GIAMBELLI, PIERI, AND TABLEAU FORMULAS VIA RAISING OPERATORS

HARRY TAMVAKIS

Abstract. We give a direct proof of the equivalence between the Giambelli and Pieri type formulas for Hall-Littlewood functions using Young's raising operators, parallel to joint work with Buch and Kresch for the Schubert classes on isotropic Grassmannians. We prove several closely related mirror identities enjoyed by the Giambelli polynomials, which lead to new recursions for Schubert classes. The raising operator approach is applied to obtain tableau formulas for the Hall-Littlewood functions, the theta polynomials of [BKT2], and related Stanley symmetric functions. Finally, we introduce the notion of a skew element w of the hyperoctahedral group and identify the set of reduced words for w with the set of standard k-tableaux on a skew shape λ/μ.

0. Introduction

The classical Schubert calculus is concerned with the algebraic structure of the cohomology ring of the Grassmannian $\mathrm{G}(m,N)$ of m-dimensional subspaces of complex affine N-space. The cohomology has a free $\mathbb{Z}$-basis of Schubert classes σ_λ, induced by the natural decomposition of $\mathrm{G}(m,N)$ into a disjoint union of Schubert cells. On the other hand, the ring is generated by the Chern classes of the universal quotient bundle $\mathcal{Q}$ over $\mathrm{G}(m,N)$, also known as the *special* Schubert classes. The theorems of Pieri [Pi] and Giambelli [G] are fundamental building blocks of the subject: the former is a rule for a product of a general Schubert class σ_λ with a special one, while the latter is a formula equating σ_λ with a polynomial in the special classes. When one expresses the Chern classes involved in terms of the Chern roots of $\mathcal{Q}$, the Schubert classes are replaced by Schur S-polynomials, thus exhibiting a link between the Schubert calculus and the ring of symmetric functions.

In a series of papers with Buch and Kresch [BKT1, BKT2], we obtained corresponding results for the Grassmannians parametrizing (non-maximal) isotropic subspaces of a vector space equipped with a nondegenerate symmetric or skew-symmetric bilinear form. Our solution of the Giambelli problem for isotropic Grassmannians uses the *raising operators* of Young [Y] in an essential way; the resulting formula interpolates naturally between a Jacobi-Trudi determinant and a Schur Pfaffian. A rather different context in which a Giambelli type formula appears that has this interpolation property is the theory of Hall-Littlewood symmetric functions [Li, M]; these objects also satisfy a Pieri rule [Mo].

The raising operator approach allows one to see *directly* that the Pieri and Giambelli results are formally *equivalent* to each other in all the above instances. This amounts to showing that *the Giambelli polynomials satisfy the Pieri rule*, for the

Date: January 20, 2011.
2000 *Mathematics Subject Classification.* Primary 05E15; Secondary 14M15, 14N15, 05E05.
The author was supported in part by NSF Grants DMS-0639033 and DMS-0901341.

converse implication then follows easily. As a consequence, working either in the context of Schubert calculus or the theory of symmetric functions, it suffices to prove either of the two theorems to establish them both. In geometry, the Pieri rule may be proved concisely by studying a triple intersection of the appropriate Schubert cells, following Hodge [H]; this is the method used in [BKT1].

When working algebraically, it is convenient to use the Giambelli formula as the starting point, and this is the point of view of the present article. One advantage of the raising operator definition is that it makes clear that the basic Giambelli polynomials, which are indexed by partitions λ, make sense when the index is any finite sequence of integers, positive or negative. When the index is a partition, it is the axis of reflection for a pair of *mirror identities*, both closely connected to the Pieri rule. The simplest example is in the case of Schur S-functions:

$$\sum_{\alpha\geq 0} s_{\lambda+\alpha} = \sum_{\mu\supset\lambda} s_\mu \quad \text{and} \quad \sum_{\alpha\geq 0} s_{\lambda-\alpha} = \sum_{\mu\subset\lambda} s_\mu$$

where the sums are over all compositions α and partitions μ obtained from λ by adding (respectively subtracting) a horizontal strip. In §4.4 we use raising operators and the mirror identities to obtain top row recursion formulas for Schubert classes on isotropic Grassmannians.

Our central thesis up to this point is that the aforementioned results may be proved without recourse to their realizations in terms of symmetric functions. The main application of the mirror identities, however, lies in the latter theory, where they can be used to obtain *reduction formulas* for the number of variables x which appear in the argument of a symmetric polynomial. These in turn lead directly to *tableau formulas* for the polynomials in question. The tableau formulas suggest the introduction of new symmetric functions indexed by *skew Young diagrams*, and one can then study their properties. In this way, many important aspects of the classical theory of symmetric functions, as presented in the first three chapters of [M], may be established by entirely different methods. Moreover, the intention is to apply these techniques in a hitherto unexplored situation.

Our main result is a tableau formula for the theta polynomials of [BKT2], which are the Billey-Haiman type C Schubert polynomials [BH] for Grassmannian elements of the hyperoctahedral group B_∞ (the union of the Weyl groups for the root systems of type B_n or C_n). The theorem states that for any k-strict partition λ, we have

$$\Theta_\lambda(x\,;y) = \sum_U 2^{n(U)}(xy)^U \tag{1}$$

where the sum is over all k-bitableaux U of shape λ (see §5.3 for the precise definitions). It bears emphasis that our proof of (1) does not use inner products, tableau correspondences, or algorithms such as jeu de taquin and the like. We show that formula (1) (and its counterpart in the theory of Schur and Hall-Littlewood functions) follows essentially from the definition of $\Theta_\lambda(x\,;y)$ using raising operators, once the Pieri rule is established. The result is surprising because the Pieri rule for the product $\vartheta_p \cdot \Theta_\lambda$ involves partitions μ which do not contain the diagram of λ.

We are led to introduce symmetric functions $F^{(k)}_{\lambda/\mu}(x)$, defined for any pair $\mu \subset \lambda$ of k-strict partitions by the equation

$$F^{(k)}_{\lambda/\mu}(x) = \sum_T 2^{n(T)} x^T$$

where the sum is over all k-tableaux T of skew shape λ/μ. When $k = 0$, the $F^{(k)}_{\lambda/\mu}$ coincide with the usual skew Schur Q-functions $Q_{\lambda/\mu}$. For general k, we prove that $F^{(k)}_{\lambda/\mu}$ either vanishes or is equal to a certain type C Stanley symmetric function F_w, and therefore is always a nonnegative integer linear combination of Schur Q-functions. The function F_w is indexed by an element w of the hyperoctahedral group which we call *skew*. We argue that the skew elements of B_∞ are directly analogous to the 321-avoiding permutations [BJS] in the symmetric group. For example, we show that the reduced words for w are in 1-1 correspondence with the standard k-tableaux of shape λ/μ.

We begin this article in §1 by giving a short proof using raising operators of the equivalence between the classical Giambelli formula and the Pieri rule. This is followed by a discussion of the analogous result for the algebra of Hall-Littlewood functions in §2, and then the more sophisticated arguments of [BKT2] in §3. The raising operators R^λ which appear in the Giambelli formulas of isotropic Schubert calculus depend on the partition λ; this leads to a more challenging and dynamic theory. In §4 we obtain the mirror identities for Hall-Littlewood functions and isotropic Grassmannian Schubert classes, and give some first applications. Section 5 contains our treatment of the various tableau formulas. In §5.1 we stop short of proving the (known) tableau formula for skew Hall-Littlewood functions, since the more difficult case of theta polynomials and the skew functions $F^{(k)}_{\lambda/\mu}(x)$ is handled in detail in §5.2 – §5.5. Finally, in §6 we define the skew elements of the hyperoctahedral group and relate the results of §5 to the Billey-Haiman Schubert polynomials and type C Stanley symmetric functions.

For simplicity we will restrict to the type A and C theories throughout the article, and not discuss the analogous results for orthogonal Grassmannians and Weyl groups here. We hope that §1 and the examples of §3.2 provide a useful introduction to [BKT2]. To emphasize that the formulas in §1 - §4 do not depend on their specific realizations in the Schubert calculus or the theory of symmetric functions, we have used the nonstandard notation U_λ, V_λ, W_λ for the basic polynomials, and in §1.4, §2.3, and §3.3 briefly discuss how these relate to other known objects.

Many thanks are due to my collaborators Anders Buch and Andrew Kresch for their inspired work on related projects.

1. The type A theory

1.1. **Raising operators.** An *integer sequence* or *integer vector* is a sequence of integers $\{\alpha_i\}_{i\geq 1}$, only finitely many of which are non-zero. The largest integer $\ell \geq 0$ such that $\alpha_\ell \neq 0$ is called the *length* of α, denoted $\ell(\alpha)$; we will identify an integer sequence of length ℓ with the vector consisting of its first ℓ terms. We let $|\alpha| = \sum \alpha_i$ and $\#\alpha$ equal the number of non-zero parts α_i of α. We write $\alpha \geq \beta$ if $\alpha_i \geq \beta_i$ for each i. An integer vector α is a *composition* if $\alpha_i \geq 0$ for all i and a *partition* if $\alpha_i \geq \alpha_{i+1} \geq 0$ for all i. As is customary, we represent a partition λ by its Young diagram of boxes, which has λ_i boxes in row i. We write $\mu \subset \lambda$ instead of $\mu \leq \lambda$ for the containment relation between two Young diagrams; in this case the set-theoretic difference $\lambda \smallsetminus \mu$ is called a skew diagram and denoted λ/μ.

For two integer sequences α, β such that $|\alpha| = |\beta|$, we say that α *dominates* β and write $\alpha \succeq \beta$ if $\alpha_1 + \cdots + \alpha_i \geq \beta_1 + \cdots + \beta_i$ for each i. Given any integer sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ and $i < j$, we define

$$R_{ij}(\alpha) = (\alpha_1, \ldots, \alpha_i + 1, \ldots, \alpha_j - 1, \ldots);$$

a raising operator R is any monomial in these R_{ij}'s. Note that we have $R\,\alpha \succeq \alpha$ for any integer sequence α; conversely, if $\alpha \succeq \beta$, there exists a raising operator R such that $\alpha = R\,\beta$. See [M, I.1] for more information.

1.2. **The Giambelli formula.** Consider the polynomial ring $A = \mathbb{Z}[u_1, u_2, \ldots]$ where the u_i are countably infinite commuting independent variables. We regard A as a graded ring with each u_i having graded degree i, and adopt the convention here and throughout the paper that $u_0 = 1$ while $u_r = 0$ for $r < 0$. For each integer vector α, set $u_\alpha = \prod_i u_{\alpha_i}$; then A has a free $\mathbb{Z}$-basis consisting of the monomials u_λ for all partitions λ.

Given a raising operator R, define $R\,u_\alpha = u_{R\alpha}$. Here we view the raising operator R as acting on the index α, and not on the monomial u_α itself. Thus, if the components of α are a permutation of the components of β, it may happen that $R\,u_\alpha \neq R\,u_\beta$ even though $u_\alpha = u_\beta$ as elements of A. Notice that if $\alpha_\ell < 0$ for $\ell = \ell(\alpha)$, then $R\,u_\alpha = 0$ in A for any raising operator R.

For any integer vector α, define U_α by the *Giambelli formula*

$$U_\alpha := \prod_{i<j} (1 - R_{ij})\, u_\alpha. \tag{2}$$

Although the product in (2) is infinite, if we expand it into a series of terms we see that only finitely many of the summands are non-zero; hence, U_α is well defined. For any partition λ, we clearly have

$$U_\lambda = u_\lambda + \sum_{\mu \succ \lambda} a_{\lambda\mu} u_\mu$$

where $a_{\lambda\mu} \in \mathbb{Z}$ and the sum is over partitions μ which strictly dominate λ. We deduce that the U_λ for λ a partition form another $\mathbb{Z}$-basis of A.

We have $U_{(r,s)} = (1 - R_{12})\, u_{(r,s)} = u_r u_s - u_{r+1} u_{s-1}$, hence $U_{(r,s)} = -U_{(s-1,r+1)}$ for any $r, s \in \mathbb{Z}$. This property has a straightforward generalization.

Lemma 1. *Let α and β be integer vectors. Then for any $r, s \in \mathbb{Z}$ we have*

$$U_{(\alpha,r,s,\beta)} = -U_{(\alpha,s-1,r+1,\beta)}\,.$$

We postpone the proof until §2, where we derive a more general result in the context of the Hall-Littlewood theory (Lemma 3).

1.3. **The Pieri rule.** For any $d \geq 1$ define the raising operator R^d by

$$R^d = \prod_{1 \leq i < j \leq d} (1 - R_{ij}).$$

For $p > 0$ and any partition λ of length ℓ, we compute

$$u_p \cdot U_\lambda = u_p \cdot R^\ell\, u_\lambda = R^{\ell+1} \cdot \prod_{i=1}^{\ell} (1 - R_{i,\ell+1})^{-1}\, u_{(\lambda,p)}$$

$$= R^{\ell+1} \cdot \prod_{i=1}^{\ell} (1 + R_{i,\ell+1} + R_{i,\ell+1}^2 + \cdots)\, u_{(\lambda,p)}$$

and therefore

$$u_p \cdot U_\lambda = \sum_{\nu \in \mathcal{N}} U_\nu, \tag{3}$$

where $\mathcal{N} = \mathcal{N}(\lambda, p)$ is the set of all compositions $\nu \geq \lambda$ such that $|\nu| = |\lambda| + p$ and $\nu_j = 0$ for $j > \ell + 1$.

Call a composition $\nu \in \mathcal{N}$ *bad* if there exists a $j > 1$ such that $\nu_j > \lambda_{j-1}$, and let X be the set of all bad compositions. Define an involution $\iota : X \to X$ as follows: for $\nu \in X$, choose j maximal such that $\nu_j > \lambda_{j-1}$, and set

$$\iota(\nu) = (\nu_1, \ldots, \nu_{j-2}, \nu_j - 1, \nu_{j-1} + 1, \nu_{j+1}, \ldots, \nu_{\ell+1}).$$

Lemma 1 implies that $U_\nu + U_{\iota(\nu)} = 0$ for every $\nu \in X$, therefore all bad indices may be omitted from the sum in (3). We are left with the *Pieri rule*

$$u_p \cdot U_\lambda = \sum_{\nu} U_\nu \tag{4}$$

summed over all partitions ν with $|\nu| = |\lambda| + p$ such that $\nu_1 \geq \lambda_1 \geq \nu_2 \geq \lambda_2 \geq \cdots$. In the language of Young diagrams, this condition means that $\nu \supset \lambda$ and the skew diagram ν/λ is a *horizontal p-strip.*

Conversely, suppose that we are given a family $\{T_\lambda\}$ of elements of A, one for each partition λ, such that $T_p = u_p$ for every integer $p \geq 0$ and the T_λ satisfy the Pieri rule $T_p \cdot T_\lambda = \sum_\nu T_\nu$, with the sum over ν as in (4). We claim then that

$$T_\lambda = U_\lambda = \prod_{i<j} (1 - R_{ij})\, u_\lambda$$

for every partition λ. To see this, note that the Pieri rule implies that

$$U_\lambda + \sum_{\mu \succ \lambda} a_{\lambda\mu}\, U_\mu = u_{\lambda_1} \cdots u_{\lambda_\ell} = T_\lambda + \sum_{\mu \succ \lambda} a_{\lambda\mu}\, T_\mu$$

for some constants $a_{\lambda\mu} \in \mathbb{Z}$. The claim now follows by induction on λ. We thus see that *the Giambelli formula and Pieri rule are formally equivalent to each other.*

1.4. **The Grassmannian and symmetric functions.** Equation (2) may be written in the more familiar form

$$U_\alpha = \det(u_{\alpha_i + j - i})_{i,j}. \tag{5}$$

We will prove that (2) and (5) are equivalent, following [M, I.3]. Suppose the length of α is ℓ and let $\rho = (0, 1, \ldots, \ell - 1)$. We work in the ring $\mathbb{Z}[x_1, x_1^{-1}, \ldots, x_\ell, x_\ell^{-1}]$ and let $R\, x^\alpha = x^{R\alpha}$ for any raising operator R. The key is to use the *Vandermonde identity*

$$\det(x_i^{j-1})_{1 \leq i,j \leq \ell} = \prod_{1 \leq i < j \leq \ell} (x_j - x_i)$$

which implies that

$$\prod_{1 \leq i < j \leq \ell} (1 - R_{ij})\, x^\alpha = \prod_{1 \leq i < j \leq \ell} (1 - x_i x_j^{-1})\, x^\alpha = x^{\alpha - \rho} \det(x_i^{j-1})_{1 \leq i,j \leq \ell}$$

$$= \sum_{\sigma \in S_\ell} (-1)^\sigma x_1^{\alpha_1 + \sigma(1) - 1} \cdots x_\ell^{\alpha_\ell + \sigma(\ell) - \ell} = \det(x_i^{\alpha_i + j - i})_{1 \leq i,j \leq \ell}.$$

One now applies the $\mathbb{Z}$-module homomorphism $\mathbb{Z}[x_1, x_1^{-1}, \ldots, x_\ell, x_\ell^{-1}] \to A$ sending x^α to u_α for each α to both ends of the above equation to obtain the result.

The two standard realizations of the Pieri and Giambelli formulas in the literature are the cohomology ring of the Grassmannian $\mathrm{G} = \mathrm{G}(m, N)$ and the ring Λ of symmetric functions. The ring $\mathrm{H}^*(\mathrm{G}, \mathbb{Z})$ has a free $\mathbb{Z}$-basis of Schubert classes σ_λ, one for each partition λ whose diagram is contained in an $m \times (N-m)$ rectangle $\mathcal{R}(m, N-m)$. There is a ring epimorphism $\phi : A \to \mathrm{H}^*(\mathrm{G}, \mathbb{Z})$ sending the generators u_p to the special Schubert classes σ_p for $1 \leq p \leq N-m$ and to zero for $p > N-m$. The map ϕ satisfies $\phi(U_\lambda) = \sigma_\lambda$ if $\lambda \subset \mathcal{R}(m, N-m)$, and $\phi(U_\lambda) = 0$ otherwise. For further details, the reader may consult e.g. [Fu, Chapter 9.4].

Let $x = (x_1, x_2, \ldots)$ be an infinite set of commuting variables, and for each $p \geq 0$ let $h_p = h_p(x)$ be the p-th *complete symmetric function*, that is, formal sum of all monomials in the x_i of degree p. The ring $\Lambda = \mathbb{Z}[h_1, h_2, \ldots]$ may be identified with the ring of symmetric functions in the variables x_i. There is a ring isomorphism $A \to \Lambda$ sending u_p to h_p for all p. For any partition λ, the element U_λ is mapped to the Schur S-function $s_\lambda(x)$. The equation

$$s_\lambda(x) = \det(h_{\lambda_i+j-i}(x))_{1 \leq i,j \leq \ell(\lambda)}$$

expressing the Schur functions in terms of the complete symmetric functions is called the Jacobi-Trudi identity (see e.g. [M, I.(3.4)]).

2. The Hall-Littlewood theory

2.1. The Giambelli formula. Let $v_1, v_2, \ldots$ be an infinite family of commuting variables, with v_i having degree i for all i. As before let $v_0 = 1$, $v_r = 0$ for $r < 0$, $v_\alpha = \prod_i v_{\alpha_i}$ for each integer sequence α, and $R\, v_\alpha = v_{R\alpha}$ for any raising operator R. Let t be a formal variable and consider the graded polynomial ring $A_t = \mathbb{Z}[t][v_1, v_2, \ldots]$.

For any integer vector α, define $V_\alpha \in A_t$ by the *Giambelli formula*

$$V_\alpha := \prod_{i<j} \frac{1-R_{ij}}{1-tR_{ij}}\, v_\alpha. \tag{6}$$

Let $\alpha = (\alpha_1, \ldots, \alpha_{\ell-1})$ be an arbitrary integer vector and $r \in \mathbb{Z}$. Expanding the raising operator product in (6)

$$\prod_{1 \leq i<j \leq \ell} \frac{1-R_{ij}}{1-tR_{ij}} = \prod_{1 \leq i<j<\ell} \frac{1-R_{ij}}{1-tR_{ij}} \prod_{i=1}^{\ell-1} \frac{1-R_{i\ell}}{1-tR_{i\ell}}$$

along the last (i.e., the ℓ-th) component of (α, r) gives

$$V_{(\alpha,r)} = \sum_\gamma t^{|\gamma|-\#\gamma}(t-1)^{\#\gamma} V_{\alpha+\gamma} v_{r-|\gamma|}, \tag{7}$$

summed over all compositions $\gamma \in \mathbb{N}^{\ell-1}$, where $\mathbb{N} = \{0, 1, \ldots\}$. Equation (7) may be used to give a recursive definition of V_α. The v_λ and V_λ for λ a partition form two free $\mathbb{Z}[t]$-bases of A_t.

Proposition 1. *Suppose that we have an equation in* A_t

$$\sum_\nu a_\nu V_\nu = \sum_\nu b_\nu V_\nu$$

where the sums are over all integer vectors $\nu = (\nu_1, \ldots, \nu_\ell)$, while a_ν and b_ν are polynomials in $\mathbb{Z}[t]$, only finitely many of which are non-zero. Then we have

$$\sum_\nu a_\nu V_{(\mu,\nu)} = \sum_\nu b_\nu V_{(\mu,\nu)}$$

for any integer vectors μ.

Proof. It suffices to show that if $\sum_\nu c_\nu V_\nu = 0$ for some $c_\nu \in \mathbb{Z}[t]$, then $\sum_\nu c_\nu V_{(\mu,\nu)} = 0$. We will prove that $\sum_\nu c_\nu V_{(p,\nu)} = 0$ for any integer p; the desired result then follows by induction.

Upon expanding the raising operators in the definition of V_ν, the equation $\sum_\nu c_\nu V_\nu = 0$ becomes $\sum_\alpha \widetilde{c}_\alpha v_\alpha = 0$ for some coefficients $\widetilde{c}_\alpha \in \mathbb{Z}[t]$. We therefore must show that $\sum_\alpha \widetilde{c}_\alpha \Psi\, v_{(p,\alpha)} = 0$, where $\Psi = \displaystyle\prod_{j=2}^{\ell+1} \frac{1 - R_{1,j}}{1 - tR_{1,j}}$. For any integer sequence $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ and every permutation τ in the symmetric group S_ℓ, define $\tau(\alpha) = (\alpha_{\tau(1)}, \ldots, \alpha_{\tau(\ell)})$. By exchanging rows, we may assume that $\sum_\alpha \widetilde{c}_\alpha v_\alpha$ and $\sum_\alpha \widetilde{c}_\alpha \Psi\, v_{(p,\alpha)}$ are both summed over *decreasing* integer sequences α, because

$$\Psi\, v_{(p,\alpha)} = \sum_{\gamma \geq 0} t^{|\gamma| - \#\gamma} (t-1)^{\#\gamma} v_{p+|\gamma|} v_{\alpha-\gamma} = \Psi\, v_{(p,\tau(\alpha))}$$

for every $\tau \in S_\ell$. If α has a negative component then clearly $v_\alpha = \Psi\, v_{(p,\alpha)} = 0$. Moreover, since the v_α for α a partition form a $\mathbb{Z}[t]$-basis of A_t, it follows from $\sum_\alpha \widetilde{c}_\alpha v_\alpha = 0$ that $\widetilde{c}_\alpha = 0$ for all partitions α. We therefore have $\sum_\alpha \widetilde{c}_\alpha \Psi\, v_{(p,\alpha)} = 0$, as desired. $\square$

For any integers r and s, we claim that the equation

$$V_{(r,s)} + V_{(s-1,r+1)} = t\,(V_{(r+1,s-1)} + V_{(s,r)}) \tag{8}$$

holds in the ring A_t. Indeed, (8) follows from the identity

$$\frac{1 - R_{12}}{1 - tR_{12}} = (1 - R_{12}) + t\,\frac{1 - R_{12}}{1 - tR_{12}}\, R_{12}$$

and the fact that $(1 - R_{12})(v_{(r,s)} + v_{(s-1,r+1)}) = 0$ in A_t.

Lemma 2. *For $c \in \mathbb{Z}$ and $d \geq 1$, we have*

$$V_{(c,c+d)} + (1-t) \sum_{i=1}^{d-1} V_{(c+i,c+d-i)} = t\, V_{(c+d,c)}\,. \tag{9}$$

Proof. We use induction on d. When $d = 1$ the result follows by setting $s = r + 1$ in (8). If $d > 1$, the inductive hypothesis gives

$$(1-t) \sum_{i=1}^{d-1} V_{(c+i,c+d-i)} = V_{(c+d-1,c+1)} - t\, V_{(c+1,c+d-1)}.$$

The identity (9) thus reduces to

$$V_{(c,c+d)} + V_{(c+d-1,c+1)} - t\, V_{(c+1,c+d-1)} = t\, V_{(c+d,c)},$$

and this follows directly from equation (8). $\square$

We can generalize the identity (8) as follows.

Lemma 3. *Let α and β be integer vectors. Then for any $r,s\in\mathbb{Z}$ we have*

$$V_{(\alpha,r,s,\beta)}+V_{(\alpha,s-1,r+1,\beta)}=t\,(V_{(\alpha,r+1,s-1,\beta)}+V_{(\alpha,s,r,\beta)}) \tag{10}$$

in the ring A_t.

Proof. By Proposition 1, we may assume that α is empty. If $\beta=(\beta',b)$ has positive length, where $b\in\mathbb{Z}$, we set $\mu=(r,s,\beta')$ and the identity follows by induction, since

$$V_{(\mu,b)}=\sum_{\gamma}t^{|\gamma|-\#\gamma}(t-1)^{\#\gamma}V_{\mu+\gamma}v_{b-|\gamma|}.$$

Finally, if both α and β are empty, the result is true by (8). ☐

Notice that Lemma 1 is the specialization of Lemma 3 at $t=0$. Using Lemma 3 with $\alpha=\emptyset$ and arguing as in the proof of Lemma 2 gives the following result.

Corollary 1. *For $c\in\mathbb{Z}$, $d\geq 1$, and β any integer vector, we have*

$$V_{(c,c+d,\beta)}+(1-t)\sum_{i=1}^{d-1}V_{(c+i,c+d-i,\beta)}=t\,V_{(c+d,c,\beta)}\,.$$

2.2. **The Pieri rule.** Let $\lambda\subset\mu$ be two partitions such that μ/λ is a horizontal strip, and let J be the set of integers $c\geq 1$ such that μ/λ does not (respectively does) have a box in column c (respectively column $c+1$). Define

$$\psi_{\mu/\lambda}(t)=\prod_{c\in J}(1-t^{m_c(\lambda)}),$$

where $m_c(\lambda)$ denotes the number of parts of λ that are equal to c.

For any $d\geq 1$ define the raising operator R_t^d by

$$R_t^d=\prod_{1\leq i<j\leq d}\frac{1-R_{ij}}{1-tR_{ij}}.$$

Given a partition λ of length ℓ, we compute

$$\begin{aligned}v_p\cdot V_\lambda=v_p\cdot R_t^\ell\,v_\lambda&=R_t^{\ell+1}\cdot\prod_{i=1}^{\ell}\frac{1-tR_{i,\ell+1}}{1-R_{i,\ell+1}}\,v_{(\lambda,p)}\\&=R_t^{\ell+1}\cdot\prod_{i=1}^{\ell}(1+(1-t)R_{i,\ell+1}+(1-t)R_{i,\ell+1}^2+\cdots)\,v_{(\lambda,p)}\end{aligned}$$

and therefore

$$v_p\cdot V_\lambda=\sum_{\nu\in\mathcal{N}}(1-t)^{\#(\nu-\lambda)}\,V_\nu, \tag{11}$$

where $\mathcal{N}=\mathcal{N}(\lambda,p)$ is the set of compositions $\nu\geq\lambda$ such that $|\nu|=|\lambda|+p$ and $\nu_j=0$ for $j>\ell+1$, as in §1.3.

For any integer composition ν, set $\nu^*=(\nu_2,\nu_3,\ldots)$. Define

$$\mathcal{N}^*=\{\nu\in\mathcal{N}\mid\nu_j\leq\lambda_{j-1}\text{ for }j>2\}$$

and for $\nu\in\mathcal{N}^*$, let

$$T_\nu=(1-t)^{\#(\nu_1-\lambda_1)}\psi_{\nu^*/\lambda^*}(t)\,V_\nu\,.$$

Using Proposition 1 and induction on the length of λ gives

$$\sum_{\nu\in\mathcal{N}}(1-t)^{\#(\nu-\lambda)}\,V_\nu=\sum_{\nu\in\mathcal{N}^*}T_\nu. \tag{12}$$

Note that if $\nu \in \mathcal{N}^*$ satisfies $\nu_2 < \lambda_1$, then $T_\nu = \psi_{\nu/\lambda}(t)\, V_\nu$. Therefore

$$\sum_{\nu \in \mathcal{N}^*} T_\nu = \sum_{\substack{\nu \in \mathcal{N}^* \\ \nu_2 < \lambda_1}} \psi_{\nu/\lambda}(t)\, V_\nu + \sum_{\substack{\nu \in \mathcal{N}^* \\ \nu_2 \geq \lambda_1}} T_\nu. \tag{13}$$

We claim that for each fixed $d \geq 0$,

$$\sum_{\substack{\nu \in \mathcal{N}^* \\ \nu_2 \geq \lambda_1 \\ \nu_1+\nu_2 = 2\lambda_1+d}} T_\nu \;=\; \sum_{\substack{\nu \in \mathcal{N}^* \\ \nu_2 = \lambda_1 \\ \nu_1 = \lambda_1+d}} \psi_{\nu/\lambda}(t)\, V_\nu. \tag{14}$$

Equation (14) is justified by using Corollary 1 to fix all rows except the first two. We then apply the identity

$$V_{(c,c+d)} + (1-t)\sum_{i=1}^{d-1} V_{(c+i,c+d-i)} + (1-t)\, V_{(c+d,c)} = V_{(c+d,c)} \tag{15}$$

when $\nu \smallsetminus \lambda$ has a box in column $c = \lambda_1$, and the identity

$$(1-t^m)\, V_{(c,c+d)} + (1-t)(1-t^m)\sum_{i=1}^{d-1} V_{(c+i,c+d-i)} + (1-t)\, V_{(c+d,c)} = (1-t^{m+1})\, V_{(c+d,c)} \tag{16}$$

for $m \geq 1$, otherwise. The identities (15) and (16) follow directly from Lemma 2, proving the claim.

Summing (14) over all $d \geq 0$, applying the result to (13), and taking (11) and (12) into account, we finally obtain the *Pieri rule*

$$v_p \cdot V_\lambda = \sum_\mu \psi_{\mu/\lambda}(t)\, V_\mu, \tag{17}$$

summed over all partitions $\mu \supset \lambda$ with $|\mu| = |\lambda| + p$ and μ/λ a horizontal p-strip.

Conversely, suppose that we are given a family $\{T_\lambda\}$ of elements of A_t, one for each partition λ, such that $T_p = v_p$ for every integer $p \geq 0$ and the T_λ satisfy the Pieri rule $T_p \cdot T_\lambda = \sum_\mu \psi_{\mu/\lambda}(t)\, T_\mu$, with the sum over μ and $\psi_{\mu/\lambda}(t)$ as in (17). It follows then as in §1.3 that

$$T_\lambda = V_\lambda = \prod_{i<j} \frac{1-R_{ij}}{1-tR_{ij}}\, v_\lambda.$$

We have established that the Giambelli formula (6) and Pieri rule (17) are formally equivalent to each other.

2.3. **Hall-Littlewood symmetric functions.** Let $x = (x_1, x_2, \ldots)$ be as in §1.4, define the formal power series $q_r(x\,;t)$ by the generating equation

$$\prod_{i=1}^{\infty} \frac{1-x_i t z}{1-x_i z} = \sum_{r=0}^{\infty} q_r(x\,;t) z^r$$

and set $\Gamma_t = \mathbb{Z}[t][q_1, q_2, \ldots]$. There is a $\mathbb{Z}[t]$-linear ring isomorphism $A_t \to \Gamma_t$ sending v_r to $q_r(x\,;t)$ for all $r \geq 1$. For any partition λ, the element V_λ is mapped to the *Hall-Littlewood function* $Q_\lambda(x\,;t)$. The raising operator formula (6) for Hall-Littlewood functions is due to Littlewood [Li], who also obtained Lemma 3 in this setting. The Pieri rule (17) for the functions $Q_\lambda(x\,;t)$ was first proved by Morris [Mo]; an alternative proof may be found in [M, III.5].

If Λ denotes the ring of symmetric functions from §1.4, then the $\mathbb{Z}[t]$-linear ring isomorphism $A_t \to \Lambda[t]$ sending v_r to $h_r(x)$ for every $r \geq 1$ maps V_λ to the *modified Hall-Littlewood function* $Q'_\lambda(x\,;t)$ of Lascoux, Leclerc, and Thibon [LLT]. Now specialize $t=-1$ in the definition of the ring A_t and its distinguished basis V_λ, and consider the isomorphism $A_{-1} \to \Lambda$ obtained by mapping v_r to the r-th elementary symmetric function $e_r(x)$ for every $r \geq 1$. The basis element V_λ is then mapped to the $\widetilde{Q}$-polynomial $\widetilde{Q}_\lambda(x)$ of Pragacz and Ratajski [PR].

3. The type C theory: Isotropic Grassmannians

Fix an integer $k \geq 0$. The aim of this section is to explain the main conclusions of [BKT1, BKT2] regarding the Giambelli and Pieri formulas for the Grassmannian $\mathrm{IG} = \mathrm{IG}(n-k, 2n)$ of isotropic $(n-k)$-dimensional subspaces of $\mathbb{C}^{2n}$, equipped with a symplectic form. We provide an exposition of those aspects relevant to the present article, including examples, and refer the reader to the original papers for detailed proofs.

3.1. **The Giambelli formula.** We consider an infinite family $w_1, w_2, \ldots$ of commuting variables, with w_i of degree i for all i, and set $w_0 = 1$, $w_r = 0$ for $r < 0$, and $w_\alpha = \prod_i w_{\alpha_i}$ as before. Let $I^{(k)} \subset \mathbb{Z}[w_1, w_2, \ldots]$ be the ideal generated by the relations

$$\frac{1-R_{12}}{1+R_{12}}\, w_{(r,r)} = w_r^2 + 2\sum_{i=1}^{r}(-1)^i w_{r+i}w_{r-i} = 0 \quad \text{for } r > k. \tag{18}$$

Define the graded ring $B^{(k)} = \mathbb{Z}[w_1, w_2, \ldots]/I^{(k)}$. All equations in $B^{(k)}$, such as the Giambelli and Pieri formulas, will be valid only up to the elements of $I^{(k)}$.

Let $\Delta^\circ = \{(i,j) \in \mathbb{N}\times\mathbb{N} \mid 1 \leq i < j\}$, equipped with the partial order $\leq$ defined by $(i',j') \leq (i,j)$ if and only if $i' \leq i$ and $j' \leq j$. A finite subset D of Δ° is a *valid set of pairs* if it is an order ideal, i.e., $(i,j) \in D$ implies $(i',j') \in D$ for all $(i',j') \in \Delta^\circ$ with $(i',j') \leq (i,j)$.

A partition λ is *k-strict* if all its parts greater than k are distinct. Given a k-strict partition λ, we define a set of pairs $\mathcal{C}(\lambda)$ by

$$\mathcal{C}(\lambda) = \{(i,j) \in \Delta^\circ \mid \lambda_i + \lambda_j > 2k + j - i \ \text{ and } \ j \leq \ell(\lambda)\}.$$

It is easy to check that $\mathcal{C}(\lambda)$ is a valid set of pairs. Conversely, assuming $k > 0$, let D be any valid set of pairs D and set $d_i = \#\{j \mid (i,j) \in D\}$. Then the prescription

$$\lambda_i = \begin{cases} k+1+d_i & \text{if } d_i > 0, \\ k & \text{if } d_i = 0 \end{cases}$$

for $1 \leq i \leq d_1 + 1$ defines a k-strict partition λ such that $\mathcal{C}(\lambda) = D$. The white dots in Figure 1 illustrate a typical valid set of pairs.

For any valid set of pairs D, we define the raising operator

$$R^D = \prod_{i<j}(1-R_{ij}) \prod_{i<j\,:\,(i,j)\in D}(1+R_{ij})^{-1}.$$

If λ is a k-strict partition, then set $R^\lambda := R^{\mathcal{C}(\lambda)}$. For any such λ, define $W_\lambda \in B^{(k)}$ by the *Giambelli formula*

$$W_\lambda := R^\lambda\, w_\lambda. \tag{19}$$

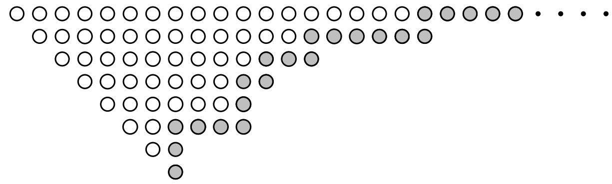


FIGURE 1. A set of pairs (in white) and its outside rim (in grey).

In contrast to (2) and (6), we see that the raising operator R^λ in the Giambelli definition (19) depends on λ. When $\lambda_i + \lambda_j \leq 2k + j - i$ for all $i < j$, equation (19) becomes $W_\lambda = \prod_{i<j}(1 - R_{ij})\, w_\lambda$, while when $\lambda_i + \lambda_j > 2k + j - i$ for $1 \leq i < j \leq \ell(\lambda)$, the formula is equivalent to $W_\lambda = \prod_{i<j} \dfrac{1 - R_{ij}}{1 + R_{ij}}\, w_\lambda$.

Example 1. In the ring $B^{(2)}$ we have

$$\begin{aligned}
W_{621} &= \frac{1 - R_{12}}{1 + R_{12}} \frac{1 - R_{13}}{1 + R_{13}} (1 - R_{23})\, w_{621} \\
&= (1 - 2R_{12} + 2R_{12}^2 - \cdots)(1 - 2R_{13} + 2R_{13}^2 - \cdots)(1 - R_{23})\, w_{621} \\
&= (1 - 2R_{12} + 2R_{12}^2 - 2R_{12}^3)(1 - 2R_{13} - R_{23})\, w_{621} \\
&= w_{621} - w_{63} - 2w_{711} + 4w_{81} - 2w_9 \\
&= w_6 w_2 w_1 - w_6 w_3 - 2w_7 w_1^2 + 4w_8 w_1 - 2w_9.
\end{aligned}$$

From equation (18) we deduce that either the partition λ is k-strict, or w_λ is a $\mathbb{Z}$-linear combination of the w_μ such that μ is k-strict and $\mu \succ \lambda$. It follows that the w_λ for λ k-strict span $B^{(k)}$ as an abelian group. A dimension counting argument shows that in fact, the w_λ for λ k-strict form a $\mathbb{Z}$-basis of $B^{(k)}$. From the definition (6) it follows that for any partition λ, W_λ is of the form

$$W_\lambda = w_\lambda + \sum_{\mu \succ \lambda} a_{\lambda\mu}\, w_\mu$$

with coefficients $a_{\lambda\mu} \in \mathbb{Z}$. We deduce that

$$W_\lambda = w_\lambda + \sum_{\mu \succ \lambda} b_{\lambda\mu}\, w_\mu$$

with the sum restricted to k-strict partitions $\mu \succ \lambda$. Therefore, the W_λ as λ runs over k-strict partitions form a $\mathbb{Z}$-basis of $B^{(k)}$.

More generally, given a valid set of pairs D and an integer sequence α, we denote $R^D\, w_\alpha$ by W_α^D. We say that D is the *denominator set* of W_α^D. If $r, s \in \mathbb{Z}$, then

$$W_{(r,s)}^{\emptyset} = -W_{(s-1,r+1)}^{\emptyset}$$

while if $D \neq \emptyset$ and $r + s > 2k$, then

$$W_{(r,s)}^{D} = -W_{(s,r)}^{D}$$

in the ring $B^{(k)}$. To generalize these equations, care is required, as e.g. the direct analogue of Lemma 1 for the W_α^D fails. Suppose D is a valid set of pairs and $j \geq 1$. We say that the pair $(j, j+1)$ is *D-tame* if

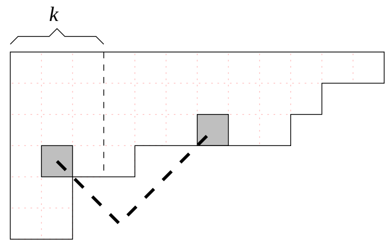


FIGURE 2. Two k-related boxes in a k-strict Young diagram

(i) $(j, j+1) \notin D$ and for all $h < j$, $(h, j) \notin D$ if and only if $(h, j+1) \notin D$, or

(ii) $(j, j+1) \in D$ and for all $h > j+1$, $(j, h) \in D$ if and only if $(j+1, h) \in D$.

Lemma 4 ([BKT2]). *Let $\alpha = (\alpha_1, \ldots, \alpha_{j-1})$ and $\beta = (\beta_{j+2}, \ldots, \beta_\ell)$ be integer vectors, and assume that $(j, j+1)$ is D-tame.*

(a) *If $(j, j+1) \notin D$, then for any $r, s \in \mathbb{Z}$ we have*

$$W^D_{(\alpha,r,s,\beta)} = -W^D_{(\alpha,s-1,r+1,\beta)} .$$

(b) *If $(j, j+1) \in D$, then for any $r, s \in \mathbb{Z}$ such that $r + s > 2k$, we have*

$$W^D_{(\alpha,r,s,\beta)} = -W^D_{(\alpha,s,r,\beta)} .$$

Example 2. The ring $B := B^{(0)}$ has free $\mathbb{Z}$-basis consisting of the W_λ for λ a strict partition, and the Giambelli formula (19) is equivalent to the identity

$$W_\lambda = \prod_{i<j} \frac{1 - R_{ij}}{1 + R_{ij}}\, w_\lambda. \tag{20}$$

Lemma 4 in this case gives $W_{(\alpha,r,s,\beta)} = -W_{(\alpha,s,r,\beta)}$ whenever $r + s > 0$.

3.2. **The Pieri rule.** We begin our analysis in the same way as in the previous sections. For $p > 0$ and any k-strict partition λ of length ℓ, we find that

$$w_p \cdot W_\lambda = \sum_{\nu \in \mathcal{N}} W^{\mathcal{C}(\lambda)}_\nu, \tag{21}$$

where $\mathcal{N} = \mathcal{N}(\lambda, p)$ is the set of all compositions $\nu \geq \lambda$ such that $|\nu| = |\lambda| + p$ and $\nu_j = 0$ for $j > \ell + 1$. The problem with the right hand side of (21) is two-fold: first, the compositions $\nu \in \mathcal{N}$ are not k-strict partitions, and second, the denominator set $\mathcal{C}(\lambda)$ is the same for each term in the sum, and will have to be modified so as to agree with the summands in the eventual Pieri rule. Rather than give the complete argument, we will state the Pieri rule which results and then discuss some features of the proof.

We let $[r, c]$ denote the box in row r and column c of a Young diagram. Suppose that $c \leq k < c'$. We say that the boxes $[r, c]$ and $[r', c']$ are *k-related* if $c + c' = 2k + 2 + r - r'$. In the diagram of Figure 2, the two grey boxes are k-related.

Given two partitions λ and μ with $\lambda \subset \mu$, the skew Young diagram μ/λ is called a *vertical strip* if it does not contain two boxes in the same row. For any two k-strict partitions λ and μ, let α_i (respectively β_i) denote the number of boxes of λ (respectively μ) in column i, for $1 \leq i \leq k$. We have a relation $\lambda \to \mu$ if μ can be obtained by removing a vertical strip from the first k columns of λ and adding a horizontal strip to the result, so that for each i with $1 \leq i \leq k$,

(1) if $\beta_i = \alpha_i$, then the box $[\alpha_i, i]$ is k-related to at most one box of $\mu \smallsetminus \lambda$; and

(2) if $\beta_i < \alpha_i$, then the boxes $[\beta_i, i], \ldots, [\alpha_i, i]$ must each be k-related to exactly one box of $\mu \smallsetminus \lambda$, and these boxes of $\mu \smallsetminus \lambda$ must all lie in the same row.

If $\lambda \to \mu$, we let $\mathbb{A}$ be the set of boxes of $\mu \smallsetminus \lambda$ in columns $k+1$ and higher which are *not* mentioned in (1) or (2). Define the connected components of $\mathbb{A}$ by agreeing that two boxes in $\mathbb{A}$ are connected if they share at least a vertex. Then define $N(\lambda, \mu)$ to be the number of connected components of $\mathbb{A}$ which do not have a box in column $k+1$. Finally, we can state the Pieri rule for $B^{(k)}$: Given any k-strict partition λ and integer $p \geq 0$,

$$w_p \cdot W_\lambda = \sum_{\substack{\lambda \to \mu \\ |\mu| = |\lambda| + p}} 2^{N(\lambda,\mu)} \, W_\mu \,. \tag{22}$$

The *outside rim* $\partial \mathcal{C}$ of $\mathcal{C} = \mathcal{C}(\lambda)$ is the set of pairs $(i,j) \in \Delta^\circ \smallsetminus \mathcal{C}$ such that $i = 1$ or $(i-1, j-1) \in \mathcal{C}$; an example of these sets is displayed in Figure 1. The Young diagrams of the partitions μ which appear in (22) do not all contain the diagram of λ, in contrast to the Pieri rule of §2.2 for Hall-Littlewood functions. However, the denominator sets $\mathcal{C}(\mu)$ for the partitions μ with $\lambda \to \mu$ all contain $\mathcal{C}$, are contained in $\mathcal{C} \cup \partial \mathcal{C}$, and are empty beyond column $\ell + 1$.

To prove (22), our task is to show that

$$\sum_{\nu \in \mathcal{N}(\lambda, p)} W_\nu^{\mathcal{C}(\lambda)} = \sum_{\substack{\lambda \to \mu \\ |\mu| = |\lambda| + p}} 2^{N(\lambda,\mu)} \, W_\mu^{\mathcal{C}(\mu)} \,. \tag{23}$$

It is clear that we need a mechanism to modify the denominator sets of the terms $W_\nu^{\mathcal{C}(\lambda)}$ on the left hand side of (23). This is based on the observation that if D is a denominator set and $(i,j) \in \Delta^\circ \smallsetminus D$, then

$$W_\alpha^D = W_\alpha^{D \cup (i,j)} + W_{R_{ij}\alpha}^{D \cup (i,j)} \,. \tag{24}$$

Equation (24) follows directly from the raising operator identity

$$1 - R_{ij} = \frac{1 - R_{ij}}{1 + R_{ij}} + \frac{1 - R_{ij}}{1 + R_{ij}} \, R_{ij}.$$

The following three detailed examples illustrate how Lemma 4 and (24) may be used repeatedly to obtain (23). For simplicity, the commas are omitted from the notation for integer vectors and pairs.

Example 3. The following chain of equalities holds in $B^{(1)}$.

$$\begin{aligned}
w_1 \cdot W_{3211} &= W_{32111}^{12} + W_{3212}^{12} + W_{3221}^{12} + W_{3311}^{12} + W_{4211}^{12} \\
&= W_{32111}^{12} + \left(W_{3221}^{12,13} + W_{4211}^{12,13} \right) + \left(W_{4211}^{12,13} + W_{5201}^{12,13} \right) \\
&= W_{32111}^{12} + (W_{3221}^{12,13,23} + W_{3311}^{12,13,23}) + 2\, W_{4211}^{12,13} + (W_{5201}^{12,13,14} + W_{6200}^{12,13,14}) \\
&= W_{32111}^{\mathcal{C}(32111)} + 2\, W_{4211}^{\mathcal{C}(4211)} + W_{62}^{\mathcal{C}(62)}.
\end{aligned}$$

Observe that Lemma 4(a) is used to show that $W_{3212}^{12} = W_{5201}^{12,13,14} = 0$, while Lemma 4(b) implies that $W_{3311}^{12} = W_{3221}^{12,13,23} = W_{3311}^{12,13,23} = 0$.

Example 4. The following chain of equalities holds in $B^{(1)}$.

$$\begin{aligned}
w_3 \cdot W_{21} &= W^{\emptyset}_{51} + W^{\emptyset}_{42} + W^{\emptyset}_{411} + W^{\emptyset}_{33} + W^{\emptyset}_{24} \\
&\quad + W^{\emptyset}_{321} + W^{\emptyset}_{312} + W^{\emptyset}_{231} + W^{\emptyset}_{222} + W^{\emptyset}_{213} \\
&= \left(W^{12}_{51} + W^{12}_{6}\right) + \left(W^{12}_{42} + W^{12}_{51}\right) + \left(W^{12}_{411} + W^{12}_{501}\right) + \left(W^{12}_{33} + W^{12}_{42}\right) \\
&\quad + \left(W^{12}_{24} + W^{12}_{33}\right) + \left(W^{12}_{321} + W^{12}_{411}\right) + \left(W^{12}_{231} + W^{12}_{321}\right) \\
&= W^{12}_{6} + 2\,W^{12}_{51} + W^{12}_{42} + \left(W^{12,13}_{411} + W^{12,13}_{51}\right) + \left(W^{12,13}_{501} + W^{12,13}_{6}\right) \\
&\quad + W^{12}_{321} + \left(W^{12,13}_{411} + W^{12,13}_{51}\right) \\
&= 2\,W^{\mathcal{C}(6)}_{6} + 4\,W^{\mathcal{C}(51)}_{51} + W^{\mathcal{C}(42)}_{42} + 2\,W^{\mathcal{C}(411)}_{411} + W^{\mathcal{C}(321)}_{321}.
\end{aligned}$$

Here we use Lemma 4(a) to see that

$$W^{\emptyset}_{312} = W^{\emptyset}_{222} + W^{\emptyset}_{213} = W^{12,13}_{501} = 0,$$

while Lemma 4(b) gives

$$W^{12}_{33} = W^{12}_{42} + W^{12}_{24} = W^{12}_{231} + W^{12}_{321} = 0.$$

Example 5. Let $k = 0$. We give a complete cancellation scheme along the above lines assuming that *the integer p is less than or equal to λ_ℓ*. For any integers $d, e \geq 1$ define the raising operator $R^{[d,e]}$ by

$$R^{[d,e]} = \prod_{1\leq i<j\leq d} (1 - R_{ij}) \prod_{1\leq i<j\leq e} (1 + R_{ij})^{-1}.$$

We compute that

$$w_p \cdot W_\lambda = R^{[\ell,\ell]}\, w_{(\lambda,p)} = \sum_{\nu\in\mathcal{N}} R^{[\ell+1,\ell]}\, w_\nu = \sum_{\nu\in\mathcal{N}} R^{[\ell+1,\ell+1]} \prod_{i=1}^{\ell} (1 + R_{i,\ell+1})\, w_\nu$$

and therefore

$$w_p \cdot W_\lambda = \sum_{(\nu,\gamma)\in\mathcal{N}'(\lambda,p)} W_{\nu+\gamma} \tag{25}$$

where $\mathcal{N}'(\lambda, p)$ denotes the set of all pairs (ν, γ) of integer vectors of length at most $\ell + 1$ with $\nu \in \mathcal{N}$, $\nu + \gamma \geq \lambda$, $|\nu + \gamma| = |\lambda| + p$, and $\gamma_i \in \{0, 1\}$ for $1 \leq i \leq \ell$.

For every pair $(\nu, \gamma) \in \mathcal{N}'(\lambda, p)$, define $\mu = \nu + \gamma$. Call a pair (ν, γ) *bad* if there exists a $j > 1$ such that

$$\text{(i) } \mu_j = \mu_{j-1} \quad \text{or} \quad \text{(ii) } \mu_j > \lambda_{j-1} \quad \text{or} \quad \text{(iii) } \mu_j = \lambda_{j-1} \text{ and } \gamma_j = 0.$$

Let X be the set of all bad pairs in $\mathcal{N}'(\lambda, p)$. We next define an involution $\iota : X \to X$ as follows. For $(\nu, \gamma) \in X$, choose j maximal such that (i), (ii), or (iii) holds. If $\mu_j = \mu_{j-1}$, we let $(\nu', \gamma') = (\nu, \gamma)$. Otherwise, let $\varpi \in S_\ell$ be the transposition $(j-1, j)$ and define (ν', γ') by setting $(\nu'_i, \gamma'_i) = (\nu_{\varpi(i)}, \gamma_{\varpi(i)})$ for $1 \leq i \leq \ell$. Finally, set $\iota(\nu, \gamma) = (\nu', \gamma')$ for each $(\nu, \gamma) \in X$. Lemma 4 applied to rows $j-1$ and j gives $W_{\nu+\gamma} = -W_{\nu'+\gamma'}$ for every $(\nu, \gamma) \in X$, therefore all bad indices may be omitted from the sum in (25).

We are left with pairs (ν, γ) such that $\mu = \nu + \gamma \supset \lambda$ is a strict partition with μ/λ a horizontal strip. Observe that every connected component C of μ/λ which does not lie in row $\ell + 1$ contributes a multiplicity of 2 to the sum (25). Indeed, if

C lies in rows r through s with $r \leq s$, then condition (iii) implies that $\gamma_i = 1$ for $r < i \leq s$, while γ_r can be either 0 or 1. We have therefore proved the Pieri rule

$$w_p \cdot W_\lambda = \sum_\mu 2^{N(\lambda,\mu)}\, W_\mu$$

summed over all strict partitions $\mu \supset \lambda$ with $|\mu| = |\lambda| + p$ and μ/λ a horizontal p-strip. The exponent $N(\lambda, \mu)$ equals the number of connected components of μ/λ which do not meet the first column.

In the general case, there is a substitution algorithm by which the left hand side of (23) *evolves* into the right hand side. Although Examples 3, 4, and 5 appear encouraging, they are still far from a precise formulation of this algorithm for a general Pieri product $w_p \cdot W_\lambda$. It is instructive to try to prove along these lines that the Pieri rule of Example 5 holds without the simplifying assumption that $p \leq \lambda_\ell$. The complete proof for arbitrary $k \geq 0$ is given in [BKT2]; we note here that for technical reasons, the argument is performed in type B, that is, for odd orthogonal Grassmannians. Each summand $W_\nu^{\mathcal{C}(\lambda)}$ for $\nu \in \mathcal{N}$ in (23) gives rise to a *tree* of successor terms, with the branching given by repeated substitutions using (24). In this way, we obtain the *substitution forest*; the roots of the trees in the forest are the $W_\nu^{\mathcal{C}(\lambda)}$ for $\nu \in \mathcal{N}$. By construction, the sum of all the roots is equal to the sum of all the leaves. When the algorithm terminates, the set of leaves contains the terms $W_\mu^{\mathcal{C}(\mu)}$ for $\lambda \to \mu$, each appearing exactly $2^{N(\lambda,\mu)}$ times; the remaining leaves are either zero or cancel in pairs, as dictated by Lemma 4.

Once we know that the Giambelli formula (19) used to define W_λ satisfies the Pieri rule (22), the same reasoning as in the previous two sections establishes that the two results are formally equivalent.

3.3. **The isotropic Grassmannian and theta polynomials.** Let the vector space $E = \mathbb{C}^{2n}$ be equipped with a nondegenerate skew-symmetric bilinear form. A subspace Σ of E is *isotropic* if the form vanishes when restricted to Σ. The dimensions of such isotropic subspaces Σ range from 0 to n; when $\dim(\Sigma) = n$ we say that Σ is a *Lagrangian* subspace.

Choose $n \geq k \geq 0$ and let $\mathrm{IG} = \mathrm{IG}(n-k, 2n)$ denote the Grassmannian parametrizing isotropic subspaces of E. Its cohomology ring $\mathrm{H}^*(\mathrm{IG}, \mathbb{Z})$ has a free $\mathbb{Z}$-basis of Schubert classes σ_λ, one for each k-strict partition λ whose diagram is contained in the $(n-k)\times(n+k)$ rectangle $\mathcal{R}(n-k, n+k)$. Following [BKT1, BKT2], the special Schubert classes σ_p for IG are the Chern classes of the universal quotient bundle over IG, as in §1.4. There is a ring epimorphism $\psi : B^{(k)} \to \mathrm{H}^*(\mathrm{IG}, \mathbb{Z})$ sending the generators w_p to the special Schubert classes σ_p for $1 \leq p \leq n + k$ and to zero for $p > n + k$. For any k-strict partition λ, we have $\psi(W_\lambda) = \sigma_\lambda$ if $\lambda \subset \mathcal{R}(n-k, n+k)$, and $\psi(W_\lambda) = 0$ otherwise.

Let $x = (x_1, x_2, \ldots)$ as in §1.4 and set $y = (y_1, \ldots, y_k)$. Define the formal power series $\vartheta_r(x\,; y)$ by the generating equation

$$\prod_{i=1}^{\infty} \frac{1 + x_i t}{1 - x_i t} \prod_{j=1}^{k} (1 + y_j t) = \sum_{r=0}^{\infty} \vartheta_r(x\,; y) t^r.$$

Set $\Gamma^{(k)} = \mathbb{Z}[\vartheta_1, \vartheta_2, \ldots]$ to be the ring of theta polynomials. There is a ring isomorphism $B^{(k)} \to \Gamma^{(k)}$ sending w_p to ϑ_p for all p. For any k-strict partition λ, the element W_λ is mapped to the *theta polynomial* $\Theta_\lambda(x\,;y)$ of [BKT2]. These polynomials agree with the Schubert polynomials of type C defined by Billey and Haiman [BH] indexed by a Grassmannian permutation of the hyperoctahedral group; see [BKT2, §6] and §6 of the present paper for further information.

We next discuss this theory when $k=0$. If α has length ℓ and $m>0$ is the least integer such that $2m\geq \ell$, then equation (20) may be written in its original form

$$W_\alpha = \mathrm{Pfaffian}(W_{\alpha_i,\alpha_j})_{1\leq i<j\leq 2m}. \tag{26}$$

To see this, one may argue as in §1.4, this time using Schur's identity [S, §IX]

$$\prod_{1\leq r<s\leq 2m}\frac{x_r-x_s}{x_r+x_s} = \mathrm{Pfaffian}\left(\frac{x_r-x_s}{x_r+x_s}\right)_{1\leq r,s\leq 2m}.$$

In the theory of symmetric functions, the ring $\Gamma:=\Gamma^{(0)}$ is called the ring of Schur Q-functions. For any strict partition λ, the element W_λ is mapped to the Schur Q-function $Q_\lambda(x)$, which Schur [S] defined using the Pfaffian equation (26).

Let ϕ denote the ring homomorphism $A_{-1}\to\Gamma$ sending v_p to $q_p(x)$ for every $p\geq 1$. Then $\phi(V_\lambda)=Q_\lambda(x)$ whenever λ is *strict*, and $\phi(V_\lambda)=0$ for non-strict partitions λ. The Pieri rule for the products $q_p\cdot Q_\lambda$ may therefore be obtained by specializing Morris' rule (17) for the Hall-Littlewood symmetric functions when $t=-1$. The Pieri and Giambelli formulas for the Lagrangian Grassmannian $\mathrm{LG}(n,2n)$ were proved by Hiller and Boe [HB] and Pragacz [Pra], respectively.

4. Mirror identities and recursion formulas

4.1. Let λ be any partition of length ℓ. Our proof of the Pieri rule for V_λ in §2.2 began with the equation

$$v_p\cdot V_\lambda = v_p\cdot R\,v_\lambda = R\,v_{(\lambda,p)}$$

for an appropriate raising operator R, and relied on the identity

$$\sum_{\alpha\geq 0}(1-t)^{\#\alpha}\,V_{\lambda+\alpha} = \sum_\mu \psi_{\mu/\lambda}(t)\,V_\mu \tag{27}$$

where the first sum is over all compositions α of length at most $\ell+1$, and the second over partitions $\mu\supset\lambda$ such that μ/λ is a horizonal strip. On the other hand, we may write $v_p\cdot R\,v_\lambda = R'\,v_{(p,\lambda)}$ for a raising operator R' which agrees with R up to a shift of its indices, and attempt a similar analysis. When p is sufficiently large, we are led to the *mirror identity* to (27), which is an analogous formula for the sum $\displaystyle\sum_{\alpha\geq 0}(1-t)^{\#\alpha}\,V_{\lambda-\alpha}$.

We will require an auxiliary result which stems from [BKT3].

Lemma 5. *Let P_r be the set of partitions μ with $|\mu|=r$, and let m be a positive integer. Then the $\mathbb{Z}[t]$-linear map*

$$\phi\ :\ \bigoplus_{r=0}^{\lfloor\frac{m-1}{2}\rfloor}\bigoplus_{\mu\in P_r}\mathbb{Z}[t]\to A_t$$

which, for given r and $\mu\in P_r$, sends the corresponding basis element to $v_{m-r}\,V_\mu$, is injective.

Proof. The Pieri rule (17) implies that the image of ϕ is contained in the linear span of the elements $V_{(m-r,\mu)}$ for $0 \leq r < \frac{m}{2}$ and μ in P_r. Now the linear map ϕ is represented by a block triangular matrix with invertible diagonal matrices as the blocks along the diagonal, and hence is an isomorphism onto its image. $\square$

Suppose μ is a partition with $\mu \subset \lambda$ such that λ/μ is a horizontal strip. Let I denote the set of integers $c \geq 1$ such that λ/μ does (respectively does not) have a box in column c (respectively column $c+1$). Define

$$\varphi_{\lambda/\mu}(t) = \prod_{c \in I}(1 - t^{m_c(\lambda)})$$

and observe that for $p \geq \lambda_1$, the Pieri rule (17) for $v_p \cdot V_\lambda$ may be written in the form

$$v_p \cdot V_\lambda = \sum_{r \geq 0} \sum_{\substack{\mu \subset \lambda \\ |\mu| = |\lambda| - r}} \varphi_{\lambda/\mu}(t)\, V_{(p+r,\mu)}. \tag{28}$$

Theorem 1. *For any partition λ, we have*

$$\sum_{\alpha \geq 0} (1-t)^{\#\alpha}\, V_{\lambda - \alpha} = \sum_{\mu \subset \lambda} \varphi_{\lambda/\mu}(t)\, V_\mu, \tag{29}$$

where the first sum is over all compositions α, and the second over partitions $\mu \subset \lambda$ such that λ/μ is a horizontal strip.

Proof. Choose $p > |\lambda|$ and let $\ell = \ell(\lambda)$. Expanding the Giambelli formula with respect to the first row gives

$$v_p \cdot V_\lambda = R_t^{\ell+1} \prod_{j=2}^{\ell+1} \frac{1 - tR_{1j}}{1 - R_{1j}}\, v_{(p,\lambda)} = R_t^{\ell+1} \prod_{j=2}^{\ell+1} (1 + (1-t)R_{1j} + (1-t)R_{1j}^2 + \cdots)\, v_{(p,\lambda)}$$

and therefore

$$v_p \cdot V_\lambda = \sum_{\alpha \geq 0} (1-t)^{\#\alpha}\, V_{(p+|\alpha|, \lambda - \alpha)}. \tag{30}$$

We compare (28) with (30), and claim that for every integer $r \geq 0$,

$$\sum_{|\alpha| = r} (1-t)^{\#\alpha}\, V_{(p+r,\lambda-\alpha)} = \sum_{\substack{\mu \subset \lambda \\ |\mu| = |\lambda| - r}} \varphi_{\lambda/\mu}(t)\, V_{(p+r,\mu)}. \tag{31}$$

The proof is by induction on r, with the case $r = 0$ being a tautology. For the induction step, suppose that we have for some $r > 0$ that

$$\sum_{s \geq r} \sum_{|\alpha| = s} (1-t)^{\#\alpha}\, V_{(p+s,\lambda-\alpha)} = \sum_{s \geq r} \sum_{\substack{\mu \subset \lambda \\ |\mu| = |\lambda| - s}} \varphi_{\lambda/\mu}(t)\, V_{(p+s,\mu)}. \tag{32}$$

Expanding the Giambelli formula with respect to the first component, we obtain $v_{p+s}\, V_{\lambda-\alpha}$ as the leading term of $V_{(p+s,\lambda-\alpha)}$, while $v_{p+s}\, V_\mu$ is the leading term of $V_{(p+s,\mu)}$. Using these in (32) together with Lemma 5 proves that

$$\sum_{|\alpha| = r} (1-t)^{\#\alpha}\, V_{\lambda-\alpha} = \sum_{\substack{\mu \subset \lambda \\ |\mu| = |\lambda| - r}} \varphi_{\lambda/\mu}(t)\, V_\mu. \tag{33}$$

Proposition 1 now shows that (31) is true, completing the induction. Since we have simultaneously checked that (33) holds for every integer $r \geq 0$, the proof of Theorem 1 is complete. $\square$

4.2. Let λ be any k-strict partition of length ℓ. In this section, we will obtain the mirror identity to the following version of (23):

$$\sum_{\alpha\geq 0} W^{\mathcal{C}(\lambda)}_{\lambda+\alpha} = \sum_{\lambda\to\mu} 2^{N(\lambda,\mu)}\, W^{\mathcal{C}(\mu)}_{\mu}$$

where the first sum is over all compositions α of length at most $\ell+1$, and the second over k-strict partitions μ with $\lambda\to\mu$.

Proposition 2. *Let* $\Psi = \prod_{j=2}^{\ell+1}\dfrac{1-R_{1j}}{1+R_{1j}}$, *and suppose that we have an equation*

$$\sum_\nu a_\nu w_\nu = \sum_\nu b_\nu w_\nu$$

in $B^{(k)}$, *where the sums are over all* $\nu=(\nu_1,\ldots,\nu_\ell)$, *while* a_ν *and* b_ν *are integers only finitely many of which are non-zero. Then we have*

$$\sum_\nu a_\nu \Psi\, w_{(p,\nu)} = \sum_\nu b_\nu \Psi\, w_{(p,\nu)}$$

in the ring $B^{(k)}$, *for any integer* p.

Proof. It suffices to show that $\sum_\nu c_\nu w_\nu = 0$ implies that $\sum_\nu c_\nu \Psi\, w_{(p,\nu)} = 0$. We may assume by interchanging rows that the sum is over partitions ν, because

$$\Psi\, w_{(p,\nu)} = \Psi\, w_{(p,\tau(\nu))}$$

for every permutation τ in the symmetric group S_ℓ, and $w_\nu = \Psi\, w_{(p,\nu)} = 0$ whenever ν has a negative component. Recall that the w_ν for k-strict partitions ν form a $\mathbb{Z}$-basis for $B^{(k)}$. Using the relations (18) and induction on the dominance order, we see that for any partition ν, either ν is k-strict, or w_ν is a $\mathbb{Z}$-linear combination of the w_μ such that μ is k-strict and $\mu\succ\nu$.

It follows that we can identify the sum $\sum_\nu c_\nu w_\nu$ with a $\mathbb{Z}$-linear combination of relations of the form $\dfrac{1-R_{h,h+1}}{1+R_{h,h+1}}\, w_\nu$, where $h\geq 1$ and ν is a partition such that $\nu_h=\nu_{h+1}>k$. Therefore it will suffice to show that any such relation $\dfrac{1-R_{h,h+1}}{1+R_{h,h+1}}\, w_\nu = 0$ implies that $\Psi\,\dfrac{1-R_{h,h+1}}{1+R_{h,h+1}}\, w_{(p,\nu)}=0$. For this, it is enough to check that

$$\frac{1-R_{1,h}}{1+R_{1,h}}\cdot\frac{1-R_{1,h+1}}{1+R_{1,h+1}}\cdot\frac{1-R_{h,h+1}}{1+R_{h,h+1}}\, w_{(p,\nu)} = 0.$$

But we have

$$\frac{1-R_{1,h}}{1+R_{1,h}}\cdot\frac{1-R_{1,h+1}}{1+R_{1,h+1}}\cdot\frac{1-R_{h,h+1}}{1+R_{h,h+1}} = \frac{1-R_{1,h}}{1+R_{1,h}} - \frac{1-R_{1,h+1}}{1+R_{1,h+1}} + \frac{1-R_{h,h+1}}{1+R_{h,h+1}}$$

and since $\nu_h=\nu_{h+1}$, the result is clear. □

The next result is an easy consequence of the Pieri rule (22).

Lemma 6 ([BKT3])**.** *Let* λ *and* ν *be* k*-strict partitions such that* $\nu_1>\max(\lambda_1,\ell(\lambda)+2k)$ *and* $p\geq 0$. *Then the coefficient of* W_ν *in the Pieri product* $w_p\cdot W_\lambda$ *is equal to the coefficient of* $W_{(\nu_1+1,\nu_2,\nu_3,\ldots)}$ *in the product* $w_{p+1}\cdot W_\lambda$.

We apply Lemma 6 to make the following important definition.

Definition 1. Let λ and μ be k-strict partitions with $\mu \subset \lambda$, and choose any $p \geq \max(\lambda_1+1, \ell(\lambda)+2k)$. If $|\lambda| = |\mu|+r$ and $\lambda \to (p+r,\mu)$, then we write $\mu \rightsquigarrow \lambda$ and say that λ/μ is a *k-horizontal strip*. We define $n(\lambda/\mu) := N(\lambda,(p+r,\mu))$; in other words, the numbers $n(\lambda/\mu)$ are the exponents that appear in the Pieri product

$$w_p \cdot W_\lambda = \sum_{r,\mu} 2^{n(\lambda/\mu)}\, W_{(p+r,\mu)} \tag{34}$$

with the sum over integers $r \geq 0$ and k-strict partitions $\mu \subset \lambda$ with $|\mu| = |\lambda| - r$.

Note that a k-horizontal strip λ/μ is a pair of partitions λ and μ with $\mu \rightsquigarrow \lambda$. As such it depends on λ and μ and not only on the difference $\lambda \smallsetminus \mu$. A similar remark applies to the integer $n(\lambda/\mu)$ and the polynomials $\varphi_{\lambda/\mu}(t)$, $\psi_{\lambda/\mu}(t)$ of §4.1. Observe also that $n(\lambda/\lambda) = 0$ and $n(\lambda/\mu) \geq 1$ whenever $\lambda \neq \mu$. We will study the relation $\mu \rightsquigarrow \lambda$ and the integer $n(\lambda/\mu)$ in more detail in §5.3.

Theorem 2. *For λ any k-strict partition we have*

$$\sum_{\alpha\geq 0} 2^{\#\alpha}\, W^{\mathcal{C}(\lambda)}_{\lambda-\alpha} = \sum_{\mu\rightsquigarrow\lambda} 2^{n(\lambda/\mu)}\, W_\mu, \tag{35}$$

where the first sum is over all compositions α.

Proof. The argument is essentially the same as that in the proof of Theorem 1. We choose $p > |\lambda| + 2k$ and prove by induction that for each $r \geq 0$,

$$\sum_{|\alpha|=r} 2^{\#\alpha}\, W^{\widehat{\mathcal{C}}}_{(p+r,\lambda-\alpha)} = \sum_{\substack{\mu\rightsquigarrow\lambda \\ |\mu|=|\lambda|-r}} 2^{n(\lambda/\mu)}\, W_{(p+r,\mu)},$$

where $\widehat{\mathcal{C}} = \mathcal{C}((p,\lambda))$. Clearly we have an analogue of Lemma 5 which holds for the algebra $B^{(k)}$, and we use Proposition 2 as a substitute for Proposition 1. The point is that for such integers p and r, the raising operator expressions $R^{\widehat{\mathcal{C}}}$ and $R^{(p+r,\mu)}$ both contain the product $\Psi = \prod_{j=2}^{\ell+1} \frac{1-R_{1j}}{1+R_{1j}}$, where $\ell = \ell(\lambda)$. $\square$

4.3. We now give some consequences of the previous mirror identities for the Schur S- and Q-functions. Given any integer sequence ν, let $s_\nu(x)$ and $Q_\nu(x)$ denote the Schur functions defined in §2.3 and §3.3, respectively. When $t = 0$, Theorem 1 specializes to the following well known result.

Corollary 2. *For any partition λ, we have*

$$\sum_{\alpha\geq 0} s_{\lambda-\alpha}(x) = \sum_{\mu} s_\mu(x),$$

where the first sum is over all compositions α and the second over all partitions $\mu \subset \lambda$ such that λ/μ is a horizontal strip.

The *rim* of a partition λ is the set of boxes $[r,c]$ of its Young diagram such that box $[r+1,c+1]$ lies outside the diagram of λ. If we choose $k > |\lambda|$ in Theorem 2, then we have $\mathcal{C}(\lambda) = \emptyset$ and can thus deduce the following result.

Corollary 3. *For any partition λ, we have*

$$\sum_{\alpha\geq 0} 2^{\#\alpha}\, s_{\lambda-\alpha}(x) = \sum_{\mu} 2^{n(\lambda/\mu)}\, s_\mu(x),$$

where the first sum is over all compositions α and the second over all partitions $\mu \subset \lambda$ such that λ/μ is contained in the rim of λ, and $n(\lambda/\mu)$ equals the number of edge-connected components of λ/μ.

The *shifted diagram* of a strict partition λ, denoted $\mathcal{S}(\lambda)$, is obtained from the usual Young diagram by shifting the i-th row $(i-1)$ squares to the right, for every $i > 1$. Moreover, when μ is a strict partition with $\mu \subset \lambda$, we set $\mathcal{S}(\lambda/\mu) = \mathcal{S}(\lambda) \smallsetminus \mathcal{S}(\mu)$. We introduce this notation to emphasize the similarity between Corollary 3 and the next result, which follows by setting $k = 0$ in Theorem 2.

Corollary 4. *For any strict partition λ, we have*

$$\sum_{\alpha \geq 0} 2^{\#\alpha}\, Q_{\lambda-\alpha}(x) = \sum_{\mu} 2^{n(\lambda/\mu)}\, Q_{\mu}(x),$$

where the first sum is over all compositions α and the second over all strict partitions $\mu \subset \lambda$ such that $\mathcal{S}(\lambda/\mu)$ is contained in the rim of $\mathcal{S}(\lambda)$, and $n(\lambda/\mu)$ equals the number of edge-connected components of $\mathcal{S}(\lambda/\mu)$.

4.4. We next show how the raising operator formalism we have developed may be used to obtain a recursion formula for the basis elements W_λ, expressed in terms of the length of λ. This question came up naturally during our work on Giambelli formulas for the *quantum* cohomology ring of isotropic Grassmannians [BKT3, §1.3].

Throughout this subsection λ is a partition of length ℓ and $p \geq \lambda_1$ is an integer. For completeness, we begin with the kind of recursion we have in mind for the elements $U_{(p,\lambda)}$ of §1. We claim that for any partition λ and $p \geq \lambda_1$, we have

$$U_{(p,\lambda)} = \sum_{r,\mu} (-1)^r\, u_{p+r}\, U_\mu \tag{36}$$

where the sum is over integers $r \geq 0$ and partitions μ obtained from λ by removing a vertical strip with r boxes. Indeed, Applying the definition (2) gives

$$U_{(p,\lambda)} = \prod_{j>i>1} (1 - R_{ij}) \prod_{j>1} (1 - R_{1j})\, u_{(p,\lambda)} = \sum_{r,\nu} (-1)^r\, u_{p+r}\, U_\nu$$

where the sum is over integers $r \geq 0$ and compositions ν obtained from λ by removing r boxes, no two in the same row. If ν is not a partition, then we must have $\nu_{j+1} = \nu_j + 1$ for some j; hence $U_\nu = 0$ by Lemma 1. The result follows.

The analogue of equation (36) for the $W_{(p,\lambda)}$ when p and λ are arbitrary appears complicated. The next theorem gives a top row recursion for sufficiently large p.

Theorem 3. *For any k-strict partition λ and $p \geq \max(\lambda_1 + 1, \ell(\lambda) + 2k)$, we have*

$$W_{(p,\lambda)} = \sum_{r,\mu} (-1)^r\, 2^{n(\lambda/\mu)} w_{p+r}\, W_\mu, \tag{37}$$

where the sum is over $r \geq 0$ and $\mu \rightsquigarrow \lambda$ with $|\mu| = |\lambda| - r$.

Proof. The Giambelli formula for $W_{(p,\lambda)}$ implies that

$$W_{(p,\lambda)} = R^{\overline{\mathcal{C}}} \prod_{j=2}^{\ell+1} \frac{1 - R_{1j}}{1 + R_{1j}}\, w_{(p,\lambda)} = \sum_{\alpha} (-1)^{|\alpha|} 2^{\#\alpha}\, w_{p+|\alpha|} W^{\mathcal{C}(\lambda)}_{\lambda-\alpha}$$

where $\overline{\mathcal{C}}$ denotes the image of $\mathcal{C}(\lambda)$ under the map which sends (i,j) to $(i+1,j+1)$. We obtain

$$W_{(p,\lambda)} = \sum_{r\geq 0}(-1)^r w_{p+r} \sum_{|\alpha|=r} 2^{\#\alpha}\, W_{\lambda-\alpha}^{\mathcal{C}(\lambda)}$$

and then apply the mirror identity (35) to finish the proof. □

The reader should compare the stable Pieri rule (34) with the recursion formula (37). This is illustrated in the next example.

Example 6. For $k=2$ and $\lambda=(7,4,2,1)$, we have

$$\begin{aligned} w_7 \cdot W_{4,2,1} = {} & W_{7,4,2,1} + 2\,W_{8,3,2,1} + 2\,W_{8,4,2} + 2\,W_{9,2,2,1} + 2\,W_{9,3,1,1} + 4\,W_{9,3,2} \\ & + 2\,W_{10,2,1,1} + 4\,W_{10,2,2} + 4\,W_{10,3,1} + 4\,W_{11,2,1} + 2\,W_{11,3} + 2\,W_{12,2}\,, \end{aligned}$$

and hence

$$\begin{aligned} W_{7,4,2,1} = {} & w_7\,W_{4,2,1} - w_8\,(2\,W_{3,2,1} + 2\,W_{4,2}) + w_9\,(2\,W_{2,2,1} + 2\,W_{3,1,1} + 4\,W_{3,2}) \\ & - w_{10}\,(2\,W_{2,1,1} + 4\,W_{2,2} + 4\,W_{3,1}) + w_{11}\,(4\,W_{2,1} + 2\,w_3) - 2\,w_{12}\,w_2\,. \end{aligned}$$

Setting $k=0$ in Theorem 3 produces the following recursion formula for Schur Q-functions.

Corollary 5. *For any strict partition λ and $p > \lambda_1$, we have*

$$Q_{(p,\lambda)}(x) = \sum_{r,\mu}(-1)^r\, 2^{n(\lambda/\mu)} q_{p+r}(x)\, Q_\mu(x) \tag{38}$$

where the sum is over $r\geq 0$ and strict partitions $\mu \subset \lambda$ such that λ/μ is a horizontal r-strip, and $n(\lambda/\mu)$ equals the number of connected components of λ/μ.

Observe that (38) is a generalization of Schur's identity

$$Q_{a,b}(x) = q_a(x)\,q_b(x) - 2\,q_{a+1}(x)\,q_{b-1}(x) + 2\,q_{a+2}(x)\,q_{b-2}(x) - \cdots$$

for any integers a, b with $a > b \geq 0$.

It would interesting to prove a top row recursion formula for the W_λ polynomials analogous to (37) in the general case, as well as for the V_λ polynomials (in the Hall-Littlewood theory). Note that both of these recursions would have to interpolate between formula (36), which arises when k is sufficiently large for the W_λ and when $t=0$ for the V_λ, and formula (38), which arises when $k=0$ or $t=-1$, respectively.

5. Reduction formulas and tableaux

5.1. The mirror identity (29) may be applied in the theory of Hall-Littlewood functions $Q_\lambda(x\,;t)$ of §2.3 to obtain a *reduction formula*; what is being reduced is the number of x variables used in the argument of $Q_\lambda(x\,;t)$. Let $\tilde{x} = (x_2, x_3, \ldots)$ and observe that $q_p(x\,;t) = \sum_{i=0}^p q_i(x_1\,;t)\,q_{p-i}(\tilde{x}\,;t)$. Therefore, for any integer sequence λ, we obtain

$$q_\lambda(x\,;t) = \sum_{\alpha\geq 0} q_\alpha(x_1\,;t)\,q_{\lambda-\alpha}(\tilde{x}\,;t) = \sum_{\alpha\geq 0} x_1^{|\alpha|}\,(1-t)^{\#\alpha}\,q_{\lambda-\alpha}(\tilde{x}\,;t) \tag{39}$$

summed over all compositions α. If R denotes any raising operator, we have

$$R\,q_\lambda(x\,;t) = q_{R\lambda}(x\,;t) = \sum_{\alpha\geq 0} q_\alpha(x_1\,;t)\,q_{R\lambda-\alpha}(\tilde{x}\,;t) = \sum_{\alpha\geq 0} q_\alpha(x_1\,;t)\,R\,q_{\lambda-\alpha}(\tilde{x}\,;t).$$

Applying the Giambelli formula to (39), we thus deduce that for any partition λ, we have

$$Q_\lambda(x\,;t) = \sum_{\alpha\geq 0} x_1^{|\alpha|}(1-t)^{\#\alpha}Q_{\lambda-\alpha}(\tilde{x}\,;t) = \sum_{p=0}^{\infty} x_1^p \sum_{|\alpha|=p}(1-t)^{\#\alpha}Q_{\lambda-\alpha}(\tilde{x}\,;t) \tag{40}$$

and hence, using (29), the reduction formula

$$Q_\lambda(x\,;t) = \sum_{p=0}^{\infty} x_1^p \sum_{\substack{\mu\subset\lambda \\ |\mu|=|\lambda|-p}} \varphi_{\lambda/\mu}(t)\, Q_\mu(\tilde{x}\,;t) \tag{41}$$

with the second sum over partitions $\mu \subset \lambda$ such that λ/μ is a horizontal p-strip. Repeated application of the reduction equation (41) results in the *tableau formula* of [M, III.(5.11)] for the Hall-Littlewood functions.

There is an alternative approach to the proof of (41) which applies standard symmetric function theory, as in e.g. [M, III.5]. If $x' = (x_1', x_2', \ldots)$ is a second set of commuting variables, then the equation

$$Q_\lambda(x, x'\,;t) = \sum_\mu Q_{\lambda/\mu}(x\,;t)\, Q_\mu(x'\,;t)$$

summed over partitions $\mu \subset \lambda$ may be used to define the *skew Hall-Littlewood functions* $Q_{\lambda/\mu}(x\,;t)$. In particular, this gives

$$Q_\lambda(x_1, \tilde{x}\,;t) = \sum_\mu Q_{\lambda/\mu}(x_1\,;t)\, Q_\mu(\tilde{x}\,;t).$$

According to [M, III.(5.14)], we have

$$Q_{\lambda/\mu}(x_1\,;t) = \varphi_{\lambda/\mu}(t)\, x_1^{|\lambda-\mu|} \tag{42}$$

and hence (41) is established. By comparing with (40), we can apply this reasoning to prove the mirror identity (29) for the Hall-Littlewood functions. Note however that the proof of (42) uses the inner product of [M, III.4], and the point here is that this is not needed in order to establish Theorem 1 (which of course does not involve the variables x). The raising operator approach will be exploited further in the next subsections when we study theta polynomials.

5.2. Let $y = (y_1, \ldots, y_k)$ and consider the theta polynomials $\vartheta_r(x\,;y)$ and $\Theta_\lambda(x\,;y)$ defined in §3.3, so that $\Theta_\lambda = R^\lambda\, \vartheta_\lambda$ for any k-strict partition λ. We compute that

$$\sum_{r=0}^{\infty} \vartheta_r(x\,;y)t^r = \frac{1+x_1t}{1-x_1t}\prod_{i=2}^{\infty}\frac{1+x_it}{1-x_it}\prod_{j=1}^{k}(1+y_jt) = \sum_{i=0}^{\infty} 2^{\#i}\, x_1^i \sum_{r=0}^{\infty}\vartheta_r(\tilde{x}\,;y)$$

and therefore, for any k-strict partition λ,

$$\vartheta_\lambda(x\,;y) = \sum_{\alpha\geq 0} x_1^{|\alpha|}\, 2^{\#\alpha}\, \vartheta_{\lambda-\alpha}(\tilde{x}\,;y) \tag{43}$$

summed over all compositions α. Applying the raising operator R^λ to both sides of (43) produces

$$\Theta_\lambda(x\,;y) = \sum_{\alpha\geq 0} x_1^{|\alpha|}\, 2^{\#\alpha}\, \Theta^{\mathcal{C}(\lambda)}_{\lambda-\alpha}(\tilde{x}\,;y) = \sum_{p=0}^{\infty} x_1^p \sum_{|\alpha|=p} 2^{\#\alpha}\, \Theta^{\mathcal{C}(\lambda)}_{\lambda-\alpha}(\tilde{x}\,;y),$$

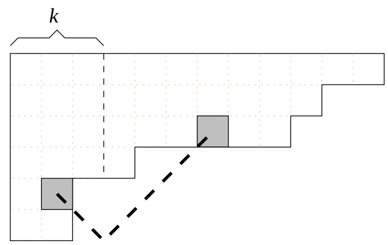


FIGURE 3. Two k'-related boxes in a Young diagram

where $\Theta_{\lambda-\alpha}^{\mathcal{C}(\lambda)} = R^\lambda\,\vartheta_{\lambda-\alpha}$ by definition. We now use the mirror identity (35) to deduce the next result.

Theorem 4. *For any k-strict partition λ, we have the reduction formula*

$$\Theta_\lambda(x\,;y) = \sum_{p=0}^{\infty} x_1^p \sum_{\substack{\mu \rightsquigarrow \lambda \\ |\mu|=|\lambda|-p}} 2^{n(\lambda/\mu)}\,\Theta_\mu(\tilde{x}\,;y). \tag{44}$$

5.3. In this subsection we will apply the reduction formula (44) to obtain a tableau description of the theta polynomials Θ_λ, where the tableaux in question are fillings of the Young diagram of λ. We say that boxes $[r,c]$ and $[r',c']$ are *k'-related* if $|c-k-\frac{1}{2}|+r = |c'-k-\frac{1}{2}|+r'$ (we think of k' as being equal to $k-\frac{1}{2}$). For example, the two grey boxes in the diagram of Figure 3 are k'-related. We call box $[r,c]$ a *left box* if $c \leq k$ and a *right box* if $c > k$.

If $\mu \subset \lambda$ are two k-strict partitions such that λ/μ is a k-horizontal strip, we define $\lambda_0 = \mu_0 = \infty$ and agree that the diagrams of λ and μ include all boxes $[0,c]$ in row zero. We let R (respectively $\mathbb{A}$) denote the set of right boxes of μ (including boxes in row zero) which are bottom boxes of λ in their column and are (respectively are not) k'-related to a left box of λ/μ.

Lemma 7. *A pair $\mu \subset \lambda$ of k-strict partitions forms a k-horizontal strip λ/μ if and only if (i) λ/μ is contained in the rim of λ, and the right boxes of λ/μ form a horizontal strip; (ii) no two boxes in R are k'-related; and (iii) if two boxes of λ/μ lie in the same column, then they are k'-related to exactly two boxes of R, which both lie in the same row. The integer $n(\lambda/\mu)$ is equal to the number of connected components of $\mathbb{A}$ which do not have a box in column $k+1$.*

Proof. We have $\mu \rightsquigarrow \lambda$ if and only if $\lambda \to (p+r,\mu)$ for any $p > |\lambda|+2k$, where $r = |\lambda-\mu|$. Observe that a box of $(p+r,\mu) \smallsetminus \lambda$ corresponds to a box of μ which is a bottom box of λ in its column. The fact that $\mu \rightsquigarrow \lambda$ is characterized by conditions (i)–(iii) and that $n(\lambda/\mu) = N(\lambda,(p+r,\mu))$ is computed as claimed is now an easy translation of the definitions in §3.2. □

Let λ and μ be any two k-strict partitions with $\mu \subset \lambda$.

Definition 2. a) A *k-tableau* T of shape λ/μ is a sequence of k-strict partitions

$$\mu = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^r = \lambda$$

such that λ^i/λ^{i-1} is a k-horizontal strip for $1 \leq i \leq r$. We represent T by a filling of the boxes in λ/μ with positive integers which is weakly increasing along each row and down each column, such that for each i, the boxes in T with entry i form the skew diagram λ^i/λ^{i-1}. A *standard k-tableau* on λ/μ is a k-tableau T of shape λ/μ such that the entries $1, 2, \ldots, |\lambda - \mu|$ each appear once in T. For any k-tableau T we define

$$n(T) = \sum_i n(\lambda^i/\lambda^{i-1}) \quad \text{and} \quad x^T = \prod_i x_i^{m_i}$$

where m_i denotes the number of times that i appears in T.

b) Let $\mathbf{P}$ denote the ordered alphabet $\{1' < 2' < \cdots < k' < 1 < 2 < \cdots\}$. The symbols $1', \ldots, k'$ are said to be *marked*, while the rest are *unmarked*. A *k-bitableau* U of shape λ is a filling of the boxes in λ with elements of $\mathbf{P}$ which is weakly increasing along each row and down each column, such that (i) the marked entries are strictly increasing along each row, and (ii) the unmarked entries form a k-tableau T. We define

$$n(U) = n(T) \quad \text{and} \quad (xy)^U = x^T \prod_{j=1}^{k} y_j^{n_j}$$

where n_j denotes the number of times that j' appears in U.

Theorem 5. *For any k-strict partition λ, we have*

$$\Theta_\lambda(x\,;y) = \sum_U 2^{n(U)} (xy)^U \tag{45}$$

where the sum is over all k-bitableaux U of shape λ

Proof. Let m be a positive integer, $x^{(m)} = (x_1, \ldots, x_m)$, and let $\Theta_\lambda(x^{(m)}\,;y)$ be the result of substituting $x_i = 0$ for $i > m$ in $\Theta_\lambda(x\,;y)$. It follows from equation (44) that

$$\Theta_\lambda(x^{(m)}\,;y) = \sum_{p=0}^{\infty} x_m^p \sum_{\substack{\mu \rightsquigarrow \lambda \\ |\mu| = |\lambda| - p}} 2^{n(\lambda/\mu)}\, \Theta_\mu(x^{(m-1)}\,;y). \tag{46}$$

Iterating equation (46) m times produces

$$\Theta_\lambda(x^{(m)}\,;y) = \sum_{\mu, T} 2^{n(T)}\, x^T \Theta_\mu(0\,;y)$$

where the sum is over all k-strict partitions $\mu \subset \lambda$ and k-tableau T of shape λ/μ with no entries greater than m, and $\Theta_\mu(0\,;y)$ is obtained from $\Theta_\mu(x\,;y)$ by substituting $x_i = 0$ for all i. The raising operator definition of Θ_μ gives

$$\Theta_\mu(0\,;y) = R^\mu\, e_\mu(y) \tag{47}$$

where $e_\mu = \prod_i e_{\mu_i}(y)$ and $e_r(y)$ denotes the r-th elementary symmetric polynomial in y. Since $e_r(y) = 0$ for $r > k$, we deduce from (47) that $\Theta_\mu(0\,;y) = 0$ unless μ is contained in the first k columns. But in this case we have $\mathcal{C}(\mu) = \emptyset$ and

$$\Theta_\mu(0\,;y) = \prod_{i<j} (1 - R_{ij})\, e_\mu(y) = s_{\mu'}(y)$$

where μ' is the partition conjugate to μ. The combinatorial definition of Schur S-functions [M, I.(5.12)] states that

$$s_{\mu'}(y) = \sum_S y^S \tag{48}$$

summed over all semistandard Young tableaux S of shape μ' with entries from 1 to k. In the spirit of this article, the identity (48) may be derived using raising operators, by iterating the specialization of equation (41) at $t = 0$. We conclude that

$$\Theta_\lambda(x^{(m)}\,;y) = \sum_U 2^{n(U)}\,(xy)^U$$

summed over all k-bitableaux U of shape λ with no entries greater than m. The result follows by letting $m \to \infty$. □

Example 7. Let $k = 1$, $\lambda = (3,1)$, and consider the alphabet $\mathbf{P}_{1,2} = \{1' < 1 < 2\}$. There are twelve k-bitableaux T of shape λ with entries in $\mathbf{P}_{1,2}$. The bitableau $T = \begin{matrix} 1'\,1\,2 \\ 1 \end{matrix}$ satisfies $n(T) = 3$, the seven bitableaux

$$\begin{matrix} 1\,1\,2 \\ 1 \end{matrix} \qquad \begin{matrix} 1\,1\,2 \\ 2 \end{matrix} \qquad \begin{matrix} 1\,2\,2 \\ 1 \end{matrix} \qquad \begin{matrix} 1\,2\,2 \\ 2 \end{matrix} \qquad \begin{matrix} 1'\,1\,2 \\ 2 \end{matrix} \qquad \begin{matrix} 1'\,2\,2 \\ 1 \end{matrix} \qquad \begin{matrix} 1'\,1\,2 \\ 1' \end{matrix}$$

satisfy $n(T) = 2$, while the four bitableaux

$$\begin{matrix} 1'\,1\,1 \\ 1 \end{matrix} \qquad \begin{matrix} 1'\,2\,2 \\ 2 \end{matrix} \qquad \begin{matrix} 1'\,1\,1 \\ 1' \end{matrix} \qquad \begin{matrix} 1'\,2\,2 \\ 1' \end{matrix}$$

satisfy $n(T) = 1$. We deduce from Theorem 5 that

$$\begin{aligned} \Theta_{3,1}(x_1,x_2\,;y_1) &= (4x_1^3x_2 + 8x_1^2x_2^2 + 4x_1x_2^3) + (2x_1^3 + 8x_1^2x_2 + 8x_1x_2^2 + 2x_2^3)\,y_1 \\ &\quad + (2x_1^2 + 4x_1x_2 + 2x_2^2)\,y_1^2 \\ &= Q_{3,1}(x_1,x_2) + (Q_3(x_1,x_2) + Q_{2,1}(x_1,x_2))\;y_1 + Q_2(x_1,x_2)\,y_1^2. \end{aligned}$$

5.4. Notice that (45) may be rewritten as

$$\Theta_\lambda(x\,;y) = \sum_{\mu,T} 2^{n(T)}\,x^T\,s_{\mu'}(y)$$

with the sum over all partitions $\mu \subset \lambda$ and k-tableau T of shape λ/μ. This motivates the following definition.

Definition 3. For λ and μ any two k-strict partitions with $\mu \subset \lambda$, let

$$F^{(k)}_{\lambda/\mu}(x) = \sum_T 2^{n(T)}\,x^T$$

where the sum is over all k-tableaux T of shape λ/μ.

Corollary 6. *Let $x = (x_1, x_2, \ldots)$ and $x' = (x'_1, x'_2, \ldots)$ be two sets of variables, and let λ be any k-strict partition. Then we have*

$$\Theta_\lambda(x,x'\,;y) = \sum_{\mu\subset\lambda} F^{(k)}_{\lambda/\mu}(x)\,\Theta_\mu(x'\,;y)\,, \tag{49}$$

$$\Theta_\lambda(x\,;y) = \sum_{\mu\subset\lambda} F^{(k)}_{\lambda/\mu}(x)\,s_{\mu'}(y)\,, \tag{50}$$

and

$$F^{(k)}_\lambda(x,x') = \sum_{\mu\subset\lambda} F^{(k)}_{\lambda/\mu}(x)\,F^{(k)}_\mu(x'), \tag{51}$$

where the sums are over all k-strict partitions $\mu \subset \lambda$.

Proof. Let m be a positive integer, $x^{(m)} = (x_1,\ldots,x_m)$, and let $F^{(k)}_{\lambda/\mu}(x^{(m)})$ (respectively $\Theta_\lambda(x^{(m)}\,;y)$) be the result of substituting $x_i = 0$ for $i > m$ in $F^{(k)}_{\lambda/\mu}(x)$ (respectively $\Theta_\lambda(x\,;y)$). Applying equation (46) as in the proof of Theorem 5 gives

$$\Theta_\lambda(x^{(m)},x'\,;y) = \sum_{\mu\subset\lambda} F^{(k)}_{\lambda/\mu}(x^{(m)})\,\Theta_\mu(x'\,;y).$$

Now let $m\to\infty$ to obtain (49). Equations (50) and (51) are deduced from (49) by substituting $x' = 0$ and $y = 0$, respectively. $\square$

The polynomials $\Theta_\mu(x'\,;y)$ for μ a k-strict partition form a free $\mathbb{Z}$-basis for the ring $\Gamma^{(k)}(x'\,;y)$ of theta polynomials in x' and y. The identity (49) states that $F^{(k)}_{\lambda/\mu}(x)$ is the coefficient of $\Theta_\mu(x'\,;y)$ when $\Theta_\lambda(x,x'\,;y)$ is expanded in this basis. We deduce that $F^{(k)}_{\lambda/\mu}(x)$ is a *symmetric function* in the variables x. The identity (51) may be further generalized as follows.

Corollary 7. *For any two k-strict partitions λ and μ with $\mu\subset\lambda$, we have*

$$F^{(k)}_{\lambda/\mu}(x,x') = \sum_{\nu} F^{(k)}_{\lambda/\nu}(x)\,F^{(k)}_{\nu/\mu}(x'), \tag{52}$$

where the sum is over all k-strict partitions ν with $\mu\subset\nu\subset\lambda$.

Proof. Let $z = (z_1,z_2,\ldots)$ be a third set of variables. We have

$$\begin{aligned}\sum_{\mu\subset\lambda} F^{(k)}_{\lambda/\mu}(x,x')\,\Theta_\mu(z\,;y) = \Theta_\lambda(x,x',z\,;y) &= \sum_{\nu} F^{(k)}_{\lambda/\nu}(x)\,\Theta_\nu(x',z\,;y)\\ &= \sum_{\mu,\nu} F^{(k)}_{\lambda/\nu}(x)\,F^{(k)}_{\nu/\mu}(x')\,\Theta_\mu(z\,;y).\end{aligned}$$

We now equate the coefficients of $\Theta_\mu(z\,;y)$ at either end of the above equalities. $\square$

In the next section, we will show that the k-strict partitions ν which contribute positively to the sum in equation (52) correspond to the reduced factorizations of a certain *skew element* in the hyperoctahedral group.

Example 8. a) Suppose that $k = 0$ or, more generally, that $\mu_i \geq \min(k,\lambda_i)$ for all i. Then Definition 3 becomes the tableau based definition of skew Schur Q-functions [M, III.(8.16)], hence $F^{(k)}_{\lambda/\mu}(x) = Q_{\lambda/\mu}(x)$.

b) Suppose that $\mathcal{C}(\lambda) = \emptyset$, so in particular $\lambda_i > k$ implies that $i = 1$. One then checks that $n(\lambda/\mu)$ is equal to the number of edge-connected components of λ/μ. For any partition $\mu\subset\lambda$, Worley [W, §2.7] has shown that

$$\sum_T 2^{n(T)}\,x^T = S_{\lambda/\mu}(x),$$

where the symmetric function $S_{\lambda/\mu}(x)$ satisfies

$$S_{\lambda/\mu}(x) = \det(q_{\lambda_i-\mu_j+j-i}(x))_{i,j}.$$

We therefore have $F^{(k)}_{\lambda/\mu}(x) = S_{\lambda/\mu}(x)$ and

$$\Theta_\lambda(x\,;y) = \sum_{\mu\subset\lambda} S_{\lambda/\mu}(x)\,s_{\mu'}(y),$$

in agreement with [BKT2, §5.5].

c) It is clear from Definition 3 that in general, the skew function $F^{(k)}_{\lambda/\mu}(x)$ depends on both λ and μ, and not only on the difference $\lambda \smallsetminus \mu$. For instance, we have $F^{(1)}_{(3,2)/(3)}(x) = 0$, while $F^{(1)}_{(r,2)/(r)}(x) = Q_2(x)$ for any $r > 3$. On the other hand, $F^{(1)}_{(5,4,1,1)/(4,3)}(x) = F^{(1)}_{(\lambda,5,4,1,1)/(\lambda,4,3)}(x) = 0$ for any strict partition λ with $\lambda_\ell > 5$. Criteria for the vanishing of $F^{(k)}_{\lambda/\mu}(x)$ are given in Proposition 6 and Corollary 10.

d) Suppose that there is only one variable x. Then we have

$$F^{(k)}_{\lambda/\mu}(x) = \begin{cases} 2^{n(\lambda/\mu)}\, x^{|\lambda-\mu|} & \text{if } \lambda/\mu \text{ is a } k\text{-horizontal strip,} \\ 0 & \text{otherwise.} \end{cases}$$

The reader should compare this to (42).

5.5. Observe that we have

$$\vartheta_p(x,x';y) = \sum_{i=0}^{p} q_i(x)\,\vartheta_{p-i}(x';y)$$

and therefore, for any k-strict partition λ,

$$\vartheta_\lambda(x,x';y) = \sum_{\alpha\geq 0} q_\alpha(x)\,\vartheta_{\lambda-\alpha}(x';y)$$

summed over all compositions $\alpha \geq 0$. It follows that

$$\Theta_\lambda(x,x';y) = \sum_{\alpha\geq 0} q_\alpha(x)\,\Theta^{\mathcal{C}(\lambda)}_{\lambda-\alpha}(x';y). \tag{53}$$

For each fixed composition α, the product $q_\alpha(x) = \prod_i q_{\alpha_i}(x)$ is a nonnegative linear combination of Schur Q-functions. In addition, $\Theta^{\mathcal{C}(\lambda)}_{\lambda-\alpha}(x\,;y)$ is a $\mathbb{Z}$-linear combination of the $\Theta_\mu(x'\,;y)$ for k-strict partitions μ. By comparing (53) with (49), we deduce that for every two k-strict partitions λ, μ with $\mu \subset \lambda$, the function $F^{(k)}_{\lambda/\mu}(x)$ is a linear combination of Schur Q-functions with integer coefficients. In the next section, we will prove that these coefficients are *nonnegative* integers.

6. Stanley symmetric functions and skew elements

6.1. Let B_n be the hyperoctahedral group of signed permutations on the set $\{1,\ldots,n\}$, and $B_\infty = \cup_n B_n$. We will adopt the notation where a bar is written over an entry with a negative sign. The group B_∞ is generated by the simple transpositions $s_i = (i, i+1)$ for $i > 0$, and the sign change $s_0(1) = \overline{1}$. Every element $w \in B_\infty$ can be expressed as a product of simple reflections s_i; any such expression of minimal length is called a *reduced word* for w. The length of w, denoted $\ell(w)$, is the length of any reduced word for w. A factorization $w = uv$ in B_∞ is *reduced* if $\ell(w) = \ell(u) + \ell(v)$.

Following [FS] and [FK1, FK2], we will use the nilCoxeter algebra $\mathcal{B}_n$ of the hyperoctahedral group B_n to define type C Stanley symmetric functions and Schubert polynomials. $\mathcal{B}_n$ is the free associative algebra with unity generated by the elements $u_0, u_1, \ldots, u_{n-1}$ modulo the relations

$$\begin{array}{rcll} u_i^2 & = & 0 & i \geq 0\,; \\ u_i u_j & = & u_j u_i & |i-j| \geq 2\,; \\ u_i u_{i+1} u_i & = & u_{i+1} u_i u_{i+1} & i > 0\,; \\ u_0 u_1 u_0 u_1 & = & u_1 u_0 u_1 u_0. & \end{array}$$

For any $w \in B_n$, choose a reduced word $s_{a_1} \cdots s_{a_\ell}$ for w and define $u_w = u_{a_1} \ldots u_{a_\ell}$. Since the last three relations listed are the Coxeter relations for B_n, it is clear that u_w is well defined, and that the u_w for $w \in B_n$ form a free $\mathbb{Z}$-basis of $\mathcal{B}_n$.

Let ω be an indeterminate and define

$$A_i(\omega) = (1 + \omega u_{n-1})(1 + \omega u_{n-2}) \cdots (1 + \omega u_i) \ ;$$

$$C(\omega) = (1 + \omega u_{n-1}) \cdots (1 + \omega u_1)(1 + 2\omega u_0)(1 + \omega u_1) \cdots (1 + \omega u_{n-1}).$$

Set $x = (x_1, x_2, \ldots)$ and consider the formal product $C(x) := C(x_1)C(x_2)\cdots$. Arguing as in [FK2, Prop. 4.2], we see that the relation $C(x_i)C(x_j) = C(x_j)C(x_i)$ holds for all indices i and j. We deduce that the functions $F_w(x)$ in the formal power series expansion

$$C(x) = \sum_{w \in B_n} F_w(x)\, u_w \tag{54}$$

are symmetric functions in x. The F_w are the *type C Stanley symmetric functions*, introduced and studied in [BH, FK2, L].

Let $y = (y_1, y_2, \ldots)$. The Billey-Haiman *type C Schubert polynomials* $\mathfrak{C}_w(x\,;y)$ for $w \in B_n$ are defined by expanding the formal product

$$C(x)A_1(y_1)A_2(y_2) \cdots A_{n-1}(y_{n-1}) = \sum_{w \in B_n} \mathfrak{C}_w(x\,;y)\, u_w. \tag{55}$$

The above definition is equivalent to the one in [BH], as is shown in [FK2, §7]. One checks that $\mathfrak{C}_w$ is stable under the natural inclusion of B_n in B_{n+1}, and hence well defined for $w \in B_\infty$. We also deduce the following result from (54) and (55).

Proposition 3. *Let $w \in B_\infty$ and $x' = (x'_1, x'_2, \ldots)$. Then we have*

$$F_w(x, x') = \sum_{uv=w} F_u(x)F_v(x') \tag{56}$$

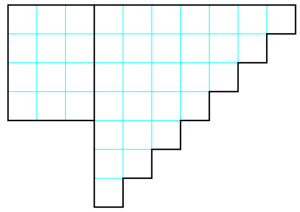
and

$$\mathfrak{C}_w(x, x'\,;y) = \sum_{uv=w} F_u(x)\, \mathfrak{C}_v(x'\,;y) \tag{57}$$

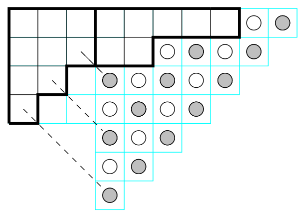
where the sums are over all reduced factorizations $uv = w$ in B_∞.

6.2. We say that an element $w \in B_\infty$ is *k-Grassmannian* if it is Grassmannian with respect to the simple reflection s_k, i.e., if $\ell(ws_i) = \ell(w) + 1$ for all $i \neq k$. Given any k-strict partition λ, we will define a k-Grassmannian element $w_\lambda \in B_\infty$. If $\mathcal{P}(k, n)$ denotes the set of k-strict partitions whose Young diagrams fit inside an $(n - k) \times (n + k)$ rectangle, then $w_\lambda \in B_n$ for any n such that $\lambda \in \mathcal{P}(k, n)$.

The signed permutation $w_\lambda = (w_1, \ldots, w_n)$ has a unique descent at k, that is, $w(i) < w(i + 1)$ whenever $i \neq k$ and the first k entries of w_λ are positive. For $\lambda \in \mathcal{P}(k, n)$ we let λ^1 be the strict partition formed by the boxes of λ in columns $k + 1$ through $k + n$. The negative entries of w_λ are then given by the parts of λ^1. Consider the shape $\Pi(k, n)$ obtained by attaching an $(n - k) \times k$ rectangle to the left side of a staircase partition with n rows. When $n = 7$ and $k = 3$, this looks as follows.

$\Pi(k,n) \quad =$ 

The diagram of λ can be placed inside $\Pi(k,n)$ so that the northwest corners of λ and $\Pi(k,n)$ coincide. The boxes of the staircase partition which are outside λ then fall into south-west to north-east diagonals. The first k (respectively, the last $n-k-\ell(\lambda^1)$) entries of w_λ are the lengths of the diagonals which are (respectively, are not) k-related to one of the bottom boxes in the first k columns of λ. For example, the partition $\lambda = (8,5,2,1) \in \mathcal{P}(3,7)$ results in the element $w_\lambda = 147\overline{5}\overline{2}36$.

$\lambda \ =$ 

Observe that w_λ is stable under the inclusion of W_n into W_{n+1}, and thus is well defined as an element of W_∞.

Conversely, for any k-Grassmannian element $w \in B_n$ there exist strict partitions u, ζ, v of lengths k, r, and $n-k-r$, respectively, so that

$$w = (u_k, \ldots, u_1, \overline{\zeta}_1, \ldots, \overline{\zeta}_r, v_{n-k-r}, \ldots, v_1).$$

Define α_i for $1 \leq i \leq k$ by

$$\alpha_i = u_i + i - k - 1 + \#\{j \mid \zeta_j > u_i\}.$$

Then $w = w_\lambda$ for the partition $\lambda \in \mathcal{P}(k,n)$ with $\lambda^1 = \zeta$ and such that the lengths of the first k columns of λ are given by $\alpha_1, \ldots, \alpha_k$.

It was proved in [BKT2, §6] that

$$\Theta_\lambda(x\,;y) = \mathfrak{C}_{w_\lambda}(x\,;y) \tag{58}$$

for any k-strict partition λ.

6.3. The following definition can be formulated for any Coxeter group, but the name is justified by its application in the case of the finite classical Weyl groups.

Definition 4. An element $w \in B_\infty$ is called *skew* if there exists a k-strict partition λ (for some k) and a reduced factorization $w_\lambda = ww'$ in B_∞.

Note that if we have a reduced factorization $w_\lambda = ww'$ in B_n for some partition $\lambda \in \mathcal{P}(k,n)$, then the right factor w' is k-Grassmannian, and therefore equal to w_μ for some partition $\mu \in \mathcal{P}(k,n)$.

Proposition 4. *Suppose that w is a skew element of B_∞, and let λ and μ be k-strict partitions such that the factorization $w_\lambda = ww_\mu$ is reduced. Then we have $\mu \subset \lambda$ and $F_w(x) = F^{(k)}_{\lambda/\mu}(x)$.*

Proof. By combining (58) with (49) and (57), we see that

$$\sum_{\mu\subset\lambda} F^{(k)}_{\lambda/\mu}(x)\,\Theta_\mu(x'\,;y) = \Theta_\lambda(x,x'\,;y) = \sum_{uv=w_\lambda} F_u(x)\,\mathfrak{C}_v(x'\,;y) \tag{59}$$

where the second sum is over all reduced factorizations $uv = w_\lambda$. In any such factorization, the right factor v is equal to w_ν for some k-strict partition ν, and therefore $\mathfrak{C}_v(x'\,;y) = \Theta_\nu(x'\,;y)$. Since the $\Theta_\nu(x'\,;y)$ for ν a k-strict partition form a $\mathbb{Z}$-basis for the ring $\Gamma^{(k)}(x'\,;y)$ of theta polynomials in x' and y, the desired result follows immediately. $\square$

The Pieri rule (22) illustrates that the order relation on k-strict partitions given by the inclusion of Young diagrams is not compatible with the Bruhat order on B_∞. However, Proposition 4 shows that the *weak* Bruhat order on the k-Grassmannian elements of B_∞ respects the inclusion of k-strict diagrams.

Definition 5. Let λ and μ be k-strict partitions in $\mathcal{P}(k,n)$ with $\mu\subset\lambda$. We say that (λ,μ) is a *compatible pair* if there is a reduced word for w_λ whose last $|\mu|$ entries form a reduced word for w_μ; equivalently, if we have $\ell(w_\lambda w_\mu^{-1}) = |\lambda-\mu|$. In other words, (λ,μ) is a compatible pair if w_λ exceeds w_μ in the weak Bruhat order.

Corollary 8. *Let (λ,μ) be a compatible pair. Then there is a 1-1 correspondence between reduced factorizations of $w_\lambda w_\mu^{-1}$ and k-strict partitions ν with $\mu\subset\nu\subset\lambda$ such that (λ,ν) and (ν,μ) are compatible pairs.*

Proof. Suppose that $w_\lambda = ww_\mu$ and let $w = w'w''$ be a reduced factorization. Then $w''w_\mu$ is k-Grassmannian and hence equal to w_ν for a unique k-strict partition ν. Proposition 4 implies that $\mu\subset\nu\subset\lambda$, and clearly the pairs (λ,ν) and (ν,μ) are compatible. The converse is obvious. $\square$

Theorem 6. *Let λ and μ be k-strict partitions in $\mathcal{P}(k,n)$ with $\mu\subset\lambda$. Then the following conditions are equivalent:* (a) $F^{(k)}_{\lambda/\mu}(x)\neq 0$*;* (b) (λ,μ) *is a compatible pair;* (c) *there exists a standard k-tableau on λ/μ. If any of these conditions holds, then $F^{(k)}_{\lambda/\mu}(x) = F_{w_\lambda w_\mu^{-1}}(x)$.*

Proof. Equation (59) may be rewritten in the form

$$\sum_{\mu\subset\lambda} F^{(k)}_{\lambda/\mu}(x)\,\Theta_\mu(x'\,;y) = \sum_{\mu} F_{w_\lambda w_\mu^{-1}}(x)\,\Theta_\mu(x'\,;y) \tag{60}$$

where the second sum is over all $\mu\subset\lambda$ such that (λ,μ) is a compatible pair. It follows that

$$F^{(k)}_{\lambda/\mu}(x) = \begin{cases} F_{w_\lambda w_\mu^{-1}}(x) & \text{if } (\lambda,\mu) \text{ is a compatible pair,} \\ 0 & \text{otherwise} \end{cases}$$

and hence that (a) and (b) are equivalent. Suppose now that (λ,μ) is a compatible pair with $|\lambda| = |\mu|+1$, so that $w_\lambda = s_i w_\mu$ for some $i\geq 0$. Observe that if x is a single variable, then $F_{s_i}(x) = 2x$, and therefore $F^{(k)}_{\lambda/\mu}(x) = 2x\neq 0$. We deduce from Example 8(d) that λ/μ must be a k-horizontal strip. Using Corollary 8, it follows that there is a 1-1 correspondence between reduced words for $w_\lambda w_\mu^{-1}$ and sequences of k-strict partitions

$$\mu = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^r = \lambda$$

such that $|\lambda^i| = |\lambda^{i-1}|+1$ and λ^i/λ^{i-1} is a k-horizontal strip for $1 \leq i \leq r = |\lambda - \mu|$. The latter objects are exactly the standard k-tableaux on λ/μ. This shows that (b) implies (c), and the converse is also clear. □

The previous results show that the non-zero terms in equations (49) and (51) correspond exactly to the terms in equations (57) and (56), respectively, when $w = w_\lambda$ is a k-Grassmannian element of B_∞.

Corollary 9. *Let $w \in B_\infty$ be a skew element and (λ, μ) be a compatible pair such that $w_\lambda = ww_\mu$. Then the number of reduced words for w is equal to the number of standard k-tableaux on λ/μ and to the coefficient of $x_1x_2\cdots x_r$ in $2^{-r}F^{(k)}_{\lambda/\mu}(x)$, where $r = |\lambda - \mu| = \ell(w)$.*

Example 9. Let λ be a k-strict partition, λ^1 be defined as in §6.2, $\lambda^2 = \lambda \smallsetminus \lambda^1$, and $\mu \subset \lambda^2$. We can form a standard k-tableau on λ/μ by filling the boxes of λ^2/μ, going down the columns from left to right, and then filling the boxes of λ^1, going across the rows from top to bottom. If $\mu \subset \lambda$ but $\mu_1 > k$, this procedure does not always work, see for instance Example 8(c). When $k = 3$, $\lambda = (8, 6, 5, 2)$, and $\mu = \emptyset$, the 3-tableau on λ which results is

$$\begin{array}{cccccccc}
1 & 5 & 9 & 12 & 13 & 14 & 15 & 16 \\
2 & 6 & 10 & 17 & 18 & 19 & & \\
3 & 7 & 11 & 20 & 21 & & & \\
4 & 8 & & & & & &
\end{array}$$

which corresponds to the reduced word

$$s_1\, s_0\, s_2\, s_1\, s_0\, s_4\, s_3\, s_2\, s_1\, s_0\, s_3\, s_2\, s_1\, s_5\, s_4\, s_3\, s_2\, s_6\, s_5\, s_4\, s_3$$

for the Grassmannian element $w_\lambda = 167\overline{532}4 \in B_7$.

A sequence $(i_1, \ldots, i_m)$ is called *unimodal* if for some r with $0 \leq r \leq m$, we have $i_1 > i_2 > \cdots > i_r < i_{r+1} < \cdots < i_m$. An element $w \in B_\infty$ is unimodal if it has a reduced word $s_{i_1} \cdots s_{i_m}$ such that $(i_1, \ldots, i_m)$ is a unimodal sequence. A tableau T has *content* given by the composition α if α_i of the entries of T are equal to i, for each $i \geq 1$. We can now state the following generalization of Corollary 9.

Proposition 5. *Let $w \in B_\infty$ be a skew element and (λ, μ) be a compatible pair such that $w_\lambda = ww_\mu$. Then there is a 1-1 correspondence between reduced factorizations $u_1 \cdots u_r$ of w into unimodal elements u_i and k-tableaux T of shape λ/μ with r distinct entries. The lengths of the u_i agree with the content of T, and the number of reduced words for w obtained by concatenating unimodal reduced words for $u_1, \ldots, u_r$ of the corresponding lengths is equal to $2^{n(T)-r}$.*

Proof. From the definition of F_w in §6.1 it follows that if there is only one variable x and $w \neq 1$, then we have $F_w(x) = 2\, n_w\, x^{\ell(w)}$, where n_w denotes the number of unimodal reduced words for w. Moreover, for each $m \geq 1$ we have

$$F_w(x_1, \ldots, x_m) = \sum_{u_1\cdots u_m = w} F_{u_1}(x_1) \cdots F_{u_m}(x_m) \tag{61}$$

summed over all reduced factorizations $u_1 \cdots u_m$ for w. The result follows by comparing (61) with Definition 3 and using Example 8(d) and Corollary 8. □

Let $\lambda \in \mathcal{P}(k,n)$ and $w_\lambda \in B_n$ be the corresponding k-Grassmannian element. There are analogues of Corollary 9 and Proposition 5 for $\Theta_\lambda(x\,;y)$ and the k-bitableaux of shape λ. We say that a permutation $v \in S_n$ is *decreasing* if v has a reduced word $s_{i_1}\cdots s_{i_m}$ such that $i_1 > \cdots > i_m$. Then the k-bitableaux of shape λ correspond to reduced factorizations $u_1\cdots u_r v_1 \cdots v_s$ of w_λ with the $u_i \in B_n$ unimodal and the $v_j \in S_n$ decreasing. We leave the details to the reader.

According to [BH] and [L], the type C Stanley symmetric function F_w is a nonnegative integer linear combination of Schur Q-functions. Together with Theorem 6, this implies the following result.

Corollary 10. *For any two k-strict partitions λ, μ with $\mu \subset \lambda$, the function $F^{(k)}_{\lambda/\mu}(x)$ is a nonnegative integer linear combination of Schur Q-functions.*

6.4. We say that a permutation ϖ is *fully commutative* if any reduced word for ϖ can be obtained from any other by a sequence of braid relations that only involve commuting generators. It follows from [Ste, Thm. 4.2] that a permutation ϖ is fully commutative if and only if there exists a Grassmannian permutation ϖ_λ and a reduced factorization $\varpi_\lambda = \varpi\varpi'$ for some permutation ϖ'. In other words, the skew elements of the symmetric group are exactly the fully commutative elements.

Following [FS], the expansion of the formal product

$$A_1(x_1)A_1(x_2)\cdots = \sum_{\varpi\in S_n} G_\varpi(x)\, u_\varpi$$

may be used to define the type A Stanley symmetric functions $G_\varpi(x)$ for $\varpi \in S_n$. Stanley [Sta] introduced G_ϖ to study the set of reduced words for ϖ (he actually worked with $G_{\varpi^{-1}}$). It is shown in [BJS, §2] that if $\varpi \in S_n$ is fully commutative (or equivalently, *321-avoiding*) then $G_\varpi = s_{\lambda/\mu}$ is a skew Schur function, and the number of reduced words for ϖ is equal to the number of standard tableaux on λ/μ. Our present definition and study of skew elements in the hyperoctahedral group is therefore completely analogous to this established theory for the symmetric group.

Any fully commutative element of B_n, in the sense of [Ste], is a skew element. However the converse is emphatically false, for example the 1-Grassmannian element $w_{(4,1)} = 2\overline{3}1$ is not fully commutative. The three reduced words

$$s_1\, s_2\, s_1\, s_0\, s_1 \qquad s_2\, s_1\, s_2\, s_0\, s_1 \qquad s_2\, s_1\, s_0\, s_2\, s_1$$

for $2\overline{3}1$ correspond respectively to the three standard 1-tableaux

1234	1245	1345
5	3	2

on the diagram $\lambda = (4,1)$.

References

[BH] S. Billey and M. Haiman : *Schubert polynomials for the classical groups*, J. Amer. Math. Soc. **8** (1995), 443–482.

[BJS] S. Billey, W. Jockusch, and R. P. Stanley : *Some combinatorial properties of Schubert polynomials*, J. Algebraic Combin. **2** (1993), no. 4, 345–374.

[BKT1] A. S. Buch, A. Kresch, and H. Tamvakis : *Quantum Pieri rules for isotropic Grassmannians*, Invent. Math. **178** (2009), 345–405.

[BKT2] A. S. Buch, A. Kresch, and H. Tamvakis : *A Giambelli formula for isotropic Grassmannians*, Preprint (2008), available at arXiv:0811.2781.

[BKT3] A. S. Buch, A. Kresch, and H. Tamvakis : *Quantum Giambelli formulas for isotropic Grassmannians*, Preprint (2008), available at arXiv:0812.0970.

[FK1] S. Fomin and A. N. Kirillov : *The Yang-Baxter equation, symmetric functions, and Schubert polynomials*, Discrete Math. **153** (1996), 123–143.

[FK2] S. Fomin and A. N. Kirillov : *Combinatorial B_n-analogs of Schubert polynomials*, Trans. Amer. Math. Soc. **348** (1996), 3591–3620.

[FS] S. Fomin and R. P. Stanley : *Schubert polynomials and the nil-Coxeter algebra*, Adv. in Math. **103** (1994), 196–207.

[Fu] W. Fulton : *Young Tableaux*, Cambridge University Press, Cambridge, 1997.

[G] G. Z. Giambelli : *Risoluzione del problema degli spazi secanti*, Mem. R. Accad. Sci. Torino (2) **52** (1902), 171–211.

[HB] H. Hiller and B. Boe : *Pieri formula for SO_{2n+1}/U_n and Sp_n/U_n*, Adv. in Math. **62** (1986), 49–67.

[H] W. V. D. Hodge: *The intersection formulae for a Grassmannian variety*, J. London Math. Soc. **17** (1942), 48–64.

[L] T. K. Lam : *B_n Stanley symmetric functions*, Discrete Math. **157** (1996), 241–270.

[LLT] A. Lascoux, B. Leclerc, and J.-Y. Thibon: *Fonctions de Hall-Littlewood et polynômes de Kostka-Foulkes aux racines de l'unité*, C. R. Acad. Sci. Paris Sér. I Math. **316** (1993), no.1, 1–6.

[Li] D. E. Littlewood : *On certain symmetric functions*, Proc. Lond. Math. Soc. (3) **11** (1961), 485–498.

[M] I. Macdonald : *Symmetric Functions and Hall Polynomials*, Second edition, Clarendon Press, Oxford, 1995.

[Mo] A. O. Morris : *A note on the multiplication of Hall functions*, J. London Math. Soc. **39** (1964), 481–488.

[Pra] P. Pragacz : *Algebro-geometric applications of Schur S- and Q-polynomials*, Séminare d'Algèbre Dubreil-Malliavin 1989-1990, Springer Lecture Notes in Math. 1478 (1991), 130–191.

[PR] P. Pragacz and J. Ratajski : *Formulas for Lagrangian and orthogonal degeneracy loci; $\widetilde{Q}$-polynomial approach*, Compositio Math. **107** (1997), 11–87.

[Pi] M. Pieri : *Sul problema degli spazi secanti. Nota 1^a*, Rend. Ist. Lombardo (2) **26** (1893), 534–546.

[S] I. Schur : *Über die Darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen*, J. reine angew. Math. **139** (1911), 155–250.

[Sta] R. P. Stanley : *On the number of reduced decompositions of elements of Coxeter groups*, European J. Combin. **5** (1984), 359–372.

[Ste] J. R. Stembridge : *On the fully commutative elements of Coxeter groups*, J. Algebraic Combin. **5** (1996), 353–385.

[T] H. Tamvakis : *Quantum cohomology of isotropic Grassmannians*, Geometric Methods in Algebra and Number Theory, 311–338, Progress in Math. 235, Birkhäuser, 2005.

[W] D. R. Worley : *A theory of shifted Young tableaux*, Ph.D. thesis, MIT, 1984.

[Y] A. Young : *On quantitative substitutional analysis VI*, Proc. Lond. Math. Soc. (2) **34** (1932), 196–230.

University of Maryland, Department of Mathematics, 1301 Mathematics Building, College Park, MD 20742, USA

E-mail address: `harryt@math.umd.edu`