f(1) \in X_\mu(F_\bullet)$, and $f(\infty) \in X_\nu(G_\bullet)$, for given isotropic flags $E_\bullet$, $F_\bullet$, and $G_\bullet$ in general position. As in [BKT1], we will show that these morphisms are maps to a Lagrangian Grassmannian $\mathrm{LG}(d,2d)$ contained in IG whose image curves pass through three general points of LG.

Following [Bu], we define the *kernel* of a rational map $f : \mathbb{P}^1 \to \mathrm{IG}$ as the intersection of all the subspaces $\Sigma_t = f(t)$ in the image of f, and the *span* of f as the linear span of these subspaces in V:

$$\operatorname{Ker}(f) = \bigcap \Sigma_t \subset V \quad\text{and}\quad \operatorname{Span}(f) = \sum \Sigma_t \subset V\,.$$

If f has degree d then $\dim\operatorname{Ker}(f) \geq m-d$ and $\dim\operatorname{Span}(f) \leq m+d$ (see [Bu, Lemma 1]). Notice that $\operatorname{Ker}(f) \subset \operatorname{Span}(f) \subset \operatorname{Ker}(f)^\perp$.

For any integer $d \leq m$, let Y_d be the variety parametrizing pairs (A,B) of subspaces of V such that $A \subset B \subset A^\perp$, $\dim A = m-d$, and $\dim B = m+d$. Since Y_d is a $\mathrm{G}(2d, 2k+2d)$-bundle over $\mathrm{IG}(m-d, 2n)$, we compute that $\dim(Y_d) = \dim(\mathrm{IG}) + d(n+1+k) - 3d(d+1)/2$. The algebraic group Sp_{2n} acts on the ambient vector space V, and hence also on Y_d, but this action is not transitive. Let T_d be the variety of triples (A, Σ, B) of subspaces of V such that $(A,B) \in Y_d$, $\Sigma \in \mathrm{IG}$, and $A \subset \Sigma \subset B$. Let $\pi : T_d \to Y_d$ be the projection.

For each k-strict partition λ, define a subvariety $Y_\lambda \subset Y_d$ by the prescription

$$Y_\lambda(E_\bullet) = \{(A,B) \in Y_d \mid \exists\, \Sigma \in X_\lambda(E_\bullet) : A \subset \Sigma \subset B\}\,.$$

Let $T_\lambda(E_\bullet) = \{(A,\Sigma,B) \in T_d \mid \Sigma \in X_\lambda(E_\bullet)\}$. Then we have $\pi(T_\lambda(E_\bullet)) = Y_\lambda(E_\bullet)$. Since the general fibers of π are isomorphic to Lagrangian Grassmannians $\mathrm{LG}(d, 2d)$, it follows from this that the codimension of $Y_\lambda(E_\bullet)$ in Y_d is at least $|\lambda| - d(d+1)/2$. The following lemma implies that this codimension is obtained if and only if λ contains the staircase partition $\rho_d = (d, d-1, \ldots, 2, 1)$ with d rows.

Lemma 1.3. *The restricted projection $\pi : T_\lambda(E_\bullet) \to Y_\lambda(E_\bullet)$ is generically one to one when $\rho_d \subset \lambda$, and has fibers of positive dimension when $\rho_d \not\subset \lambda$.*

Proof. We can assume that $E_r = \mathrm{Span}\{e_1, \ldots, e_r\}$ where $\{e_1, \ldots, e_{2n}\}$ is a standard symplectic basis for V, i.e. $(e_i, e_j) = 0$ when $i + j \neq 2n+1$. We will drop the flag $E_\bullet$ from the notation and write simply X_λ, Y_λ, and T_λ. Set $p_j = p_j(\lambda)$ for $1 \leq j \leq m$, $p_0 = 0$, $p_{m+1} = 2n+1$, and $\#_j = \#\{i < j \mid p_i + p_j > 2n+1\}$. Then we have $p_j = n + k + 1 - \lambda_j + \#_j$ by Proposition 4.3. We also define

$$s_j = \max(m + d + p_j - 2n - j, 0) = \max(d + 1 - j - \lambda_j + \#_j, 0).$$

For (A,B) in Y_λ, we set $A_j = A \cap E_{p_j}$ and $B_j = B \cap E_{p_j}$. By definition there exists Σ in X_λ, with $A \subset \Sigma \subset B$. We set $\Sigma_j = \Sigma \cap E_{p_j}$, so that $\dim(\Sigma_j) \geq j$. Now,

$$\dim(A_j) \geq j - d, \tag{5}$$

$$\dim(B_j) \geq j + s_j, \tag{6}$$

since A is of codimension d in Σ, and B_j is the intersection of spaces of dimension $m+d$ and p_j, while also $\Sigma_j \subset B_j$. For j such that equality holds in (6) we have $j \leq \dim(\Sigma_{j-\varepsilon}^\perp \cap B_j) = j + s_j - \dim(\Sigma_{j-\varepsilon}) + \dim(\Sigma_{j-\varepsilon} \cap B_j^\perp)$ for $\varepsilon \in \{0,1\}$, since $(\Sigma_{j-\varepsilon}^\perp \cap B_j)^\perp = \Sigma_{j-\varepsilon} + B_j^\perp$. Therefore, equality in (6) implies

$$\dim(B_j \cap B_j^\perp) \geq j - s_j, \tag{7}$$

$$\dim(B_{j-1} \cap B_j^\perp) \geq j - 1 - s_j. \tag{8}$$

Define U_λ to be the open subset of points (A,B) in Y_λ satisfying equality in (5) for $j \geq d$ and in (6), (7), (8) for $1 \leq j \leq d$. We will show that if $\rho_d \subset \lambda$, then $U_\lambda \neq \emptyset$, and for $(A,B) \in U_\lambda$ there is a unique point $\Sigma \in X_\lambda$ with $A \subset \Sigma \subset B$.

For the nonemptiness, we have from $\lambda_j \geq d + 1 - j$ that $s_j \leq \#_j$. For $j > d$ and $s_j > 0$ we also have $s_j \leq d + 1 - j + \#_j \leq \#\{i \leq d \mid p_i + p_j > 2n+1\}$. We can therefore choose a permutation $\pi \in \mathfrak{S}_d$ such that for all $1 \leq j \leq m+1$ and $i \leq s_j$,

$$\pi(i) < j \quad \text{and} \quad p_{\pi(i)} + p_j \geq 2n + 2 + s_j - i\,.$$

Now we define $A = \mathrm{Span}(e_{p_{d+1}}, \ldots, e_{p_m})$ and $B = \mathrm{Span}(e_{p_1}, \ldots, e_{p_m}, u_1, \ldots, u_d)$, where $u_i = e_{2n+1-p_{\pi(i)}} + e_{p_j - 1 - s_j + i}$ for $s_{j-1} < i \leq s_j$. Equality holds in (5) for $j \geq d$. Notice that $2n + 1 - p_{\pi(i)} \leq p_j - 1 - s_j + i$ and $p_{j-1} < p_j - 1 - s_j + i$ when $s_{j-1} < i \leq s_j$. It follows that $B_j = \mathrm{Span}(e_{p_1}, \ldots, e_{p_j}, u_1, \ldots, u_{s_j})$, so equality in

(6) holds for all j. By the pairings $(e_{p_{\pi(i)}}, u_i) \neq 0$ and $(e_{p_{\pi(i)}}, u_{i'}) = 0$ for $i' > i$, we have that the restriction of the bilinear form to the span of $e_{p_{\pi(1)}}, \ldots, e_{p_{\pi(s_j)}}, u_1, \ldots, u_{s_j}$ is nondegenerate, for any j. So $\dim(B_j \cap B_j^\perp) \leq \dim(B_j) - 2s_j = j - s_j$. Since B_{j-1} contains the first $s_j + s_{j-1}$ of these vectors, $\dim(B_{j-1} \cap B_j^\perp) \leq \dim(B_{j-1}) - (s_j + s_{j-1}) = j - 1 - s_j$. Equality in (7) and (8) is established.

Let $(A, B) \in U_\lambda$ and $0 \leq j \leq d$. We claim that there exists a unique isotropic subspace $\Sigma_j \subset B_j$ of dimension j, such that

$$\dim(\Sigma_j \cap B_i) \geq i \tag{9}$$

for $1 \leq i \leq j$. We prove this by induction on j, the base case $j = 0$ being clear. The equalities (6) and (7) imply that a maximal isotropic subspace of B_j has dimension j. If Σ_j satisfies (9) then the induction hypothesis implies that $\Sigma_j \cap B_{j-1} = \Sigma_{j-1}$, so it is enough to show that Σ_{j-1} extends to a unique isotropic subspace of dimension j in B_j, i.e. $\dim(\Sigma_{j-1}^\perp \cap B_j) = j$. This follows from equality (8) because $\dim(\Sigma_{j-1}^\perp \cap B_j) = s_j + 1 + \dim(\Sigma_{j-1} \cap B_j^\perp) \leq j$. Now $\Sigma_d \cap A = 0$ by the equality (5) for $j = d$, hence, if we set $\Sigma = \Sigma_d + A$, then $\Sigma \in X_\lambda$. Conversely, for any $\Sigma \in X_\lambda$ with $A \subset \Sigma \subset B$ we have $\dim(\Sigma \cap B_i) \geq i$ for all i, therefore $\Sigma \cap B_d = \Sigma_d$ and $\Sigma = \Sigma_d + A$.

We finally show that when $\rho_d \not\subset \lambda$ we have $\dim(Y_\lambda) < \dim(T_\lambda)$. Let $X_\lambda^\circ \subset X_\lambda$ be the Schubert cell. It is enough to show that if $(A, B) \in Y_\lambda$ satisfies $A_d = 0$, then there exists $\Sigma' \in X_\lambda \smallsetminus X_\lambda^\circ$ with $A \subset \Sigma' \subset B$. If this is false, then we can choose $\Sigma \in X_\lambda^\circ$ with $A \subset \Sigma \subset B$. Set $\Sigma_i = \Sigma \cap E_{p_i}$ for $1 \leq i \leq m$. For some j we have $\lambda_j \leq d - j$ which implies that $s_j > \#_j$, hence $\dim(\Sigma_{j-1}^\perp \cap E_{p_j}) \geq p_j - \#_j > p_j - s_j$. Using (6) we obtain $\dim(\Sigma_{j-1}^\perp \cap B_j) > j$, so there exists a j-dimensional isotropic extension Σ_j' of Σ_{j-1} contained in $B \cap E_{p_j - 1}$. For $i = j+1, \ldots, d$ we now choose an i-dimensional isotropic extension Σ_i' of Σ_{i-1}' contained in B_i. This is possible because B_i already contains the i-dimensional isotropic subspace Σ_i. The subspace $\Sigma' = \Sigma_d' + A$ then satisfies $\Sigma' \in X_\lambda \smallsetminus X_\lambda^\circ$ and $A \subset \Sigma' \subset B$. □

The next proposition is used to reconstruct rational maps from their kernel and span. Special cases are equivalent to computing the Gromov-Witten invariant counting curves through three general points in maximal isotropic Grassmannians, which was first done in [KT1, Cor. 8] and [KT2, Cor. 7]. Although these results suffice to handle the case of IG, for the non maximal orthogonal Grassmannians we will need to work more generally.

Consider a vector space $W \cong \mathbb{C}^{2d}$ with an arbitrary bilinear form $(\ ,\ )$, and let $\overline{\mathrm{G}}(d, W)$ denote the Grassmannian of dimension d subspaces of W which are isotropic with respect to $(\ ,\ )$. There is a closed embedding ι of $\overline{\mathrm{G}}(d, W)$ into the type A Grassmannian $\mathrm{G}(d, W)$, and we say that a morphism $f : \mathbb{P}^1 \to \overline{\mathrm{G}}(d, W)$ has degree d if the composite $\iota \circ f$ has degree d. We say that two points U_1, U_2 of $\overline{\mathrm{G}}(d, W)$ are in general position if the intersection $U_1 \cap U_2$ is trivial.

Proposition 1.1. *Let U_1, U_2, and U_3 be three points of $\overline{\mathrm{G}}(d, W)$ which are pairwise in general position. Then there is a unique morphism $f : \mathbb{P}^1 \to \overline{\mathrm{G}}(d, W)$ of degree d such that $f(0) = U_1$, $f(1) = U_2$, and $f(\infty) = U_3$.*

Proof. For any basis $\{v_1, \ldots, v_d\}$ of U_2, write $v_i = u_i + w_i$ with $u_i \in U_1$ and $w_i \in U_3$. As in the proof of [BKT1, Prop. 1], one shows that the map $f(s : t) = \mathrm{Span}(su_1 + tw_1, \ldots, su_d + tw_d\}$ is the unique one satisfying the condition in the proposition. Here $(s:t)$ are the homogeneous coordinates on $\mathbb{P}^1$. For any $1 \leq i,j \leq d$ the quadratic form $q_{ij}(s,t) := (su_i + tw_i, su_j + tw_j)$ vanishes at $(1,0)$, $(1,1)$, and $(0,1)$, and hence vanishes identically. It follows that the subspaces of W in the image of f are all isotropic. □

Example 1.2. We have $\langle \sigma_{\rho_m}, [\text{point}], [\text{point}] \rangle_m = 1$ for all isotropic Grassmannians $\mathrm{IG} = \mathrm{IG}(m,2n)$. In fact, it follows from Lemma 1.3 that Y_{ρ_m} has codimension zero in Y_m, so $Y_{\rho_m} = Y_m$. Let $U, V \in \mathrm{IG}$ be general points. By setting $(A,B) = (0, U \oplus V) \in Y_m$, the lemma implies that exactly one point $\Sigma \in X_{\rho_m}$ satisfies $\Sigma \subset U \oplus V$. Note that the span of any curve counted by the Gromov-Witten invariant must be contained in $U \oplus V$, and therefore the curve is itself contained in $\mathrm{LG}(m, U \oplus V) \subset \mathrm{IG}$. The claim now follows from Proposition 1.1.

The following theorem generalizes [BKT1, Thm. 2]. The vanishing statement in part (d) was proved for Lagrangian Grassmannians in [KT1].

Theorem 1.3. *Let $d \geq 0$ and choose $\lambda, \mu, \nu \in \mathcal{P}(k,n)$ such that $|\lambda| + |\mu| + |\nu| = \dim(\mathrm{IG}) + d(n+1+k)$. Let X_λ, X_μ, and X_ν be Schubert varieties of $\mathrm{IG}(m,2n)$ in general position, and let Y_λ, Y_μ, Y_ν be the associated subvarieties of Y_d.*

(a) *The subvarieties Y_λ, Y_μ, and Y_ν intersect transversally in Y_d, and $Y_\lambda \cap Y_\mu \cap Y_\nu$ is finite. For each point $(A,B) \in Y_\lambda \cap Y_\mu \cap Y_\nu$ we have $A = B \cap B^\perp$.*

(b) *The assignment $f \mapsto (\mathrm{Ker}(f), \mathrm{Span}(f))$ gives a bijection of the set of rational maps $f : \mathbb{P}^1 \to \mathrm{IG}$ of degree d such that $f(0) \in X_\lambda$, $f(1) \in X_\mu$, $f(\infty) \in X_\nu$, with the points of the intersection $Y_\lambda \cap Y_\mu \cap Y_\nu$ in Y_d.*

(c) *We have $\langle \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle_d = \int_{Y_d} [Y_\lambda] \cdot [Y_\mu] \cdot [Y_\nu]$.*

(d) *If λ does not contain the staircase partition ρ_d, then $\langle \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle_d = 0$.*

Proof. For integers $0 \leq e_1, e_2 \leq d$ and $r \geq 0$ we let $Y^r_{e_1,e_2}$ be the variety of pairs (A,B) such that $A \subset B \subset A^\perp \subset V$, $\dim(A) = m - e_1$, $\dim(B) = m + e_2$, and $\dim(B \cap B^\perp) = m - e_1 + r$. These varieties have a transitive action of Sp_{2n}, and Y_d is the union of the varieties $Y^r_{d,d}$ for even numbers $r \leq \min(2d, 2k)$. In general, $Y^r_{e_1,e_2}$ is empty unless $r \leq \min(e_1 + e_2, 2k + e_1 - e_2)$ and $e_1 + e_2 - r$ is even. We also set

$$Y^r_\lambda = \{(A,B) \in Y^r_{e_1,e_2} \mid \exists \Sigma \in X_\lambda : A \subset \Sigma \subset B\}.$$

Our first goal is to prove that $Y^r_\lambda \cap Y^r_\mu \cap Y^r_\nu$ is empty unless $e_1 = e_2 = d$ and $r = 0$, in which case the intersection is a finite set of points.

The smooth map $Y^r_{e_1,e_2} \to \mathrm{IF}(m - e_1, m - e_1 + r; 2n)$ given by $(A,B) \mapsto (A, B \cap B^\perp)$ has fibers isomorphic to open subsets of the type A Grassmannian $\mathrm{G}(e_1 + e_2 - r, 2k + 2e_1 - 2r)$, so we obtain

$$\dim(Y^r_{e_1,e_2}) = \frac{1}{2}(n^2 + 2nk - 3k^2 + m + 2me_1 - e_1^2 + 4e_2k - 2e_2^2 - e_1 + r - r^2).$$

Define the variety $T^r = \{(A, \Sigma, B) \mid (A,B) \in Y^r_{e_1,e_2}, \Sigma \in \mathrm{IG}, A \subset \Sigma \subset B\}$, and let $\pi_1 : T^r \to \mathrm{IG}$ and $\pi_2 : T^r \to Y^r_{e_1,e_2}$ be the projections. Then $Y^r_\lambda = \pi_2(\pi_1^{-1}(X_\lambda))$. Given a point $(A,B) \in Y^r_{e_1,e_2}$, the fiber $\pi_2^{-1}((A,B))$ is the union of the varieties

$$Q_s = \{\Sigma \in \mathrm{IG} \mid A \subset \Sigma \subset B,\ \dim(\Sigma \cap B \cap B^\perp) = m - e_1 + s\}$$

for all integers s. Notice that the dimension of an isotropic subspace of B is at most equal to $m + (r - e_1 + e_2)/2$, and if $\Sigma \in Q_s$ then $\Sigma + (B \cap B^\perp)$ is such a subspace of dimension $m+r-s$. This implies that Q_s is empty unless $2s \geq r+e_1-e_2$. Since the map $Q_s \to \mathrm{IG}(e_1-s, B/(B\cap B^\perp))$ given by $\Sigma \mapsto (\Sigma + (B\cap B^\perp))/(B\cap B^\perp)$ has fibers isomorphic to open subsets of $\mathrm{G}(e_1, e_1+r-s)$, it follows that when Q_s is not empty, we have

$$\dim Q_s = \frac{1}{2}(2e_1e_2 - e_1^2 + e_1 + 2e_1 s - 2e_2 s + 2rs - 3s^2 - s)\,.$$

Choose s such that Q_s has the same dimension as the fibers of π_2. Then the codimension of Y^r_λ in $Y^r_{e_1,e_2}$ is at least $|\lambda| - \dim Q_s$. It follows that $\mathrm{codim}(Y^r_\lambda) + \mathrm{codim}(Y^r_\mu) + \mathrm{codim}(Y^r_\nu) - \dim(Y^r_{e_1,e_2})$ is greater than or equal to the number $\Delta(e_1,e_2,r,s) := \dim \mathrm{IG} + d(n+1+k) - \dim(Y^r_{e_1,e_2}) - 3\dim Q_s$. A computation shows that $\Delta(e_1,e_2,r,s)$ is equal to

$$(r-3s)(r-3s-1)/2 + (d-e_1) + (n+k)(d-e_2) + (m+e_2-2e_1+3s)(e_2-e_1)\,.$$

Suppose $Y^r_\lambda \cap Y^r_\mu \cap Y^r_\nu \neq \emptyset$. Since the action of Sp_{2n} on $Y^r_{e_1,e_2}$ is transitive, it follows from the Kleiman-Bertini theorem that $\Delta(e_1,e_2,r,s) \leq 0$. If $e_2 \geq e_1$ then this immediately implies that $e_1 = e_2 = d$, so we may assume that $e_2 < e_1$. If $(A,B) \in Y^r_\lambda \cap Y^r_\mu \cap Y^r_\nu$ then we can find a subspace $B' \subset A^\perp$ containing B, such that $\dim B' = m + e_1$. Since (A,B') must be a point in the intersection of modified Schubert varieties in some variety $Y^{r'}_{e_1,e_1}$, it follows that $\Delta(e_1,e_1,r',s') \leq 0$, so $e_1 = d$. Still assuming $e_2 < e_1$, we can take a subspace $B'' \subset A^\perp$ containing B, such that $\dim B'' = m + d - 1$. Since (A,B'') lies in the intersection of modified Schubert varieties in some space $Y^{r''}_{d,d-1}$, it follows that

$$\Delta(d,d-1,r'',s'') = (r''-3s'')(r''-3s''-1)/2 + 2k + 1 - 3s'' + d$$

is non-positive, and since $r'' \leq 2k+1$, this number is also greater than or equal to $(r''-3s'')(r''-3s''+1)/2 + d$. But this number is positive, which gives a contradiction. We conclude that $e_1 = e_2 = d$, so $\Delta(d,d,r,s) = (r-3s)(r-3s-1)/2 \leq 0$. This is possible only if $r = 3s$ or $r = 3s+1$. Since $2s \geq r + e_1 - e_2 = r$, we finally obtain $r = s = 0$ as required.

It is now clear that part (a) is true. In fact, the intersection $Y_\lambda \cap Y_\mu \cap Y_\nu$ is contained in the open Sp_{2n}-orbit $Y^0_{d,d}$ of Y_d, so this intersection is transverse by the Kleiman-Bertini theorem. Furthermore, since $\mathrm{codim}(Y_\lambda) \geq |\lambda| - \dim Q_0 = |\lambda| - d(d+1)/2$, we deduce that the intersection is finite.

Let $f : \mathbb{P}^1 \to \mathrm{IG}$ be a rational map of degree d such that $f(0) \in X_\lambda$, $f(1) \in X_\mu$, and $f(\infty) \in X_\nu$. Then $(\mathrm{Ker}(f), \mathrm{Span}(f))$ must be a point of an intersection $Y^r_\lambda \cap Y^r_\mu \cap Y^r_\nu$ in some space $Y^r_{e_1,e_2}$, necessarily with $e_1 = e_2 = d$, so we must have $\dim \mathrm{Ker}(f) = m - d$ and $\dim \mathrm{Span}(f) = m + d$. This shows that the map $f \mapsto (\mathrm{Ker}(f), \mathrm{Span}(f))$ of part (b) is well defined. In particular we have $d \leq m$. On the other hand, let $(A_0, B_0) \in Y_\lambda \cap Y_\mu \cap Y_\nu \subset Y_d$ be any point. To establish (b), we must show that there is a unique rational map $f : \mathbb{P}^1 \to \mathrm{IG}$ of degree d such that $f(0) \in X_\lambda$, $f(1) \in X_\mu$, $f(\infty) \in X_\nu$, and $(\mathrm{Ker}(f), \mathrm{Span}(f)) = (A_0, B_0)$.

Setting $W = B_0/A_0$ we can identify $\overline{\mathrm{G}}(d,W)$ with $\{\Sigma \in \mathrm{IG} \mid A_0 \subset \Sigma \subset B_0\}$. Since $B_0 \cap B_0^\perp = A_0$ by part (a), we have $\overline{\mathrm{G}}(d,W) \cong \mathrm{LG}(d,2d)$. Let $P \in X_\lambda \cap \overline{\mathrm{G}}(d,W)$, $Q \in X_\mu \cap \overline{\mathrm{G}}(d,W)$, and $R \in X_\nu \cap \overline{\mathrm{G}}(d,W)$. We claim that $P \cap Q = A_0$. Otherwise there exists $A_0 \subset A_0' \subset P \cap Q$ with $\dim A_0' = m - d + 1$. Now define

$$Y' = \{(A, A', B) \mid (A,B) \in Y^0_{d,d},\ A \subset A' \subset B,\ \dim A' = m - d + 1\}\,.$$

Notice that the action of Sp_{2n} on this variety is transitive, and that $\dim Y' = \dim Y_d + 2d - 1$. We also set $Y'_\lambda = \{(A, A', B) \in Y' \mid \exists \Sigma \in X_\lambda : A' \subset \Sigma \subset B\}$. Our assumptions show that $(A_0, A'_0, B_0) \in Y'_\lambda \cap Y'_\mu \cap \pi^{-1}(Y_\nu)$ where $\pi : Y' \to Y_d$ is the projection. Set $T' = \{(A, A', \Sigma, B) \mid (A, A', B) \in Y', \Sigma \in \mathrm{IG}, A' \subset \Sigma \subset B\}$ and let $\pi'_1 : T' \to \mathrm{IG}$ and $\pi'_2 : T' \to Y'$ be the projections. The fibers of π'_2 are isomorphic to $\mathrm{LG}(d-1, 2d-2)$, so they have dimension $d(d-1)/2$. Since $Y'_\lambda = \pi'_2({\pi'_1}^{-1}(X_\lambda))$, this implies that $\mathrm{codim}(Y'_\lambda) \geq |\lambda| - d(d-1)/2$. It follows that

$$\mathrm{codim}(Y'_\lambda) + \mathrm{codim}(Y'_\mu) + \mathrm{codim}(\pi^{-1}(Y_\nu)) \geq$$
$$|\lambda| + |\mu| + |\nu| - 2d(d-1)/2 - d(d+1)/2 = \dim(Y_d) + 2d > \dim(Y')\,.$$

Since this implies that $Y'_\lambda \cap Y'_\mu \cap \pi^{-1}(Y_\nu) = \emptyset$, we conclude that $P \cap Q = A_0$ as claimed.

Notice that the intersection $X_\lambda \cap \overline{\mathrm{G}}(d, W)$ must have dimension zero, since otherwise one could choose the point $P \in X_\lambda \cap \overline{\mathrm{G}}(d, W)$ such that $P \cap Q \supsetneq A_0$. By Lemma 1.3 this implies that $\rho_d \subset \lambda$ (which proves (d)), and $X_\lambda \cap \overline{\mathrm{G}}(d, W)$ must furthermore be a single point. We conclude that $X_\lambda \cap \overline{\mathrm{G}}(d, W) = \{P\}$, $X_\mu \cap \overline{\mathrm{G}}(d, W) = \{Q\}$, $X_\nu \cap \overline{\mathrm{G}}(d, W) = \{R\}$, and $P \cap Q = Q \cap R = R \cap P = A_0$. Now by Proposition 1.1 there is a unique rational map $f : \mathbb{P}^1 \to Z$ of degree d such that $f(0) \in X_\lambda$, $f(1) \in X_\mu$, and $f(\infty) \in X_\nu$. This proves (b), and (a) and (b) together imply (c). □

Set $X^+ = \mathrm{IG}(m+1, 2n+2)$. The following proposition was proved in [KT1, Prop. 4] for Lagrangian Grassmannians. As in loc. cit., the proof is based on an explicit correspondence between lines in $X = \mathrm{IG}(m, 2n)$ and points in X^+, but the underlying geometry is more challenging in the general case.

Proposition 1.2. *Let λ, μ, ν be k-strict partitions such that $|\lambda|+|\mu|+|\nu| = \dim X + (n+k+1)$ and $\ell(\lambda) + \ell(\mu) + \ell(\nu) \leq 2m+1$. Then*

$$\langle \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle_1 = \frac{1}{2} \int_{\mathrm{IG}(m+1,2n+2)} [X^+_\lambda] \cdot [X^+_\mu] \cdot [X^+_\nu]\,.$$

Proof. Let $E_\bullet, F_\bullet, G_\bullet \subset V = \mathbb{C}^{2n}$ be isotropic flags in general position. Let $H = \mathbb{C}^2$ be a two-dimensional symplectic vector space, and choose generic elements $e, f, g \in H$. Set $V^+ = V \oplus H$ and let $E^+_\bullet, F^+_\bullet, G^+_\bullet \subset V^+$ be the augmented flags given by $E^+_p = E_{p-1} \oplus \mathbb{C}e$, $F^+_p = F_{p-1} \oplus \mathbb{C}f$, and $G^+_p = G_{p-1} \oplus \mathbb{C}g$. We will prove that the assignment $\Sigma^+ \mapsto (\Sigma^+ \cap V, (\Sigma^+ + H) \cap V)$ gives a well defined 2-1 map from $X^+_\lambda(E^+_\bullet) \cap X^+_\mu(F^+_\bullet) \cap X^+_\nu(G^+_\bullet)$ onto the intersection $Y_\lambda(E_\bullet) \cap Y_\mu(F_\bullet) \cap Y_\nu(G_\bullet) \subset Y_1$. Notice that $X^+_\lambda(E^+_\bullet) = \{\Sigma^+ \in X^+ \mid \dim(\Sigma^+ \cap (E_{p_j(\lambda)} + \mathbb{C}e)) \geq j,\ \forall\, 1 \leq j \leq \ell(\lambda)\}$.

The group $G = \mathrm{Sp}(V) \times \mathrm{SL}(H)$ acts on X^+ and our choices imply that the varieties $X^+_\lambda, X^+_\mu, X^+_\nu$ are in general position for this action. It is easily seen that the G-action on X^+ has the following four orbits.

$O_1 = \mathrm{IG}(m+1, V)$
$O_2 = \mathrm{IG}(m, V) \times \mathrm{IG}(1, H)$
$O_3 = \{\Sigma^+ \in X^+ \mid \dim(\Sigma^+ \cap V) = m, \Sigma^+ \cap H = 0\}$
$O_4 = \{\Sigma^+ \in X^+ \mid \dim(\Sigma^+ \cap V) = m-1\}$

We start by showing that the intersection $X^+_\lambda \cap X^+_\mu \cap X^+_\nu$ is contained in the open orbit O_4. Notice at first that $X^+_\lambda(E_\bullet) \cap \mathrm{IG}(m+1, V)$ is the Schubert variety in $\mathrm{IG}(m+1, V)$ given by the partition $\overline{\lambda} = (\lambda_1 - 1, \ldots, \lambda_{\ell(\lambda)} - 1)$ obtained by deleting the first column of λ. Since $|\overline{\lambda}|+|\overline{\mu}|+|\overline{\nu}| > |\lambda|+|\mu|+|\nu|-2m-2 = \dim \mathrm{IG}(m+1,V)$, it follows that $X^+_\lambda \cap X^+_\mu \cap X^+_\nu \cap \mathrm{IG}(m+1,V) = \emptyset$.

Now assume $\Sigma^+ \in X^+_\lambda \cap X^+_\mu \cap X^+_\nu \cap O_2$. Then $\Sigma^+ = \Sigma \oplus \mathbb{C}h$ where $\Sigma \in X$ and $0 \neq h \in H$. We may assume that $h \notin \mathbb{C}f$ and $h \notin \mathbb{C}g$, which implies that $\Sigma \in X_\mu(F_\bullet) \cap X_\nu(G_\bullet)$. Define the k-strict partition η by $\eta_j = \lambda_{j+1}$ for $j < \lambda_1 - 2k$; $\eta_j = \lambda_{j+1} - 1$ for $\lambda_1 - 2k \leq j < \ell(\lambda)$; and $\eta_j = 0$ for $j \geq \ell(\lambda)$. Since $\lambda_1 + \lambda_j \leq 2k + j - 1$ implies that $j > \lambda_1 - 2k$, it follows that $p_{j+1}(\lambda) \leq p_j(\eta)$ for each $j < \ell(\lambda)$. Since $\dim(\Sigma \cap E_{p_{j+1}(\lambda)}) \geq j$ for all j, this implies that $\Sigma \in X_\eta(E_\bullet)$. But $|\eta|+|\mu|+|\nu| \geq |\lambda|+|\mu|+|\nu|-n-k > \dim X$, so $X_\eta \cap X_\mu \cap X_\nu = \emptyset$, a contradiction.

We finally check that no point of $X^+_\lambda \cap X^+_\mu \cap X^+_\nu$ is contained in O_3. We have a smooth map $\pi : O_3 \to X$ given by $\pi(\Sigma^+) = \Sigma^+ \cap V$. The fiber over a point $\Sigma \in X$ is an open subset of $\mathbb{P}(H \oplus \Sigma^\perp/\Sigma)$, so $\dim O_3 = \dim X + 2k + 1 = \dim X^+ - m$. Now suppose $\Sigma^+ \in X^+_\lambda \cap X^+_\mu \cap X^+_\nu \cap O_3$. If $\Sigma^+ \not\subset V \oplus \mathbb{C}e$, then the dimension of $\Sigma^+ \cap (V \oplus \mathbb{C}e)$ is at most m. But $\Sigma^+ \cap V$ has dimension m, so we must have $\Sigma^+ \cap (V \oplus \mathbb{C}e) = \Sigma^+ \cap V$. In particular we obtain $\Sigma^+ \cap (E_i \oplus \mathbb{C}e) = (\Sigma^+ \cap V) \cap E_i$ for each i, so $\Sigma^+ \cap V \in X_\lambda$ and $\Sigma^+ \in \pi^{-1}(X_\lambda)$. Now notice that Σ^+ cannot be contained in both $V \oplus \mathbb{C}e$ and $V \oplus \mathbb{C}f$, so by permuting λ, μ, ν, we may assume that $\Sigma^+ \not\subset V \oplus \mathbb{C}f$ and $\Sigma^+ \not\subset V \oplus \mathbb{C}g$. This implies that $\Sigma^+ \in \pi^{-1}(X_\mu) \cap \pi^{-1}(X_\nu)$. Since $\pi^{-1}(X_\lambda \cap X_\mu \cap X_\nu) = \emptyset$, we then deduce that $\Sigma^+ \subset V \oplus \mathbb{C}e$. However, all components of the variety $Z = \{\Sigma^+ \in X^+_\lambda \mid \Sigma^+ \subset V \oplus \mathbb{C}e\}$ have dimension strictly smaller than the dimension of X^+_λ, so the codimension in O_3 of each of the components of $Z \cap O_3$ is at least $|\lambda| - m + 1$. Since $|\lambda| - m + 1 + |\mu| + |\nu| = \dim X + 2k + 2 > \dim O_3$ we conclude that $\pi^{-1}(X_\mu \cap X_\nu) \cap Z \cap O_3 = \emptyset$, a contradiction. This verifies that $X^+_\lambda \cap X^+_\mu \cap X^+_\nu \subset O_4$.

Notice that for $\Sigma^+ \in O_4$ we must have $\Sigma^+ \cap H = 0$, since otherwise $\Sigma^+ \cap H = \mathbb{C}h$ for some $0 \neq h \in H$, and $\Sigma^+ \subset (\mathbb{C}h)^\perp = V \oplus \mathbb{C}h$. This implies that $\dim(\Sigma^+ \cap V) \geq m$, a contradiction. Since $\dim(\Sigma^+ + H) = m + 3$ and $(\Sigma^+ + H) + V = V \oplus H$, it follows that $\dim((\Sigma^+ + H) \cap V) = m + 1$. Thus we have a well defined map $\phi : O_4 \to Y_1$ given by $\Sigma^+ \mapsto (\Sigma^+ \cap V, (\Sigma^+ + H) \cap V)$.

If $\Sigma^+ \in X^+_\lambda \cap O_4$ then $\dim(\Sigma^+ \cap (E_{p_j(\lambda)} \oplus \mathbb{C}e)) \geq j$ implies that $\dim((\Sigma^+ \cap V) \cap E_{p_j(\lambda)}) \geq j - 1$ for each j. Since $e \notin \Sigma^+$ we also have $\dim((\Sigma^+ + H) \cap (E_{p_j(\lambda)} \oplus \mathbb{C}e)) \geq j + 1$, which implies that $\dim((\Sigma^+ + H) \cap E_{p_j(\lambda)}) \geq j$. It follows from this that $\phi(X^+_\lambda \cap O_4) \subset Y_\lambda$, and we conclude that ϕ gives a well defined map $X^+_\lambda \cap X^+_\mu \cap X^+_\nu \to Y_\lambda \cap Y_\mu \cap Y_\nu \subset Y_1$.

Now let $(A, B) \in Y_\lambda(E_\bullet) \cap Y_\mu(F_\bullet) \cap Y_\nu(G_\bullet)$ be given. We must prove that $\phi^{-1}((A, B)) \cap X^+_\lambda \cap X^+_\mu \cap X^+_\nu$ contains exactly two points. We know from Theorem 1.3 (a) that $A = B \cap B^\perp$, so $W = B/A \oplus H$ is a symplectic space of dimension 4, and $\mathrm{LG}(2, W)$ is identified with the subset of $\Sigma^+ \in X^+$ such that $A \subset \Sigma^+ \subset B \oplus H$. Since $(A, B) \in Y_\lambda(E_\bullet)$ we have $\dim(A \cap E_{p_j(\lambda)}) \geq j - 1$ and $\dim(B \cap E_{p_j(\lambda)}) \geq j$ for all j. By the proof of Theorem 1.3, there exist unique points $\Sigma_1 \in X_\lambda(E_\bullet)$, $\Sigma_2 \in X_\mu(F_\bullet)$, $\Sigma_3 \in X_\nu(G_\bullet)$ such that $A \subset \Sigma_i \subset B$ for each i, and these points are pairwise distinct. Set $\widetilde{E} = \Sigma_1 \oplus \mathbb{C}e$, $\widetilde{F} = \Sigma_2 \oplus \mathbb{C}f$, and $\widetilde{G} = \Sigma_3 \oplus \mathbb{C}g$. Then there are exactly two points $\Sigma^+ \in X^+$, such that $A \subset \Sigma^+ \subset B \oplus H$ and $\dim(\Sigma^+ \cap \widetilde{E}) \geq m$, $\dim(\Sigma^+ \cap \widetilde{F}) \geq m$, $\dim(\Sigma^+ \cap \widetilde{G}) \geq m$. This is true because exactly two isotropic planes of $\mathrm{LG}(2, W)$ are incident to all of the subspaces $\Sigma_1/A \oplus \mathbb{C}e$, $\Sigma_2/A \oplus \mathbb{C}f$, and $\Sigma_3/A \oplus \mathbb{C}g$. Since $\dim(\widetilde{E} \cap (E_{p_j(\lambda)} \oplus \mathbb{C}e)) \geq j+1$ and $\dim(\Sigma^+ \cap \widetilde{E}) \geq m$, we obtain $\dim(\Sigma^+ \cap (E_{p_j(\lambda)} + \mathbb{C}e)) \geq j$, so $\Sigma^+ \in X^+_\lambda$. Similarly we have $\Sigma^+ \in X^+_\mu \cap X^+_\nu$.

On the other hand, if $\Sigma^+ \in X^+_\lambda \cap X^+_\mu \cap X^+_\nu$ is such that $A \subset \Sigma^+ \subset B \oplus H$, then $A = \Sigma^+ \cap V$ and $B = (\Sigma^+ + H) \cap V$. We claim that $\Sigma_1 = (\Sigma^+ + \mathbb{C}e) \cap V$. In fact, the isotropic m-plane $\Sigma_1' = (\Sigma^+ + \mathbb{C}e) \cap V$ satisfies $A \subset \Sigma_1' \subset B$. Since $\dim((\Sigma^+ + \mathbb{C}e) \cap (E_{p_j(\lambda)} \cap \mathbb{C}e)) \geq j+1$ we get $\dim(\Sigma_1' \cap E_{p_j(\lambda)}) \geq j$ for each j, so $\Sigma_1' = \Sigma_1$ must be the unique isotropic m-plane from $X_\lambda(E_\bullet)$ which lies between A and B. It follows from the claim that $\Sigma^+ + \widetilde{E} \subset \Sigma^+ + \mathbb{C}e$, so $\dim(\Sigma^+ \cap \widetilde{E}) \geq m$. Similarly we see that $\dim(\Sigma^+ \cap \widetilde{F}) \geq m$ and $\dim(\Sigma^+ \cap \widetilde{G}) \geq m$, which finally establishes the required 2 to 1 map. □

Example 1.3. The length condition in Proposition 1.2 is essential. For example, for $X = \mathrm{IG}(2,8)$ and $X^+ = \mathrm{IG}(3,10)$ we have $\langle \sigma_{1,1}, \sigma_{4,1}, \sigma_{6,5} \rangle_1 = 0$ and

$$\int_{X^+} [X^+_{1,1}] \cdot [X^+_{4,1}] \cdot [X^+_{6,5}] = 1.$$

1.5. **Quantum cohomology.** The (small) quantum cohomology ring $\mathrm{QH}^*(\mathrm{IG})$ is a $\mathbb{Z}[q]$-algebra which is isomorphic to $\mathrm{H}^*(\mathrm{IG}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$ as a module over $\mathbb{Z}[q]$. The degree of the formal variable q here is given by

$$\deg(q) = \int_{\mathrm{IG}} c_1(T_{\mathrm{IG}}) \cdot \sigma_{(1)^\vee} = n + k + 1\,.$$

The ring structure on $\mathrm{QH}^*(\mathrm{IG})$ is determined by the relation

$$\sigma_\lambda \cdot \sigma_\mu = \sum \langle \sigma_\lambda, \sigma_\mu, \sigma_{\nu^\vee} \rangle_d \, \sigma_\nu \, q^d,$$

the sum over $d \geq 0$ and k-strict partitions ν with $|\nu| = |\lambda| + |\mu| - (n+k+1)d$. For any partition ν, define ν^* by removing the first row of ν, that is, $\nu^* = (\nu_2, \nu_3, \ldots)$.

Theorem 1.4 (Quantum Pieri rule for IG)**.** *For any k-strict partition $\lambda \in \mathcal{P}(k,n)$ and integer $p \in [1, n+k]$, we have*

$$\sigma_p \cdot \sigma_\lambda = \sum_{\lambda \to \mu} 2^{N(\lambda,\mu)} \, \sigma_\mu + \sum_{\lambda \to \nu} 2^{N(\lambda,\nu)-1} \, \sigma_{\nu^*} \, q$$

in the quantum cohomology ring of $\mathrm{IG}(n-k, 2n)$*. The first sum is over partitions $\mu \in \mathcal{P}(k,n)$ such that $|\mu| = |\lambda| + p$, and the second sum is over partitions $\nu \in \mathcal{P}(k, n+1)$ with $|\nu| = |\lambda| + p$ and $\nu_1 = n+k+1$.*

Proof. The first sum is dictated by the classical Pieri rule (Theorem 1.1). Theorem 1.3 (d) implies that the coefficient of q^d in the product $\sigma_p \, \sigma_\lambda$ vanishes for all $d \geq 2$. By Proposition 1.2, the coefficient of $\sigma_\nu \, q$ in the product is equal to

$$\langle \sigma_p, \sigma_\lambda, \sigma_{\nu^\vee} \rangle_1 = \frac{1}{2} \int_{X^+} [X^+_p] \cdot [X^+_\lambda] \cdot [X^+_{\nu^\vee}]\,,$$

where $\nu^\vee$ denotes the dual partition of ν with respect to the Grassmannian $\mathrm{IG} = \mathrm{IG}(m, 2n)$. One checks that the result of dualizing $\nu^\vee$ with respect to $X^+ = \mathrm{IG}(m+1, 2n+2)$ is $\nu^+ = (n+k+1, \nu_1, \ldots, \nu_m)$ (information on dual partitions is given in Section 4.4). Finally, it follows from Theorem 1.1 that the coefficient of $\sigma_\nu \, q$ is equal to $2^{N(\lambda,\nu^+)-1}$ if $\lambda \to \nu^+$, and otherwise this coefficient is zero. □

Example 1.4. In the quantum ring of $\mathrm{IG}(4,12)$ we have $\sigma_4 \cdot \sigma_{(5,3,2,2)} = 4\sigma_{(8,4,2,2)} + 2\sigma_{(7,5,2,2)} + 2\sigma_{(7,4,3,2)} + \sigma_{(6,5,3,2)} + \sigma_{(8,4,3,1)} + 2\sigma_{(4,2,1)}\, q + 2\sigma_{(3,2,2)}\, q + \sigma_{(3,2,1,1)}\, q$. The q-terms can be found by expanding the product $[X^+_4] \cdot [X^+_{(5,3,2,2)}]$ in $\mathrm{H}^*(\mathrm{IG}(5,14))$.

Theorem 1.5 (Ring presentation)**.** *The quantum cohomology ring* $\mathrm{QH}^*(\mathrm{IG})$ *is presented as a quotient of the polynomial ring* $\mathbb{Z}[\sigma_1, \dots, \sigma_{n+k}, q]$ *modulo the relations*

$$\det(\sigma_{1+j-i})_{1\leq i,j\leq r} = 0, \qquad n-k+1 \leq r \leq n+k \tag{10}$$

and

$$\sigma_r^2 + 2\sum_{i=1}^{n+k-r} (-1)^i \sigma_{r+i}\sigma_{r-i} = (-1)^{n+k-r}\sigma_{2r-n-k-1}\, q, \qquad k+1 \leq r \leq n. \tag{11}$$

Proof. According to [ST] (see also [FP, Sec. 10]), we only need to check how the classical relations in Theorem 1.2 deform under the quantum product. The relations (10) are true in $\mathrm{QH}^*(\mathrm{IG})$ because the degree of q is greater than $n+k$. Using (2) for $X^+ = \mathrm{IG}(m+1, 2n+2)$ we obtain the identity $[X_r^+]^2 + \sum_{i=1}^{n+k-r}(-1)^i [X^+_{r+i}][X^+_{r-i}] = (-1)^{n+k-r}[X^+_{n+k+1}][X^+_{2r-n-k-1}] = (-1)^{n+k-r}[X^+_{(n+k+1,2r-n-k-1)}]$ in $\mathrm{H}^*(X^+)$. If $2r < n+k+1$ then the right hand side is understood to be zero. The quantum Pieri rule for IG now implies that the q-terms of (11) agree. □

1.6. **Computing Gromov-Witten invariants.** For any partition $\lambda = (\lambda_1, \dots, \lambda_\ell)$ in $\mathcal{P}(k,n)$, iterating the quantum Pieri rule as in the proof of Theorem 1.2(b) gives

$$\sigma_{\lambda_1} \cdots \sigma_{\lambda_\ell} = \sigma_\lambda + \sum_{d,\mu} c_{d,\mu}\sigma_\mu q^d$$

in $\mathrm{QH}^*(\mathrm{IG})$, where $c_{d,\mu} \in \mathbb{Z}$ and the partitions μ in the sum satisfy $\mu \succ \lambda$ or $|\mu| < |\lambda|$. Therefore our quantum Pieri formula can be used recursively to identify a given Schubert class with a polynomial in the special Schubert classes and q. These expressions together with the quantum Pieri rule can then be used to evaluate any quantum product $\sigma_\lambda \cdot \sigma_\mu$, and hence any Gromov-Witten invariant $\langle \sigma_\lambda, \sigma_\mu, \sigma_\nu\rangle_d$ on IG, as in the following example.

Example 1.5. In $\mathrm{QH}^*(\mathrm{IG}(3,10))$ we have

$$\sigma_4\sigma_2^2 = \sigma_{(4,2,2)} + \sigma_{(4,3,1)} + 3\sigma_{(5,2,1)} + 4\sigma_{(6,1,1)} + 3\sigma_{(5,3)} + 5\sigma_{(6,2)} + 8\sigma_{(7,1)} + 2q.$$

This, combined with other quantum Pieri products, leads to the expression

$$\sigma_{(4,2,2)} = \sigma_4\sigma_2^2 - \sigma_4\sigma_3\sigma_1 - \sigma_5\sigma_2\sigma_1 + \sigma_5\sigma_3 + 2\sigma_7\sigma_1 - q.$$

Using this identity we can evaluate the product

$$\sigma_{(4,2,2)} \cdot \sigma_{(5,3,1)} = (\sigma_4\sigma_2^2 - \sigma_4\sigma_3\sigma_1 - \sigma_5\sigma_2\sigma_1 + \sigma_5\sigma_3 + 2\sigma_7\sigma_1 - q) \cdot \sigma_{(5,3,1)}$$
$$= \sigma_{(7,6,4)} + 4\sigma_{(7,2)}\, q + \sigma_{(5,3,1)}\, q + \sigma_{(7,1,1)}\, q + 2\sigma_{(6,2,1)}\, q + 3\sigma_{(6,3)}\, q + \sigma_{(5,4)}\, q + \sigma_1\, q^2.$$

Since the Poincaré dual of σ_1 is $\sigma_{(7,6,4)}$, we obtain $\langle \sigma_{(4,2,2)}, \sigma_{(5,3,1)}, \sigma_{(7,6,4)}\rangle_2 = 1$.

2. The Grassmannian $\mathrm{OG}(n-k, 2n+1)$

2.1. **Schubert classes.** Consider a vector space $V \cong \mathbb{C}^{2n+1}$ and a nondegenerate symmetric bilinear form on V. For each $m = n-k < n$, the odd orthogonal Grassmannian $\mathrm{OG} = \mathrm{OG}(m, 2n+1)$ parametrizes the m-dimensional isotropic subspaces in V. The algebraic variety OG has dimension $2m(n-m) + m(m+1)/2$, the same as the dimension of $\mathrm{IG}(m, 2n)$. Moreover, the Schubert varieties in OG are indexed by the same set of k-strict partitions $\mathcal{P}(k,n)$ as for IG.

An isotropic flag $F_\bullet$ of V is a complete flag $0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_{2n+1} = V$ such that $F_{n+i} = F_{n+1-i}^\perp$ for all $1 \leq i \leq n+1$. For each such flag and $\lambda \in \mathcal{P}(k,n)$, define the Schubert variety

$$X_\lambda(F_\bullet) = \{\Sigma \in \mathrm{OG} \mid \dim(\Sigma \cap F_{\overline{p}_j(\lambda)}) \geq j, \quad \forall\, 1 \leq j \leq \ell(\lambda)\}\,,$$

where

$$\overline{p}_j(\lambda) = n + k - \lambda_j + \#\{i < j : \lambda_i + \lambda_j \leq 2k + j - i\} + \begin{cases} 2 & \text{if } \lambda_j \leq k, \\ 1 & \text{if } \lambda_j > k. \end{cases}$$

The codimension of this variety is equal to $|\lambda|$. We define $\tau_\lambda \in \mathrm{H}^{2|\lambda|}(\mathrm{OG}) = \mathrm{H}^{2|\lambda|}(\mathrm{OG}, \mathbb{Z})$ to be the cohomology class dual to the cycle given by $X_\lambda(F_\bullet)$. These Schubert classes form a $\mathbb{Z}$-basis for the cohomology ring of OG.

2.2. **Classical Pieri rule.** The following comparison of structure constants between IG and OG is well known; in fact this extends to the complete flag varieties in types B and C; see [BS, Sec. 3.1]. For each k-strict partition λ, let $\ell_k(\lambda)$ denote the number of parts λ_i which are strictly greater than k. Consider λ and μ in $\mathcal{P}(k,n)$ and suppose that

$$\sigma_\lambda \sigma_\mu = \sum_\nu e_{\lambda,\mu}^\nu \, \sigma_\nu$$

in $\mathrm{H}^*(\mathrm{IG}(n-k, 2n))$. Then for the corresponding formula

$$\tau_\lambda \tau_\mu = \sum_\nu f_{\lambda,\mu}^\nu \, \tau_\nu$$

in $\mathrm{H}^*(\mathrm{OG}(n-k, 2n+1))$, we have

$$f_{\lambda,\mu}^\nu = 2^{\ell_k(\nu) - \ell_k(\lambda) - \ell_k(\mu)} \, e_{\lambda,\mu}^\nu. \tag{12}$$

It follows that the map $\mathrm{H}^*(\mathrm{OG}, \mathbb{Z}) \to \mathrm{H}^*(\mathrm{IG}, \mathbb{Q})$ sending τ_λ to $2^{-\ell_k(\lambda)}\sigma_\lambda$ is an isomorphism onto its image.

The *special* Schubert varieties for OG are the varieties $X_p = X_{(p)}(F_\bullet)$, for $1 \leq p \leq n+k$. These are defined by a single Schubert condition as follows: let

$$\varepsilon(p) = n + k - p + \begin{cases} 2 & \text{if } p \leq k, \\ 1 & \text{if } p > k. \end{cases} \tag{13}$$

Then $X_p(F_\bullet) = \{\, \Sigma \in \mathrm{OG} \mid \Sigma \cap F_{\varepsilon(p)} \neq 0 \,\}$. We let τ_p denote the cohomology class of X_p. The classical Pieri rule for OG will involve the same relation $\lambda \to \mu$ and set $\mathbb{A} \subset \mu \smallsetminus \lambda$ that appeared in Definition 1.3. For each λ and μ with $\lambda \to \mu$, we let $N'(\lambda, \mu)$ equal the number (respectively, one less than the number) of components of $\mathbb{A}$, if $p \leq k$ (respectively, if $p > k$).

Theorem 2.1 (Pieri rule for $\mathrm{OG}(m, 2n+1)$)**.** *For any k-strict partition λ and integer $p \in [1, n+k]$, we have*

$$\tau_p \cdot \tau_\lambda = \sum_{\lambda \to \mu,\ |\mu| = |\lambda| + p} 2^{N'(\lambda,\mu)} \, \tau_\mu. \tag{14}$$

Proof. The result follows immediately from Theorem 1.1 and (12). $\square$

Example 2.1. For $\mathrm{OG}(4, 13)$ we have $k = 2$ and $n = 6$ as in Example 1.1. Using the diagrams displayed there, we obtain

$$\tau_4 \cdot \tau_{(5,3,2,2)} = 2\tau_{(8,4,2,2)} + \tau_{(7,5,2,2)} + 2\tau_{(7,4,3,2)} + \tau_{(6,5,3,2)} + \tau_{(8,4,3,1)}.$$

2.3. **Presentation of** $\mathrm{H}^*(\mathrm{OG},\mathbb{Z})$**.** If $\mathcal{S}$ (respectively $\mathcal{Q}$) denotes the tautological subbundle (respectively, quotient bundle) over $\mathrm{OG}(n-k,2n+1)$, then one has that

$$c_p(\mathcal{Q}) = \begin{cases} \tau_p & \text{if } p \leq k, \\ 2\tau_p & \text{if } p > k. \end{cases} \tag{15}$$

Let $\delta_p = 1$, if $p \leq k$, and $\delta_p = 2$, otherwise.

Theorem 2.2. a) *The cohomology ring* $\mathrm{H}^*(\mathrm{OG}(n-k,2n+1),\mathbb{Z})$ *is presented as a quotient of the polynomial ring* $\mathbb{Z}[\tau_1,\ldots,\tau_{n+k}]$ *modulo the relations*

$$\det(\delta_{1+j-i}\tau_{1+j-i})_{1\leq i,j\leq r} = 0, \qquad n-k+1 \leq r \leq n, \tag{16}$$

$$\sum_{p=k+1}^{r} (-1)^p \tau_p \det(\delta_{1+j-i}\tau_{1+j-i})_{1\leq i,j\leq r-p} = 0, \qquad n+1 \leq r \leq n+k, \tag{17}$$

and

$$\tau_r^2 + \sum_{i=1}^{r} (-1)^i \delta_{r-i}\tau_{r+i}\tau_{r-i} = 0, \qquad k+1 \leq r \leq n. \tag{18}$$

b) *The monomials* $\tau^\lambda = \prod_i \tau_{\lambda_i}$ *with* $\lambda \in \mathcal{P}(k,n)$ *form a* $\mathbb{Z}$*-basis for* $\mathrm{H}^*(\mathrm{OG},\mathbb{Z})$.

Proof. The statement (b) is proved exactly as the corresponding part of Theorem 1.2, in particular the special Schubert classes τ_p generate $\mathrm{H}^*(\mathrm{OG})$. We will use Lemma 1.1, and begin by noting that the rank of $\mathrm{H}^*(\mathrm{OG})$ as a $\mathbb{Z}$-module is the same as that of $\mathrm{H}^*(\mathrm{IG}(n-k,2n))$. The Whitney sum formula shows that the relations (16) hold in the cohomology ring of OG. The remaining relations follow easily from the Pieri rule. For (17), one observes that for each r,

$$\det(\delta_{1+j-i}\tau_{1+j-i})_{1\leq i,j\leq r} = \tau_{(1^r)}.$$

This proves that condition (i) of Lemma 1.1 holds.

To prove (ii), let K be any field of characteristic different than 2. For each r, set $h_r = \det(c_{1+j-i})_{1\leq i,j\leq r}$. Using the change of variables in (15), we see that it suffices to show that the quotient of the ring $K[c_1,\ldots,c_{n+k}]$ modulo the relations

$$h_r = 0, \quad n-k+1 \leq r \leq n, \qquad \sum_{p=k+1}^{r} (-1)^p c_p h_{r-p} = 0, \quad n+1 \leq r \leq n+k,$$

and

$$c_r^2 + 2\sum_{i=1}^{r} (-1)^i c_{r+i}c_{r-i} = 0, \qquad k+1 \leq r \leq n$$

is finite dimensional. By Lemma 1.2, it will suffice to show that the relations $h_r = 0$ for $n+1 \leq r \leq n+k$ are also true in this ring. The formal identity

$$h_{n+1} = \left(\sum_{p=1}^{k} (-1)^{p+1} c_p h_{n+1-p}\right) + \left(\sum_{p=k+1}^{n+1} (-1)^{p+1} c_p h_{n+1-p}\right)$$

and the first line of relations prove that $h_{n+1} = 0$; the vanishing of h_r for $r > n+1$ is established similarly.

Next, suppose that $\mathrm{char}(K) = 2$. We will show that each generator τ_p is a nilpotent element of the quotient of $K[\tau_1,\ldots,\tau_{n+k}]$ modulo relations (16), (17), and (18). In characteristic 2, the equations (16) assert the vanishing of Schur determinants in $\tau_1, \dots, \tau_k$ for $r = n-k+1, \dots, n$, and we know by Lemma 1.2 that this implies $\tau_1, \dots, \tau_k$ are nilpotent. Now rewrite (17) as

$$\tau_r = \sum_{p=k+1}^{r-1} (-1)^{p+1} \tau_p \det(\delta_{1+j-i}\tau_{1+j-i})_{1\le i,j\le r-p}, \qquad n+1 \le r \le n+k.$$

As all determinants on the right hand side are nilpotent, so is each τ_r for $r = n+1, \dots, n+k$. Finally, the relations (18) are used to complete the proof of (a). $\square$

2.4. **Gromov-Witten invariants.** In this and the following section we assume that $k > 0$, or equivalently $m < n$. Given a degree $d \ge 0$ and partitions $\lambda, \mu, \nu \in \mathcal{P}(k,n)$ such that $|\lambda|+|\mu|+|\nu| = \dim(\mathrm{OG})+d(n+k)$, we define the Gromov-Witten invariant $\langle \tau_\lambda, \tau_\mu, \tau_\nu \rangle_d$ to be the number of rational maps $f \colon \mathbb{P}^1 \to \mathrm{OG}$ of degree d such that $f(0) \in X_\lambda(E_\bullet)$, $f(1) \in X_\mu(F_\bullet)$, and $f(\infty) \in X_\nu(G_\bullet)$, for given isotropic flags $E_\bullet$, $F_\bullet$, and $G_\bullet$ in general position. Here a rational map of degree d to OG is a morphism $f : \mathbb{P}^1 \to \mathrm{OG}$ such that $\int f_*[\mathbb{P}^1] \cdot \tau_1 = d$. We define the kernel and span of such a map as in Section 1.4; again we have $\dim \mathrm{Ker}(f) \ge m-d$ and $\dim \mathrm{Span}(f) \le m+d$ for any map $f : \mathbb{P}^1 \to \mathrm{OG}$ of degree d.

In the orthogonal Lie types, the basic model for degree d maps to OG will have target $\mathrm{OG}(d, 2d)$, when d is even, but a different space $\overline{\mathrm{OG}} = \overline{\mathrm{OG}}(d, 2d)$ when d is odd. The variety $\overline{\mathrm{OG}}$ parametrizes isotropic subspaces of dimension d in a vector space $W \cong \mathbb{C}^{2d}$ equipped with a degenerate symmetric bilinear form such that $\mathrm{Ker}(W \to W^*)$ has dimension 2.

For any integer $d \le m$ the variety Y_d parametrizes pairs (A, B) of subspaces of V with $A \subset B \subset A^\perp$, $\dim A = m-d$, and $\dim B = m+d$, as in Section 1.4. Moreover, for any Schubert variety X_λ in OG, the subvariety Y_λ of Y_d consists of the set of pairs $(A,B) \in Y_d$ such that there exists a $\Sigma \in X_\lambda$ with $A \subset \Sigma \subset B$. Here we compute that $\dim(Y_d) = \dim(\mathrm{OG}) + d(n+k) - 3d(d-1)/2$, and we'll show that the codimension of Y_λ in Y_d is at least equal to $|\lambda| - d(d-1)/2$. When $d = m+1$, the role of Y_d is played by the Grassmannian $Y_{m+1} := \mathrm{G}(2m+1, V)$ of all subspaces $B \subset V$ with $\dim B = 2m+1$, and the varieties $Y_\lambda \subset Y_{m+1}$ are defined in the same way.

Let T_d be the variety of triples (A, Σ, B) such that $(A,B) \in Y_d$, $\Sigma \in \mathrm{OG}$, and $A \subset \Sigma \subset B$. Let $\pi : T_d \to Y_d$ be the projection. For $\lambda \in \mathcal{P}(k,n)$ we define $T_\lambda(E_\bullet) = \{(A, \Sigma, B) \in T_d \mid \Sigma \in X_\lambda(E_\bullet)\}$. We have $\pi(T_\lambda(E_\bullet)) = Y_\lambda(E_\bullet)$.

When $d \le m$, we let Y'_d denote the variety parametrizing triples (A, A', B) of subspaces of V with $(A,B) \in Y_d$ and A' isotropic with $A \subset A' \subset B$ and $\dim A' = \dim A + 1$. There is a projection map $\varphi : Y'_d \to Y_d$. Moreover, to each Schubert variety X_λ in OG there corresponds a subvariety Y'_λ of Y'_d, defined as the locus of (A, A', B) such that there exists a $\Sigma \in X_\lambda$ with $A' \subset \Sigma \subset B$. Let T'_d denote the variety of nested spaces (A, A', Σ, B) with $\Sigma \in \mathrm{OG}$ and $B \subset A^\perp$. There is a projection map $\pi' \colon T'_d \to Y'_d$. We define T'_λ to consist of (A, A', Σ, B) in T'_d with $\Sigma \in X_\lambda$. Then $\pi'(T'_\lambda) = Y'_\lambda$.

Attached to each $(A, B) \in Y_d$ there is an invariant $r = \dim(B \cap B^\perp) - \dim(A)$. The quantity r measures the degeneracy of the induced bilinear form on B/A. The general point of Y_d has $r = 0$, while the $r = 2$ locus of Y_d is a locally closed subvariety of Y_d. For $\lambda \in \mathcal{P}(k,n)$ we define $N_d = N_d(\lambda) = \#\{j \le m \mid \lambda_j = d - j \le k\}$.

Lemma 2.1. (a) *The restricted projection* $\pi : T_\lambda(E_\bullet) \to Y_\lambda(E_\bullet)$ *is generically* 2^{N_d}-*to*-1 *when* $\rho_{d-1} \subset \lambda$ *and has fibers of positive dimension when* $\rho_{d-1} \not\subset \lambda$.

When $d \le m$, *we furthermore have:*

(b) *The restriction of* π *over the* $r = 2$ *locus of* $Y_\lambda(E_\bullet)$ *is generically unramified* 2^{N_d}-*to*-1 *when* $(\rho_{d-1}, 1) \subset \lambda$ *and has fibers of positive dimension when* $(\rho_{d-1}, 1) \not\subset \lambda$.

(c) *The map* $\pi' \colon T'_\lambda(E_\bullet) \to Y'_\lambda(E_\bullet)$ *is generically* 1-*to*-1 *when* $\rho_{d-1} \subset \lambda$.

(d) *The restriction of* π' *to the* $r = 2$ *locus is generically* 1-*to*-1 *when* $(\rho_{d-1}, 1) \subset \lambda$.

Proof. We can assume that $E_r = \mathrm{Span}\{e_1, \ldots, e_r\}$ where $\{e_1, \ldots, e_{2n+1}\}$ is a standard orthogonal basis for V, i.e. $(e_i, e_j) = 0$ when $i + j \ne 2n + 2$. We write X_λ for $X_\lambda(E_\bullet)$, Y_λ for $Y_\lambda(E_\bullet)$, and T_λ for $T_\lambda(E_\bullet)$. Set $p_j = \bar{p}_j(\lambda)$ for $1 \le j \le m$, $p_0 = 0$, $p_{m+1} = 2n + 2$, $\#_j = \#\{i < j \,|\, p_i + p_j > 2n + 2\}$, and $s_j = \max(m + d + p_j - 2n - 1 - j, 0)$. We first prove statements (a)–(d) in the case $d \le m$, then we state the modifications necessary to obtain (a) in case $d = m + 1$.

For (A, B) in Y_λ, we set $A_j = A \cap E_{p_j}$ and $B_j = B \cap E_{p_j}$. By definition there exists Σ in X_λ with $A \subset \Sigma \subset B$. We set $\Sigma_j = \Sigma \cap E_{p_j}$, so that $\dim(\Sigma_j) \ge j$. Now,

$$\dim(A_j) \ge j - d, \tag{19}$$

$$\dim(B_j) \ge j + s_j, \tag{20}$$

since A is of codimension d in Σ, and B_j is the intersection of spaces of dimension $m + d$ and p_j, while also $\Sigma_j \subset B_j$. For j such that equality holds in (20) we have $j \le \dim(\Sigma_{j-\varepsilon}^\perp \cap B_j) = j + s_j - \dim(\Sigma_{j-\varepsilon}) + \dim(\Sigma_{j-\varepsilon} \cap B_j^\perp)$ for $\varepsilon \in \{0, 1\}$, since $(\Sigma_{j-\varepsilon}^\perp \cap B_j)^\perp = \Sigma_{j-\varepsilon} + B_j^\perp$. So, equality in (20) implies

$$\dim(B_j \cap B_j^\perp) \ge j - s_j, \tag{21}$$

$$\dim(B_{j-1} \cap B_j^\perp) \ge j - 1 - s_j. \tag{22}$$

When $\lambda_j = d - j \le k$ we have $s_j = 1 + \#_j$, and hence $\dim(\Sigma_{j-1} \cap B_j^\perp) \ge j - 1 - \#_j = j - s_j$, which gives us the following strengthening of (22):

$$\dim(B_{j-1} \cap B_j^\perp) \ge \begin{cases} j - s_j, & \text{when } \lambda_j = d - j \le k, \\ j - 1 - s_j, & \text{otherwise.} \end{cases} \tag{23}$$

Define U_λ to be the open subset of Y_λ of (A, B) satisfying equality in (19) for $j \ge d$ and in (20), (21), (23) for $1 \le j \le d$. We show: U_λ contains a point in the $r = 0$ locus when $\rho_{d-1} \subset \lambda$ and contains a point in the $r = 2$ locus when $(\rho_{d-1}, 1) \subset \lambda$. Then we show for (A, B) in U_λ that there are precisely 2^{N_d} possible $\Sigma \in X_\lambda$ satisfying $A \subset \Sigma \subset B$.

The condition $\lambda_j \ge d - j$ implies $s_j = 1 + \#_j$ when $\lambda_j = d - j \le k$, and $s_j \le \#_j$ otherwise. For $j > d$ we also have $s_j \le \#\{i \le d \,|\, p_i + p_j > 2n + 2\}$. We can therefore choose $\pi \in \mathfrak{S}_d$ such that for all $1 \le j \le m + 1$ and $i \le s_j$ we have

$$\pi(i) \le \begin{cases} j, & \text{when } \lambda_j = d - j \le k, \\ j - 1, & \text{otherwise,} \end{cases} \tag{24}$$

$$p_{\pi(i)} + p_j \ge 2n + 3 + s_j - i. \tag{25}$$

Further, we dictate that among all permutations satisfying (24), (25), π is chosen so that $\pi^{-1}(d)$ is as large as possible.

We exhibit a point in the $r = 0$ locus of U_λ. Set $A = \mathrm{Span}(e_{p_{d+1}}, \ldots, e_{p_m})$ and $B = \mathrm{Span}(e_{p_1}, \ldots, e_{p_m}, u_1, \ldots, u_d)$ where $u_i = e_{2n+2-p_{\pi(i)}} + e_{p_j - 1 - s_j + i}$ for $s_{j-1} < i \le s_j$. Equality holds in (19) for $j \ge d$ and, by (25) equality holds in

(20) for all j. The pairings $(e_{p_{\pi(i)}}, u_i) = 1$ and $(e_{p_{\pi(i)}}, u_{i'}) = 0$ for $i' > i$ imply the nondegeneracy of the restriction of $(\ ,\ )$ to $\mathrm{Span}(e_{p_{\pi(1)}}, \ldots, e_{p_{\pi(s_j)}}, u_1, \ldots, u_{s_j})$ for any j. So (A, B) lies in the $r = 0$ locus of Y_λ, and from (24) we have equality for all j of (21) and (23), hence $(A, B) \in U_\lambda$.

Under the further assumption that $\ell(\lambda) \geq d$, we exhibit a point in the $r = 2$ locus of U_λ. We set $i_0 = \pi^{-1}(d)$, define j_0 to satisfy $s_{j_0-1} < i_0 \leq s_{j_0}$, and set $t_0 = p_{j_0} - 1 - s_{j_0} + i_0$ By (25), we have $t_0 \geq 2n + 2 - p_d$. Now we distinguish two cases.

First is the case $t_0 > 2n + 2 - p_d$. Then we define A and B as above, but with $u_{i_0} = e_{t_0}$. By (24), $j_0 > d$, and we have $t_0 > p_{j_0-1} \geq p_d$. The computation of $\dim(B_j)$ goes through as before. We have the pairing $(e_{p_{\pi(i)}}, u_i) = 1$ for $i \neq i_0$, and $(e_{p_{\pi(i)}}, u_{i'}) = 0$ for all i and i' with $i' > i$. Since $s_d < i_0$, the argument for equality in (21), (23) for $j \leq d$ goes through as before. Now we claim $(e_{p_j}, u_{i_0}) = 0$ for all $j \leq d$. So we have e_{p_d}, $u_{i_0} \in B^\perp$, while a direct argument shows that the bilinear form restricted to $\mathrm{Span}(e_{p_1}, \ldots, e_{p_{d-1}}, u_1, \ldots, u_{i_0-1}, u_{i_0+1}, \ldots, u_d)$ is nondegenerate. Hence (A, B) lies in the $r = 2$ locus of U_λ.

Suppose, to the contrary, there exists $c \leq d$ with $(e_{p_c}, u_{i_0}) \neq 0$. Then $p_c = 2n + 2 - t_0$, and $c < d$. We set $i_1 = \pi^{-1}(c)$ and define j_1 so $s_{j_1-1} < i_1 \leq s_{j_1}$. Then

$$p_{j_1} - 1 - s_{j_1} + i_1 \geq 2n + 2 - p_c = p_{j_0} - 1 - s_{j_0} + i_0, \tag{26}$$

hence the inequality is strict, and $i_1 > i_0$. Now, if we define $\tilde{\pi} \in \mathfrak{S}_d$ by $\tilde{\pi}(i) = \pi(i)$ for $i \notin \{i_0, i_1\}$, with $\tilde{\pi}(i_0) = c$ and $\tilde{\pi}(i_1) = d$, then $\tilde{\pi}$ satisfies (24), (25). The only inequality needed to be checked is $p_c + p_{j_0} \geq 2n + 3 + s_{j_0} - i_0$, and this holds by (26). Since $\tilde{\pi}$ satisfies $\tilde{\pi}^{-1}(d) > \pi^{-1}(d)$, we have reached a contradiction.

In case $t_0 = 2n + 2 - p_d$ we define A and B as above, but with $u_{i_0} = e_c$, where c is defined to be the smallest positive integer not equal to p_j or to $2n + 2 - p_j$ for any j. A general remark is that if equality in (25) holds for some $i \leq s_j$, then $\pi(i') > \pi(i)$ for all $i' < i$. Indeed, if $i' < i$ then

$$0 \leq p_{\pi(i')} + p_j - 2n - 3 - s_j + i' < p_{\pi(i')} + p_j - 2n - 3 - s_j + i = p_{\pi(i')} - p_{\pi(i)}.$$

In this case we have such equality for $i = i_0$. Hence $i_0 = 1$. The $n + d + 1 - k - j_0$ integers

$$p_1, \ldots, p_d,\ 2n + 2 - p_{j_0}, \ldots, 2n + 2 - p_m$$

are distinct and less than or equal to $2n + 2 - t_0$. But $n + d + 1 - k - j_0 = 2n + 2 - p_{j_0} + s_{j_0} = 2n + 2 - t_0$, so $p_1, \ldots, p_d$ must be the first d integers not equal to $2n + 2 - p_j$ for any $j > d$. An inductive argument shows that $\pi(i) = d + 1 - i$ for $i = 1, \ldots, d$, with equality in (25) for each i. As before, we have e_{p_d}, $u_{i_0} \in B^\perp$, and (A, B) lies in the $r = 2$ locus of U_λ.

Given $(A, B) \in U_\lambda$, we claim that for $j \leq d$ are precisely $2^{\#\{i \leq j \mid \lambda_i = d - i \leq k\}}$ isotropic spaces $\Sigma_j \subset B_j$ such that $\dim(\Sigma_j \cap B_i) \geq i$ for $i \leq j$. We prove this by induction on j, the base case $j = 0$ being clear. The equalities (20) and (21) imply that a maximal isotropic subspace of B_j has dimension j. So, using the induction hypothesis, we have that $\dim(\Sigma_j \cap B_i) \geq i$ for $i \leq j$ implies $\Sigma_j \cap B_{j-1} = \Sigma_{j-1}$. So, it is enough to show that Σ_{j-1} extends in precisely two ways to an isotropic subspace of dimension j in B_j when $\lambda_j = d - j \leq k$, and extends uniquely to such a subspace otherwise. If $\lambda_j = d - j \leq k$, then we have $\dim(\Sigma_{j-1}^\perp \cap B_j) = s_j + 1 + \dim(\Sigma_{j-1} \cap B_j^\perp) = j + 1$, and $\Sigma_{j-1}^\perp \cap B_j \cap (\Sigma_{j-1}^\perp \cap B_j)^\perp = \Sigma_{j-1} + (B_j \cap B_j^\perp) = \Sigma_{j-1}$ (since $B_j \cap B_j^\perp = \Sigma_{j-1} \cap B_j^\perp$), so $(\Sigma_{j-1}^\perp \cap B_j)/\Sigma_{j-1}$ is two-dimensional with induced nondegenerate symmetric bilinear form. Otherwise, we have $\dim(\Sigma_{j-1}^{\perp} \cap B_j) \le j$, so the extension is unique. By the equality (19), $\Sigma_d \cap A = 0$, so if we set $\Sigma = \Sigma_d + A$ then $\Sigma \in X_\lambda$. For any $\Sigma \in X_\lambda$ with $A \subset \Sigma \subset B$ we have $\dim(\Sigma \cap B_i) \ge i$ for all i, hence $\Sigma \cap B_d = \Sigma_d$ and $\Sigma = \Sigma_d + A$.

By the same argument, with families of vector spaces over local Artinian $\mathbb{C}$-algebras, we see that the morphism $T_\lambda \to Y_\lambda$ is unramified (hence étale) over U_λ.

Now suppose $\rho_{d-1} \not\subset \lambda$. We want to show that $\dim(Y_\lambda) < \dim(T_\lambda)$. Let $X_\lambda^\circ \subset X_\lambda$ be the Schubert cell. It suffices to show that if $(A, B) \in Y_\lambda$ with $A_d = 0$, there exists $\Sigma' \in X_\lambda \smallsetminus X_\lambda^\circ$ with $A \subset \Sigma' \subset B$. If this is false, then we can choose $\Sigma \in X_\lambda^\circ$ with $A \subset \Sigma \subset B$. Set $\Sigma_i = \Sigma \cap E_{p_i}$ for $1 \le i \le m$. We have $\lambda_j < d - j$ for some j with $\lambda_j \le k$. Then $s_j > 1 + \#_j$, hence $\dim(\Sigma_{j-1}^{\perp} \cap E_{p_j}) \ge p_j - \#_j > p_j - s_j + 1$. Using (20) we obtain $\dim(\Sigma_{j-1}^{\perp} \cap B_j) > j + 1$, so there exists a j-dimensional isotropic extension Σ_j' of Σ_{j-1} contained in $B \cap E_{p_j-1}$. For $i = j + 1, \ldots, d$ we now choose an i-dimensional isotropic extension Σ_i' of Σ_{i-1}' contained in B_i. The subspace $\Sigma' = \Sigma_d' + A$ then satisfies $\Sigma' \in X_\lambda \smallsetminus X_\lambda^\circ$ and $A \subset \Sigma' \subset B$, and statement (a) is proved.

To complete the proof of (b) it suffices by semi-continuity of fiber dimensions to treat the case that $\rho_{d-1} \subset \lambda$ and $\ell(\lambda) = d - 1$. Generically on the $r = 2$ locus of Y_λ, the intersection of $B \cap B^\perp$ with $E_{p_{d-1}}$ is trivial, since $\lambda_{d-1} \ge 1$. So it suffices to show that if (A, B) lies in the $r = 2$ locus of Y_λ and satisfies $A_d = 0$ and $B^\perp \cap B_{d-1} = 0$ then there exists $\Sigma' \in X_\lambda \smallsetminus X_\lambda^\circ$ with $A \subset \Sigma' \subset B$. If this is false, then we can choose $\Sigma \in X_\lambda^\circ$ with $A \subset \Sigma \subset B$. Since $\lambda_d = 0$, the assumptions imply $\dim(B^\perp \cap B_d) = 2$. We set $\Sigma_{d-1} = \Sigma \cap E_{p_{d-1}}$. Since $B^\perp \cap B_d$ meets Σ_{d-1} trivially, there exists an isotropic extension Σ_d' of Σ_{d-1} of dimension d contained in $B \cap E_{p_d-1}$. The space $\Sigma' = \Sigma_d' + A$ then satisfies $\Sigma' \in X_\lambda \smallsetminus X_\lambda^\circ$.

Statements (c) and (d) follow from (a) and (b). If $(A, B) \in U_\lambda$ then the generic A' with $(A, A', B) \in Y_\lambda'$ is contained in a unique Σ with $A \subset \Sigma \subset B$ and $\Sigma \in X_\lambda$.

In case $d = m + 1$, then exactly as above we have (20), and equality in (20) implies (21) and (23). We define $U_\lambda \subset Y_\lambda$ by equality in (20), (21), (23) for $1 \le j \le m$. When $\rho_{d-1} \subset \lambda$, the construction of a point in the $r = 0$ locus of U_λ is as above, except that we take $u_{i_0} = e_{d+1-i_0} + e_{2n+1-d+i_0}$ where $i_0 = \pi^{-1}(d)$, and this satisfies $(e_{p_j}, u_{i_0}) = 0$ for $1 \le j \le m$ (if for some c we have $(e_{p_c}, u_{i_0}) \ne 0$, then $p_c = d + 1 - i_0$, and this contradicts the maximality of $\pi^{-1}(d)$, as argued above). Then $B = \mathrm{Span}(e_{p_1}, \ldots, e_{p_m}, u_1, \ldots, u_d)$ lies in the $r = 0$ locus of U_λ, since $(u_{i_0}, u_{i_0}) \ne 0$. The argument that exhibits precisely $2^{\#\{i \le j \mid \lambda_i = d-i \le k\}}$ isotropic $\Sigma_j \subset B_j$ such that $\dim(\Sigma_j \cap B_i) \ge i$ for $i \le j$ goes through for $j \le m$, leading to $\Sigma = \Sigma_m \in X_\lambda$. The argument when $\rho_{d-1} \not\subset \lambda$ is unchanged except that i ranges from $j + 1$ to m in the construction of Σ_i', and then $\Sigma' = \Sigma_m'$ lies in $X_\lambda \smallsetminus X_\lambda^\circ$. $\square$

To state our main theorem relating Gromov-Witten invariants to classical intersection numbers, we need some further notation. We let Z_d° be the $r = 2$ locus of Y_d. Define Z_d to be the variety parametrizing triples (A, N, B) of subspaces such that $A \subset N \subset B \subset N^\perp$, $\dim A = m - d$, $\dim N = m - d + 2$, and $\dim B = m + d$. For $\lambda \in \mathcal{P}(k, n)$, define a subvariety $Z_\lambda \subset Z_d$ by setting

$$Z_\lambda(E_\bullet) = \{(A, N, B) \in Z_d \mid \exists\, \Sigma \in X_\lambda(E_\bullet) : A \subset \Sigma \subset B\}\,.$$

Notice that there is a dense open subset of Z_d (where $\dim(B \cap B^\perp) = \dim(A) + 2$) that is isomorphic to Z_d°.

Let Z'_d denote the variety parametrizing 4-tuples (A, A', N, B) of subspaces of V with $(A, N, B) \in Z_d$ and A' isotropic with $A \subset A' \subset B$ and $\dim A' = \dim A + 1$. There is a projection map $\psi : Z'_d \to Z_d$. For $\lambda \in \mathcal{P}(k, n)$, define a subvariety $Z'_\lambda \subset Z'_d$ as the locus of (A, A', N, B) such that there exists a $\Sigma \in X_\lambda$ with $A' \subset \Sigma \subset B$.

In contrast to the symplectic case, the map $T_\lambda \to Y_\lambda$ can be finite-to-one for orthogonal Grassmannians, so it is important to distinguish between the image $Y_\lambda = \pi(T_\lambda)$ and the image class $\pi_*[T_\lambda]$. For this purpose we set $\upsilon_\lambda = \pi_*[T_\lambda]$ and $\upsilon'_\lambda = \pi'_*[T'_\lambda]$. We analogously define ζ_λ and ζ'_λ to be the image classes, taking values in the homology (or Chow groups) of Z_d and Z'_d respectively, of the variety of (A, N, Σ, B) and (A, A', N, Σ, B) respectively, with $\Sigma \in X_\lambda$. The function sending a rational map $f : \mathbb{P}^1 \to \mathrm{OG}$ to the pair $(\mathrm{Ker}(f), \mathrm{Span}(f))$ in general will be 2^{M_d}-to-1, where $M_d = M_d(\lambda, \mu, \nu)$ is a multiplicity defined by

$$(27)\quad M_d(\lambda^1, \lambda^2, \lambda^3) = \sum_{i=1}^{3} N_d(\lambda^i) - \begin{cases} \min\big(\#\{\, i \mid N_d(\lambda^i) \geq 1 \,\}, 2\big) & \text{if } d \leq m, \\ 0 & \text{if } d = m+1. \end{cases}$$

For each pair $(A, B) \in Y_d$, we identify the Grassmannian $\overline{\mathrm{G}}(d, B/A)$ with the locus of isotropic Σ of dimension m such that $A \subset \Sigma \subset B$. Let $r(d)$ be 0 when d is even and 2 when d is odd. When $d \leq m$, let $S_d = S_d(\lambda, \mu, \nu)$ be the set of pairs (A, B) in the $r = r(d)$ locus of $Y_\lambda \cap Y_\mu \cap Y_\nu$ such that if two or three among the sets $X_\lambda \cap \overline{\mathrm{G}}(d, B/A)$, $X_\mu \cap \overline{\mathrm{G}}(d, B/A)$, $X_\nu \cap \overline{\mathrm{G}}(d, B/A)$ have cardinality one, the corresponding two or three subspaces of B/A are in pairwise general position. When $d = m + 1$ is even, let $S_d = Y_\lambda \cap Y_\mu \cap Y_\nu$, and when $d = m + 1$ is odd, let $S_d = \emptyset$.

Theorem 2.3. *Let $d \geq 0$ and choose $\lambda, \mu, \nu \in \mathcal{P}(k, n)$ such that $|\lambda| + |\mu| + |\nu| = \dim(\mathrm{OG}) + d(n + k)$. Let X_λ, X_μ, and X_ν be Schubert varieties of* $\mathrm{OG}(m, 2n + 1)$ *in general position, with associated subvarieties and classes in Y_d, Y'_d, Z_d, and Z'_d.*

(a) *The subvarieties Y_λ, Y_μ, and Y_ν intersect transversally in Y_d, and their intersection is finite. For each point $(A, B) \in Y_\lambda \cap Y_\mu \cap Y_\nu$ we have $A = B \cap B^\perp$ or* $\dim(B \cap B^\perp) = \dim A + 2$.

(b) *The assignment $f \mapsto (\mathrm{Ker}(f), \mathrm{Span}(f))$ gives a 2^{M_d}-to-1 association between rational maps $f : \mathbb{P}^1 \to \mathrm{OG}$ of degree d such that $f(0) \in X_\lambda$, $f(1) \in X_\mu$, $f(\infty) \in X_\nu$ and the subset S_d of $Y_\lambda \cap Y_\mu \cap Y_\nu$.*

(c) *When $d \leq m$ is even, the Gromov-Witten invariant $\langle \tau_\lambda, \tau_\mu, \tau_\nu \rangle_d$ is equal to*

$$\int_{Y_d} \upsilon_\lambda \cdot \upsilon_\mu \cdot \upsilon_\nu - \frac{1}{2} \int_{Y'_d} (\varphi^* \upsilon_\lambda \cdot \upsilon'_\mu \cdot \upsilon'_\nu + \upsilon'_\lambda \cdot \varphi^* \upsilon_\mu \cdot \upsilon'_\nu + \upsilon'_\lambda \cdot \upsilon'_\mu \cdot \varphi^* \upsilon_\nu)$$
$$- \int_{Z_d} \zeta_\lambda \cdot \zeta_\mu \cdot \zeta_\nu + \frac{1}{2} \int_{Z'_d} (\psi^* \zeta_\lambda \cdot \zeta'_\mu \cdot \zeta'_\nu + \zeta'_\lambda \cdot \psi^* \zeta_\mu \cdot \zeta'_\nu + \zeta'_\lambda \cdot \zeta'_\mu \cdot \psi^* \zeta_\nu).$$

When $d \leq m$ is odd, the Gromov-Witten invariant $\langle \tau_\lambda, \tau_\mu, \tau_\nu \rangle_d$ is equal to

$$\int_{Z_d} \zeta_\lambda \cdot \zeta_\mu \cdot \zeta_\nu - \frac{1}{2} \int_{Z'_d} (\psi^* \zeta_\lambda \cdot \zeta'_\mu \cdot \zeta'_\nu + \zeta'_\lambda \cdot \psi^* \zeta_\mu \cdot \zeta'_\nu + \zeta'_\lambda \cdot \zeta'_\mu \cdot \psi^* \zeta_\nu).$$

When $d = m + 1$ is even, we have

$$\langle \tau_\lambda, \tau_\mu, \tau_\nu \rangle_d = \int_{\mathrm{G}(2m+1, 2n+1)} \upsilon_\lambda \cdot \upsilon_\mu \cdot \upsilon_\nu.$$

(d) *We have* $\langle \tau_\lambda, \tau_\mu, \tau_\nu \rangle_d = 0$ *if* λ *does not contain* ρ_{d-1} *when* d *is even, or does not contain* $(\rho_{d-1}, 1)$ *when* d *is odd.*

Proof. Given integers $0 \leq e_1 \leq m$, $0 \leq e_2 \leq d \leq m+1$, and $r \geq 0$, we let $Y^r_{e_1,e_2}$ be the variety of pairs (A, B) such that $A \subset B \subset A^\perp \subset V$, $\dim(A) = m - e_1$, $\dim(B) = m + e_2$, and $\dim(B \cap B^\perp) = m - e_1 + r$. In general, $Y^r_{e_1,e_2}$ is empty unless $r \leq \min(e_1 + e_2, 2k + e_1 - e_2)$. These varieties have a transitive action of SO_{2n+1}, and, when $d \leq m$, Y_d is the union of the varieties $Y^r_{d,d}$ for all numbers r with $0 \leq r \leq \min(2d, 2k)$. Set

$$Y^r_\lambda = \{(A, B) \in Y^r_{e_1,e_2} \mid \exists\, \Sigma \in X_\lambda : A \subset \Sigma \subset B\}\,.$$

We claim that $Y^r_\lambda \cap Y^r_\mu \cap Y^r_\nu$ is empty unless (i) $e_1 = e_2 = d$ and $r \in \{0, 2\}$, or (ii) $e_1 = m$, $e_2 = d = m + 1$, and $r = 0$. In both of these cases, $Y^r_\lambda \cap Y^r_\mu \cap Y^r_\nu$ will be a finite set of points.

Following the proof of Theorem 1.3 we compute that the dimension of $Y^r_{e_1,e_2}$ is equal to

$$\frac{1}{2}(n^2 + 2nk - 3k^2 + n - k + 2ne_1 - 2ke_1 - e_1^2 + 4e_2k - 2e_2^2 + e_1 + 2e_2 - r - r^2)\,.$$

Define the variety $T^r = \{(A, \Sigma, B) \mid (A, B) \in Y^r_{e_1,e_2}, \Sigma \in \mathrm{OG}, A \subset \Sigma \subset B\}$, and let $\pi_1 : T^r \to \mathrm{OG}$ and $\pi_2 : T^r \to Y^r_{e_1,e_2}$ be the projections. Then $Y^r_\lambda = \pi_2(\pi_1^{-1}(X_\lambda))$. Given a point $(A, B) \in Y^r_{e_1,e_2}$, the fiber $\pi_2^{-1}((A, B))$ is the union of the varieties

$$Q_s = \{\Sigma \in \mathrm{OG} \mid A \subset \Sigma \subset B,\ \dim(\Sigma \cap B \cap B^\perp) = m - e_1 + s\} \tag{28}$$

for all integers s. We have that Q_s is empty unless $2s \geq r + e_1 - e_2$. The map $Q_s \to \mathrm{OG}(e_1 - s, B/(B \cap B^\perp))$ given by $\Sigma \mapsto (\Sigma + (B \cap B^\perp))/(B \cap B^\perp)$ has fibers isomorphic to open subsets of $\mathrm{G}(e_1, e_1 + r - s)$. It follows that

$$\dim Q_s = \frac{1}{2}(2e_1e_2 - e_1^2 - e_1 + 2e_1s - 2e_2s + 2rs - 3s^2 + s)\,,$$

whenever Q_s is not empty.

If s is such that Q_s has the same dimension as the fibers of π_2, then the codimension of Y^r_λ in $Y^r_{e_1,e_2}$ is at least $|\lambda| - \dim Q_s$. It follows that $\mathrm{codim}(Y^r_\lambda) + \mathrm{codim}(Y^r_\mu) + \mathrm{codim}(Y^r_\nu) - \dim(Y^r_{e_1,e_2})$ is greater than or equal to the number $\Delta'(e_1, e_2, r, s) := \dim \mathrm{OG} + d(n+k) - \dim(Y^r_{e_1,e_2}) - 3\dim Q_s$. A computation shows that $\Delta'(e_1, e_2, r, s)$ is equal to

$$(r - 3s)(r - 3s + 1)/2 + (n + k)(d - e_2) + (n - k - 1 + e_2 - 2e_1 + 3s)(e_2 - e_1)\,.$$

Suppose $Y^r_\lambda \cap Y^r_\mu \cap Y^r_\nu \neq \emptyset$; then the Kleiman-Bertini theorem implies that $\Delta'(e_1, e_2, r, s) \leq 0$. If $e_2 > e_1$ then we must have $e_2 = e_1 + 1 = d = m + 1$ and $r = s = 0$. When $e_2 \leq e_1$ an argument as in the proof of Theorem 1.3 shows that $e_1 = e_2 = d$. But then $2\Delta'(d, d, r, s) = (r - 3s)(r - 3s + 1) \leq 0$, which implies that $r = 3s$ or $r = 3s - 1$. Combining this with $2s \geq r$, we get $(r, s) = (0, 0)$ or $(r, s) = (2, 1)$. So $Y_\lambda \cap Y_\mu \cap Y_\nu$ is finite and contained in the union of the $r = 0$ and $r = 2$ loci, and $Y^0_\lambda \cap Y^0_\mu \cap Y^0_\nu$ and $Y_\lambda \cap Y_\mu \cap Y_\nu \cap Z^\circ_d$ are smooth. The transversality assertion (a) is, however, stronger. Its proof uses the following result.

Lemma 2.2. *Let* $f \colon Y \to X$ *be a finite flat morphism of schemes of finite type over an algebraically closed field* k*. Suppose that for any closed points* $y \in Y$ *and* $x \in X$ *with* $f(y) = x$ *there exists a (smooth, connected, quasi-projective) pointed curve* (C, p) *and a map* $C \to X$ *sending* p *to* x *with the property that the projection*

$\pi\colon C \times_X Y \to C$ admits a section s such that the restriction of π to $s(C \smallsetminus \{p\})$ is unramified. Then f is unramified over a nonempty open subset of X.

Proof. Denote by d the degree of f, and for $x \in X$ let $n_f(x)$ be the number of geometric points of $f^{-1}(x)$. By [EGA, IV.15.5.1] the function n_f is lower semicontinuous. The morphism f is unramified over a nonempty open subset of X if and only if $n_f(x) = d$ for some $x \in X$. So, suppose that the largest value of n_f is smaller than d, and let $x \in X$ be a closed point at which $n_f(x)$ achieves this value. Since $n_f(x) < d$ the fiber $f^{-1}(x)$ contains a non-reduced point y. Let $C \to X$, with π and s, be as in the statement of the lemma. Since C is smooth and π is flat, the irreducible component $s(C)$ of $C \times_X Y$ must intersect some other irreducible component. Letting D denote the scheme-theoretic closure of $C \times_X Y \smallsetminus s(C)$ in $C \times_X Y$, we have a morphism $C \amalg D \to C \times_X Y$ of schemes flat over C, that is an isomorphism over $C \smallsetminus \{p\}$. Denoting by ψ the morphism $D \to C$ we have $n_\psi(p) = n_\pi(p)$, hence for general $p' \in C$ we have $n_\pi(p') = 1 + n_\psi(p') \geq 1 + n_\psi(p) = 1 + n_\pi(p)$. So there exists $x' \in X$ with $n_f(x') > n_f(x)$, and we have reached a contradiction. $\square$

We observe that $Y_{(\rho_{d-1},1)}(E_\bullet) = Y_{(1^d)}(E_\bullet)$. One containment is obvious. The equality now follows from the fact that $Y_{(\rho_{d-1},1)}(E_\bullet)$ is a variety of codimension 1 in Y_d, while $Y_{(1^d)}(E_\bullet)$ is a variety but is not the whole of Y_d, since for general (A,B) we have $\dim(B \cap E_{n+k+d}) = 2d-1$ with $(\ ,\ )$ restricted to $B \cap E_{n+k+d}$ nondegenerate, so there can be no isotropic subspace of $B \cap E_{n+k+d}$ of dimension d. We have containments

$$Z_d^\circ \subset Y_{(1^d)}(E_\bullet) \subset Y_d$$

and hence an induced morphism of conormal bundles

$$\mathcal{N}_{Y_{(1^d)}/Y_d}|_{Z_d^\circ} \to \mathcal{N}_{Z_d^\circ/Y_d}.$$

Let us define $W(E_\bullet)$ to be the open subset of Z_d° consisting of (A,B) where $A \cap E_{n+k+d+1} = 0$ and the morphism of conormal bundles is nondegenerate.

Fix $\lambda^0 = (\rho_{d-1},1)$, and set $p_j^0 = \bar{p}_j(\lambda^0)$ for $1 \leq j \leq m$. Fix some permutation in $\mathfrak{S}_d$ satisfying the conditions stated in the proof of Lemma 2.1. Then the proof explicitly constructs a point (A,B) in the $r = 2$ locus of U_{λ^0}, such that $\Sigma := \mathrm{Span}(e_{p_1},\ldots,e_{p_m})$ is contained in B, and $A \cap E_{n+k+d+1} = 0$. Since $\Sigma \in X_{\lambda^0}^\circ$ and $T_{\lambda^0} \to Y_{\lambda^0}$ is étale over U_{λ^0}, the point (A,B) is a nonsingular point of $Y_{\lambda^0} = Y_{(1^d)}$. Hence we have $(A,B) \in W(E_\bullet)$. Given an arbitrary orthogonal basis $e'_1, \ldots, e'_{2n+1}$ of V, with corresponding isotropic flag $E'_\bullet$, the same construction yields a point

$$(A',B') \in W(E'_\bullet). \tag{29}$$

Now suppose $\lambda \supset \lambda^0$, and set $p_j = \bar{p}_j(\lambda)$ for $1 \leq j \leq m$. Since $\ell(\lambda) \geq d$, there exists an orthogonal basis $e'_1, \ldots, e'_{2n+1}$ consisting of basis vectors of V, such that

$$\begin{aligned} e'_{p_j^0} &= e_{p_j}, && \text{for } 1 \leq j \leq d, \\ e'_{n+k+d+1} &= e_c, && \text{for some } c > p_d, \\ e'_{n+k+d+1+j} &= e_{p_{d+j}}, && \text{for } 1 \leq j \leq m-d. \end{aligned}$$

Given such a basis, with corresponding isotropic flag $E'_\bullet = E'_\bullet(\lambda)$, we have

$$(A',B') \in Y_\lambda(E_\bullet), \tag{30}$$

$$Y_\lambda(E_\bullet) \subset Y_{(1^d)}(E'_\bullet), \tag{31}$$

the latter a consequence of the fact that $E_{p_d} \subset E'_{n+k+d}$. Notice that $Y_\lambda(E_\bullet) \cap W(E'_\bullet)$ is an open subset of the $r = 2$ locus on $Y_\lambda(E_\bullet)$, and is nonempty by (29) and (30).

In the situation where $Y^2_\lambda \cap Y^2_\mu \cap Y^2_\nu \neq \emptyset$ we must have $\lambda \supset \lambda^0$ by Lemma 2.1 (b). Letting $Z^\circ_\lambda(E_\bullet, E'_\bullet(\lambda)) = Y_\lambda(E_\bullet) \cap W(E'_\bullet(\lambda))$, we may assume by Kleiman-Bertini that $Y^2_\lambda \cap Y^2_\mu \cap Y^2_\nu$ is contained in the intersection of the translates of $Z^\circ_\lambda(E_\bullet, E'_\bullet(\lambda))$, $Z^\circ_\mu(E_\bullet, E'_\bullet(\mu))$, and $Z^\circ_\nu(E_\bullet, E'_\bullet(\nu))$; the assertions thus far concerning $Y_\lambda \cap Y_\mu \cap Y_\nu$ are valid on the translates coming from a nonempty open subset $U \subset \mathrm{SO}_{2n+1} \times \mathrm{SO}_{2n+1} \times \mathrm{SO}_{2n+1}$. We now verify the condition in Lemma 2.2 for the map $Y \to U$ where Y is the family of intersections of translates of Y_λ, Y_μ, and Y_ν. Suppose $y \in Y$ maps to $x \in U$, where x and y are closed points. If y is a reduced point in the fiber over x then we can take $C \to U$ to be a constant map. So we suppose y to be a non-reduced point in the fiber over x, which implies that y lies in the $r = 2$ locus, and we proceed to construct (C, p) and $C \to U$ as required.

Without loss of generality the first component of x is the identity element of SO_{2n+1}. So we have $(A, B) \in Z^\circ_\lambda(E_\bullet, E'_\bullet(\lambda))$. Set $N = B \cap B^\perp \cap E'_{n+k+d+1}(\lambda)$. Since $(A, B) \in W(E'_\bullet(\lambda))$ we have $\dim(N) = 2$. Let M be a vector space of dimension 4 containing N and contained in $A^\perp$, such that the restriction of $(\, , \,)$ to M is nondegenerate. Then $\mathbb{C}^{2n+1}$ is the direct sum of M and $M^\perp$.

Every element of GL_N extends uniquely to an element of SO_M, which we regard as an element of SO_{2n+1} by letting it act trivially on $M^\perp$. This way we may regard SL_N as a subgroup of SO_{2n+1}. Now SL_N acts on Y_d fixing (A, B). The subvariety Z°_d of Y_d is SL_N-invariant. So, SL_N acts on the conormal space to Z°_d in Y_d at (A, B). The component of the fixed point locus for the SL_N-action containing (A, B) is contained in Z°_d, hence the the conormal space to Z°_d at (A, B), when decomposed into irreducible SL_N-representations, has no trivial factors. A three-dimensional representation of SL_N without trivial factors is irreducible. From the orbit structure we know that the conormal vector given by a defining equation for $Y_{(1^d)}(E'_\bullet)$ inside Y_d, which is well-defined up to scale and nonzero since $(A, B) \in W(E'_\bullet(\lambda))$, has SL_N-orbit not contained in any proper linear subspace of $\mathcal{N}_{Z^\circ_d/Y_d,(A,B)}$.

We make a similar analysis of copies of SL_2 sitting inside the second and third factors of SO_{2n+1}, as above but with $E_\bullet$ and $E'_\bullet(\lambda)$ replaced by corresponding translates of $E_\bullet$ and $E'_\bullet(\mu)$ or $E'_\bullet(\nu)$. So we have u, v, $w \in \mathcal{N}_{Z^\circ_d/Y_d,(A,B)}$ and a triple of actions of SL_2 on $\mathcal{N}_{Z^\circ_d/Y_d,(A,B)}$ such that images of u, v, and w under a generic triple of group elements span $\mathcal{N}_{Z^\circ_d/Y_d,(A,B)}$. We have, then, a map

$$\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2 \to \mathrm{SO}_{2n+1} \times \mathrm{SO}_{2n+1} \times \mathrm{SO}_{2n+1}$$

sending the identity element to x and a dense open subset into U, such that (A, B) is a point in every fiber of the image and an unramified point in the fiber over the generic point of the image (the latter assertion follows from (31) and the fact that the images of u, v, and w generically span $\mathcal{N}_{Z^\circ_d/Y_d,(A,B)}$). Taking C to be a general curve in $\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2$ containing the identity, with p equal to identity, the condition of Lemma 2.2 is satisfied. So, part (a) is established.

For $d \le m$, we consider a pair $(A, B) \in Y^r_\lambda \cap Y^r_\mu \cap Y^r_\nu$. When $r = 0$, $\overline{\mathrm{G}}(d, B/A)$ has two connected components. When $r = 2$, let $N = B \cap B^\perp$ and call an isotropic Σ with $A \subset \Sigma \subset B$ valid if $\dim(\Sigma \cap N) = 1$. The locus of all valid Σ also splits into two connected components, depending on the family of the isotropic $(d-1)$-dimensional space $(\Sigma + N)/N$ in B/N. We also call any point of $\overline{\mathrm{G}}(d, B/A)$ valid when $r = 0$. Let $P \in X_\lambda \cap \overline{\mathrm{G}}(d, B/A)$, $Q \in X_\mu \cap \overline{\mathrm{G}}(d, B/A)$, and $R \in X_\nu \cap \overline{\mathrm{G}}(d, B/A)$. It is easy to construct examples where $P \cap Q \neq A$; however a dimension counting argument as in the proof of Theorem 1.3 shows that $0 \leq \dim(P \cap Q) - \dim(A) \leq 1$.

Consider a morphism $f : \mathbb{P}^1 \to \mathrm{OG}$ of degree d such that $f(0) \in X_\lambda$, $f(1) \in X_\mu$, and $f(\infty) \in X_\nu$. Then $(\mathrm{Ker}(f), \mathrm{Span}(f))$ must be a point of an intersection $Y^r_\lambda \cap Y^r_\mu \cap Y^r_\nu$ in some space $Y^r_{e_1,e_2}$; thus $d \leq m+1$. Moreover, we must have $(e_1, e_2) = (d, d)$ and $(r, s) \in \{(0,0), (2,1)\}$, if $d \leq m$, or $(e_1, e_2) = (m, m+1)$ and $r = s = 0$, if $d = m+1$. The next result can be justified by an argument similar to the proof of [Bu, Lemma 1].

Lemma 2.3. *Let $f : \mathbb{P}^1 \to \mathrm{G}(c, c+d)$ be a morphism of degree d. Assume that $c \leq d$, $\mathrm{Ker}(f) = 0$, and $\mathrm{Span}(f) = \mathbb{C}^{c+d}$. Then any two distinct c-dimensional subspaces in the image of f are in general position.*

When $d = m+1$ is odd, then $\overline{\mathrm{G}}(d-1, B/A)$ has no rational curves of degree d, and statements (b) and (d) in this case reduce to trivialities. When $d \leq m$ and $r \neq r(d)$ (that is, d is odd and $r = 0$ or d is even and $r = 2$) then there exists no triple of distinct valid points of $\overline{\mathrm{G}}(d, B/A)$ in general position. By Lemma 2.3, now, the assignment in (b) maps any f to a point $(\mathrm{Ker}(f), \mathrm{Span}(f))$ in $S_d(\lambda, \mu, \nu)$.

Suppose $d \leq m$. We claim that the set S_d in the statement of (b) is precisely the set of points (A, B) in $Y_\lambda \cap Y_\mu \cap Y_\nu$ for which there exists a triple of points in $\overline{\mathrm{G}}(d, B/A)$ in general position, one in each Schubert variety X_λ, X_μ, X_ν, and that the number of such triples is precisely 2^{M_d}. This claim follows easily, once we observe that when $N_d(\lambda) \geq 1$ the 2^{N_d} points of $X_\lambda \cap \overline{\mathrm{G}}(d, B/A)$ are distributed equally amongst the two components of valid $\Sigma \in \overline{\mathrm{G}}(d, B/A)$. Now, part (b) is established by invoking Proposition 1.1.

When $d = m+1$ is even, then $\overline{\mathrm{G}}(d-1, B/A) \cong \mathrm{OG}(d, 2d)$ and all triples are in general position. The number of triples is 2^{M_d} by Lemma 2.1 (a). So (b) is true in this case. Statement (c) in this case follows immediately.

In case $d \leq m$, statement (c) follows from (b) by keeping track of the possible distributions of triples among the two connected components of valid points of $\overline{\mathrm{G}}(d, B/A)$, and by noting that the degrees in Lemma 2.1 parts (a) and (b) are equal, and the degrees in Lemma 2.1 parts (c) and (d) are equal. Statement (d) follows from Lemma 2.1. $\square$

Example 2.2. On $\mathrm{OG}(4, 11)$ we have $\langle \tau_{643}, \tau_{6431}, \tau_{6432} \rangle_4 = 1$. For general isotropic flags $F_\bullet$, $G_\bullet$, the intersection $Y_{6431}(F_\bullet) \cap Y_{6432}(G_\bullet)$ has two irreducible components, one with the (generically) unique lifts to $X_{6431}(F_\bullet)$ and to $X_{6432}(G_\bullet)$ in general position, and one with the lifts both containing $F_7 \cap G_5$. A general translate of Y_{643} intersects each of these components in a single point, hence $\#(Y_\lambda \cap Y_\mu \cap Y_\nu) = 2$ and (since $N_4(643) = 1$) we have

$$\int_{Y_4} \upsilon_{643} \cdot \upsilon_{6431} \cdot \upsilon_{6432} = 4.$$

The set $S_d(643, 6431, 6432)$ is just a single point, and the formula in Theorem 2.3(c) gives $\langle \tau_{643}, \tau_{6431}, \tau_{6432} \rangle_4 = 4 - (1/2) \cdot 6$. (Notice, since $\ell(643) < 4$, the intersection $Y_\lambda \cap Y_\mu \cap Y_\nu$ is contained in the $r = 0$ locus, and there is no contribution from the integrals over Z_4 and Z'_4.)

We have seen that the parameter space of lines on OG is the orthogonal two-step flag variety $Z_1 = \mathrm{OF}(m-1, m+1; 2n+1)$. It follows that

$$\langle \tau_\lambda, \tau_\mu, \tau_\nu \rangle_1 = \int_{Z_1} [Z_\lambda] \cdot [Z_\mu] \cdot [Z_\nu] \tag{32}$$

where Z_λ, Z_μ, and Z_ν are the associated Schubert varieties in Z_1.

For any Young diagram λ, let $\overline{\lambda}$ denote the diagram obtained by deleting the leftmost column of λ; hence $\overline{\lambda}_i = \max\{\lambda_i - 1, 0\}$. Given any Schubert variety X_λ in $\mathrm{OG}(m, V)$, we will consider an associated Schubert variety $X_{\overline{\lambda}}$ in $\mathrm{OG}(m+1, V)$, with cohomology class $\tau_{\overline{\lambda}}$. The following result characterizes the degree 1 quantum correction terms in the quantum Pieri rule, and is an exact analogue of what happens for type A Grassmannians. What is different from type A is that there will also be degree 2 quantum correction terms in the Pieri rule for OG.

Proposition 2.1. *For any integer $p \in [1, n+k]$ and λ, $\mu \in \mathcal{P}(k,n)$ with $|\lambda| + |\mu| + p = \dim \mathrm{OG} + n + k$, we have*

$$\langle \tau_\lambda, \tau_\mu, \tau_p \rangle_1 = \int_{\mathrm{OG}(n-k+1, 2n+1)} \tau_{\overline{\lambda}} \cdot \tau_{\overline{\mu}} \cdot \tau_{p-1}. \tag{33}$$

Proof. Consider the three-step flag variety $U = \mathrm{OF}(m-1, m, m+1; 2n+1)$, with its natural projections $\pi_1 : U \to \mathrm{OG}$ and $\pi_2 : U \to Z_1$. Note that for every $\lambda \in \mathcal{P}(k,n)$ and isotropic flag $F_\bullet$, we have $Z_\lambda = \pi_2(\pi_1^{-1}(X_\lambda))$. We have the following commutative diagram

$$\begin{array}{ccccc} U & \xrightarrow{\varphi_2} & \mathrm{OF}(m, m+1; 2n+1) & \xrightarrow{\varphi_1} & \mathrm{OG}(m, 2n+1) \\ \big\downarrow{\scriptstyle \pi_2} & & \big\downarrow{\scriptstyle \pi} & & \\ Z_1 & \xrightarrow{\xi} & \mathrm{OG}(m+1, 2n+1) & & \end{array}$$

where every arrow is a natural smooth projection and $\pi_1 = \varphi_1 \varphi_2$. Observe that for each λ we have $\pi(\varphi_1^{-1}(X_\lambda(F_\bullet))) = X_{\overline{\lambda}}(F_\bullet)$. We now apply (32) and use the projection formula repeatedly to obtain

$$\begin{aligned}
\langle \tau_\lambda, \tau_\mu, \tau_p \rangle_1 &= \int_{Z_1} [Z_\lambda] \cdot [Z_\mu] \cdot [Z_{(p)}] \\
&= \int_{Z_1} \pi_{2*}\pi_1^*\tau_\lambda \cdot \pi_{2*}\pi_1^*\tau_\mu \cdot \xi^*\tau_{p-1} \\
&= \int_U \pi_2^*\pi_{2*}\pi_1^*\tau_\lambda \cdot \pi_1^*\tau_\mu \cdot \varphi_2^*\pi^*\tau_{p-1} \\
&= \int_{\mathrm{OF}(m,m+1;2n+1)} \varphi_{2*}\pi_2^*\pi_{2*}\pi_1^*\tau_\lambda \cdot \varphi_1^*\tau_\mu \cdot \pi^*\tau_{p-1} \\
&= \int_{\mathrm{OF}(m,m+1;2n+1)} \pi^*\pi_*\varphi_1^*\tau_\lambda \cdot \varphi_1^*\tau_\mu \cdot \pi^*\tau_{p-1} \\
&= \int_{\mathrm{OG}(m+1;2n+1)} \pi_*\varphi_1^*\tau_\lambda \cdot \pi_*\varphi_1^*\tau_\mu \cdot \tau_{p-1} \\
&= \int_{\mathrm{OG}(m+1,2n+1)} \tau_{\overline{\lambda}} \cdot \tau_{\overline{\mu}} \cdot \tau_{\overline{p}}.
\end{aligned}$$

□

2.5. **Quantum cohomology.** The quantum cohomology ring $\mathrm{QH}^*(\mathrm{OG})$ is defined similarly to that for IG, but the degree of q here is $n+k$. The quantum Pieri formula for non-maximal orthogonal Grassmannians involves both linear and quadratic q terms. This is surprising since the corresponding Pieri rule for the maximal isotropic Grassmannian $\mathrm{OG}(n, 2n+1)$ (which is isomorphic to $\mathrm{OG}(n+1, 2n+2)$) has only linear q terms. Note that the degree of q on $\mathrm{OG}(n, 2n+1)$ is $2n$, which is twice the expected value, since its degree on $\mathrm{OG}(n-k, 2n+1)$ equals $n+k$ when $k>0$. In fact, when $k>0$, the q^2 terms in the quantum Pieri formula for $\mathrm{OG}(n-k, 2n+1)$ behave like the q terms in the analogous formula for $\mathrm{OG}(n, 2n+1)$, while the q terms on $\mathrm{OG}(n-k, 2n+1)$ behave (and may be computed) like the q terms in the quantum Pieri rule for the usual (type A) Grassmannian.

Geometrically, this jump in q-degree is explained by the degree doubling phenomenon on the maximal orthogonal Grassmannian, whereby lines are mapped to conics in projective space under the Plücker embedding (since the first Chern class of the universal quotient bundle $\mathcal{Q}$ is twice a Schubert class). Algebraically, it is simplest to understand the transition between the maximal and non-maximal isotropic cases by considering the variety $\mathrm{OG}(n+1-k, 2n+2)$ for $k \geq 0$, as explained in the Introduction. Formally, the single quantum parameter q on $\mathrm{OG}(n+1, 2n+2)$ is replaced with two square roots q_1 and q_2 on $\mathrm{OG}(n, 2n+2)$, which in turn are identified on $\mathrm{OG}(n-1, 2n+2)$. Compare Theorems 3.4 and A.1 for further details.

To formulate the quantum Pieri rule for OG, we require some more notation. Let $\mathcal{P}'(k, n+1)$ be the set of $\nu \in \mathcal{P}(k, n+1)$ for which $\ell(\nu) = n+1-k$, $2k \leq \nu_1 \leq n+k$, and the number of boxes in the second column of ν is at most $\nu_1 - 2k + 1$. For any $\nu \in \mathcal{P}'(k, n+1)$, we let $\widetilde{\nu} \in \mathcal{P}(k, n)$ be the partition obtained by removing the first row of ν as well as $n+k-\nu_1$ boxes from the first column. That is,

$$\widetilde{\nu} = (\nu_2, \nu_3, \ldots, \nu_r), \text{ where } r = \nu_1 - 2k + 1.$$

Recall also from Section 1.5 that for any partition λ, we set $\lambda^* = (\lambda_2, \lambda_3, \ldots)$.

Theorem 2.4 (Quantum Pieri rule for OG)**.** *For any k-strict partition $\lambda \in \mathcal{P}(k, n)$ and integer $p \in [1, n+k]$, we have*

$$\tau_p \cdot \tau_\lambda = \sum_{\lambda \to \mu} 2^{N'(\lambda,\mu)}\, \tau_\mu + \sum_{\lambda \to \nu} 2^{N'(\lambda,\nu)}\, \tau_{\widetilde{\nu}}\, q \;+ \sum_{\lambda^* \to \rho} 2^{N'(\lambda^*,\rho)}\, \tau_{\rho^*}\, q^2 \tag{34}$$

in the quantum cohomology ring $\mathrm{QH}^*(\mathrm{OG}(n-k, 2n+1))$*. Here* (i) *the first sum is classical, as in (14),* (ii) *the second is over $\nu \in \mathcal{P}'(k, n+1)$ with $\lambda \to \nu$ and $|\nu| = |\lambda| + p$, and* (iii) *the third sum is empty unless $\lambda_1 = n+k$, and over $\rho \in \mathcal{P}(k, n)$ such that $\rho_1 = n+k$, $\lambda^* \to \rho$, and $|\rho| = |\lambda| - n - k + p$.*

Our proof of Theorem 2.4 will require several auxiliary results. Let $\mu^\vee$ be the dual partition to μ in $\mathcal{P}(k, n)$. From the the picture of $\mu^\vee$ given in Section 4.4 it is straightforward to prove that $\ell(\mu^\vee) = n-k$ if and only if $\mu_1 < \ell(\mu) + 2k$. Proposition 2.1 shows that for $\lambda, \mu \in \mathcal{P}(k, n)$, the coefficient of $\tau_\mu\, q$ in the Pieri product $\tau_p\, \tau_\lambda$ in $\mathrm{QH}^*(\mathrm{OG}(n-k, 2n+1))$ is equal to the coefficient of $\tau_{(\overline{\mu^\vee})^\vee}$ in the product $\tau_{p-1}\, \tau_{\overline{\lambda}}$ in $\mathrm{H}^*(\mathrm{OG}(n+1-k, 2n+1))$, when $\ell(\lambda) = \ell(\mu^\vee) = n-k$, and equals 0, otherwise. By computing the dual partition in $\mathcal{P}(k-1, n)$ to $\overline{\mu^\vee}$, we deduce the following result.

Proposition 2.2. *Consider λ, $\mu \in \mathcal{P}(k, n)$ with $|\mu| + n + k = |\lambda| + p$, for $1 \leq p \leq n+k$. If $\ell(\lambda) = n-k$ and $\mu_1 < \ell(\mu) + 2k$ then, in* $\mathrm{QH}^*(\mathrm{OG})$*, the coefficient of*

$\tau_\mu\, q$ in $\tau_p\, \tau_\lambda$ is equal to the coefficient of $\tau_{(\ell(\mu)+2k-1,\,\overline{\mu})}$ in $\tau_{p-1}\, \tau_{\overline{\lambda}} \in \mathrm{H}^*(\mathrm{OG}(n+1-k, 2n+1))$. Otherwise, the coefficient vanishes.

To characterize the q^2 terms that occur in Pieri products we follow the model of the maximal orthogonal Grassmannian $\mathrm{OG}(n, 2n+1)$, and idea in the proof of [BKT1, Thm. 6]. We will need the next proposition.

Proposition 2.3. *Suppose $\lambda \in \mathcal{P}(k,n)$ with $\lambda_1 < n+k$. Then, for any p, the coefficient of q^2 in the quantum Pieri product $\tau_p\, \tau_\lambda$ vanishes.*

Proof. For degree reasons we may assume $k \leq n-2$. Degree 2 maps $f : \mathbb{P}^1 \to \mathrm{OG}$ are studied by considering their image conics $C \subset \mathrm{OG}$. Let $X_p(E_\bullet)$, $X_\lambda(F_\bullet)$, and $X_\mu(G_\bullet)$ be given, for some $\mu \in \mathcal{P}(k,n)$, and consider the following conditions.

(i) There are only finitely many conics C in OG incident to $X_p(E_\bullet)$, $X_\lambda(F_\bullet)$, and $X_\mu(G_\bullet)$;
(ii) No such conic is incident to the intersection of any two of these varieties;
(iii) Each conic C has $\dim \mathrm{Ker}(C) = m-2$ and $\dim \mathrm{Span}(C) = m+2$;
(iv) The induced bilinear form on $\mathrm{Span}(C)/\mathrm{Ker}(C)$ is always nondegenerate.

We claim that conditions (i)–(iv) imply that the set of conics incident to $X_p(E_\bullet)$, $X_\lambda(F_\bullet)$, and $X_\mu(G_\bullet)$ is empty. Conditions (i)–(iv) hold in particular when $|\lambda| + |\mu| + p = \dim \mathrm{OG} + 2(n+k)$ and $E_\bullet$, $F_\bullet$, and $G_\bullet$ are general (by dimension reasoning, plus Theorem 2.3).

The first step in the proof of the claim is a reduction to the case

$$p = n+k, \qquad \lambda \in \{\lambda^0, \lambda^1\}, \qquad \mu = (n+k, \ldots, 2k+1),$$

where $\lambda^0 = (n+k-1, \ldots, 2k)$ and $\lambda^1 = (n+k-1, \ldots, 2k+1)$. Assume (i)–(iv) and let C denote a conic in OG satisfying the incidence conditions. Recall that $X_p(E_\bullet) = \{\, \Sigma \mid \Sigma \cap E_{\varepsilon(p)} \neq 0 \,\}$. If $\Pi \in X_p(E_\bullet) \cap C$ and $0 \neq x \in \Pi \cap E_{\varepsilon(p)}$, then $\{\, \Sigma \mid x \in \Sigma \,\}$ is a translate of X_{n+k} contained in $X_p(E_\bullet)$. Let $\Sigma \in X_\lambda(F_\bullet) \cap C$. If $\ell(\lambda) = m$, we define $\widetilde{F}_{m+1} = (\Sigma^\perp \cap F_i) + \Sigma$ with i chosen so this space has dimension $m+1$, and then $X_{\lambda^0}(\widetilde{F}_\bullet) \subset X_\lambda(F_\bullet)$. If $\ell(\lambda) < m$, then put $R = \Sigma \cap F_j$ with j chosen so that $\dim R = m-1$, and define $\widetilde{F}_m = (R^\perp \cap F_i) + R$ for i such that this has dimension m. Then $X_{\lambda^1}(\widetilde{F}_\bullet) \subset X_\lambda(F_\bullet)$. Set $A = \mathrm{Ker}(C)$ and $B = \mathrm{Span}(C)$.

Case 1. Suppose $\lambda = \lambda^0$. The three Schubert varieties are the point G_m, the locus of Σ containing x, and the locus of $\Sigma \subset F_{m+1}$. The space B contains x, G_m, and an m-dimensional subspace of F_{m+1} containing A. By conditions (ii) and (iii), the space $(G_m + \langle x \rangle) \cap F_{m+1}$ must contain some vector t not in A, and applying condition (iv) also, we see that for general $y \in F_{m+1}$, the space $B' := G_m + \langle x, y \rangle$ has dimension $n-k+2$ with the symmetric form nondegenerate on B'/A. Let C' be the component containing G_m of the space of maximal isotropic spaces containing A and contained in B'. Set $\Sigma' = A + \langle t, y \rangle$; then $\Sigma' \subset F_{m+1}$ and $\Sigma' \in C'$, so C' is a conic incident to the three Schubert varieties. There are infinitely many such conics C', so we have a contradiction to (i).

Case 2. Suppose $\lambda = \lambda^1$. The three Schubert varieties are the point G_m, the locus of Σ containing x, and the locus of Σ with $\dim(\Sigma \cap F_m) \geq m-1$. Let Σ_1 be a point of C in the last of these varieties. We may suppose $\Sigma_1 \neq F_m$, for otherwise we can reduce to Case 1. So $M := \Sigma_1 \cap F_m$ has dimension $m-1$. If $A \subset F_m$ then there exists isotropic Σ' of dimension m containing $A + \langle x \rangle$ and contained in $(x^\perp \cap F_m) + \langle x \rangle$, and we quickly get a contradiction to (ii). So $A \not\subset F_m$, and hence $\dim(A \cap M) = m-3$. If, now, $F_m \subset B$, i.e., $\dim(F_m \cap G_m) = m-2$, then the component containing G_m of the space of maximal isotropic spaces containing $F_m \cap G_m$ and contained in B is a conic, and as before we reduce to Case 1. So $A_0 := F_m \cap G_m$ has dimension $m-3$. With $B_0 := F_m + G_m$ there are two possibilities for the bilinear form on B_0/A_0: it could be nondegenerate or it could have 2-dimensional kernel. In the former case, the infinite family of spaces A' of dimension $m-2$ with $A_0 \subset A' \subset G_m \cap x^\perp$ gives rise to an infinite family of conics C' with $\mathrm{Ker}(C') = A'$ and $\mathrm{Span}(C') = (A')^\perp \cap B_0$, contradicting (i). In the latter case there is a family of conics with kernel A and varying span contradicting (i). $\square$

Proof of Theorem 2.4. Theorem 2.3 (d) implies that the quantum Pieri product $\tau_p\, \tau_\lambda$ contains at most quadratic q terms. We begin by studying the q-linear terms in this product. For dimension reasons, the right hand side of (33) vanishes when either λ or μ has length less than $n-k$.

The classical Pieri rule for OG (Theorem 2.1) implies that for $\lambda,\, \mu \in \mathcal{P}(k,n)$ with $\ell(\lambda) = n-k$ and $\mu_1 < \ell(\mu)+2k$, we have $\overline{\lambda} \to (\ell(\mu)+2k-1,\, \overline{\mu})$ in $\mathcal{P}(k-1,n)$ if and only if $\lambda \to (\ell(\mu)+2k, \mu, 1^{n-k-\ell(\mu)})$ in $\mathcal{P}(k,n+1)$, with the same coefficients N'. It follows by Proposition 2.2 that for $\lambda,\, \mu \in \mathcal{P}(k,n)$, the coefficient of $\tau_\mu\, q$ in the quantum product $\tau_p\, \tau_\lambda$ in $\mathrm{QH}^*(\mathrm{OG})$ is equal to the coefficient of $\tau_{(\ell(\mu)+2k,\mu,1^{n-k-\ell(\mu)})}$ in the cup product $\tau_p\, \tau_\lambda$ in $\mathrm{H}^*(\mathrm{OG}(n+1-k, 2n+3))$ when $\ell(\lambda) = n-k$ and $\mu_1 < \ell(\mu)+2k$, and is 0 otherwise. Observe that the condition $\ell(\lambda) = n-k$ may be omitted, since when $\ell(\lambda) < n-k$, the product $\tau_p\, \tau_\lambda$ in $\mathrm{H}^*(\mathrm{OG}(n+1-k, 2n+3))$ involves no terms indexed by partitions of length $n+1-k$. Notice that $\nu \mapsto \widetilde{\nu}$ induces a 1-1 map $\mathcal{P}'(k,n+1) \to \mathcal{P}(k,n)$ with image $\{\mu \in \mathcal{P}(k,n) \,:\, \mu_1 < \ell(\mu)+2k\}$, and the inverse of this map is given by $\mu \mapsto (\ell(\mu)+2k, \mu, 1^{n-k-\ell(\mu)})$. Combining these facts, we see that the coefficient of $\tau_{\widetilde{\nu}}\, q$ on the right hand side of (34) is equal to the coefficient of τ_ν in the product $\tau_p\, \tau_\lambda$ in $\mathrm{H}^*(\mathrm{OG}(n+1-k, 2n+3))$, for $\nu \in \mathcal{P}'(k,n+1)$, and these are all the linear q terms.

We next show that the basic relation $\tau_{n+k}^2 = q$ holds in $\mathrm{QH}^*(\mathrm{OG})$. Note that τ_{n+k}^2 vanishes in cohomology, and the coefficient of q vanishes by Proposition 2.2. So $\tau_{n+k}^2 = c\, q^2$ for some $c \in \mathbb{Z}$. That $c = 1$ can be shown geometrically by exhibiting a unique conic on OG passing through a point and two general translates of X_{n+k}. An alternative argument uses associativity of the quantum product. We have $\tau_{n+k}\, \tau_{(1^{n-k})} = \tau_{(1^{n-k})}\, q$ by Proposition 2.2. Hence $\tau_{n+k}^2\, \tau_{(1^{n-k})} = \tau_{(1^{n-k})}\, q^2$, and $c = 1$.

According to Proposition 2.2 the term $\tau_\mu\, q$ appears in a Pieri product only when $\mu_1 < \ell(\mu)+2k$, and in particular, $\mu_1 < n+k$. Proposition 2.3 asserts that whenever $\lambda_1 < n+k$, the product $\tau_p\, \tau_\lambda$ carries only such degree one quantum correction terms $\tau_\mu\, q$. One now completes the proof of the Theorem as follows. It suffices to consider products $\tau_p\, \tau_\lambda$ when $\lambda_1 = n+k$. In this case we have an equation $\tau_\lambda = \tau_{n+k}\, \tau_{\lambda^*}$ in $\mathrm{QH}^*(\mathrm{OG})$. If $p = n+k$, then $\tau_{n+k}\, \tau_\lambda = \tau_{n+k}^2\, \tau_{\lambda^*} = \tau_{\lambda^*}\, q^2$, and the quantum Pieri formula is verified. If $p < n+k$, then we write $\tau_p\, \tau_\lambda = \tau_{n+k}\, (\tau_p\, \tau_{\lambda^*})$. By Proposition 2.3, the product in parentheses receives only linear quantum correction terms, and hence is known by Proposition 2.2. As the quantum Pieri rule for multiplication by τ_{n+k} has already been proved, it remains only to show that the result agrees with the formula in the theorem, and this is easily checked. $\square$

Example 2.3. In the quantum cohomology ring of OG$(4, 13)$ we have

$$\tau_4 \cdot \tau_{(5,3,2,2)} = 2\tau_{(8,4,2,2)} + \tau_{(7,5,2,2)} + 2\tau_{(7,4,3,2)} + \tau_{(6,5,3,2)} + \tau_{(8,4,3,1)} \\ +2\tau_{(4,2,2)}\, q + \tau_{(4,3,1)}\, q + 2\tau_{(3,2,2,1)}\, q + 2\tau_{(4,2,1,1)}\, q.$$

On the same Grassmannian we also have

$$\tau_5 \cdot \tau_{(8,4,1,1)} = \tau_{(8,6,4,1)} + 2\tau_{(8,7,3,1)} + \tau_{(8,7,4)} + \tau_{(7,2,1,1)}\, q + 2\tau_{(6,3,1,1)}\, q \\ +\tau_{(5,4,1,1)}\, q + \tau_{(1,1,1)}\, q^2 + 2\tau_{(2,1)}\, q^2 + 4\tau_{(3)}\, q^2.$$

Theorem 2.5 (Ring presentation)**.** *The quantum cohomology ring* $\mathrm{QH}^*(\mathrm{OG})$ *is presented as a quotient of the polynomial ring* $\mathbb{Z}[\tau_1, \ldots, \tau_{n+k}, q]$ *modulo the relations*

$$\det(\delta_{1+j-i}\tau_{1+j-i})_{1\le i,j\le r} = 0, \qquad n-k+1 \le r \le n, \tag{35}$$

$$\sum_{p=k+1}^{r} (-1)^p \tau_p \det(\delta_{1+j-i}\tau_{1+j-i})_{1\le i,j\le r-p} = 0, \qquad n+1 \le r < n+k, \tag{36}$$

$$\sum_{p=k+1}^{n+k} (-1)^p \tau_p \det(\delta_{1+j-i}\tau_{1+j-i})_{1\le i,j\le n+k-p} = q \tag{37}$$

and

$$\tau_r^2 + \sum_{i=1}^{r} (-1)^i \delta_{r-i}\tau_{r+i}\tau_{r-i} = 0, \qquad k+1 \le r \le n. \tag{38}$$

Proof. Set $h_r = \det(\delta_{1+j-i}\tau_{1+j-i})_{1\le i,j\le r}$. We have that $h_r = \tau_{(1^r)}$ for $r \le n-k$, while $h_r = 0$ for $n-k+1 \le r < n+k$, because these relations hold classically and the degree of q is $n+k$. This proves (35) and (36). The quantum Pieri rule implies that for $p > k$, $\tau_p h_{n+k-p} = 0$ unless $p = 2k$, when $\tau_{2k} h_{n-k} = q$. We deduce that (37) also holds in $\mathrm{QH}^*(\mathrm{OG})$.

We are left with proving that there are no quantum correction terms in relation (38); the result then follows from [ST]. When $n - k > 1$, this is immediate since the quantum Pieri rule does not give rise to any quantum correction terms. In the case of the quadric OG$(1, 2n+1)$ the quantum Pieri rule gives

$$\tau_n^2 - \tau_{n+1}\tau_{n-1} + \cdots + (-1)^{n-1}\tau_{2n-1}\tau_1 = c\,\tau_1\, q$$

with the coefficient $c = 1 - 2 + 2 - \cdots \pm 2 \mp 1 = 0$. □

3. The Grassmannian OG$(n+1-k, 2n+2)$

3.1. **Schubert classes.** In this section, we consider the even orthogonal Grassmannian $\mathrm{OG}' = \mathrm{OG}(m, 2n+2)$, which parametrizes the m-dimensional isotropic subspaces in a vector space $V \cong \mathbb{C}^{2n+2}$ with a nondegenerate symmetric bilinear form. The variety OG$'$ has dimension $2m(n+1-m) + m(m-1)/2$.

Two subspaces E and F of V are said to be *in the same family* if $\dim(E \cap F) \equiv (n+1) (\mathrm{mod}\, 2)$. Fix once and for all an isotropic subspace L of V with $\dim(L) = n+1$. An isotropic flag is a complete flag $F_\bullet$ of subspaces of V such that $F_{n+1+i} = F_{n+1-i}^\perp$ for all $0 \le i \le n$, and F_{n+1} and L are in the same family. As the orthogonal space $F_n^\perp / F_n$ contains only two isotropic lines, to each such flag $F_\bullet$ there corresponds an alternate isotropic flag $\widetilde{F}_\bullet$ such that $\widetilde{F}_i = F_i$ for all $i \le n$ but with $\widetilde{F}_{n+1}$ in the opposite family from F_{n+1}.

Set $k = n + 1 - m > 0$. The Schubert varieties in OG' are indexed by a set $\widetilde{\mathcal{P}}(k,n)$ which differs from that used in previous sections. To any k-strict partition λ we associate a number in $\{0,1,2\}$ called the *type* of λ, denoted $\mathrm{type}(\lambda)$. If λ has no part equal to k, then we set $\mathrm{type}(\lambda) = 0$; otherwise we have $\mathrm{type}(\lambda) = 1$ or $\mathrm{type}(\lambda) = 2$ (thus 'type' is a multi-valued function). The elements of $\widetilde{\mathcal{P}}(k,n)$ are the k-strict partitions contained in an $m \times (n+k)$ rectangle of all three possible types. For every $\lambda \in \widetilde{\mathcal{P}}(k,n)$, we define an index set $P' = \{p'_1 < \cdots < p'_m\} \subset [1, 2n+2]$ with

$$p'_j(\lambda) = n + k - \lambda_j + \#\{\, i < j \mid \lambda_i + \lambda_j \leq 2k - 1 + j - i \,\} + \begin{cases} 1 & \text{if } \lambda_j > k, \text{ or } \lambda_j = k < \lambda_{j-1} \text{ and } n + j + \mathrm{type}(\lambda) \text{ is even,} \\ 2 & \text{otherwise.} \end{cases}$$

Given any isotropic flag $F_\bullet$, each $\lambda \in \widetilde{\mathcal{P}}(k,n)$ indexes a codimension $|\lambda|$ Schubert variety $X_\lambda(F_\bullet)$ in OG', defined as the locus of $\Sigma \in \mathrm{OG}'$ such that

$$\dim(\Sigma \cap F_{p'_j}) \geq j, \ \text{ if } p'_j \neq n+2, \ \text{ and } \dim(\Sigma \cap \widetilde{F}_{n+1}) \geq j, \ \text{ if } p'_j = n+2,$$

for all j with $1 \leq j \leq \ell(\lambda)$. For each $\lambda \in \widetilde{\mathcal{P}}(k,n)$, we let τ_λ denote the cohomology class in $\mathrm{H}^{2|\lambda|}(\mathrm{OG}', \mathbb{Z})$ dual to the cycle determined by the Schubert variety indexed by λ. Each such Schubert class has a type which is the same as the type of λ; this serves to distinguish two separate classes for each partition λ with some part $\lambda_i = k$.

3.2. **Classical Pieri rule.** The *special* Schubert varieties for $\mathrm{OG}(n+1-k, 2n+2)$ are the varieties $X_1, \ldots, X_{k-1}, X_k, X'_k, X_{k+1}, \ldots, X_{n+k}$. These are defined by a single Schubert condition as follows. For $p \neq k$, we have

$$X_p(F_\bullet) = \{\, \Sigma \in \mathrm{OG}' \mid \Sigma \cap F_{\varepsilon(p)} \neq 0 \,\}$$

where $\varepsilon(p)$ is given by (13). If n is even, then

$$X_k(F_\bullet) = \{\, \Sigma \in \mathrm{OG}' \mid \Sigma \cap F_{n+1} \neq 0 \,\}$$

and

$$X'_k(F_\bullet) = \{\, \Sigma \in \mathrm{OG}' \mid \Sigma \cap \widetilde{F}_{n+1} \neq 0 \,\},$$

while the roles of F_{n+1} and $\widetilde{F}_{n+1}$ are switched if n is odd. We let τ_p denote the cohomology class of $X_p(F_\bullet)$ for $1 \leq p \leq n+k$ and τ'_k denote the cohomology class of $X'_k(F_\bullet)$; note that $\mathrm{type}(\tau_k) = 1$ and $\mathrm{type}(\tau'_k) = 2$.

The Pieri rule for OG' requires a slightly different shifting convention than that used for IG and OG. Given a k-strict partition λ, we say that the box in row r and column c of λ is *k'-related* to the box in row r' and column c' if $|c - (2k+1)/2| + r = |c' - (2k+1)/2| + r'$. Using this convention, the relation $\lambda \to \mu$ is defined as in Definition 1.3, with the added condition that $\mathrm{type}(\lambda) + \mathrm{type}(\mu) \neq 3$. Moreover, the multiplicity $N'(\lambda, \mu)$ is equal to the number (respectively, one less than the number) of components of $\mathbb{A}$, if $p \leq k$ (respectively, if $p > k$).

Let $g(\lambda, \mu)$ be the number of columns of μ among the first k which do not have more boxes than the corresponding column of λ, and

$$h(\lambda, \mu) = g(\lambda, \mu) + \max(\mathrm{type}(\lambda), \mathrm{type}(\mu)).$$

If $p \neq k$, then set $\delta_{\lambda\mu} = 1$. If $p = k$ and $N'(\lambda, \mu) > 0$, then set

$$\delta_{\lambda\mu} = \delta'_{\lambda\mu} = 1/2,$$

while if $N'(\lambda, \mu) = 0$, define

$$\delta_{\lambda\mu} = \begin{cases} 1 & \text{if } h(\lambda,\mu) \text{ is odd,} \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \delta'_{\lambda\mu} = \begin{cases} 1 & \text{if } h(\lambda,\mu) \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

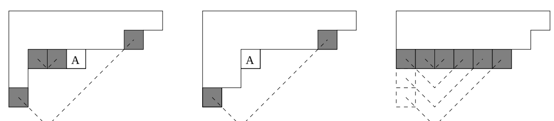
Theorem 3.1 (Pieri rule for $\mathrm{OG}(m, 2n+2)$)**.** *For any element $\lambda \in \widetilde{\mathcal{P}}(k,n)$ and integer $p \in [1, n+k]$, we have*

$$\tau_p \cdot \tau_\lambda = \sum_\mu \delta_{\lambda\mu}\, 2^{N'(\lambda,\mu)}\, \tau_\mu, \tag{39}$$

where the sum is over all $\mu \in \widetilde{\mathcal{P}}(k,n)$ with $\lambda \to \mu$ and $|\mu| = |\lambda| + p$. Furthermore, the product $\tau'_k \cdot \tau_\lambda$ is obtained by replacing $\delta_{\lambda\mu}$ with $\delta'_{\lambda\mu}$ throughout.

This theorem will be proved in Section 5.

Example 3.1. For the Grassmannian $\mathrm{OG}(5, 14)$ we have $k = 2$ and $n = 6$. For the partition $\lambda = (8, 7, 2, 1, 1)$ with $\mathrm{type}(\lambda) = 1$, we get the following partitions $\mu \in \widetilde{\mathcal{P}}(2, 6)$, all of type 0 or 1, such that $\lambda \to \mu$ and $|\mu| = |\lambda| + 2$:



We obtain $\tau_2 \cdot \tau_\lambda = \tau_{(8,7,4,1,1)} + \tau_{(8,7,3,2,1)} + \tau_{(8,7,6)}$ and $\tau'_2 \cdot \tau_\lambda = \tau_{(8,7,4,1,1)} + \tau_{(8,7,3,2,1)}$. Notice that the product $(\tau_2 + \tau'_2) \cdot \tau_\lambda$ is obtained from (39) by ignoring $\delta_{\lambda\mu}$.

3.3. **Presentation of** $\mathrm{H}^*(\mathrm{OG}', \mathbb{Z})$**.** If $\mathcal{S}$ (respectively $\mathcal{Q}$) denotes the tautological subbundle (respectively, quotient bundle) over OG', then one has that

$$c_p(\mathcal{Q}) = \begin{cases} \tau_p & \text{if } p < k, \\ \tau_k + \tau'_k & \text{if } p = k, \\ 2\tau_p & \text{if } p > k. \end{cases} \tag{40}$$

For each $r > 0$, let Δ_r denote the $r \times r$ Schur determinant

$$\Delta_r = \det(c_{1+j-i})_{1 \le i,j \le r},$$

where each variable c_p represents the Chern class $c_p(\mathcal{Q})$. For each $\lambda \in \widetilde{\mathcal{P}}(k,n)$ we define a monomial τ^λ in the special Schubert classes as follows. If λ is not of type 2, then set $\tau^\lambda = \prod_i \tau_{\lambda_i}$. If $\mathrm{type}(\lambda) = 2$ then τ^λ is defined by the same product formula, but replacing each occurrence of τ_k with τ'_k.

Theorem 3.2. a) *Define polynomials c_p using the equations (40). Then the cohomology ring $\mathrm{H}^*(\mathrm{OG}(n+1-k, 2n+2), \mathbb{Z})$ is presented as a quotient of the polynomial ring $\mathbb{Z}[\tau_1, \ldots, \tau_k, \tau'_k, \tau_{k+1}, \ldots, \tau_{n+k}]$ modulo the relations*

$$\Delta_r = 0, \qquad n - k + 1 < r \le n, \tag{41}$$

$$\tau_k \Delta_{n+1-k} = \tau'_k \Delta_{n+1-k} = \sum_{p=k+1}^{n+1} (-1)^{p+k+1} \tau_p \Delta_{n+1-p}, \tag{42}$$

$$\sum_{p=k+1}^{r} (-1)^p \tau_p \Delta_{r-p} = 0, \qquad n + 1 < r \le n + k, \tag{43}$$

and

$$\tau_r^2 + \sum_{i=1}^{r}(-1)^i \tau_{r+i} c_{r-i} = 0, \qquad k+1 \le r \le n, \tag{44}$$

$$\tau_k \tau_k' + \sum_{i=1}^{k}(-1)^i \tau_{k+i}\tau_{k-i} = 0. \tag{45}$$

b) *The monomials* τ^λ *with* $\lambda \in \widetilde{\mathcal{P}}(k,n)$ *form a* $\mathbb{Z}$*-basis for* $\mathrm{H}^*(\mathrm{OG}', \mathbb{Z})$.

Proof. We prove only part (a), as the proof of (b) is similar to that of Theorem 1.2. We first note that $\mathrm{H}^*(\mathrm{OG}')$ is a free abelian group of rank $2^{n+1-k}\binom{n+1}{k}$, and argue using the Pieri rule for OG' that the special Schubert classes generate the cohomology ring of OG' over $\mathbb{Z}$. The Whitney sum formula and the Pieri rule may be used to show that the displayed relations hold in $\mathrm{H}^*(\mathrm{OG}')$, as in the proof of Theorem 2.2. The first equality in (42) also follows from the relation $(\tau_k - \tau_k')c_{n+1-k}(\mathcal{S}) = 0$; a proof of this is given in [T, Sec. 6.1].

We proceed to apply Lemma 1.1 once again. To prove that the displayed relations form a regular sequence in $A = K[\tau_1, \ldots, \tau_k, \tau_k', \ldots, \tau_{n+k}]$ for any field K, we may assume that K is algebraically closed. We claim that the affine variety determined by all the relations is a single point (the origin). This suffices, since Hilbert's Nullstellensatz then asserts that the ideal I of relations is contained in some power of the maximal ideal at the origin, and thus A/I is a finite dimensional K-vector space. To prove the claim, we again separate cases according the the characteristic of K. In addition, one of the relations is $(\tau_k - \tau_k')\Delta_{n+1-k} = 0$, which implies $\tau_k = \tau_k'$ or $\Delta_{n+1-k} = 0$.

If $\mathrm{char}(K) \neq 2$ and $\tau_k = \tau_k'$, then the relations (41)–(45) and the power series argument in the proofs of Theorems 1.2 and 2.2 show that $\Delta_{n-k+2} = \cdots = \Delta_{2n+1} = 0$. Lemma 1.2 then implies that all the τ_i must vanish. If $\mathrm{char}(K) \neq 2$ and $\Delta_{n+1-k} = 0$, then the relations (41)–(43), (44) and the same argument may be used to show that

$$\tau_1 = \cdots = \tau_{k-1} = \tau_k + \tau_k' = \tau_{k+1} = \cdots = \tau_{n+k} = 0.$$

It follows that $\tau_k = -\tau_k'$, and now (45) gives $\tau_k = \tau_k' = 0$.

If $\mathrm{char}(K) = 2$ and $\tau_k = \tau_k'$, then the Schur determinants involving $\tau_1, \ldots, \tau_{k-1}$ in degrees $n-k+2, \ldots, n$ all vanish, so by Lemma 1.2 we get $\tau_1 = \cdots = \tau_{k-1} = 0$. Now (42) implies that $\tau_{n+1} = 0$, and then relations (43) give $\tau_{n+2} = \cdots = \tau_{n+k} = 0$. The remaining relations now show that all the τ_i vanish. Finally, if $\mathrm{char}(K) = 2$ and $\Delta_{n+1-k} = 0$, then we have determinantal relations involving $\tau_1, \ldots, \tau_{k-1}, \tau_k + \tau_k'$ in degrees $n-k+1, \ldots, n$. Therefore all these elements are zero, in particular $\tau_k = \tau_k'$, and we are reduced to the previous case. □

3.4. **Gromov-Witten invariants.** The theory here is quite similar to the case of the odd orthogonal Grassmannian of Section 2.4. However since the Picard group of the Grassmannian $\mathrm{OG}(n, 2n+2)$ has rank two, we will assume that $m < n$ in this and the following section, and discuss the quantum cohomology of $\mathrm{OG}(n, 2n+2)$ in the Appendix. When $m < n$ (or $k > 1$), the Gromov-Witten invariants $\langle \tau_\lambda, \tau_\mu, \tau_\nu \rangle_d$ are defined as in Section 2.4, for elements λ, μ, and ν of $\widetilde{\mathcal{P}}(k,n)$ such that $|\lambda| + |\mu| + |\nu| = \dim(\mathrm{OG}') + d(n+k)$.

For any integer $d \leq m$ and Schubert variety X_λ in OG', the varieties Y_d, Y'_d, Z_d, Z'_d and the associated subvarieties Y_λ, Y'_λ, Z_λ, Z'_λ, classes υ_λ, υ'_λ, ζ_λ, ζ'_λ, and quantity r are defined exactly as in Section 2.4, together with the projection maps $\varphi : Y'_d \to Y_d$ and $\psi : Z'_d \to Z_d$. When $d = m+1$, we have $Y_{m+1} = \mathrm{G}(2m+1, V)$ with subvarieties $Y_\lambda \subset Y_{m+1}$ as before. For $\lambda \in \widetilde{\mathcal{P}}(k,n)$ we define $N'_d = N'_d(\lambda) = \#\{j \leq m \,|\, \lambda_j = d-j < k\}$, and we define the quantity M'_d as in (27) but with N'_d in place of N_d. The subset $S_d(\lambda,\mu,\nu)$ of the $r = r(d)$ locus of $Y_\lambda \cap Y_\mu \cap Y_\nu$ is also defined as before.

Lemma 3.1. (a) *The restricted projection $\pi : T_\lambda(E_\bullet) \to Y_\lambda(E_\bullet)$ is generically $2^{N'_d}$-to-1 when $\rho_{d-1} \subset \lambda$ and has fibers of positive dimension when $\rho_{d-1} \not\subset \lambda$.*
When $d \leq m$, we furthermore have:
(b) *The restriction of π over the $r = 2$ locus of $Y_\lambda(E_\bullet)$ is generically unramified $2^{N'_d}$-to-1 when $(\rho_{d-1}, 1) \subset \lambda$ and has fibers of positive dimension when $(\rho_{d-1}, 1) \not\subset \lambda$.*
(c) *The map $\pi' : T'_\lambda(E_\bullet) \to Y'_\lambda(E_\bullet)$ is generically 1-to-1 when $\rho_{d-1} \subset \lambda$.*
(d) *The restriction of π' to the $r = 2$ locus is generically 1-to-1 when $(\rho_{d-1}, 1) \subset \lambda$.*

Proof. The proof of Lemma 2.1 can be copied, with the following modifications. We take $p_j = p'_j(\lambda)$, $p_{m+1} = 2n+3$, $\#_j = \{i < j \,|\, p_i + p_j > 2n+3\}$, and

$$s_j = \begin{cases} \max(d-j-k, 0), & \text{when } \lambda_j = k < \lambda_{j-1}, \\ \max(m+d+p_j-2n-2-j, 0), & \text{otherwise.} \end{cases}$$

Then $\rho_{d-1} \subset \lambda$ implies $s_j = \#_j + 1$ when $\lambda_j = d-j < k$ and $s_j \leq \#_j$ otherwise. Throughout, $\lambda_j = d-j \leq k$ should be replaced by $\lambda_j = d-j < k$, N_d by N'_d, $2n$ by $2n+1$, and A_j, B_j, Σ_j defined by intersecting with $\widetilde{E}_{n+1}$ when $p_j = n+2$.

In the argument with $\rho_{d-1} \not\subset \lambda$, an additional case $\lambda_j = k < \min(\lambda_{j-1}, d-j)$ must be considered. In this case we have $s_j > 0$, so $\dim(B_j) \geq j+1$. Hence $\dim(B \cap E_n) \geq j$, so there exists a j-dimensional isotropic extension Σ'_j of Σ_{j-1} contained in $B \cap E_n$. ☐

Theorem 3.3. *Let $d \geq 0$ and choose $\lambda, \mu, \nu \in \widetilde{\mathcal{P}}(k,n)$ such that $|\lambda| + |\mu| + |\nu| = \dim(\mathrm{OG}') + d(n+k)$. Let X_λ, X_μ, and X_ν be Schubert varieties of $\mathrm{OG}(m, 2n+2)$ in general position, with associated subvarieties and classes in Y_d, Y'_d, Z_d, and Z'_d.*
(a) *The subvarieties Y_λ, Y_μ, and Y_ν intersect transversally in Y_d, and their intersection is finite. For each point in $(A, B) \in Y_\lambda \cap Y_\mu \cap Y_\nu$ we have $A = B \cap B^\perp$ or $\dim(B \cap B^\perp) = \dim A + 2$.*
(b) *The assignment $f \mapsto (\mathrm{Ker}(f), \mathrm{Span}(f))$ gives a $2^{M'_d}$-to-1 association between rational maps $f : \mathbb{P}^1 \to \mathrm{OG}'$ of degree d such that $f(0) \in X_\lambda$, $f(1) \in X_\mu$, $f(\infty) \in X_\nu$ and the subset S_d of $Y_\lambda \cap Y_\mu \cap Y_\nu$.*
(c) *When $d \leq m$ is even, the Gromov-Witten invariant $\langle \tau_\lambda, \tau_\mu, \tau_\nu \rangle_d$ is equal to*

$$\int_{Y_d} \upsilon_\lambda \cdot \upsilon_\mu \cdot \upsilon_\nu - \frac{1}{2} \int_{Y'_d} (\varphi^* \upsilon_\lambda \cdot \upsilon'_\mu \cdot \upsilon'_\nu + \upsilon'_\lambda \cdot \varphi^* \upsilon_\mu \cdot \upsilon'_\nu + \upsilon'_\lambda \cdot \upsilon'_\mu \cdot \varphi^* \upsilon_\nu)$$
$$- \int_{Z_d} \zeta_\lambda \cdot \zeta_\mu \cdot \zeta_\nu + \frac{1}{2} \int_{Z'_d} (\psi^* \zeta_\lambda \cdot \zeta'_\mu \cdot \zeta'_\nu + \zeta'_\lambda \cdot \psi^* \zeta_\mu \cdot \zeta'_\nu + \zeta'_\lambda \cdot \zeta'_\mu \cdot \psi^* \zeta_\nu).$$

When $d \leq m$ is odd, the Gromov-Witten invariant $\langle \tau_\lambda, \tau_\mu, \tau_\nu \rangle_d$ is equal to

$$\int_{Z_d} \zeta_\lambda \cdot \zeta_\mu \cdot \zeta_\nu - \frac{1}{2} \int_{Z'_d} (\psi^* \zeta_\lambda \cdot \zeta'_\mu \cdot \zeta'_\nu + \zeta'_\lambda \cdot \psi^* \zeta_\mu \cdot \zeta'_\nu + \zeta'_\lambda \cdot \zeta'_\mu \cdot \psi^* \zeta_\nu).$$

When $d = m+1$ is even, we have

$$\langle \tau_\lambda, \tau_\mu, \tau_\nu \rangle_d = \int_{\mathrm{G}(2m+1,2n+2)} \upsilon_\lambda \cdot \upsilon_\mu \cdot \upsilon_\nu .$$

(d) *We have $\langle \tau_\lambda, \tau_\mu, \tau_\nu \rangle_d = 0$ if λ does not contain ρ_{d-1} when d is even, or does not contain $(\rho_{d-1}, 1)$ when d is odd.*

Proof. The argument is very similar to that in Theorem 2.3. Given integers $0 \leq e_1 \leq m$, $0 \leq e_2 \leq d \leq m+1$, and $r \geq 0$ we let $Y^r_{e_1,e_2}$ be the variety (or in some cases, disjoint union of two varieties) of pairs (A, B) such that $A \subset B \subset A^\perp \subset V$, $\dim(A) = m - e_1$, $\dim(B) = m + e_2$, and $\dim(B \cap B^\perp) = m - e_1 + r$. These varieties have a transitive action of SO_{2n+2}. Let

$$Y^r_\lambda = \{(A, B) \in Y^r_{e_1,e_2} \mid \exists\, \Sigma \in X_\lambda : A \subset \Sigma \subset B\} .$$

We will show as before that $Y^r_\lambda \cap Y^r_\mu \cap Y^r_\nu$ is empty unless (i) $e_1 = e_2 = d$ and $r \in \{0, 2\}$, or (ii) $e_1 = m$, $e_2 = d = m+1$, and $r = 0$.

Following the proof of Theorem 1.3 we compute that the dimension of (any component of) $Y^r_{e_1,e_2}$ is equal to

$$\frac{1}{2}(n^2 + 2nk - 3k^2 + n + 3k + 2ne_1 - 2ke_1 - e_1^2 + 4e_2k - 2e_2^2 + 3e_1 - r - r^2) .$$

The varieties Q_s defined by equation (28) have the same dimension as in the proof of Theorem 2.3. We deduce that $\operatorname{codim}(Y^r_\lambda) + \operatorname{codim}(Y^r_\mu) + \operatorname{codim}(Y^r_\nu) - \dim(Y^r_{e_1,e_2})$ is greater than or equal to the number $\Delta''(e_1, e_2, r, s) := \dim \mathrm{OG}' + d(n+k) - \dim(Y^r_{e_1,e_2}) - 3 \dim Q_s$, which is equal to

$$(r - 3s)(r - 3s + 1)/2 + (n+k)(d - e_2) + (n - k + e_2 - 2e_1 + 3s)(e_2 - e_1) .$$

The remainder of the argument is exactly the same as in the case of OG. The analysis of Y_{λ^0} can be carried out with $\lambda^0 = (\rho_{d-1}, 1)$ of arbitrary type (the analysis yields $Y_{\lambda^0} = Y_{(1^d)}$, so the type is irrelevant). $\square$

For $\lambda \in \widetilde{\mathcal{P}}(k, n)$, let $\overline{\lambda}$ denote the diagram obtained by deleting the leftmost column of λ, with the convention that $\operatorname{type}(\overline{\lambda}) = \operatorname{type}(\lambda)$.

Proposition 3.1. *For $p \in [1, n+k]$ and λ, $\mu \in \widetilde{\mathcal{P}}(k, n)$ with $|\lambda| + |\mu| + p = \dim \mathrm{OG}' + n + k$, we have*

$$\langle \tau_\lambda, \tau_\mu, \tau_p \rangle_1 = \int_{\mathrm{OG}(n+2-k,2n+2)} \tau_{\overline{\lambda}} \cdot \tau_{\overline{\mu}} \cdot \tau_{p-1} . \tag{46}$$

Moreover, when $p = k$, we have

$$\langle \tau_\lambda, \tau_\mu, \tau'_k \rangle_1 = \int_{\mathrm{OG}(n+2-k,2n+2)} \tau_{\overline{\lambda}} \cdot \tau_{\overline{\mu}} \cdot \tau'_{k-1} . \tag{47}$$

Proof. The argument is the same as the one in the proof of Proposition 2.1. $\square$

3.5. **Quantum cohomology.** As in Section 2.5, the degree of the variable q in $\mathrm{QH}^*(\mathrm{OG}')$ is $n{+}k$, and the quantum Pieri rule will involve both q and q^2 terms. Let $\widetilde{\mathcal{P}}'(k,n{+}1)$ be the set of $\nu \in \widetilde{\mathcal{P}}(k,n{+}1)$ such that $\ell(\nu)=n{+}2{-}k$, $2k{-}1 \leq \nu_1 \leq n{+}k$, and the number of boxes in the second column of ν is at most $\nu_1 - 2k + 2$. For any $\nu \in \widetilde{\mathcal{P}}'(k,n+1)$, we let $\widetilde{\nu} \in \widetilde{\mathcal{P}}(k,n)$ be the element of $\widetilde{\mathcal{P}}'(k,n)$ obtained by removing the first row of ν as well as $n+k+1-\nu_1$ boxes from the first column. That is,

$$\widetilde{\nu} = (\nu_2, \nu_3, \ldots, \nu_r), \text{ where } r = \nu_1 - 2k + 1.$$

Moreover, we have $\mathrm{type}(\widetilde{\nu}) = \mathrm{type}(\nu)$, if $\mathrm{type}(\nu)=0$, and otherwise $\mathrm{type}(\widetilde{\nu}) = 3 - \mathrm{type}(\nu)$. Finally, for any $\lambda \in \widetilde{\mathcal{P}}(k,n)$, we define λ^* with $\mathrm{type}(\lambda^*) = \mathrm{type}(\lambda)$ by $\lambda^* = (\lambda_2, \lambda_3, \ldots)$.

Theorem 3.4 (Quantum Pieri rule for OG$'$)**.** *For any k-strict partition $\lambda \in \widetilde{\mathcal{P}}(k,n)$ and integer $p \in [1, n+k]$, we have*

$$\tau_p \cdot \tau_\lambda = \sum_{\lambda \to \mu} \delta_{\lambda\mu}\, 2^{N'(\lambda,\mu)}\, \tau_\mu + \sum_{\lambda \to \nu} \delta_{\lambda\nu}\, 2^{N'(\lambda,\nu)}\, \tau_{\widetilde{\nu}}\, q \;+ \sum_{\lambda^* \to \rho} \delta_{\lambda^*\rho}\, 2^{N'(\lambda^*,\rho)}\, \tau_{\rho^*}\, q^2$$

in the quantum cohomology ring $\mathrm{QH}^*(\mathrm{OG}(n+1-k, 2n+2))$*. Here* (i) *the first sum is classical, as in (39),* (ii) *the second sum is over $\nu \in \widetilde{\mathcal{P}}'(k,n+1)$ with $\lambda \to \nu$ and $|\nu| = |\lambda| + p$, and* (iii) *the third sum is empty unless $\lambda_1 = n+k$, and over $\rho \in \widetilde{\mathcal{P}}(k,n)$ such that $\rho_1 = n+k$, $\lambda^* \to \rho$, and $|\rho| = |\lambda| - n - k + p$. Furthermore, the product $\tau_k' \cdot \tau_\lambda$ is obtained by replacing δ with δ' throughout.*

Proof. The argument is similar to the proof of Theorem 2.4. Theorem 3.3(d) implies that the quantum Pieri products $\tau_p\,\tau_\lambda$ and $\tau_k'\,\tau_\lambda$ contain at most quadratic q terms. A dimension count shows that the right hand side of equations (46) and (47) vanishes when either λ or μ has length less than $n+1-k$.

If $\mu^\vee$ is the dual partition to μ in $\widetilde{\mathcal{P}}(k,n)$ (see Sections 4.3 and 4.4), then $\ell(\mu^\vee) = n+1-k$ if and only if $\mu_1 < \ell(\mu) + 2k - 1$. Furthermore, if $k > 1$ and $\mu \in \widetilde{\mathcal{P}}(k,n)$ satisfies $\mu_1 < \ell(\mu) + 2k - 1$, then $(\overline{\mu^\vee})^\vee = (\ell(\mu) + 2k - 2, \overline{\mu})^\dagger$, where the dagger $\dagger$ means that $\mathrm{type}((\ell(\mu)+2k-2,\overline{\mu})^\dagger) = \mathrm{type}(\mu)$, if $\mathrm{type}(\mu) = 0$, and otherwise $\mathrm{type}((\ell(\mu)+2k-2,\overline{\mu})^\dagger) = 3 - \mathrm{type}(\mu)$. We deduce the following result.

Proposition 3.2. *Consider λ, $\mu \in \widetilde{\mathcal{P}}(k,n)$ with $|\mu| + n + k = |\lambda| + p$, for $1 \leq p \leq n+k$. If $\ell(\lambda) = n+1-k$ and $\mu_1 < \ell(\mu) + 2k - 1$ then, in* $\mathrm{QH}^*(\mathrm{OG}')$*, the coefficient of $\tau_\mu\, q$ in $\tau_p\,\tau_\lambda$ is equal to the coefficient of $\tau_{(\ell(\mu)+2k-2,\,\overline{\mu})^\dagger}$ in $\tau_{p-1}\,\tau_{\overline{\lambda}} \in$* $\mathrm{H}^*(\mathrm{OG}(n+2-k, 2n+2))$*. Otherwise, the coefficient vanishes.*

Theorem 3.1 implies that for λ, $\mu \in \widetilde{\mathcal{P}}(k,n)$ with $\ell(\lambda) = n+1-k$ and $\mu_1 < \ell(\mu) + 2k - 1$, we have $\overline{\lambda} \to (\ell(\mu) + 2k - 2,\, \overline{\mu})^\dagger$ in $\widetilde{\mathcal{P}}(k-1,n)$ if and only if $\lambda \to (\ell(\mu) + 2k - 1, \mu, 1^{n+1-k-\ell(\mu)})^\dagger$ in $\widetilde{\mathcal{P}}(k,n+1)$, with the same coefficients (δ and N' numbers). It follows that for λ, $\mu \in \widetilde{\mathcal{P}}(k,n)$, the coefficient of $\tau_\mu\, q$ in the quantum product $\tau_p\,\tau_\lambda$ in $\mathrm{QH}^*(\mathrm{OG}')$ is equal to the coefficient of $\tau_{(\ell(\mu)+2k-1,\mu,1^{n+1-k-\ell(\mu)})^\dagger}$ in the cup product $\tau_p\,\tau_\lambda$ in $\mathrm{H}^*(\mathrm{OG}(n+2-k, 2n+4))$ when $\mu_1 < \ell(\mu) + 2k - 1$ (and $\ell(\lambda) = n+1-k$), and is 0 otherwise. In addition, $\nu \mapsto \widetilde{\nu}$ induces a 1-1 map $\widetilde{\mathcal{P}}'(k,n+1) \to \widetilde{\mathcal{P}}(k,n)$ with image $\{\mu \in \widetilde{\mathcal{P}}(k,n) \;:\; \mu_1 < \ell(\mu) + 2k - 1\}$, and the inverse of this map is given by $\mu \mapsto (\ell(\mu) + 2k - 1, \mu, 1^{n+1-k-\ell(\mu)})^\dagger$. We deduce that the coefficient of $\tau_{\widetilde{\nu}}\, q$ in Theorem 3.4 is equal to the coefficient of τ_ν in the product $\tau_p\,\tau_\lambda$ in $\mathrm{H}^*(\mathrm{OG}(n+2-k,2n+4))$, for $\nu \in \widetilde{\mathcal{P}}'(k,n+1)$, and these are all the linear q terms. The linear q terms in $\tau'_k\,\tau_\lambda$ are handled similarly.

The rest of the argument is the same as in the proof of Theorem 2.4, using the basic relation $\tau_{n+k}^2 = q^2$ in $\mathrm{QH}^*(\mathrm{OG}')$. □

Example 3.2. In the quantum cohomology ring of $\mathrm{OG}(5,14)$ we have $\tau_2 \cdot \tau_{(8,7,2,1,1)} = \tau_{(8,7,4,1,1)} + \tau_{(8,7,3,2,1)} + \tau_{(8,7,6)} + \tau'_{(7,2,2,1,1)}\, q + \tau_{(2,1,1,1)}\, q^2 + \tau_{(2,2,1)}\, q^2 + \tau_{(3,1,1)}\, q^2$ and $\tau'_2 \cdot \tau_{(8,7,2,1,1)} = \tau_{(8,7,4,1,1)} + \tau_{(8,7,3,2,1)} + \tau_{(7,3,1,1,1)}\, q + \tau_{(2,1,1,1)}\, q^2 + \tau_{(2,2,1)}\, q^2 + \tau_{(3,1,1)}\, q^2$. The Schubert class $\tau'_{(7,2,2,1,1)}$ has type 2 while all remaining classes in these expansions have types 0 or 1.

Theorem 3.5 (Ring presentation)**.** *The quantum cohomology ring* $\mathrm{QH}^*(\mathrm{OG}')$ *is presented as a quotient of the polynomial ring* $\mathbb{Z}[\tau_1,\ldots,\tau_k,\tau'_k,\tau_{k+1},\ldots,\tau_{n+k},q]$ *modulo the relations*

$$\Delta_r = 0, \qquad n-k+1 < r \leq n, \tag{48}$$

$$\tau_k \Delta_{n+1-k} = \tau'_k \Delta_{n+1-k} = \sum_{p=k+1}^{n+1} (-1)^{p+k+1} \tau_p \Delta_{n+1-p}, \tag{49}$$

$$\sum_{p=k+1}^{r} (-1)^p \tau_p \Delta_{r-p} = 0, \qquad n+1 < r < n+k, \tag{50}$$

$$\sum_{p=k+1}^{n+k} (-1)^p \tau_p \Delta_{n+k-p} = -q, \tag{51}$$

and

$$\tau_r^2 + \sum_{i=1}^{r} (-1)^i \tau_{r+i} c_{r-i} = 0, \qquad k+1 \leq r \leq n, \tag{52}$$

$$\tau_k \tau'_k + \sum_{i=1}^{k} (-1)^i \tau_{k+i} \tau_{k-i} = 0, \tag{53}$$

where the variables c_p *are defined by (40).*

Proof. We proceed as in the proof of Theorem 2.5. The relations (48)–(50) are true because they hold classically and the degree of q is $n+k$. The quantum Pieri rule implies that only the $p = 2k-1$ summand in (51) gives a non-zero quantum correction, equal to $-q$. The remaining relations are easily checked to hold in $\mathrm{QH}^*(\mathrm{OG}')$ when $k < n$, but the case $k = n$ yields the quadric $\mathrm{OG}(1,2n+2)$, which must be checked separately. In this case the degree of q is $2n$, and we must verify (53) using the quantum Pieri rule. The coefficient of q in $\tau_n \tau'_n$ is 1 when n is even, and 0 when n is odd, while the ith term in the sum, for $1 \leq i \leq n-1$, contributes $(-1)^i q$. So, (53) holds in $\mathrm{QH}^*(\mathrm{OG}(1,2n+2))$. □

Remark. The method of computing Gromov-Witten invariants explained in Section 1.6 carries over to types B and D, using polynomial expressions in the special Schubert classes of the orthogonal Grassmannians.

4. Schubert varieties in isotropic Grassmannians

We begin this section by giving a uniform description of the Schubert varieties in isotropic Grassmannians X, parametrizing them using *index sets*. These sets record the attitude of a subspace in X with respect to a fixed isotropic flag, and are important ingredients in our proof of the classical Pieri formula for X.

Let $V \cong \mathbb{C}^N$ be a complex vector space equipped with a non-degenerate skew-symmetric or symmetric bilinear form $(\ ,\ )$. Given a non-negative integer $m \leq N/2$, we let X denote the Grassmannian of m-dimensional isotropic subspaces of V,

$$X = \{\Sigma \subset V \mid \dim(\Sigma) = m \text{ and } (\ ,\ )|_\Sigma \equiv 0\}\,.$$

This variety has a transitive action of the group $G = \mathrm{Sp}(V)$ or $G = \mathrm{SO}(V)$ of linear automorphisms preserving the form on V. There is a single exception: when $m = N/2$ and the form is symmetric, then the space of isotropic subspaces has two isomorphic connected components, each a single $\mathrm{SO}(V)$ orbit.

Let $F_\bullet$ be an isotropic flag of V. The Schubert varieties in X relative to the flag $F_\bullet$ are the orbit closures for the action of the stabilizer $B \subset G$ of $F_\bullet$. We proceed to give an elementary description of these varieties.

For any subset $H \subset V$ we let $\langle H \rangle \subset V$ denote the linear span of H. We will say that a basis $\{e_1, \dots, e_N\}$ of V is a *standard basis* with respect to the isotropic flag $F_\bullet$, if $(e_i, e_j) = 0$ for $i + j \neq N + 1$, $(e_i, e_{N+1-i}) = 1$, for $1 \leq i \leq \lfloor (N+1)/2 \rfloor$, and $F_p = \langle e_1, \dots, e_p \rangle$ for each p.

Lemma 4.1. *Given any isotropic subspace $\Sigma \subset V$ and an isotropic flag $F_\bullet \subset V$, there exists a standard basis $\{e_1, \dots, e_N\}$ of V with respect to $F_\bullet$ such that $\Sigma = \langle \{e_1, \dots, e_N\} \cap \Sigma \rangle$.*

Proof. Choose a vector $0 \neq e_1 \in F_1$. We choose a second vector $e_N \in V \smallsetminus F_{N-1}$ such that $(e_1, e_N) = 1$ as follows. If $e_1 \in \Sigma$ then choose $e_N \in V \smallsetminus F_{N-1}$. If $e_1 \notin \Sigma$ and $\Sigma \subset F_{N-1}$ then choose $e_N \in \Sigma^\perp \smallsetminus F_{N-1}$. Finally, if $e_1 \notin \Sigma$ and $\Sigma \not\subset F_{N-1}$ then choose $e_N \in \Sigma \smallsetminus F_{N-1}$.

Set $V' = \langle e_1, e_N \rangle^\perp$, $\Sigma' = \Sigma \cap V'$, and $F'_i = F_{i+1} \cap V'$. By induction we can find a basis $\{e'_1, \dots, e'_{N-2}\}$ for V' satisfying the requirement of the lemma with respect to Σ' and $F'_\bullet$. We obtain the required basis for V by setting $e_i = e'_{i-1}$ for $1 < i < N$. $\square$

For any point $\Sigma \in X$ we define a subset $P(\Sigma) \subset [1, N]$ of cardinality m by

$$P(\Sigma) = \{p \in [1, N] \mid \Sigma \cap F_p \supsetneq \Sigma \cap F_{p-1}\}\,.$$

Notice that $P(\Sigma') = P(\Sigma)$ for any point Σ' in the orbit $B.\Sigma \subset X$. On the other hand, it follows from Lemma 4.1 that any point $\Sigma' \in X$ such that $P(\Sigma') = P(\Sigma)$ must be in this orbit. We call a subset $P \subset [1, N]$ of cardinality m an *index set* if for all $i, j \in P$ we have $i + j \neq N + 1$. Any set $P(\Sigma)$ is an index set, since no vector in $F_i \smallsetminus F_{i-1}$ is orthogonal to a vector in $F_{N+1-i} \smallsetminus F_{N-i}$. On the other hand, given any index set P we can construct a point $\Sigma \in X$ with $P(\Sigma) = P$. In fact, if $\{e_1, \dots, e_N\}$ is a standard basis for V with respect to $F_\bullet$, then $\Sigma = \langle e_i : i \in P \rangle$ has this property. In other words, the B-orbits (or Schubert cells) in X correspond 1-1 to the index sets P. We let $X^\circ_P(F_\bullet)$ denote the Schubert cell given by P, that is

$$X^\circ_P(F_\bullet) = \{\Sigma \in X \mid P(\Sigma) = P\}\,.$$

The Schubert variety $X_P(F_\bullet)$ is defined as the closure of the Schubert cell $X^\circ_P(F_\bullet)$; we let $|P|$ be its codimension in X. For each index set $P = \{p_j\}$, let $[X_P] \in$

$H^{2|P|}(X,\mathbb{Z})$ denote the cohomology class Poincaré dual to the cycle defined by $X_P(F_\bullet)$.

4.1. **Type C.** Let $V \cong \mathbb{C}^{2n}$ be a symplectic vector space and $X = \mathrm{IG}(m,2n)$. Given another index set $Q = \{q_1 < \cdots < q_m\}$ we write $Q \leq P$ if $q_j \leq p_j$ for each j.

Proposition 4.1. *For any index set $P = \{p_1 < p_2 < \cdots < p_m\} \subset [1,2n]$ we have*

$$X_P(F_\bullet) = \{\Sigma \in \mathrm{IG} \mid \dim(\Sigma \cap F_{p_j}) \geq j, \quad \forall\, 1 \leq j \leq m\}\,. \tag{54}$$

Proof. The set on the right hand side of (54) is closed and equals the union of the orbits X_Q° for all index sets Q such that $Q \leq P$. We must show that each of these orbits is contained in the closure of X_P°. Assuming $Q < P$, it is enough to construct an index set P' such that $Q \leq P' < P$ and $X_{P'}^\circ \subset \overline{X_P^\circ}$.

Choose j minimal such that $q_j < p_j$ and fix a standard basis $\{e_1,\ldots,e_{2n}\}$ of V with respect to $F_\bullet$. If $2n+1-q_j \notin P$ or $q_j + p_j = 2n+1$, then set $P' = \{q_1,\ldots,q_j,p_{j+1},\ldots,p_m\}$, and define a morphism $\mathbb{P}^1 \to \mathrm{IG}$ by

$$[s:t] \mapsto \Sigma_{[s:t]} = \langle e_{p_1},\ldots,e_{p_{j-1}}, s e_{q_j} + t e_{p_j}, e_{p_{j+1}},\ldots,e_{p_m}\rangle.$$

Since $\Sigma_{[1:0]} \in X_{P'}^\circ$ and $\Sigma_{[s:t]} \in X_P^\circ$ for $t \neq 0$, it follows that $X_{P'}^\circ \subset \overline{X_P^\circ}$.

If $2n+1-q_j \in P$ and $q_j + p_j \neq 2n+1$, set $P' = (P \smallsetminus \{p_j, 2n+1-q_j\}) \cup \{q_j, 2n+1-p_j\}$. We then use the morphism $\mathbb{P}^1 \to \mathrm{IG}$ given by

$$[s:t] \mapsto \Sigma_{[s:t]} = \langle e_p : p \in P \cap P'\rangle \oplus \langle s e_{q_j} + t e_{p_j}, s e_{2n+1-p_j} \pm t e_{2n+1-q_j}\rangle,$$

where the sign is chosen so that $\Sigma_{[s:t]}$ is isotropic, to show that $X_{P'}^\circ \subset \overline{X_P^\circ}$, as required. $\square$

Define the dual index set $P^\vee = \{p_j^\vee\}$ by setting $p_j^\vee = 2n+1-p_{m+1-j}$.

Proposition 4.2. *For any index sets P and Q, we have*

$$\int_{\mathrm{IG}} [X_P] \cdot [X_Q] = \delta_{Q,P^\vee}\,.$$

Proof. Let $F_\bullet$ and $G_\bullet$ be general isotropic flags in V, and assume $P = \{p_1 < \cdots < p_m\}$ and $Q = \{q_1 < \cdots < q_m\}$ are index sets such that $X_P^\circ(F_\bullet) \cap X_Q^\circ(G_\bullet) \neq \emptyset$. For any point Σ in this intersection we have $\dim(\Sigma \cap F_{p_{m+1-j}}) \geq m+1-j$ and $\dim(\Sigma \cap G_{q_j}) \geq j$, which implies that $F_{p_{m+1-j}} \cap G_{q_j} \neq 0$. It follows from this that $q_j \geq 2n+1-p_{m+1-j}$ for each j. Notice also that $X_P^\circ(F_\bullet) \cap X_{P^\vee}^\circ(G_\bullet) = \{\Sigma_0\}$, where $\Sigma_0 = \bigoplus_j (F_{p_j} \cap G_{2n+1-p_{m+1-j}}) \subset V$. It follows that $\int [X_P]\cdot[X_{P^\vee}] = 1$, and that $X_P^\circ(F_\bullet)$ and $X_{P^\vee}^\circ(G_\bullet)$ have complementary dimensions in IG. If $Q \neq P^\vee$ then $Q > P^\vee$, so $X_{P^\vee}^\circ(G_\bullet)$ must be a proper closed subset of $X_Q^\circ(G_\bullet)$ and the intersection $X_P^\circ(F_\bullet) \cap X_Q^\circ(G_\bullet)$ has positive dimension. $\square$

Set $k = n-m$. We establish a bijection between the index sets P and the set $\mathcal{P}(k,n)$ of k-strict partitions λ contained in an $m \times (n+k)$ rectangle, i.e. $\ell(\lambda) \leq m$ and $\lambda_1 \leq n+k$. Let $P = \{p_1 < p_2 < \cdots < p_m\}$ be any index set. Choose $0 \leq s \leq m$ such that $p_s \leq n < p_{s+1}$, where $p_0 = 0$ and $p_{m+1} = 2n+1$. Write

$$[n+1,2n] \smallsetminus \{2n+1-p_1,\ldots,2n+1-p_s\} = \{r_1 < r_2 < \cdots < r_{n-s}\}$$

and choose indices $1 \leq t_{s+1} < \cdots < t_m \leq n-s$ so that $p_j = r_{t_j}$ for $s+1 \leq j \leq m$. The bijection maps P to the k-strict partition $\lambda = (\lambda_1,\ldots,\lambda_m)$ given by

$$\lambda_j = \begin{cases} n+k+1-p_j & \text{if } 1 \leq j \leq s, \\ k+j-s-t_j & \text{if } s+1 \leq j \leq m. \end{cases}$$

Since the subsets of $\{r_1,\dots,r_{n-s}\}$ of cardinality $m-s$ correspond 1-1 to the partitions $(\lambda_{s+1},\dots,\lambda_m)$ contained in an $(m-s)\times k$ rectangle, it follows that the assignment $P\mapsto\lambda$ gives a bijection between the index sets P and the k-strict partitions λ in $\mathcal{P}(k,n)$. We use the notation $X_\lambda(F_\bullet)\equiv X_P(F_\bullet)$ for the Schubert varieties in IG relative to the isotropic flag $F_\bullet$.

Proposition 4.3. *Let $P=\{p_1<\cdots<p_m\}\subset[1,2n]$ be an index set and $\lambda=(\lambda_1,\dots,\lambda_m)$ the corresponding k-strict partition. For $1\le i\le j\le m$ we have*
(i) *$\lambda_j\le k$ if and only if $p_j>n$;*
(ii) *$\lambda_i+\lambda_j\le 2k+j-i$ if and only if $p_i+p_j>2n+1$;*
(iii) *$\lambda_j=n+k+1-p_j+\#\{i<j:p_i+p_j>2n+1\}$; and*
(iv) *$p_j=n+k+1-\lambda_j+\#\{i<j:\lambda_i+\lambda_j\le 2k+j-i\}$.*

Proof. The first point (i) is clear from the definitions. Let s, $\{r_1<\cdots<r_{n-s}\}$, and $\{t_{s+1}<\cdots<t_m\}$ be chosen as above. For $j>s$ we have $t_j=\#\{i\le n-s:r_i\le p_j\}=p_j-n-\#\{i\le s:2n+1-p_i<p_j\}$, so we obtain $\lambda_j=n+k-p_j+j-s+\#\{i\le s:p_i+p_j>2n+1\}$, which implies (iii). Point (ii) is clearly true if $p_j\le n$ or $p_i>n$, so assume that $p_i\le n<p_j$. Then $\lambda_i+\lambda_j=2n+2k+2-(p_i+p_j)+\#\{l<j:p_l+p_j>2n+1\}$, and (ii) follows because $\#\{l<j:p_l+p_j>2n+1\}$ is smaller than or equal to $j-i+(p_i+p_j-2n-2)$ when $p_i+p_j>2n+1$, while it is greater than or equal to $j-i-1-(2n-p_i-p_j)$ when $p_i+p_j<2n+1$. Finally (iv) follows from (ii) and (iii). ∎

We note that if $\lambda_j=0$ then the Schubert condition $\dim(\Sigma\cap F_{p_j})\ge j$ is redundant. In fact, we have $p_j=n+k+1+\#\{i<j:p_i+p_j>2n+1\}$. If $p_1+p_j>2n+1$ then $p_j=n+k+j$, so the Schubert condition holds automatically. Otherwise choose $i\le j$ maximal so that $p_i+p_j<2n+1$ and set $C=\Sigma\cap F_{p_i}$. Then $p_j=n+k+j-i$. We claim that the above Schubert condition is a consequence of the condition $\dim(C)\ge i$. In fact, since $\Sigma\subset C^\perp$ and $F_{p_j}\subset F_{2n-p_i}=F_{p_i}^\perp\subset C^\perp$, we obtain $\dim(\Sigma\cap F_{p_j})\ge m+p_j-(2n-i)=j$ as required.

Corollary 4.1. *For each partition $\lambda\in\mathcal{P}(k,n)$, the Schubert variety indexed by λ is given by*

$$X_\lambda(F_\bullet)=\{\Sigma\in X\mid \dim(\Sigma\cap F_{p_j(\lambda)})\ge j,\quad\forall\, 1\le j\le\ell(\lambda)\}$$

where $p_j(\lambda):=n+k+1-\lambda_j+\#\{i<j:\lambda_i+\lambda_j\le 2k+j-i\}$.

One can also check that the Schubert condition $\dim(\Sigma\cap F_{p_j})\ge j$ is redundant if $\lambda_j=\lambda_{j+1}+1\ge k+2$, or if $\lambda_j=\lambda_{j+1}<2k+j-\lambda_1$. However, the following example shows that a Schubert condition $\dim(\Sigma\cap F_{p_j})\ge j$ may be necessary, even if $\lambda_j=\lambda_{j+1}$.

Example 4.1. Let $X=\mathrm{IG}(3,8)$ and $\lambda=(3,1,1)$. Then $X_\lambda(F_\bullet)$ is the variety of points $\Sigma\in X$ such that $\dim(\Sigma\cap F_3)\ge 1$, $\dim(\Sigma\cap F_5)\ge 2$, and $\dim(\Sigma\cap F_7)\ge 3$. One may check that $\{\Sigma\in X\mid \Sigma\cap F_3\neq 0 \text{ and } \Sigma\subset F_7\}=X_\lambda(F_\bullet)\cup X_5(F_\bullet)$.

Choose a standard basis $\{e_1,\dots,e_{2n}\}$ for V with respect to $F_\bullet$. The B-orbit given by P is an affine space with coordinates given using $m\times 2n$ matrices $A=\{a_{jr}\}$ in which some entries a_{jr} are free parameters and others are determined by the free entries. The matrix corresponds to the subspace $\Sigma\subset V$ spanned by its rows. Each entry a_{j,p_j} of A is equal to 1, while $a_{jr}=0$ for $r>p_j$. We also set $a_{j,p_i}=0$ for $i<j$. If $r<p_j$ and $r\notin\{p_i,2n+1-p_i\}$ for all $i<j$ then a_{jr} is a free variable.

Finally, for each $i < j$ such that $p_i + p_j > 2n+1$, the entry $a_{j,2n+1-p_i}$ is uniquely determined from the free variables by the requirement that rows i and j in A are orthogonal. For example, if $m = 3$, $n = 5$, and $P = \{3, 5, 9\}$ then the matrix A has the shape

$$A = \begin{bmatrix} * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & 1 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & 0 & = & * & = & 1 & 0 \end{bmatrix}.$$

Since the number of free variables $*$ in row j of A is equal to $p_j - j - \#\{i < j : p_i + p_j > 2n+1\} = n + k + 1 - j - \lambda_j$, it follows that the dimension of $X_\lambda(F_\bullet)$ is $m(n + k + 1) - m(m + 1)/2 - |\lambda|$.

Proposition 4.4. *The dimension of* $\mathrm{IG}(m, 2n)$ *is* $2m(n - m) + m(m + 1)/2$*, and the codimension of* $X_\lambda(F_\bullet)$ *is* $|\lambda|$*.*

For a k-strict partition λ we define the dual partition $\lambda^\vee$ to be the unique k-strict partition such that $p_j(\lambda^\vee) = 2n + 1 - p_{m+1-j}(\lambda)$. Proposition 4.3 implies that

$$\begin{aligned} \lambda^\vee_{m+1-j} = 2k + 1 - \lambda_j &+ \#\{i < j : \lambda_i + \lambda_j \leq 2k + j - i\} \\ &+ \#\{i > j : \lambda_i + \lambda_j > 2k + i - j\}. \end{aligned}$$

The relationship between the diagrams of λ and $\lambda^\vee$ is described in Section 4.4. If $[X_\lambda] \in H^{2|\lambda|}(\mathrm{IG})$ denotes the cohomology class which corresponds to $X_\lambda(F_\bullet)$, then Proposition 4.2 gives

$$\int_{\mathrm{IG}} [X_\lambda] \cdot [X_\mu] = \delta_{\mu,\lambda^\vee}.$$

4.2. **Type B.** The situation here is very similar to that in the previous type C section, so we will point out the main differences and leave the details to the reader. We equip $V \cong \mathbb{C}^{2n+1}$ with a non-degenerate symmetric bilinear form $(\ ,\ )$, and let $X = \mathrm{OG}(m, 2n + 1)$. This algebraic variety has the same dimension as $\mathrm{IG}(m, 2n)$.

For every $\Sigma \in \mathrm{OG}$, the index set

$$P = P(\Sigma) = \{p_1 < p_2 < \cdots < p_m\} \subset [1, 2n + 1]$$

satisfies $n + 1 \notin P$, and the closure of the Schubert cell $X^\circ_P(F_\bullet)$ in OG is the Schubert variety

$$X_P(F_\bullet) = \{\Sigma \in \mathrm{OG} \mid \dim(\Sigma \cap F_{p_j}) \geq j, \ \ \forall\, 1 \leq j \leq m\}.$$

Note that Proposition 4.2 holds for OG as well as IG.

The index sets $\overline{P} \subset [1, 2n + 1]$ for OG are in bijection with the index sets $P \subset [1, 2n]$ for IG and the k-strict partitions in $\mathcal{P}(k, n)$. For any $\lambda \in \mathcal{P}(k, n)$, the corresponding index set $\overline{P} = \{\overline{p}_1 < \cdots < \overline{p}_m\}$ satisfies

$$\overline{p}_j(\lambda) = \begin{cases} p_j(\lambda) + 1 & \text{if } \lambda_j \leq k, \\ p_j(\lambda) & \text{if } \lambda_j > k, \end{cases}$$

where $p_j(\lambda) = n+k+1-\lambda_j+\#\{i < j : \lambda_i+\lambda_j \leq 2k+j-i\}$ are the indices used in the previous subsection. The dual index set $\overline{P}^\vee$ in type B satisfies $\overline{p}^\vee_j = 2n+2-p_{m+1-j}$.

Let $\{e_1, \ldots, e_{2n+1}\}$ be a standard basis of V with respect to the isotropic flag $F_\bullet$. We may then represent the Schubert cell indexed by $\overline{P}$ by an $m \times (2n + 1)$ matrix $A = \{a_{jr}\}$ whose rows span the subspaces in the cell, as in type C. Each entry $a_{j,\overline{p}_j}$ of A is equal to 1, $a_{jr} = 0$ for $r > \overline{p}_j$, while $a_{j,\overline{p}_i} = 0$ for $i < j$. If $r < \overline{p}_j$ and $r \notin \{\overline{p}_i, 2n + 2 - \overline{p}_i\}$ for all $i \leq j$ then a_{jr} is a free variable. Finally, for each

$i \leq j$ such that $\overline{p}_i + \overline{p}_j > 2n+2$, the entry $a_{j,2n+2-\overline{p}_i}$ is determined from the free variables and the isotropicity condition on the rows of A. For example, if $m = 3$, $n = 5$, and $\overline{P} = \{3, 5, 10\}$ then the matrix A has the shape

$$A = \begin{bmatrix} * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & = & 0 & * & 0 & * & = & * & = & 1 & 0 \end{bmatrix}.$$

By counting the number of free entries $*$ in A, we see that the codimension of $X_\lambda(F_\bullet)$ in OG is equal to $|\lambda|$.

4.3. **Type D.** Let $V \cong \mathbb{C}^{2n+2}$ be a complex vector space equipped with a nondegenerate symmetric bilinear form $(\ ,\ )$. For any nonnegative integer $m < n+1$, let $\mathrm{OG}' = \mathrm{OG}(m, 2n+2)$. The index sets $P = \{p_j\}$ parametrize the B-orbits $X_P^\circ(F_\bullet)$ in OG', as before. However the closures of these orbits behave differently. Given another index set $Q = \{q_j\}$, we will write $Q \preceq P$ when $q_j \leq p_j$ for each j and if $p_i = n+2$ for some i then $q_i \neq n+1$. The next result shows that the Schubert variety $X_P(F_\bullet)$ is equal to the union of the orbits $X_Q^\circ(F_\bullet)$ for all index sets Q such that $Q \preceq P$.

Proposition 4.5. *Let $P = \{p_1 < p_2 < \cdots < p_m\} \subset [1, 2n+2]$ be an index set. If $n+2 \notin P$, then*

$$X_P(F_\bullet) = \{\Sigma \in \mathrm{OG}' \mid \dim(\Sigma \cap F_{p_j}) \geq j, \quad \forall\, 1 \leq j \leq m\},$$

while if $n+2 \in P$, then

$$\begin{aligned} X_P(F_\bullet) &= \{\Sigma \in \mathrm{OG}' \mid \Sigma \cap F_n = \Sigma \cap F_{n+1},\ \dim(\Sigma \cap F_{p_j}) \geq j,\ \forall\, 1 \leq j \leq m\} \\ &= \{\Sigma \in \mathrm{OG}' \mid \dim(\Sigma \cap F_{p_j}) \geq j,\ \text{if } p_j \neq n+2,\ \dim(\Sigma \cap \widetilde{F}_{n+1}) \geq j,\ \text{if } p_j = n+2\}. \end{aligned}$$

Proof. The sets displayed on the right hand sides of the above equations are closed. Following the proof of Proposition 4.1, it will suffice to construct, for every index set $Q \prec P$ an index set P' such that $Q \preceq P' \prec P$ and $X_{P'}^\circ \subset \overline{X_P^\circ}$.

Choose j minimal such that $q_j < p_j$ and fix a standard basis $\{e_1, \ldots, e_{2n+2}\}$ of V with respect to $F_\bullet$. If $2n+3-q_j \notin P$ we set $P' = \{q_1, \ldots, q_j, p_{j+1}, \ldots, p_m\}$, and use the morphism $\mathbb{P}^1 \to \mathrm{OG}'$ given by

$$[s:t] \mapsto \Sigma_{[s:t]} = \langle e_{p_1}, \ldots, e_{p_{j-1}}, s e_{q_j} + t e_{p_j}, e_{p_{j+1}}, \ldots, e_{p_m}\rangle,$$

as in the symplectic case. If $q_j + p_j = 2n+3$, use the same set P' and the morphism

$$[s:t] \mapsto \Sigma_{[s:t]} = \langle e_{p_1}, \ldots, e_{p_{j-1}}, s^2 e_{q_j} + st(e_{n+1} + e_{n+2}) - t^2 e_{p_j}, e_{p_{j+1}}, \ldots, e_{p_m}\rangle.$$

Finally, if $2n+3-q_j \in P$ and $q_j + p_j \neq 2n+3$, set $P' = (P \smallsetminus \{p_j, 2n+3-q_j\}) \cup \{q_j, 2n+3-p_j\}$, and use the morphism

$$[s:t] \mapsto \Sigma_{[s:t]} = \langle e_p : p \in P \cap P'\rangle \oplus \langle s e_{q_j} + t e_{p_j}, s e_{2n+3-p_j} - t e_{2n+3-q_j}\rangle. \qquad \square$$

In type D, for any index set P, the dual index set $P^\vee$ is defined by

$$p_j^\vee = \begin{cases} 2n+3-p_{m+1-j} & \text{if } n \text{ is odd or } p_j \notin \{n+1, n+2\}, \\ p_j & \text{if } n \text{ is even and } p_j \in \{n+1, n+2\}. \end{cases}$$

The next proposition is then proved exactly as in type C.

Proposition 4.6. *For any index sets P and Q, we have*

$$\int_{\mathrm{OG}'} [X_P] \cdot [X_Q] = \delta_{Q,P^\vee}.$$

We say that each index set P has a *type*, which is a number $\mathrm{type}(P) \in \{0,1,2\}$. If $P \cap \{n+1, n+2\} = \emptyset$, then $\mathrm{type}(P) = 0$. Otherwise, $\mathrm{type}(P)$ is equal to 1 plus the parity mod 2 of the codimension of $\Sigma \cap F_{n+1}$ in F_{n+1}, for all Σ in the Schubert cell $X_P^\circ(F_\bullet)$. Equivalently, $\mathrm{type}(P)$, when non-zero, is equal to 1 plus the parity of the number of integers in $[1, n+1] \smallsetminus P$.

Set $k = n+1-m$. We will define a different set $\widetilde{\mathcal{P}}(k,n)$ of indices for the Schubert classes in OG' which makes their codimension apparent. As with index sets, we agree that any k-strict partition λ has a type in $\{0,1,2\}$. If $\lambda_j = k$ for some j, then $\mathrm{type}(\lambda) \in \{1,2\}$; otherwise, $\mathrm{type}(\lambda) = 0$. The elements of $\widetilde{\mathcal{P}}(k,n)$ are the k-strict partitions of all possible types which are contained in an $m \times (n+k)$ rectangle.

Given an index set $P' = \{p'_1 < p'_2 < \cdots < p'_m\} \subset [1, 2n+2]$, the corresponding element $\lambda \in \widetilde{\mathcal{P}}(k,n)$ satisfies $\mathrm{type}(\lambda) = \mathrm{type}(P')$, and its underlying partition is obtained by a prescription similar to that in Section 4.1. Choose s such that $p'_s \leq n+1 < p'_{s+1}$, and write

$$[n+2, 2n+2] \smallsetminus \{2n+3-p'_1, \ldots, 2n+3-p'_s\} = \{r_1 < r_2 < \cdots < r_{n+1-s}\}.$$

Next, choose indices $1 \leq t_{s+1} < \cdots < t_m \leq n+1-s$ so that $p'_j = r_{t_j}$ for $s+1 \leq j \leq m$. Then P' maps to the k-strict partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ given by

$$\lambda_j = \begin{cases} n+k+1-p'_j & \text{if } 1 \leq j \leq s, \\ k+j-s-t_j & \text{if } s+1 \leq j \leq m. \end{cases}$$

Arguing in the same way as in Proposition 4.3, we prove

Proposition 4.7. *Let $P' = \{p'_1 < \cdots < p'_m\} \subset [1, 2n+2]$ be an index set and $\lambda = (\lambda_1, \ldots, \lambda_m)$ the corresponding element of $\widetilde{\mathcal{P}}(k,n)$. For $1 \leq i \leq j \leq m$ we have*

(i) *$\lambda_j \leq k$ if and only if $p'_j > n$;*

(ii) *$\lambda_j = k < \lambda_{j-1}$ if and only if $p'_j \in \{n+1, n+2\}$;*

(iii) *For $i < j$, $\lambda_i + \lambda_j \leq 2k-1+j-i$ if and only if $p'_i + p'_j > 2n+3$; and*

(iv) $\lambda_j = \begin{cases} n+k+1-p'_j & \textit{if } p'_j \leq n+1, \\ n+k+2-p'_j + \#\{i<j : p'_i + p'_j > 2n+3\} & \textit{if } p'_j > n+1. \end{cases}$

Conversely, for every $\lambda \in \widetilde{\mathcal{P}}(k,n)$, the associated index set $P' \subset [1, 2n+2]$ satisfies

$$p'_j = n+k-\lambda_j + \#\{\, i<j \mid \lambda_i + \lambda_j \leq 2k-1+j-i \,\}$$
$$+ \begin{cases} 1 & \text{if } \lambda_j > k, \text{ or } \lambda_j = k < \lambda_{j-1} \text{ and } n+j+\mathrm{type}(\lambda) \text{ is even,} \\ 2 & \text{otherwise.} \end{cases}$$

The conditions defining the Schubert variety $X_\lambda(F_\bullet)$ indexed by an element λ of $\widetilde{\mathcal{P}}(k,n)$ are given in Section 3.1.

Let $\{e_1, \ldots, e_{2n+2}\}$ be a standard basis of V with respect to the isotropic flag $F_\bullet$. The Schubert cells indexed by a set P' may then be represented by an $m \times (2n+2)$ matrix $A = \{a_{jr}\}$ as before. Each entry a_{j,p'_j} of A is equal to 1, $a_{jr} = 0$ for $r > p'_j$, while $a_{j,p'_i} = 0$ for $i < j$. If $r < p'_j$ and $r \notin \{p'_i, 2n+3-p'_i\}$ for all $i \leq j$ then a_{jr} is a free variable. Finally, for each $i \leq j$ such that $p'_i + p'_j > 2n+3$, the entry $a_{j,2n+3-p'_i}$ is determined from the free variables and the isotropicity condition on the rows of A. For example, if $m = 3$, $n = 5$, and $P' = \{4, 8, 11\}$ then the matrix A has the shape

$$A = \begin{bmatrix} * & * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & 0 & = & * & * & 1 & 0 & 0 & 0 & 0 \\ * & = & * & 0 & = & * & * & 0 & = & * & 1 & 0 \end{bmatrix}.$$

It follows that the codimension of $X_\lambda(F_\bullet)$ in OG' is equal to $|\lambda|$.

For each $\lambda \in \widetilde{\mathcal{P}}(k, n)$, we define a dual element $\lambda^\vee \in \widetilde{\mathcal{P}}(k, n)$ by requiring that

$$p'_j(\lambda^\vee) = \begin{cases} 2n + 3 - p'_{m+1-j}(\lambda) & \text{if } n \text{ is odd or } p'_j(\lambda) \notin \{n+1, n+2\}, \\ p'_j(\lambda) & \text{if } n \text{ is even and } p'_j(\lambda) \in \{n+1, n+2\}. \end{cases}$$

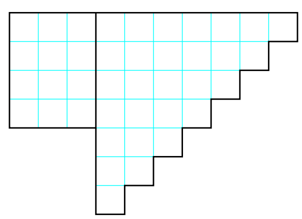
As in Proposition 4.6, for any two elements $\lambda, \mu \in \widetilde{\mathcal{P}}(k, n)$, we have

$$\int_{\mathrm{OG}'} [X_\lambda] \cdot [X_\mu] = \delta_{\mu, \lambda^\vee} .$$

4.4. **Dual partitions.** We now give a pictorial description of the relationship between a k-strict partition λ, the corresponding index set, and the Poincaré dual partition $\lambda^\vee$. We begin our discussion in types B and C.

Let Π be the diagram obtained by attaching an $m \times k$ rectangle to the left side of a staircase partition with n rows. When $n = 7$ and $k = 3$, this looks as follows.

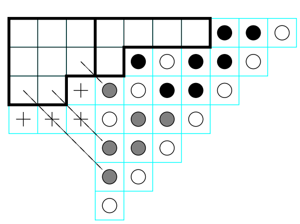
Π =



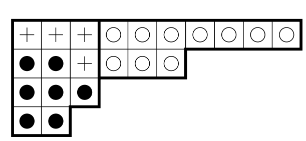
The elements of $\mathcal{P}(k, n)$ are exactly the k-strict partitions whose diagrams fit inside Π. For $\lambda \in \mathcal{P}(k, n)$ we let $\lfloor \lambda \rfloor_k$ be the set of boxes of λ in columns $k+1$ through $k+n$. If $P = \{p_1 < \cdots < p_m\}$ is the index set corresponding to λ, then the values p_i which are less than or equal to n are obtained by subtracting the number of boxes in the i-th row of $\lfloor \lambda \rfloor_k$ from $n+1$, for $1 \le i \le \ell_k(\lambda)$.

We will organize the boxes of the staircase partition which are outside λ into south-west to north-east *diagonals*. Notice that exactly $m - \ell_k(\lambda)$ of these diagonals are *not* k-related to one of the bottom boxes in the first k columns of λ. We will call these diagonals *non-related*. In type C we obtain the integers p_i which are greater than n by adding n to the length of each of the non-related diagonals; in type B we add $n+1$ to these lengths. For example, the partition $\lambda = (7, 4, 2)$ results in the (type C) index set $P = \{8-4, 8-1, 7+3, 7+7\} = \{4, 7, 10, 14\}$.

λ =



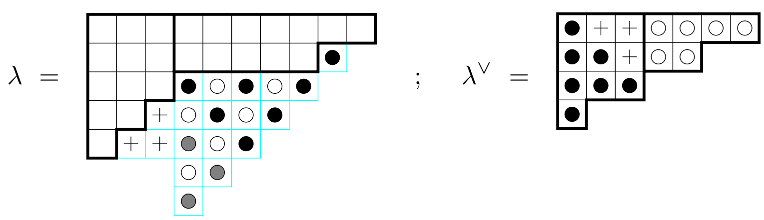
; $\lambda^\vee$ =



The parts of $\lambda^\vee$ which are larger than k are obtained by adding k to the lengths of the non-related diagonals. In other words, these lengths are the parts of $\lfloor \lambda^\vee \rfloor_k$. Finally, for $1 \leq j \leq k$, the number of boxes in the j-th column of $\lambda^\vee$ is equal to the number of boxes of the j-th column of Π which are outside λ, plus the length of the diagonal that is k-related to the bottom box of column j of λ, minus $(k+1-j)$. The dual of the partition $\lambda = (7,4,2)$ is $\lambda^\vee = (10,6,3,2)$.

Consider the analogous picture in type D. The shapes of the elements of $\widetilde{\mathcal{P}}(k,n)$ are the k-strict partitions that fit inside the diagram Π with $m = n+1-k$. Let $\lambda \in \widetilde{\mathcal{P}}(k,n)$ and let $P' = \{p'_1 < \cdots < p'_m\}$ be the corresponding index set. The values p'_i which are less than or equal to n are obtained by subtracting the parts of $\lfloor \lambda \rfloor_k$ from $n+1$. The values p'_i which are greater than $n+1$ depend on $\operatorname{type}(\lambda)$ as well as the lengths of the non-related diagonals.

The dual partition $\lambda^\vee$ can be constructed as follows. The parts of $\lfloor \lambda^\vee \rfloor_k$ consist of the lengths of the non-related diagonals, as in types B and C. For $1 \leq j \leq k$, the number of boxes in column j of $\lambda^\vee$ is equal to the number of boxes in the j-th column of Π which are outside λ, plus the length of the diagonal that is k'-related to the bottom box of column j of λ, minus $(k-j)$. For example, when $k=3$ and $n=7$, the dual of the partition $\lambda = (10,8,3,2,1)$ is $\lambda^\vee = (7,5,3,1)$.



We have $\operatorname{type}(\lambda^\vee) = 0$ if and only if $\operatorname{type}(\lambda) = 0$. Notice that when $\operatorname{type}(\lambda) > 0$, we have $\ell_k(\lambda) + \ell_k(\lambda^\vee) = n-k$, since $p'_i(\lambda) \in \{n+1, n+2\}$ for some i. As $\operatorname{type}(P')$ is equal to 1 plus the parity of the number of integers in $[1, n+1] \smallsetminus P'$, we deduce that $\ell_k(\lambda) + \operatorname{type}(\lambda) \equiv n + \ell_k(\lambda^\vee) + \operatorname{type}(\lambda^\vee) \pmod 2$. It follows that when $\operatorname{type}(\lambda) > 0$, we have $\operatorname{type}(\lambda^\vee) \equiv k + \operatorname{type}(\lambda) \pmod 2$.

4.5. **Other parametrizations.** In this short section we indicate how our notation for Schubert varieties compares to that used in previous related works, in particular [PR1], [PR2], and [T].

Let $W_n = S_n \ltimes \mathbb{Z}_2^n$ be the Weyl group for the root system C_n, thought of as a group of permutations with a sign attached to each entry. The group W_n is generated by simple reflections $s_0, \ldots, s_{n-1}$, and the symmetric group S_n is the subgroup of W_n generated by the transpositions $s_i = (i, i+1)$ for $i > 0$. If W_k denotes the parabolic subgroup of W_n generated by $\{s_i \mid i \neq k\}$, then it is well known that the set $W^{(k)} \subset W_n$ of minimal length coset representatives of W_k parametrizes the Schubert varieties in $\mathrm{IG}(n-k, 2n)$ and $\mathrm{OG}(n-k, 2n+1)$.

Pragacz and Ratajski [PR1] defined a set of partition pairs in bijection with the elements of $W^{(k)}$. The Schubert varieties are thus parametrized by the set of all partition pairs $(\alpha \,|\, \beta)$ with α contained in a $k \times (n-k)$ rectangle and β strict such that $\beta_1 \leq n$ and $\alpha_k \geq \ell(\beta)$. Each such partition pair corresponds to a unique k-strict partition $\lambda \in \mathcal{P}(k,n)$, defined by $\lambda = \alpha' + \beta$, where α' is the conjugate (or transpose) of α.

Let $\widetilde{W}_{n+1}$ be the Weyl group for the root system of type D_{n+1} and $\widetilde{W}^{(k)}$ the corresponding parameter set for the Schubert varieties in $\mathrm{OG}(n+1-k, 2n+2)$. The translation from elements of $\widetilde{W}^{(k)}$ to partitions $\lambda \in \widetilde{\mathcal{P}}(k,n)$ is similar to the above, following [T, 6.1]. However in the present work our definition of type(λ) when type(λ) $\in \{1,2\}$ differs from the convention in [T], and is essentially that introduced in [PR2]. When type(λ) > 0, we have type(λ) $= 1$ if and only if the first entry of the corresponding Weyl group element is unbarred. This relates the notation used in this paper to the cited earlier works; for further details, we refer the reader to [T, 4.1, 5.1, 6.1].

5. The classical Pieri rules

We present here our proofs of the classical Pieri rules for the Grassmannians X parametrizing isotropic subspaces in a vector space V. It is worth noting that one can achieve a uniform proof, in all three Lie types, of a rule for products with the Chern classes $c_r(\mathcal{Q})$ of the universal quotient bundle over X. To do this, we introduce the projectivization $\mathbb{P}(\mathcal{S})$ of the universal subbundle with the natural projections $\pi : \mathbb{P}(\mathcal{S}) \to X$ and $\psi : \mathbb{P}(\mathcal{S}) \to \mathbb{P}(V)$. The projection formula then gives

$$\int_X c_r(\mathcal{Q}) \cdot [X_P] \cdot [X_{Q^\vee}] = \int_{\mathbb{P}(V)} c_1(\mathcal{O}(1))^{m-1+r} \cdot \psi_* \pi^* [X_P(F_\bullet) \cap X_{Q^\vee}(G_\bullet)], \tag{55}$$

where P, Q are index sets and $F_\bullet$, $G_\bullet$ are isotropic flags in V in general position.

We claim that whenever $Y_{P,Q} := X_P(F_\bullet) \cap X_{Q^\vee}(G_\bullet)$ is non-empty, then the restriction of ψ to $\pi^{-1}(Y_{P,Q})$ either (i) has positive dimensional fibers, in which case the integral (55) vanishes, or (ii) is a birational isomorphism onto a target variety which is a complete intersection of N quadrics in $\mathbb{P}(V)$, when the integral is equal to the degree 2^N. The rule for multiplication by $c_r(\mathcal{Q})$ in the ring $\mathrm{H}^*(X)$ follows immediately, and for the Grassmannian $\mathrm{IG}(m, 2n)$, this is enough to finish the proof of the Pieri rule. In the orthogonal cases, a supplementary analysis is required to establish the rule for multiplication with the special Schubert classes; this uses the quadric in $\mathbb{P}(V)$ defined by the symmetric form.

5.1. **Types C and B.** Let $V \cong \mathbb{C}^{2n}$ be a symplectic vector space and $\mathrm{IG} = \mathrm{IG}(m, 2n)$. For any index set $P = \{p_1 < \cdots < p_m\} \subset [1, 2n]$ and isotropic flag $F_\bullet \subset V$, let $X_P(F_\bullet)$ be the Schubert variety in IG defined by (54). We set $p_0 = 0$ and $p_{m+1} = 2n+1$ for convenience.

If Q is another index set, then we have $Q \leq P$ if and only if $X_P(F_\bullet) \cap X_{Q^\vee}(G_\bullet) \neq \emptyset$ for all isotropic flags $F_\bullet$ and $G_\bullet$, where $Q^\vee$ is the index set dual to Q. When $Q \leq P$ we set $D(P,Q) = \{(j,c) \mid q_j \leq c \leq p_j\}$, and consider this as a northwest to southeast skew diagram of boxes. Define a *cut* through the diagram $D(P,Q)$ to be any integer $c \in [0, 2n]$ such that no row of $D(P,Q)$ contains boxes in both column c and column $c+1$. Equivalently, we have $p_j \leq c < q_{j+1}$ for some j. Let

$$I(P,Q) = \{c \in [0,n] : c \text{ or } 2n-c \text{ is a cut through } D(P,Q)\},$$

and let $N(P,Q)$ be the number integers $c \in I(P,Q)$ such that $c \geq 2$ and $c - 1 \notin I(P,Q)$.

Fix a symplectic basis $e_1, \ldots, e_{2n}$ of V, such that $(e_i, e_j) = \pm\delta_{i+j,2n+1}$. We let $x_1, \ldots, x_{2n} \in V^*$ be the dual basis. Given index sets P, Q with $Q \leq P$, we define $Z_{P,Q} \subset \mathbb{P}(V)$ to be the subvariety cut out by the following equations.

(a) $x_{c+1}x_{2n-c}+x_{c+2}x_{2n-c-1}+\cdots+x_d x_{2n+1-d}=0$, whenever $c<d$ are consecutive elements of $I(P,Q)$ such that $d-c\geq 2$.

(b) $x_c=0$, whenever $D(P,Q)$ has no boxes in column c, or a row of $D(P,Q)$ contains exactly one box, which is located in column $2n+1-c$.

It follows that $Z_{P,Q}$ is an irreducible complete intersection of degree $2^{N(P,Q)}$.

Let $F_\bullet$ and $G_\bullet$ be the isotropic flags defined by $F_i=\langle e_1,\dots,e_i\rangle$ and $G_i=\langle e_{2n+1-i},\dots,e_{2n}\rangle$, and set $Y_{P,Q}=X_P(F_\bullet)\cap X_{Q^\vee}(G_\bullet)\subset \mathrm{IG}$. Recall that we have natural smooth maps $\pi:\mathbb{P}(\mathcal{S})\to \mathrm{IG}$ and $\psi:\mathbb{P}(\mathcal{S})\to\mathbb{P}(V)$.

Lemma 5.1. *We have that $\psi(\pi^{-1}(Y_{P,Q}))\subset Z_{P,Q}$.*

Proof. Let $\Sigma\in Y_{P,Q}$ and let v be a non-zero vector in Σ. We must show that v satisfies the equations (a) and (b). Let c,d be consecutive elements of $I(P,Q)$ such that $|d-c|\geq 2$, and choose cuts $c'\in\{c,2n-c\}$ and $d'\in\{d,2n-d\}$ for the diagram $D(P,Q)$, as well as integers i,j such that $p_i\leq c'<q_{i+1}$ and $p_j\leq d'<q_{j+1}$. By the choice of c and d, we must have $i\neq j$. After possibly interchanging c and d, we may assume that $i<j$. Since $\dim(\Sigma\cap F_{p_i})\geq i$, $\dim(\Sigma\cap G_{2n+1-q_{i+1}}\cap F_{p_j})\geq j-i$, and $\dim(\Sigma\cap G_{2n+1-q_{j+1}})\geq m-j$, we can write $v=v_1+v_2+v_3$ where $v_1\in\Sigma\cap F_{p_i}$, $v_2\in\Sigma\cap G_{2n+1-q_{i+1}}\cap F_{p_j}$, and $v_3\in\Sigma\cap G_{2n+1-q_{j+1}}$. The equation (a) amounts to $(v_2,v_3)=0$ if $c<d$ and to $(v_1,v_2)=0$ if $c>d$.

Now consider an equation $x_c=0$ of type (b). If $D(P,Q)$ has no boxes in column c, then $p_j<c<q_{j+1}$ for some j, and the equation is true because $\Sigma=(\Sigma\cap F_{p_j})+(\Sigma\cap G_{2n+1-q_{j+1}})$. Otherwise we have $q_j=p_j=2n+1-c$ for some j, and the equation follows because $e_{2n+1-c}\in\Sigma$. □

Given two index sets $Q\leq P$, we will write $P\to Q$ if the diagram $D(P,Q)$ contains no 2×2 squares, and whenever this diagram contains two boxes in column c, it contains one box in column $2n+1-c$. In other words we have $P\to Q$ if and only if (i) $Q\leq P$, (ii) $p_j\leq q_{j+1}$ for $1\leq j\leq m-1$, and (iii) whenever $p_j=q_{j+1}$ we have $q_i<2n+1-p_j<p_i$ for some i.

Proposition 5.1. *For index sets $Q\leq P$ we have $\dim(\pi^{-1}(Y_{P,Q}))\geq\dim(Z_{P,Q})$, with equality if and only if $P\to Q$. When the latter occurs, the map $\psi:\pi^{-1}(Y_{P,Q})\to Z_{P,Q}$ is a birational isomorphism.*

Proof. Assume at first that $D(P,Q)$ contains no 2×2 squares, i.e. $p_j\leq q_{j+1}$ for each j. In this case we will show that ψ maps $\pi^{-1}(Y_{P,Q})$ onto $Z_{P,Q}$. We may assume that $m\geq 2$, and, after possibly replacing (P,Q) with $(Q^\vee,P^\vee)$, that $p_1+q_m\leq 2n+1$. It is enough to show that the image of ψ contains any vector $x=(x_1,\dots,x_{2n})$ of $Z_{P,Q}$ such that each coordinate x_c is non-zero, unless an equation (b) says otherwise. Notice that $x\in\psi(\pi^{-1}(Y_{P,Q}))$ if and only if there exists $\Sigma\in Y_{P,Q}$ with $x\in\Sigma$.

Set $P'=\{p_2,\dots,p_m\}$ and $Q'=\{q_2,\dots,q_m\}$. We claim that for some $a\in\mathbb{C}$, the vectors $v=(x_1,\dots,x_{p_1}-a,0,\dots,0)\in V$ and $v'=(0,\dots,0,a,x_{p_1+1},\dots,x_{2n})\in V$ satisfy $(v,v')=0$ and $v'\in Z_{P',Q'}$. If $p_1=q_2$ then $x_{2n+1-p_1}\neq 0$, provided that $p_1+p_m>2n+1$, so it is possible to choose a such that $(v,v')=0$, while $(v,v')=0$ for any choice of a when $p_1+p_m<2n+1$. The vector v' then satisfies the quadratic equations defining $Z_{P',Q'}$ because $x=v+v'$ satisfies the equations defining $Z_{P,Q}$. Otherwise we have $p_1<q_2$, and we can take $a=0$. Since p_1 is a cut through $D(P,Q)$, it follows from the equations defining $Z_{P,Q}$ that $(v,v')=0$.

By induction there exists $\Sigma'\in Y_{P',Q'}\subset \mathrm{IG}(m-1,V)$ such that $v'\in\Sigma'$. Since $\dim(\Sigma'\cap F_{p_{m-1}})\geq m-2$ and $v'\notin F_{p_{m-1}}$ because $x_{p_m}\neq 0$, it follows that $\Sigma'\subset$

$F_{p_{m-1}} \oplus \mathbb{C}v'$. Now since $p_1 + p_{m-1} \leq p_1 + q_m$ we obtain $p_1 + p_{m-1} < 2n+1$, so $(v, \Sigma') = 0$. Since $\Sigma' \subset G_{2n+1-q_2}$ and $x_{q_1} \neq 0$, we deduce that $\Sigma = \mathbb{C}v \oplus \Sigma'$ is a point of $\mathrm{IG}(m, V)$. Since $\dim(\Sigma' \cap F_{p_j}) \geq j-1$ for each j and $\dim(\Sigma' \cap G_{2n+1-q_j}) \geq m+1-j$ for $j \geq 2$, it follows that $\Sigma \in Y_{P,Q}$, as required.

Let $U_{P,Q} \subset Y_{P,Q}$ be the open subset of isotropic spaces $\Sigma = \langle u_1, \dots, u_m \rangle$, for which each vector u_j can be written as $u_j = a_{j,q_j} e_{q_j} + \cdots + a_{j,p_j} e_{p_j}$, with $a_{j,c} \neq 0$ unless $Z_{P,Q}$ satisfies the equation $x_c = 0$. We claim that $U_{P,Q} \neq \emptyset$. In fact, by induction on m there exists $\Sigma' = \langle u_2, \dots, u_m \rangle \in U_{P',Q'}$. If $q_1 = p_1$ then we set $u_1 = e_{p_1}$; if $p_1 + p_m > 2n+1$, so that $q_m < 2n+1-p_1 < p_m$, we furthermore replace the coordinate $a_{m,2n+1-p_1}$ of u_m with zero. Then $\Sigma = \langle u_1, \dots, u_m \rangle$ is a point of $U_{P,Q}$. Otherwise we have $q_1 < p_1$. The condition that $(u_1, u_m) = 0$ amounts to the linear equation $a_{m,2n+1-q_1} a_{1,q_1} + \cdots + a_{m,2n+1-p_1} a_{1,p_1} = 0$ in the coordinates of u_1. If $p_1 + p_m < 2n+1$ then this equation is trivial, while if $p_1 + p_m > 2n+1$, then since $\Sigma' \in U_{P',Q'}$ we have $a_{m,2n-p_1} \neq 0$ and $a_{m,2n+1-p_1} \neq 0$, so this equation has a solution for which $a_{1,c} \neq 0$ for all $q_1 \leq c \leq p_1$, and $\Sigma = \langle u_1, \dots, u_m \rangle \in U_{P,Q}$, as required.

The intersection of two Schubert cells in general position is irreducible by results of Deodhar [D, Cor. 1.2, Prop. 5.3(iv)], and this immediately implies the irreducibility of the intersection of two Schubert varieties in general position. It follows that $U_{P,Q}$ is a dense open subset of $Y_{P,Q}$. We note that in our special case, this can also be checked directly, but leave this as an exercise to the reader. We deduce that $\pi^{-1}(Y_{P,Q})$ has a dense open subset of pairs $(\mathbb{C}x, \Sigma)$, for which $\Sigma = \langle u_1, \dots, u_n \rangle \in U_{P,Q}$ with $u_j \in F_{p_j} \cap G_{2n+1-q_j}$ and $x = u_1 + \cdots + u_n$.

Suppose the diagram $D(P,Q)$ contains two boxes in some column c and no boxes in column $2n+1-c$, and let $\Sigma = \langle u_1, \dots, u_m \rangle \in U_{P,Q}$ and $x = u_1 + \cdots + u_m$ be as above. Then $c = p_j = q_{j+1}$ for some j, and $\Sigma_t = \langle u_1, \dots, u_{j-1}, u_j + t\, e_c, u_{j+1} - t\, e_c, u_{j+2}, \dots, u_m \rangle$ is a point of $Y_{P,Q}$ with $x \in \Sigma_t$ for each $t \in \mathbb{C}$. This shows that $\psi^{-1}(x)$ has positive dimension, so $\dim(\pi^{-1}(Y_{P,Q})) > \dim(Z_{P,Q})$.

On the other hand, if the diagram $D(P,Q)$ contains a box in column $2n+1-c$ whenever there are two boxes in column c, then we have $P \to Q$, and we must show that $\psi : \pi^{-1}(Y_{P,Q}) \to Z_{P,Q}$ is birational. It is enough to show that if $\Sigma = \langle u_1, \dots, u_m \rangle \in U_{P,Q}$ with $u_j \in F_{p_j} \cap G_{2n+1-q_j}$, then $\psi^{-1}(u_1 + \cdots + u_m) \cap \pi^{-1}(U_{P,Q})$ contains a single point, or equivalently, Σ is the only point of $U_{P,Q}$ containing the vector $x = u_1 + \cdots + u_m$. In fact, we claim this is true with $U_{P,Q}$ replaced by the larger open subset $W_{P,Q} \subset Y_{P,Q}$ of isotropic spaces $\Sigma = \langle u_1, \dots, u_m \rangle$, for which each vector u_j can be written as $u_j = a_{j,q_j} e_{q_j} + \cdots + a_{j,p_j} e_{p_j}$ with $a_{j,c} \neq 0$ unless $2n+1-q_1 < c < p_j$ or $Z_{P,Q}$ satisfies the equation $x_c = 0$. Suppose $\Sigma = \langle u_1, \dots, u_m \rangle \in W_{P,Q}$ and $x = u_1 + \cdots + u_m$ is also contained in $\Sigma' = \langle u'_1, \dots, u'_m \rangle \in W_{P,Q}$, where $u'_j \in F_{p_j} \cap G_{2n+1-q_j}$. Then $x = a_1 u'_1 + \cdots + a_m u'_m$ where $a_j \in \mathbb{C}$. We claim that $a_1 u'_1 = u_1$. This is clear if $p_1 < q_2$. Otherwise $p_1 = q_2$, and we can write $a_1 u'_1 = u_1 + b e_{p_1}$ for some $b \in \mathbb{C}$. Since $0 = (a_1 u'_1, x) = b\,(e_{p_1}, x) = b\, x_{2n+1-p_1}$ and $x_{2n+1-p_1} \neq 0$, it follows that $b = 0$. Now since $\langle u_2, \dots, u_m \rangle$ and $\langle u'_2, \dots, u'_m \rangle$ are points of $W_{P',Q'} \subset \mathrm{IG}(m-1, V)$, both containing $u_2 + \cdots + u_m$, it follows by induction on m that $\langle u_2, \dots, u_m \rangle = \langle u'_2, \dots, u'_m \rangle$. Therefore $\Sigma' = \Sigma$ as required.

It remains to prove that if the diagram $D(P,Q)$ contains a 2×2 square, then $\dim(\pi^{-1}(Y_{P,Q})) > \dim(Z_{P,Q})$. We do this by induction on $\dim(Y_{P,Q}) = |Q| - |P|$.

Choose j minimal such that $p_j > q_{j+1}$, i.e. j is the top row of $D(P,Q)$ containing (the upper half of) some 2×2 square. This implies that $p_{j-1} \leq q_j < q_{j+1}$.

Assume first that $2n+1-r \in P$ for all $r \in [q_{j+1}, p_j - 1]$. Then choose i such that $p_i = 2n+2-p_j$, and notice that $p_l = p_i - i + l$ for all $l \in [i, l_0]$, where $l_0 = i + p_j - q_{j+1} - 1$. This implies that $q_i < p_i$, since otherwise we would get $q_{l_0} = p_{l_0} = 2n+1-q_{j+1}$. Set $P' = (P \smallsetminus \{p_i, p_j\}) \cup \{p_i - 1, p_j - 1\}$. Then we have $\dim(Y_{P',Q}) = \dim(Y_{P,Q}) - 1$ and $\dim(Z_{P',Q}) \geq \dim(Z_{P,Q}) - 1$. We distinguish three cases. If $p_i < q_{i+1}$ then the diagram $D(P',Q)$ has no boxes in column p_i, while it has two boxes in column $p_j - 1$. If $p_i = q_{i+1}$ then $q_{j+1} \leq p_j - 2$, which implies that $D(P',Q)$ contains a 2×2 square. In both of these cases we obtain by induction that $\dim(\pi^{-1}(Y_{P',Q})) > \dim(Z_{P',Q})$. Finally, if $q_{i+1} < p_i$ then $Z_{P',Q} = Z_{P,Q}$, and by induction we have $\dim(\pi^{-1}(Y_{P',Q})) \geq \dim(Z_{P',Q})$. In all three cases we deduce that $\dim(\pi^{-1}(Y_{P,Q})) < \dim(Z_{P,Q})$ as required.

If the above assumption fails, then choose $r < p_j$ maximal such that $2n + 1 - r \notin P$. Then $r \geq q_{j+1}$. Set $P' = (P \smallsetminus \{p_j\}) \cup \{r\}$. In this case we obtain $\dim(Y_{P',Q}) = \dim(Y_{P,Q}) - 1$ and $Z_{P',Q} = Z_{P,Q}$, and the induction hypothesis shows that $\dim(\pi^{-1}(Y_{P',Q})) \geq \dim(Z_{P',Q})$. This completes the proof. □

Theorem 5.1 (Pieri rule for $\mathrm{IG}(m,2n)$)**.** *For any index set P and integer $r \in [1, n+k]$, we have*

$$\sigma_r \cdot [X_P] = \sum 2^{N(P,Q)} \, [X_Q],$$

where the sum is over all index sets Q such that $P \to Q$ and $|Q| = |P| + r$.

Proof. For any index set Q, the coefficient of $[X_Q]$ in the expansion of the product $\sigma_r \cdot [X_P]$ is equal to $\int_{\mathrm{IG}} c_r(\mathcal{Q}) \cdot [X_P] \cdot [X_{Q^\vee}]$. The result now follows from Proposition 5.1 and the projection formula, as explained earlier. □

Now suppose $V \cong \mathbb{C}^{2n+1}$ is an orthogonal space and consider $\mathrm{OG} = \mathrm{OG}(m, 2n+1)$. As noted in Section 2.2, the Pieri rule for OG is equivalent to the corresponding rule for IG. Given any two index sets P, Q we define the diagram $D(P,Q)$ and the relation $P \to Q$ as before, and set

$$I'(P,Q) = \{c \in [0,n] : c \text{ or } 2n+1-c \text{ is a cut through } D(P,Q)\} \cup \{n+1\}.$$

Moreover, let $N'(P,Q)$ be the number (respectively, one less than the number) of integers $c \in I'(P,Q)$ such that $c \geq 2$ and $c - 1 \notin I'(P,Q)$, if $r \leq k$ (respectively, if $r > k$).

Theorem 5.2 (Pieri rule for $\mathrm{OG}(m, 2n+1)$)**.** *For any index set P and integer $r \in [1, n+k]$, we have*

$$\tau_r \cdot [X_P] = \sum 2^{N'(P,Q)} \, [X_Q],$$

where the sum is over all index sets Q such that $P \to Q$ and $|Q| = |P| + r$.

5.2. **Type D.** Let $V \cong \mathbb{C}^{2n+2}$ be an orthogonal vector space and $\mathrm{OG}' = \mathrm{OG}(m, 2n+2)$. For any isotropic flag $F_\bullet \subset V$ and index set P, we have an open Schubert cell

$$X_P^\circ(F_\bullet) = \{\Sigma \in \mathrm{OG} \mid \Sigma \cap F_{p_j} \supsetneq \Sigma \cap F_{p_j - 1} \ \forall 1 \leq j \leq m\}.$$

The Schubert variety $X_P(F_\bullet)$ is the closure of this set.

We will use the order $Q \preceq P$ on the index sets defined in Section 4.3; this is equivalent to the condition $X_P(F_\bullet) \cap X_{Q^\vee}(G_\bullet) \neq \emptyset$ for all isotropic flags $F_\bullet$ and $G_\bullet$. When $Q \preceq P$ we set $D(P,Q) = \{(j,c) \mid q_j \leq c \leq p_j\}$. Define a *cut* through the diagram $D(P,Q)$ to be any integer $c \in [0, 2n+2]$ such that $p_j \leq c < q_{j+1}$ for some j. Set

$$I'(P,Q) = \{c \in [0,n] : c \text{ or } 2n+2-c \text{ is a cut through } D(P,Q)\} \cup \{n+1\},$$

and let $N(P,Q)$ be the number of integers $c \in I'(P,Q)$ such that $c \geq 2$ and $c-1 \notin I'(P,Q)$.

Given index sets $Q \preceq P$ for OG' we will write $P \to Q$ if (i) the diagram $D(P,Q)$ contains no 2×2 squares, except that a single 2×2 square is allowed across the middle (in columns $n+1$ and $n+2$), (ii) whenever column c of $D(P,Q)$ contains two boxes, column $2n+3-c$ contains at least one box, and (iii) $D(P,Q)$ cannot have exactly 3 boxes in columns $n+1$ and $n+2$.

Fix an orthogonal basis $e_1, \ldots, e_{2n+2}$ of V such that $(e_i, e_j) = \delta_{i+j,2n+3}$ and $\langle e_1, \ldots, e_{n+1}\rangle$ is in the same family as our chosen maximal isotropic subspace L. We let $x_1, \ldots, x_{2n+2} \in V^*$ be the dual basis. For index sets $Q \preceq P$ we define $Z_{P,Q} \subset \mathbb{P}(V)$ to be the subvariety cut out by the following equations.

(a) $x_{c+1}x_{2n+2-c} + x_{c+2}x_{2n+1-c} + \cdots + x_d x_{2n+3-d} = 0$, whenever $c < d$ are consecutive elements of $I'(P,Q)$ such that $d - c \geq 2$.

(b) $x_c = 0$, whenever $D(P,Q)$ has no boxes in column c; or some row of $D(P,Q)$ contains exactly one box, which is located in column $2n+3-c$; or $c \in \{n+1, n+2\}$ and only one row of $D(P,Q)$ contains a box in column c, and this row starts or terminates at column $2n+3-c$.

We see that $Z'_{P,Q}$ is an irreducible complete intersection of degree $2^{N(P,Q)}$.

Let $F_\bullet$ and $G_\bullet$ be the isotropic flags defined by $F_i = \langle e_1, \ldots, e_i\rangle$, for each i, and $G_i = \langle e_{2n+3-i}, \ldots, e_{2n+2}\rangle$ for $1 \leq i \leq n$, while

$$G_{n+1} = \begin{cases} \langle e_{n+2}, \ldots, e_{2n+2}\rangle & \text{if } n \text{ is odd,} \\ \langle e_{n+1}, e_{n+3}, \ldots, e_{2n+2}\rangle & \text{if } n \text{ is even.} \end{cases}$$

Set $Y_{P,Q} = X_P(F_\bullet) \cap X_{Q^\vee}(G_\bullet) \subset \mathrm{OG}'$. Let $\mathcal{S}$ be the tautological subbundle over OG', and let $\pi : \mathbb{P}(\mathcal{S}) \to \mathrm{OG}'$ and $\psi : \mathbb{P}(\mathcal{S}) \to \mathbb{P}(V)$ be the natural projections. Arguing as in the previous section, we prove the following.

Proposition 5.2. *For index sets $Q \preceq P$ we have $\dim(\pi^{-1}(Y_{P,Q})) \geq \dim(Z_{P,Q})$, with equality if and only if $P \to Q$. When the latter occurs, the map $\psi : \pi^{-1}(Y_{P,Q}) \to Z_{P,Q}$ is a birational isomorphism.*

Corollary 5.1. *For any index set P and integer $r \in [1, n+k]$, we have*

$$c_r(\mathcal{Q}) \cdot [X_P] = \sum 2^{N(P,Q)}\, [X_Q],$$

where the sum is over all index sets Q such that $P \to Q$ and $|Q| = |P| + r$.

We now refine Corollary 5.1 to obtain the Pieri rule for products by the special Schubert classes in OG'. Define

$$S = \{i \in [1, n+1] \ :\ q_j \leq i \leq p_j \text{ for some } j\},$$
$$S' = \{p \in P \ :\ p \geq n+2 \text{ and } 2n+3-p \in S\},$$

and set $h(P,Q) = |S| + |S'| + n$. Let $N'(P,Q) = N(P,Q)$ (respectively, $N'(P,Q) = N(P,Q) - 1$) if $r \leq k$ (respectively, if $r > k$). If $r \neq k$, then set $\delta_{PQ} = 1$. If $r = k$ and $N'(P,Q) > 0$, then set

$$\delta_{PQ} = \delta'_{PQ} = 1/2,$$

while if $N'(P,Q) = 0$, define

$$\delta_{PQ} = \begin{cases} 1 & \text{if } h(P,Q) \text{ is odd,} \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \delta'_{PQ} = \begin{cases} 1 & \text{if } h(P,Q) \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 5.3 (Pieri rule for $\mathrm{OG}(m, 2n+2)$)**.** *For any index set P and integer $r \in [1, n+k]$, we have*

$$\tau_r \cdot [X_P] = \sum \delta_{PQ}\, 2^{N'(P,Q)}\, [X_Q], \tag{56}$$

where the sum is over all index sets Q such that $P \to Q$ and $|Q| = |P| + r$. Furthermore, the product $\tau'_k\,[X_P]$ is obtained by replacing δ_{PQ} with δ'_{PQ} throughout.

Proof. When $r \neq k$, equation (56) follows immediately from (40) and Corollary 5.1. For the remaining terms we must compute the integrals

$$I_1 = \int_{\mathrm{OG}'} \tau_k \cdot [X_P] \cdot [X_{Q^\vee}] \qquad \text{and} \qquad I_2 = \int_{\mathrm{OG}'} \tau'_k \cdot [X_P] \cdot [X_{Q^\vee}].$$

The argument uses the projection formula and the quadric hypersurface $W \subset \mathbb{P}(V)$ defined by the orthogonal form on V. Let $h = c_1(\mathcal{O}_{\mathbb{P}(V)}(1)|_W)$ denote the hyperplane class in $\mathrm{H}^*(W)$, so that $h^n = e + f$, where e is the class of the ruling $\mathbb{P}(L) \subset W$, and f is the class of the opposite ruling. If $\theta : \mathbb{P}(\mathcal{S}) \to W$ denotes the natural morphism, then

$$\pi_*\theta^* e = \begin{cases} \tau_k & \text{if } n \text{ is even,} \\ \tau'_k & \text{if } n \text{ is odd,} \end{cases} \tag{57}$$

while the opposite relations hold for $\pi_*\theta^* f$.

Let $\iota : W \hookrightarrow \mathbb{P}(V)$ be the inclusion map, so that $\psi = \iota\theta$. The image of the map $\iota^* : \mathrm{H}^{2n}(\mathbb{P}(V)) \to \mathrm{H}^{2n}(W)$ consists of classes with equal coefficients in the rulings e and f. If the degree $2^{N(P,Q)}$ of $Z_{P,Q}$ is greater than one, then the class $\theta_*\pi^*[Y_{P,Q}]$ must lie in the image of ι^*. Applying the projection formula, we deduce from this and the relations (57) that $I_1 = I_2 = 2^{N(P,Q)-1}$, as required.

On the other hand, if $Z_{P,Q}$ is a linear subspace of $\mathbb{P}(V)$ lying inside the quadric W, it suffices to determine its ruling class. Note that in this situation we have $I'(P,Q) = [0, n{+}1]$ and $N(P,Q) = 0$. Let $T = \{2n{+}3{-}p \,:\, p \in S'\}$; it follows from the linear equations (b) defining $Z_{P,Q}$ that $Z_{P,Q} \cap \langle e_1, \ldots, e_{n+1}\rangle = \langle e_i \mid i \in S \smallsetminus T\rangle$. To see this, note that if $p_j \in S'$ is such that $p_j > n+2$, then $q_j = p_j$, i.e., there is only one box in row j of $D(P,Q)$. Indeed, if $q_j < p_j$, then $p_j - 1$ is not a cut, and since there is a box in column $2n+3-p_j$ of $D(P,Q)$ and $2n+3-p_j \neq p_i$ for all i, we see that $2n+2-(p_j-1)$ is not a cut either. This contradicts the fact that $I'(P,Q) = [0, n+1]$.

We deduce that if $h(P,Q)$ is odd, then $Z_{P,Q}$ and $\mathbb{P}(L)$ are in the same family, and in opposite families if $h(P,Q)$ is even. In $\mathrm{H}^*(W)$ we have the relations $e^2 = f^2 = 0$ and $ef = eh^n$, if n is odd, and $e^2 = f^2 = eh^n$ and $ef = 0$, if n is even. Using the projection formula again, we conclude that $I_1 = 1$ and $I_2 = 0$, if $h(P,Q)$ is odd, with the roles reversed if $h(P,Q)$ is even. This completes the proof. $\square$

5.3. **Proofs of Theorems 1.1 and 3.1.** We show how to derive Theorem 1.1 from Theorem 5.1, by going from index sets to partitions. It suffices to establish the equivalence of the Pieri relations "$\lambda \to \mu$" and "$P \to Q$", together with the equality of the corresponding intersection multiplicities. This is the content of the following two propositions.

Proposition 5.3. *Let P and Q be index sets for* $\mathrm{IG}(n-k,2n)$ *and let λ and μ be the corresponding k-strict partitions. Then $P \to Q$ if and only if $\lambda \to \mu$.*

Proof. The assumptions of the proposition tell us that $\lambda_j = n+k+1-p_j+|A_j|$ and $\mu_j = n+k+1-q_j+|B_j|$ for each j, where $A_j = \{i<j : p_i+p_j > 2n+1\}$ and $B_j = \{i<j : q_i+q_j > 2n+1\}$.

Assume that $P \to Q$. We first show that μ can be obtained by removing a horizontal strip from the first k columns of λ and adding a horizontal strip to the result. We need to check that $\lambda_j - 1 \leq \mu_j \leq \lambda_{j-1}$ for each j, and $\lambda_j \leq \mu_j$ when $\lambda_j > k$. To see that $\mu_j \leq \lambda_{j-1}$, notice that if $i \in B_j \smallsetminus A_{j-1}$ and $i < j-1$ then $2n+1-q_j < q_i \leq p_i < 2n+1-p_{j-1}$, which can hold for at most $q_j - p_{j-1} - 1$ integers i. If $p_{j-1} < q_j$ we therefore get $|B_j| \leq |A_{j-1}| + q_j - p_{j-1}$ and $\lambda_{j-1} - \mu_j = q_j - p_{j-1} + |A_{j-1}| - |B_j| \geq 0$. If $p_{j-1} = q_j$, then the condition $P \to Q$ implies that $q_i < 2n+1-q_j < p_i$ for some i. If $i < j-1$ then $i \in A_{j-1} \smallsetminus B_j$, while if $i \geq j-1$ then $q_{j-1}+q_j \leq q_i + q_j < 2n+1$, so $j-1 \notin B_j$. It follows that $|B_j| \leq |A_{j-1}|$ and $\mu_j \leq \lambda_{j-1}$, as required. If $\lambda_j > k$ then $q_j \leq p_j \leq n$, which implies that $\lambda_j = n+k+1-p_j \leq n+k+1-q_j = \mu_j$. In general we note that if $i < j-1$ satisfies that $i \in A_j$ and $i+1 \notin B_j$, then $2n+1-p_j < p_i \leq q_{i+1} < 2n+1-q_j$, which is true for at most $p_j - q_j - 1$ integers i. If $q_j < p_j$, this implies that $|B_j| \geq |A_j| - p_j + q_j$ and $\mu_j \geq \lambda_j$. If $q_j = p_j$ then $|B_j| \geq |A_j| - 1$, which implies that $\mu_j \geq \lambda_j - 1$, as required.

We next verify that condition (2) of Definition 1.3 holds. Assume that the box (j,λ_j) is not in μ, i.e. $\mu_j = \lambda_j - 1$. Then the above analysis shows that $q_j = p_j$ and $|B_j| = |A_j| - 1$. If we set $i = \min(A_j)$, then $A_j = [i, j-1]$, $B_j = [i+1, j-1]$, and $\lambda_j = n+k+1-p_j+j-i$. Since $A_i \subset A_j$ we see that $A_i = \emptyset$, so $\lambda_i = n+k+1-p_i$ and $\mu_i = n+k+1-q_i$. Set $c = p_j - n + k + 1$. Then $c-k-1+i = k+1-\lambda_j+j$, so the box (j,λ_j) is k-related to (i,c). Since $i \in A_j \smallsetminus B_j$ we furthermore obtain $q_i < 2n+1-p_j < p_i$, which is equivalent to $\lambda_i < c-1 < \mu_i$. We deduce that the boxes $(i,c-1)$ and (i,c) belong to $\mu \smallsetminus \lambda$, and these boxes are k-related to $(j-1,\lambda_j)$ and (j,λ_j).

For condition (2) it remains to prove that $\mu_{i+1} \leq c-3$ and $\lambda_{i-1} \geq c+1$. Notice that $q_{i+1} + q_j \neq 2n+2$; otherwise we obtain from $2n+1-p_j < p_i \leq q_{i+1}$ that $p_i = q_{i+1}$, so we must have $q_l < 2n+1-p_i < p_l$ for some l, and this would imply that $q_l < q_j = p_j \leq p_l$. We therefore obtain $c-3-\mu_{i+1} = q_{i+1}+q_j-2n-3 \geq 0$. Similarly, if $p_{i-1}+p_j = 2n$ then $p_{i-1} \leq q_i < 2n+1-q_j$ shows that $p_{i-1} = q_i$, so $q_l < 2n+1-q_i < p_l$ for some l, but this implies that $q_l \leq q_j = p_j < p_l$. It follows that $\lambda_{i-1} - c - 1 = 2n-1-p_{i-1}-p_j \geq 0$, as required.

We finally verify that condition (1) holds. Assume that λ and μ have equally many boxes in column c, where $c \leq k$, and let j be the number of boxes in this column. Then we have $\mu_{j+1} < c \leq \lambda_j$. Now suppose the box (j,c) is k-related to two boxes (i,c') and $(i-1,c'+1)$ of $\mu \smallsetminus \lambda$, where $c' > k$. Then $c+c' = 2k+2+j-i$ and $c' = \mu_i = \lambda_{i-1}$. Since $c' > k$, the latter equality implies that $p_{i-1} = q_i$. Since $\lambda_{i-1}+\lambda_j \geq c'+c > 2k+j-i$, it follows that $q_i+p_j = p_{i-1}+p_j < 2n+1$. Similarly, since $\mu_i + \mu_{j+1} \leq c'+c-1 = 2k+(j+1)-i$, we obtain $q_i + q_{j+1} > 2n+1$. But now we have $p_j < 2n+1-q_i < q_{j+1}$, so the diagram $D(P,Q)$ has no boxes in column $2n+1-q_i$, a contradiction.

Assume now that $\lambda \to \mu$. We claim that if $i \leq j$ and $\lambda_i + \lambda_j > 2k+j-i$, then $\mu_i + \mu_j > 2k+j-i$. If not, then since $\lambda_i > k$, we obtain $\lambda_i + \lambda_j - 1 \leq \mu_i + \mu_j \leq 2k+j-i < \lambda_i + \lambda_j$. Therefore $\mu_i = \lambda_i$, $\mu_j = \lambda_j - 1$, and $\lambda_i + \lambda_j = 2k+1+j-i$. It follows that box (j, λ_j) is k-related to box $(i+1, \lambda_i)$, contradicting (2) of Definition 1.3. The claim implies that $B_j \subset A_j$ for every j.

We must show that $P \to Q$, and start by checking that $Q \leq P$. If this is false, then choose j such that $q_j > p_j$, and note that $\lambda_j - \mu_j = |A_j| - |B_j| + q_j - p_j \geq 1$. The assumption $\lambda \to \mu$ then implies that $\mu_j = \lambda_j - 1$, so $A_j = B_j$ and $\lambda_j \leq k$. Furthermore, the boxes (j, λ_j) and $(j-1, \lambda_j)$ must be k-related to boxes (i, c) and $(i, c-1)$ of $\mu \smallsetminus \lambda$, with $c = 2k + 2 - \lambda_j + j - i$. We obtain $\lambda_i < c - 1 < \mu_i$, which implies that $\lambda_i + \lambda_j \leq 2k + j - i$ and $\mu_i + \mu_j > 2k + j - i$. But this means that $i \in A_j \smallsetminus B_j$, a contradiction.

Now suppose that $D(P, Q)$ contains a 2×2 square, and choose j minimal such that $p_j > q_{j+1}$. If $i \in A_j$ satisfies that $i + 1 \notin B_{j+1}$, then $2n + 1 - p_j < p_i \leq q_{i+1} < 2n + 1 - q_{j+1}$, which is true for at most $p_j - q_{j+1} - 1$ integers i. It follows that $\mu_{j+1} - \lambda_j = |B_{j+1}| - |A_j| + p_j - q_{j+1} > 0$, contradicting that $\mu \smallsetminus \lambda$ is a horizontal strip. This shows that $D(P, Q)$ contains no 2×2 squares.

Finally, suppose that $p_j = q_{j+1}$ for some $j < n - k$; we will show that there exists an i such that $q_i < 2n + 1 - p_j < p_i$. Equivalently, assuming that $\lambda_j - \mu_{j+1} = |A_j| - |B_{j+1}| \geq 0$, we must prove that

$$\lambda_j + \lambda_i \leq 2k + |j - i| \quad \text{(I1)} \qquad \text{and} \qquad \mu_{j+1} + \mu_i > 2k + |j + 1 - i| \quad \text{(I2)}$$

for some i.

If $p_j > n$, then $i = j$ satisfies (I1). Any $i \in A_j \smallsetminus B_{j+1}$ solves our problem, so assume that $A_j \subset B_{j+1}$. Since $|A_j| \geq |B_{j+1}|$, we must have $A_j = B_{j+1}$. Hence $j \notin B_{j+1}$, and thus $i = j$ also satisfies (I2).

If $p_j \leq n$, then λ_j and μ_{j+1} are both greater than k, and hence $A_j = B_{j+1} = \emptyset$. Therefore $\mu_j > \lambda_j = \mu_{j+1} > k$, and the boxes $x = (j, \lambda_j + 1)$ and $y = (j + 1, \lambda_j)$ are both in $\mu \smallsetminus \lambda$. Set $c = n + k + 1 - \lambda_j - j$. If $\lambda_{n-k} \geq c$ then the boxes x and y are k-related to the box $(n - k, c)$, and (1) of Definition 1.3 is violated. So $\lambda_{n-k} \leq c - 1 = 2k + (n - k) - j - \lambda_j$, and so $i = n - k$ satisfies (I1).

Let $i > j$ be minimal such that (I1) holds; we claim that (I2) is also true for this i. Note that $\lambda_i \leq k$. If $\lambda_j < k + i - j$, then by minimality of i we have $\lambda_{i-1} > k$, and this is impossible because λ is k-strict. It follows that $c := 2k + i - j - \lambda_j$ is at most k. The box $(i - 1, c)$ lies in λ; otherwise $\lambda_{i-1} \leq c - 1$ contradicts our choice of i. If (I2) is false, then $\mu_i < c$. Since $(i - 1, c)$ is k-related to both x and y, we again contradict Definition 1.3. □

Suppose now that P and Q are index sets for $\mathrm{IG}(n - k, 2n)$ with $P \to Q$. Recall that a *cut* is an integer $c \in [0, 2n]$ such that $p_i \leq c < q_{i+1}$ for some i. We call an element $c \in [0, 2n]$ which is not a cut a *crossing*. It is easy to see that the set of crossings is equal to $\cup_i [q_i, p_i)$. Indeed, c is a crossing if and only if $c \geq p_j$ implies $c \geq q_{j+1}$ for all j. If $q_i \leq c < p_i$ then if $c \geq p_j$ we have $j + 1 \leq i$, so $c \geq q_i \geq q_{j+1}$. Conversely, if c is a crossing then set $i = \max(j \mid q_j \leq c)$; then we have $c < p_i$.

Let $J(P, Q) = \{e \in [0, n-1] \mid n-e \text{ and } n+e \text{ are both crossings}\}$; then $N(P, Q) = \#\{e \in [1, n - 1] \mid e \in J(P, Q) \text{ and } e - 1 \notin J(P, Q)\}$. Recall that $\mathbb{A}$ is the set of boxes of $\mu \smallsetminus \lambda$ in columns $k + 1$ through $k + n$ which are *not* mentioned in (1) or (2) of Definition 1.3.

Lemma 5.2. *For $0 \leq e \leq n - 1$, we have $e \in J(P, Q)$ if and only if there exists a box in column $k + 1 + e$ of $\mathbb{A}$.*

Proof. Suppose that there exists a box B in position $(i, k + 1 + e)$ of $\mu \smallsetminus \lambda$. Then $\mu_i \geq k + 1 + e > \lambda_i$, hence $q_i \leq n - e < p_i$ and thus $n - e$ is a crossing. Suppose that B is in $\mathbb{A}$; we will show that $n+e$ is then also a crossing. Now either (i) B is k-related to a box (r,c) with $1 \leq c \leq k$ such that $\mu \smallsetminus \lambda$ has a box in position $(r+1,c)$, or else (ii) B is not related to any box (r,c) with $1 \leq c \leq k$ which is the bottom box of λ in column c.

In case (i), we have $r+k+1-c = e+i$, and claim that $q_{r+1} \leq n+e < p_{r+1}$. Indeed, since $\mu_i \geq k+1+e$ and $\mu_{r+1} \geq c$, we have $\mu_i+\mu_{r+1} > c+e+k = r+2k+1-i$. Therefore $\#\{h < r+1 \mid \mu_h + \mu_{r+1} \leq 2k+r+1-h\} \leq r-i$, hence

$$\begin{aligned} q_{r+1} &= n+k+1-\mu_{r+1} + \#\{h < r+1 \mid \mu_h + \mu_{r+1} \leq 2k+r+1-h\} \\ &\leq n+k+1-c+r-i = n+e. \end{aligned}$$

We also have $\lambda_{r+1} < c$ and $\lambda_i < k+1+e$, hence $\lambda_i + \lambda_{r+1} \leq 2k+r-i$, and by similar reasoning $p_{r+1} \geq n+k+1-(c-1)+(r+1-i) \geq n+e+2$.

In case (ii), we claim that there exists a box (r,c) of λ with $\lambda_r = c$ such that $k-\lambda_r+r = e+i$. Indeed, choose r minimal such that $k+1-\lambda_r+r > e+i$. If $k+1-\lambda_r+r \geq e+i+2$ then $\lambda_{r-1} > \lambda_r$, and the box in position $(r-1, k+r-e-i)$ is the bottom box in its column. It follows that $k-c+r = e+i$, where $c = \lambda_r$.

We claim that $q_r \leq n+e < p_r$. To see this, note that if $\mu_r < \lambda_r$ then the box $(i, k+1+e)$ does not lie in $\mathbb{A}$; hence $\mu_r \geq \lambda_r$. Also, $\mu_i \geq k+1+e$ and therefore $\mu_r + \mu_i \geq \lambda_r + k+1+e > 2k+r-i$. It follows that

$$q_r = n+k+1-\mu_r+\#\{h < r \mid \mu_h+\mu_r \leq 2k+r-h\} \leq n+k+1-\lambda_r+r-1-i = n+e.$$

We also have $\lambda_i + \lambda_r = \lambda_i + (k+r-e-i) \leq (e+k)+(k+r-e-i) = 2k+r-i$. Hence

$$p_r = n+k+1-\lambda_r + \#\{h < r \mid \lambda_h + \lambda_r \leq 2k+r-h\} > n+e.$$

It remains to show the converse. If $n-e$ is a crossing, then there exists a box $B = (i, k+1+e)$ of $\mu \smallsetminus \lambda$. Now if B does not lie in $\mathbb{A}$, then we need to show that $n+e$ is a cut, i.e., there exists a j such that $p_j \leq n+e < q_{j+1}$. Either the box (r,c) is a bottom box of λ in column c k-related to B which is also a bottom box of μ in column c, or some d boxes $(r-d+1,c),\ldots,(r,c)$ are removed from λ and B is k-related to box $(r-s,c)$, with $0 \leq s \leq d$. In the former case, set $s=0$. We have $k+1-c+r-s = e+i$ and $\lambda_{r-s} \geq c$, while $\mu_{r-s+1} < c$. Since the part of $\mu \smallsetminus \lambda$ in columns $k+1$ through $k+n$ is a horizontal strip containing B, we have $\lambda_{i-1} \geq k+1+e$ and $\mu_{i+1} < k+1+e$. Now

$$\lambda_{i-1} + \lambda_{r-s} \geq k+1+e+c = 2k+2+r-s-i,$$

hence $\lambda_{i-1} + \lambda_{r-s} > 2k + (r-s) - (i-1)$, so

$$p_{r-s} \leq n+k+1-\lambda_{r-s} + \#\{h < r-s \mid \lambda_h + \lambda_{r-s} \leq 2k+r-s-h\} \leq n+e.$$

Furthermore, $\mu_{r-s+1} + \mu_{i+1} \leq c-1+e+k = 2k+(r-s+1)-(i+1)$, thus

$$q_{r-s+1} = n+k+1-\mu_{r-s+1}+\#\{h \leq r-s \,|\, \mu_h+\mu_{r-s+1} \leq 2k+r-s+1-h\} \geq n+e+1.$$

$\square$

Lemma 5.2 immediately implies the following result.

Proposition 5.4. *If $P \to Q$ are index sets for $\mathrm{IG}(n-k,2n)$ and $\lambda \to \mu$ are the corresponding k-strict partitions, then we have $N(P,Q) = N(\lambda,\mu)$.*

We next discuss the analogous proof of Theorem 3.1.

Proposition 5.5. *Let P and Q be index sets for* $\mathrm{OG}(n+1-k, 2n+2)$ *and let λ and μ be the corresponding elements of $\widetilde{P}(k,n)$. Then $P \to Q$ if and only if $\lambda \to \mu$. In this situation we also have $N'(P,Q) = N'(\lambda,\mu)$, and if $|\mu| = |\lambda| + k$ and $N'(P,Q) = N'(\lambda,\mu) = 0$, then $h(P,Q) \equiv h(\lambda,\mu)$ mod 2.*

Proof. The equivalence of the relations $P \to Q$ and $\lambda \to \mu$ is proved in a similar way to Proposition 5.3. The condition in the definition of $P \to Q$ that $D(P,Q)$ cannot have exactly three boxes in columns $n+1$ and $n+2$ is equivalent to the condition $\mathrm{type}(\lambda) + \mathrm{type}(\mu) \neq 3$ in the definition of $\lambda \to \mu$. This follows because the type of an index set P, when non-zero, equals 1 plus the parity of the number of integers in $[1, n+1] \smallsetminus P$. We thus see that when $\mathrm{type}(P)$ and $\mathrm{type}(Q)$ are both positive, we must have $\mathrm{type}(P) = \mathrm{type}(Q)$.

One also checks as in Lemma 5.2 that in this situation, we have $N'(P,Q) = N'(\lambda,\mu)$, and we will assume this in the following. To complete the proof, we show that whenever $P \to Q$ (equivalently $\lambda \to \mu$), $|\mu| = |\lambda| + k$, and $N'(P,Q) = N'(\lambda,\mu) = 0$, we have $h(P,Q) \equiv h(\lambda,\mu)$ mod 2. We need to verify that

$$g(\lambda,\mu) + \max(\mathrm{type}(P), \mathrm{type}(Q)) + |S| + |S'| + n \equiv 0 \text{ mod } 2,$$

where the sets S and S' have been defined before Theorem 5.3.

Since $|\mu| = |\lambda| + k$ and $\mu \smallsetminus \lambda$ can have at most one box in each of the first k columns, we deduce that for each column among these where μ has the same number of boxes as the corresponding column of λ, the bottom box of μ in this column is k'-related to exactly one box of $\mu \smallsetminus \lambda$. It follows that $g(\lambda,\mu)$ is equal to the number of boxes of $\mu \smallsetminus \lambda$ in columns $k+1$ through $k+n$ minus the number of boxes in $\lambda \smallsetminus \mu$.

We claim that at least one of $\mathrm{type}(\lambda)$, $\mathrm{type}(\mu)$ must be positive. Indeed, if $\mathrm{type}(\lambda) = 0$ then the Pieri move $\lambda \to \mu$ must add a box to column k, which implies that $\mathrm{type}(\mu) > 0$. This fact has the following consequence for the diagram $D(P,Q)$: exactly one row of $D(P,Q)$ has boxes in columns $n+1$ or $n+2$, and this row begins or ends in these two columns. Moreover, if this row has boxes in both central columns, then it must contain at least one more box. It also follows from the definitions that no two boxes of $D(P,Q)$ can lie in the same column.

Using Proposition 4.7 it is easy to translate between parts $\lambda_i > k$ and elements of p_i of P less than $n+1$: they are related by the equation $\lambda_i + p_i = n+k+1$. We deduce that the number of boxes of $\mu \smallsetminus \lambda$ in columns $k+1$ through $k+n$ equals $|S|$ minus the number r of nonempty rows in the left half of $D(P,Q)$ (the part of $D(P,Q)$ which lies in the first $n+1$ columns).

Let $S'' = \{p \in S' \ : \ p > n+2\}$. In the proof of Theorem 5.3 we saw that $q_j = p_j$, for all $p_j \in S''$; moreover, since $N'(\lambda,\mu) = 0$, the set $\mathbb{A}$ is empty. Using these facts, it is straightforward to check that the elements of S'' are in 1-1 correspondence with the *removed* boxes from λ, that is, with $\lambda \smallsetminus \mu$. Combining this with the previous analysis, we see that $g(\lambda,\mu) + |S''|$ has the same parity as $|S| + r$. We are reduced then to showing that

$$n + r + \max(\mathrm{type}(P), \mathrm{type}(Q)) \equiv \begin{cases} 0 \text{ mod } 2 & \text{if } n+2 \notin S', \\ 1 \text{ mod } 2 & \text{if } n+2 \in S'. \end{cases}$$

The above is proved by a case by case analysis, according to three possibilities for $(\mathrm{type}(P), \mathrm{type}(Q))$. If $\mathrm{type}(P) = 0$ and $\mathrm{type}(Q) > 0$, then $n+2 \notin S'$. Then, if $q_j = n+1$ for some j we have $j = r$ and $\mathrm{type}(Q) \equiv 1 + (n+1) - j \equiv n + r$ mod 2, and if

$q_j = n+2$ for some j then $j = r+1$ and $\text{type}(Q) \equiv 1+(n+1)-(j-1) \equiv n+r \bmod 2$. If $\text{type}(P) > 0$ and $\text{type}(Q) = 0$, then there are two possibilities. Either $p_j = n+1$ for some j, so $n+2 \notin S'$, $j = r$, and $\text{type}(P) \equiv 1 + (n+1) - j \equiv n + r \bmod 2$, or else $p_j = n+2$ for some j, so $q_j < n+1$ and $n+2 \in S'$. Then $j = r$ and $\text{type}(P) \equiv 1 + (n+1) - (j-1) \equiv n + r + 1 \bmod 2$, as required. Finally if $\text{type}(P) = \text{type}(Q) > 0$ we must have $q_j = p_j \in \{n+1, n+2\}$ for some j while $n+2 \notin S'$, and the result again follows. □

Appendix A. Quantum cohomology of $\mathrm{OG}(n, 2n+2)$

In this section OG' will denote the Grassmannian $\mathrm{OG}(n, 2n+2)$. A presentation and Pieri rule for the classical cohomology ring of OG' were obtained in Section 3.

Given nonnegative integers d_1 and d_2, a rational map of degree (d_1, d_2) to OG' is a morphism $f : \mathbb{P}^1 \to \mathrm{OG}'$ such that

$$\int_{\mathrm{OG}'} f_*[\mathbb{P}^1] \cdot \tau_1 = d_1 \quad \text{and} \quad \int_{\mathrm{OG}'} f_*[\mathbb{P}^1] \cdot \tau_1' = d_2.$$

For three elements λ, μ, and ν in $\widetilde{\mathcal{P}}(1, n)$ such that $|\lambda| + |\mu| + |\nu| = \dim(\mathrm{OG}') + (d_1 + d_2)(n+1)$, the Gromov-Witten invariant $\langle \tau_\lambda, \tau_\mu, \tau_\nu \rangle_{d_1,d_2}$ is defined to be the number of rational maps $f : \mathbb{P}^1 \to \mathrm{OG}'$ of degree (d_1, d_2) such that $f(0) \in X_\lambda(E_\bullet)$, $f(1) \in X_\mu(F_\bullet)$, and $f(\infty) \in X_\nu(G_\bullet)$, for given isotropic flags $E_\bullet$, $F_\bullet$, and $G_\bullet$ in general position.

The parameter space of lines on OG' is the space Z of pairs (A, B) where A and B are isotropic subspaces of $\mathbb{C}^{2n+2}$ of respective dimensions $n-1$ and $n+1$. Observe that Z consists of two connected components, each isomorphic to the isotropic two-step flag variety $\mathrm{OF}' = \mathrm{OF}(n-1, n+1; 2n+2)$. One component, which we call Z_1, parametrizes lines of degree $(1, 0)$ on OG', and the other component, which we call Z_2, parametrizes lines of degree $(0, 1)$. It follows that

$$\langle \tau_\lambda, \tau_\mu, \tau_\nu \rangle_{1,0} = \int_{Z_1} [Z_\lambda] \cdot [Z_\mu] \cdot [Z_\nu]$$

where Z_λ, Z_μ, and Z_ν are the associated Schubert varieties in Z_1, defined as usual. For any $\lambda \in \widetilde{\mathcal{P}}(1, n)$, let $\overline{\lambda}$ denote the strict partition (without type) obtained by deleting the leftmost column of λ. With this convention, we have

Proposition A.1. *For any integer $p \in [1, n+1]$ and λ, $\mu \in \widetilde{\mathcal{P}}(1, n)$ with $|\lambda| + |\mu| + p = \dim \mathrm{OG}' + n + 1$, we have*

$$\langle \tau_\lambda, \tau_\mu, \tau_p \rangle_{1,0} = \int_{\mathrm{OG}(n+1, 2n+2)} \tau_{\overline{\lambda}} \cdot \tau_{\overline{\mu}} \cdot \tau_{p-1}$$

if λ and μ are both of type 0 or 1, and $\langle \tau_\lambda, \tau_\mu, \tau_p \rangle_{1,0} = 0$ if λ or μ has type 2.

A corresponding analysis applies to the degree $(0, 1)$ Gromov-Witten invariants.

The quantum cohomology ring $\mathrm{QH}^*(\mathrm{OG}')$ is a $\mathbb{Z}[q_1, q_2]$-algebra which is isomorphic to $\mathrm{H}^*(\mathrm{OG}', \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}[q_1, q_2]$ as a module over $\mathbb{Z}[q_1, q_2]$. Here both q_1 and q_2 are formal variables of degree $n+1$. The ring structure on $\mathrm{QH}^*(\mathrm{OG}')$ is determined by the relation

$$\tau_\lambda \cdot \tau_\mu = \sum \langle \tau_\lambda, \tau_\mu, \tau_{\nu^\vee} \rangle_{d_1,d_2} \, \tau_\nu \, {q_1}^{d_1} {q_2}^{d_2},$$

the sum over $d_1, d_2 \geq 0$ and $\nu \in \widetilde{\mathcal{P}}(1, n)$ with $|\nu| = |\lambda| + |\mu| - (n+1)(d_1 + d_2)$.

With the definitions of $\widetilde{\mathcal{P}}'(1,n+1)$, $\widetilde{\nu}$, and λ^* as in Section 3.5, and $\mathrm{t}(\nu) = \mathrm{type}(\nu)$, we have the following analogue of Theorem 3.4.

Theorem A.1 (Quantum Pieri rule for OG')**.** *For any 1-strict partition $\lambda \in \widetilde{\mathcal{P}}(1,n)$ and integer $p \in [1,n+1]$, we have*

$$\tau_p \cdot \tau_\lambda = \sum_{\lambda\to\mu} \delta_{\lambda\mu}\, 2^{N'(\lambda,\mu)}\, \tau_\mu + \sum_{\lambda\to\nu} \delta_{\lambda\nu}\, 2^{N'(\lambda,\nu)}\, \tau_{\widetilde{\nu}}\, q_{\mathrm{t}(\nu)} \;+ \sum_{\lambda^*\to\rho} \delta_{\lambda^*\rho}\, 2^{N'(\lambda^*,\rho)}\, \tau_{\rho^*}\, q_1q_2$$

in the quantum cohomology ring $\mathrm{QH}^*(\mathrm{OG}(n,2n+2))$. Here (i) *the first sum is classical, as in (39),* (ii) *the second sum is over $\nu \in \widetilde{\mathcal{P}}'(1,n+1)$ with $\lambda \to \nu$ and $|\nu| = |\lambda|+p$, and* (iii) *the third sum is empty unless $\lambda_1 = n+1$, and over $\rho \in \widetilde{\mathcal{P}}(1,n)$ such that $\rho_1 = n+1$, $\lambda^* \to \rho$, and $|\rho| = |\lambda| - n - 1 + p$. Furthermore, the product $\tau_1' \cdot \tau_\lambda$ is obtained by replacing δ with δ' throughout.*

Observe that if we set $q_1 = q_2 = q$ in Theorem A.1, then we obtain exactly the statement of Theorem 3.4 with $k = 1$. It should therefore not come as a surprise that the proof of Theorem A.1 is along the same lines as that of Theorem 3.4. There are no linear q terms in a quantum Pieri product $\tau_p\tau_\lambda$ unless $\ell(\lambda) = n$. All $\nu \in \widetilde{\mathcal{P}}'(1,n+1)$ have positive type, since $\nu_1 < n+2$, i.e., $\mathrm{t}(\nu) \in \{1,2\}$ for all ν in the statement of the theorem. In addition, the quadratic q terms are handled as in the proof of Theorem 2.4, using the relation $\tau_{n+1}^2 = q_1q_2$, which is easily checked directly. The assertions of Lemma 3.1 are true when $k = 1$, $d > 2$, and $\ell(\lambda) = 1$, with unaltered proof, and the argument of the proof of Theorem 3.3(d) then shows that there are no cubic or higher-degree q terms.

Theorem A.2 (Ring presentation)**.** *The quantum cohomology ring $\mathrm{QH}^*(\mathrm{OG}(n,2n+2))$ is presented as a quotient of the polynomial ring $\mathbb{Z}[\tau_1,\tau_1',\tau_2,\ldots,\tau_{n+1},q_1,q_2]$ modulo the relations*

$$\tau_1\Delta_n - q_1 = \tau_1'\Delta_n - q_2 = \sum_{p=2}^{n+1}(-1)^p\tau_p\Delta_{n+1-p},$$

$$\tau_r^2 + \sum_{i=1}^{r}(-1)^i\tau_{r+i}c_{r-i} = 0, \qquad 2 \le r \le n,$$

and

$$\tau_1\tau_1' - \tau_2 = 0,$$

where the variables c_p are defined by (40).

Proof. Let (1^r) denote the partition $(1,\ldots,1)$ of length r, and $\tau_{(1^r)}$ and $\tau'_{(1^r)}$ denote the corresponding Schubert classes of types 1 and 2, respectively. The quantum Pieri rule gives the relations

$$\tau_1\tau_{(1^n)} = \tau_{(2,1^{n-1})} + q_1, \qquad \tau_1'\tau'_{(1^n)} = \tau'_{(2,1^{n-1})} + q_2, \tag{58}$$

$$\tau_1\tau'_{(1^n)} = \tau'_{(2,1^{n-1})} + \tau_{n+1}, \qquad \tau_1'\tau_{(1^n)} = \tau_{(2,1^{n-1})} + \tau_{n+1} \tag{59}$$

in $\mathrm{QH}^*(\mathrm{OG}')$. In contrast to the case when $k > 1$, it is not true here that the Schur determinant Δ_r, for $1 \le r \le n$, is equal to a Schubert class in $\mathrm{H}^*(\mathrm{OG}',\mathbb{Z})$. However, by applying the Pieri rule to the monomials in the expansion of Δ_r, noting that $\tau_1\tau_1' = \tau_2$, we deduce that

$$\Delta_r = \tau_{(1^r)} + \tau'_{(1^r)} + \sum_\mu c_\mu\, \tau_\mu,$$

where the sum is over typed partitions μ with $\ell(\mu) < r$. The quantum Pieri rule and equations (58), (59) now easily imply all of the required relations. $\square$

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