**Quantum cohomology of homogeneous varieties: a survey**

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Let $G$ be a semisimple complex algebraic group and $P$ a parabolic subgroup of $G$. The homogeneous space $X = G/P$ is a projective complex manifold. My aim in this lecture is to survey what is known about the (small) quantum cohomology ring of $X$. Here is a brief historical introduction, with no claim of completeness. About 15 years ago, ideas from string theory and mirror symmetry led physicists to make some startling predictions in enumerative algebraic geometry (see e.g. [18, 19]). This involved the notion of *Gromov-Witten invariants*, which are certain natural intersection numbers on the moduli space of degree $d$ holomorphic maps from a compact complex curve $C$ of genus $g$ (with $n$ marked points) to $X$.

When the genus $g$ is arbitrary, computing these invariants is a rather difficult problem. The case when $X$ is a point was a conjecture of Witten, proved by Kontsevich. Later, Okounkov and Pandharipande examined the case when $X = \mathbb{P}^1$. The genus zero theory led to the so called big quantum cohomology ring, and to work on mirror symmetry by Givental, Yau, and their collaborators. I will specialize further to the case of $n = 3$ marked points, when we obtain the small quantum cohomology ring $\mathrm{QH}^*(X)$. Although much has been understood here, still many open questions remain.

## 1. Cohomology of $G/B$ and $G/P$

We begin with the Bruhat decomposition $G = \bigcup_{w\in W} BwB$, where $B$ is a Borel subgroup of $G$, and $W$ is the Weyl group. The Schubert cells in $X = G/B$ are the orbits of $B$ on $X$; their closures $Y_w = \overline{BwB/B}$ are the Schubert varieties. For each $w \in W$, let $w^\vee = w_0 w$ and $X_w = Y_{w^\vee}$, so that the complex codimension of $X_w$ is given by the length $\ell(w)$. Using Poincaré duality, we obtain the Schubert classes $\sigma_w = [X_w] \in \mathrm{H}^{2\ell(w)}(X)$, which form a free $\mathbb{Z}$-basis of $\mathrm{H}^*(X)$. This gives the additive structure of the cohomology ring.

For the multiplicative structure, if the group $W$ is generated by the simple reflections $s_i$ for $1 \leq i \leq r$, we obtain the Schubert divisor classes $\sigma_{s_i} \in \mathrm{H}^2(X)$ which generate the ring $\mathrm{H}^*(X)$. Moreover, we have Borel's presentation [2]

$$\mathrm{H}^*(G/B,\mathbb{Q}) = \mathrm{Sym}(\Lambda(B))/\mathrm{Sym}(\Lambda(B))^W_{>0}$$

where $\Lambda(B)$ denotes the character group of $B$, and $\mathrm{Sym}(\Lambda(B))^W_{>0}$ is the ideal generated by $W$-invariants of positive degree in the symmetric algebra of $\Lambda(B)$.

For any parabolic subgroup $P$, if $W_P$ is the corresponding subgroup of the Weyl group $W$, we have $\mathrm{H}^*(G/P) = \bigoplus \mathbb{Z}\,\sigma_{[w]}$, the sum over all cosets $[w] \in W/W_P$. The corresponding Borel presentation has the form

$$\mathrm{H}^*(G/P,\mathbb{Q}) = \mathrm{Sym}(\Lambda(B))^{W_P}/\mathrm{Sym}(\Lambda(B))^W_{>0}.$$

## 2. Quantum cohomology of $G/B$ and $G/P$

Let $r$ be the rank of $\mathrm{H}^2(G/P)$, and $q = (q_1, \ldots, q_r)$ a finite set of formal variables. The ring $\mathrm{QH}^*(X)$ is a graded $\mathbb{Z}[q]$-algebra which is isomorphic to $\mathrm{H}^*(X) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$ as a module over $\mathbb{Z}[q]$. The degree of each variable $q_i$ is given by $\deg(q_i) = \int_X \sigma_{s_i^\vee} \cdot c_1(TX)$. Note that our grading of cohomology classes will be with respect to their *complex* codimension.

A holomorphic map $f : \mathbb{P}^1 \to X$ has degree $d = (d_1, \ldots, d_r)$ if $f_*[\mathbb{P}^1] = \sum_i d_i \sigma_{s_i^\vee}$ in $\mathrm{H}_2(X)$. The quantum product in $\mathrm{QH}^*(X)$ is defined by

$$\sigma_u \, \sigma_v = \sum \langle \sigma_u, \sigma_v, \sigma_{w^\vee} \rangle_d \, \sigma_w \, q^d \tag{1}$$

where the sum is over $d \geq 0$ and elements $w \in W$ such that $\ell(w) = \ell(u) + \ell(v) - \sum d_i \deg(q_i)$. The nonnegative integer $\langle \sigma_u, \sigma_v, \sigma_{w^\vee} \rangle_d$ is a 3-point, genus 0 Gromov-Witten invariant, and can be defined enumeratively as the number of degree $d$ holomorphic maps $f : \mathbb{P}^1 \to X$ such that $f(0) \in \widetilde{X}_u$, $f(1) \in \widetilde{X}_v$, $f(\infty) \in \widetilde{X}_{w^\vee}$, where the tilde in $\widetilde{X}_u$, $\widetilde{X}_v$, $\widetilde{X}_{w^\vee}$ means that the respective Schubert varieties are taken to be in general position. In most cases, counting the number of such maps $f$ is equivalent to counting their images, which are degree $d$ rational curves in $X$.

Alternatively, one may realize $\langle \sigma_u, \sigma_v, \sigma_{w^\vee} \rangle_d$ as

$$\langle \sigma_u, \sigma_v, \sigma_{w^\vee} \rangle_d = \int_{\overline{M}_{0,3}(X,d)} \mathrm{ev}_1^*(\sigma_u) \, \mathrm{ev}_2^*(\sigma_v) \, \mathrm{ev}_3^*(\sigma_{w^\vee}),$$

an intersection number on Kontsevich's moduli space $\overline{M}_{0,3}(X,d)$ of stable maps. A stable map is a degree $d$ morphism $f : (C, p_1, p_2, p_3) \to X$, where $C$ is a tree of $\mathbb{P}^1$'s with three marked smooth points $p_1$, $p_2$, and $p_3$, and the stability condition is such that the map $f$ admits no automorphisms. The evaluation maps $\mathrm{ev}_i : \overline{M}_{0,3}(X,d) \to X$ are given by $\mathrm{ev}_i(f) = f(p_i)$.

One observes that each degree zero Gromov-Witten invariant

$$\langle \sigma_u, \sigma_v, \sigma_{w^\vee} \rangle_0 = \# \, \widetilde{X}_u \cap \widetilde{X}_v \cap \widetilde{X}_{w^\vee} = \int_X \sigma_u \, \sigma_v \, \sigma_{w^\vee}$$

is a classical structure constant in the cohomology ring of $X$, showing that $\mathrm{QH}^*(X)$ is a deformation of $\mathrm{H}^*(X)$. The surprising point is that the product (1) is associative; see [9] for a proof of this. We will be interested in extending the classical understanding of Schubert calculus on $G/P$ to the quantum cohomology ring.

## 3. The Grassmannian $G(m, N)$

One of the first spaces where this story was worked out was the Grassmannian $X = G(m,N) = SL_N/P_m$ of $m$ dimensional linear subspaces of $\mathbb{C}^N$. Here the Weyl groups $W = S_N$, $W_{P_m} = S_m \times S_n$, where $n = N - m$, and there is a bijection between the coset space $W/W_P$ and the set of partitions $\lambda = (\lambda_1, \ldots, \lambda_m)$ whose Young diagram is contained in an $m \times n$ rectangle $R(m,n)$. The latter objects will index the Schubert classes in $X$. Since $\mathrm{H}^2(X)$ has rank one, there is only one deformation parameter $q$, of degree $N$ in $\mathrm{QH}^*(X)$.

3.1. **Presentation** [17, Siebert and Tian]**.** There is a universal short exact sequence of vector bundles

$$0 \to S \to E \to Q \to 0$$

over $G(m,N)$, with $S$ the tautological rank $m$ subbundle of the trivial vector bundle $E$, and $Q = E/S$ the rank $n$ quotient bundle. Then

$$\mathrm{QH}^*(G(m,N)) = \mathbb{Z}[c(S), c(Q), q] / \langle c(S)c(Q) = 1 + (-1)^m q \rangle$$
$$= \mathbb{Z}[x_1, \ldots, x_m, y_1, \ldots, y_n, q]^{S_m \times S_n} / \langle e_r(x,y) = 0, r < N;\ e_N(x,y) = (-1)^m q \rangle .$$

The new relation $c_m(S^*)c_n(Q) = q$ is equivalent to $\sigma_{1^m}\sigma_n = q$, and contains the enumerative geometric statement that $\langle \sigma_{1^m}, \sigma_n, [\mathrm{pt}] \rangle_1 = 1$. This latter can be checked directly from geometry, or deduced from the Pieri rule which follows.

3.2. **Quantum Pieri rule** [1, Bertram]**.** The special Schubert classes $\sigma_p = c_p(Q)$ for $1 \le p \le n$ generate the ring $\mathrm{QH}^*(X)$. Moreover, we have

$$\sigma_p\,\sigma_\lambda = \sum_{\mu \subset R(m,n)} \sigma_\mu + \sum_{\mu \subset R(m+1,n)} \sigma_{\widehat{\mu}}\, q,$$

where both sums are over $\mu$ obtained from $\lambda$ by adding $p$ boxes, no two in a column, and $\widehat{\mu}$ is obtained from $\mu$ by removing a hook of length $N$ from its rim. This means that the only $\mu \subset R(m+1,n)$ that contribute to the second sum are those which include the northeast-most and southwest-most corners in their diagram. For example, in $X = G(3,8)$, we have $\sigma_3\,\sigma_{422} = \sigma_{542} + \sigma_{21}\,q + \sigma_{111}\,q$.

3.3. **Quantum Littlewood-Richardson numbers.** These are the Gromov-Witten invariants in the equation

$$\sigma_\lambda\,\sigma_\mu = \sum_{d,\nu} C_{\lambda,\mu}^{\nu,d}\,\sigma_\nu\,q^d \tag{2}$$

in $\mathrm{QH}^*(G(m,N))$. The quantum Pieri rule gives an algorithm to compute the quantum Littlewood-Richardson numbers $C_{\lambda,\mu}^{\nu,d}$, however not a positive combinatorial rule extending the classical one. A puzzle based conjectural rule for these numbers was given by Buch, Kresch, and the author [4], and recently a 'geometric' and positive combinatorial rule was proved by Coskun.

As one of the many combinatorial offshoots of this theory, I mention a clever reformulation of the algorithm determining the numbers $C_{\lambda,\mu}^{\nu,d}$ due to Postnikov [15]. When $d = 0$, if $s_\mu(x_1, \ldots, x_m)$ denotes the Schur polynomial in $m$ variables, and we alter the summation in (2) to be over $\mu$ instead of $\nu$, then we get

$$\sum_\mu C_{\lambda,\mu}^{\nu,0}\,s_\mu(x_1, \ldots, x_m) = \sum_{T \text{ on } \nu/\lambda} x^T$$

where the second sum is over all semistandard Young tableaux $T$ on the skew shape $\nu/\lambda$ with entries no greater than $m$. For each fixed $d \ge 0$, Postnikov defines a *toric shape* $\nu/d/\lambda$ which is a subset of the torus $T(m,n)$, the rectangle $R(m,n)$

with opposite sides identified; the role of $d$ in the description of the shape is a shift by $d$ squares in the southeast direction. One then shows that

$$\sum_{\mu} C^{\nu,d}_{\lambda,\mu}\, s_\mu(x_1,\ldots,x_m) = \sum_{T \text{ on } \nu/d/\lambda} x^T,$$

the second sum over all Young tableaux on the toric shape $\nu/d/\lambda$. One nice application of this result is determining exactly which powers $q^d$ occur in a quantum product $\sigma_\lambda\, \sigma_\mu$ with a non-zero coefficient.

## 4. Flag varieties for $SL_n$

We set $X = SL_n/B$ to be the complex manifold parametrizing complete flags of linear subspaces $0 = E_0 \subset E_1 \subset \cdots \subset E_n = \mathbb{C}^n$, with $\dim E_i = i$. We then have $\mathrm{QH}^*(X) = \bigoplus \mathbb{Z}\, \sigma_w\, q^d$, the sum over permutations $w \in S_n$ and multidegrees $d$, while $q^d = q_1^{d_1} \cdots q_{n-1}^{d_{n-1}}$, with each variable $q_i$ of degree 2.

4.1. **Presentation** [11, Givental and Kim]**.** Let $E_i$ also denote the corresponding tautological vector bundle over $X$, and $x_i = -c_1(E_i/E_{i-1})$. The Borel presentation of $\mathrm{H}^*(X)$ is a quotient of $\mathbb{Z}[x_1,\ldots,x_n]$ by the ideal generated by the elementary symmetric polynomials $e_i(x_1,\ldots,x_n)$ for $1 \le i \le n$. For the quantum cohomology ring, we have

$$\mathrm{QH}^*(X) = \mathbb{Z}[x_1,\ldots,x_n,q_1,\ldots,q_{n-1}]/\langle E_i(x,q) = 0,\ 1 \le i \le n\rangle$$

where the *quantum elementary symmetric polynomials* $E_i(x,q)$ are the coefficients of the characteristic polynomial

$$\det(A + tI_n) = \sum_{i=0}^{n} E_i(x,q)t^{n-i}$$

of the matrix

$$A = \begin{pmatrix} x_1 & q_1 & 0 & \cdots & 0 \\ -1 & x_2 & q_2 & \cdots & 0 \\ 0 & -1 & x_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & x_n \end{pmatrix}.$$

4.2. **Quantum Monk/Chevalley formula** [8, Fomin, Gelfand, and Postnikov]**.** This is a formula for the quantum product $\sigma_{s_i}\, \sigma_w$. It was extended by Peterson to any $G/B$; see section 5.2.

4.3. **Quantum cohomology of $SL_n/P$.** Ciocan-Fontanine [7] obtained analogues of the above results for any homogeneous space for $SL_n$. We remark that quantum cohomology is *not* functorial, and so one has to work on each parabolic subgroup $P$ separately. The conclusion of this discussion is that the quantum cohomology of $SL_n$ flag varieties is fairly well understood; one can also recognize each Schubert class $\sigma_w$ in the presentation of $\mathrm{QH}^*(X)$ using *quantum Schubert polynomials* [8].

## 5. Lie types other than A

5.1. **General $G/B$.** A presentation of $\mathrm{QH}^*(G/B)$ for general $G$ was given by Kim[12]. It is notable because the relations come from the integrals of motion of the Toda lattice associated to the Langlands dual group $G^\vee$. In his 1997 MIT lectures, D. Peterson announced a presentation of $\mathrm{QH}^*(G/P)$ for any parabolic subgroup $P$ of $G$. This result remains unpublished; moreover, it is difficult to relate Peterson's presentation to the Borel presentation of $\mathrm{H}^*(G/P)$ given earlier. For work in this direction when $G = SL_n$, see Rietsch [16], which includes a connection with the theory of total positivity. Recently, Cheong [6] has made a corresponding study of the Grassmannians LG and OG of maximal isotropic subspaces.

5.2. **Peterson's quantum Chevalley formula** [10]**.** Let $R$ be the root system for $G$ and $R^+$ the positive roots. For $\alpha \in R^+$ we denote by $s_\alpha$ the corresponding reflection in $W$. To any root $\alpha$ there corresponds the coroot $\alpha^\vee = 2\alpha/\langle\alpha,\alpha\rangle$ in the Cartan subalgebra of Lie($G$). For any positive coroot $\alpha^\vee$ with $\alpha^\vee = d_1\alpha_1^\vee + \cdots + d_r\alpha_r^\vee$, define $|\alpha^\vee| = \sum_i d_i$ and $q^{\alpha^\vee} = \prod_i q_i^{d_i}$. Then we have

$$\sigma_{s_i}\cdot\sigma_w = \sum_{\ell(ws_\alpha)=\ell(w)+1} \langle\omega_i,\alpha^\vee\rangle\,\sigma_{ws_\alpha} + \sum_{\ell(ws_\alpha)=\ell(w)-2|\alpha^\vee|+1} \langle\omega_i,\alpha^\vee\rangle\,\sigma_{ws_\alpha}\,q^{\alpha^\vee}$$

in $\mathrm{QH}^*(G/B)$, where the sums are over $\alpha \in R^+$ satisfying the indicated conditions, and $\omega_i$ is the fundamental weight corresponding to $s_i$. Using this result, one can recursively compute the Gromov-Witten invariants on any $G/B$ space.

5.3. **Peterson's comparison theorem** [20]**.** Every Gromov-Witten invariant $\langle\sigma_u,\sigma_v,\sigma_w\rangle_d$ on $G/P$ is equal to a corresponding number $\langle\sigma_{u'},\sigma_{v'},\sigma_{w'}\rangle_{d'}$ on $G/B$. The exact relationship between the indices is explicit, but not so easy to describe; see [20] for further details and a complete proof. Combining this with the previous result allows one to compute any Gromov-Witten invariant on any $G/P$ space.

5.4. **Grassmannians in other Lie types** [13, 14, Kresch and T.]**.** Let $X = Sp_{2n}/P_n$ be the Grassmannian $\mathrm{LG}(n,2n)$ parametrizing Lagrangian subspaces of $\mathbb{C}^{2n}$ equipped with a symplectic form. The Schubert varieties on LG are indexed by strict partitions $\lambda$ with $\lambda_1 \le n$, and the degree of $q$ this time is $n+1$.

5.4.1. *Presentation of* $\mathrm{QH}^*(\mathrm{LG})$. If $0 \to S \to E \to Q \to 0$ denotes the tautological sequence of vector bundles over LG, then we may identify $Q$ with $S^*$, and the special Schubert classes $\sigma_p = c_p(S^*)$ again generate the ring $\mathrm{QH}^*(\mathrm{LG})$. The Whitney sum formula $c_t(S)c_t(S^*) = 1$ gives the classical relations

$$(1 - \sigma_1 t + \sigma_2 t^2 - \cdots)(1 + \sigma_1 t + \sigma_2 t^2 + \cdots) = 1$$

or equivalently $\sigma_r^2 + 2\sum_{i=1}^{n-r}(-1)^i\sigma_{r+i}\sigma_{r-i} = 0$ for $1 \le r \le n$. For the quantum ring, we have the presentation

$$\mathrm{QH}^*(\mathrm{LG}) = \mathbb{Z}[\sigma_1,\ldots,\sigma_n,q]/\langle\sigma_r^2 + 2\sum_{i=1}^{n-r}(-1)^i\sigma_{r+i}\sigma_{r-i} = (-1)^{n-r}\sigma_{2r-n-1}q\rangle.$$

Observe that if we identify $q$ with $2\sigma_{n+1}$, the above equations become classical relations in the cohomology of $\mathrm{LG}(n+1,2n+2)$. Looking for the enumerative geometry which lies behind this algebraic fact, we find that

$$\langle \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle_1 = \int_{\mathrm{IG}(n-1,2n)} \sigma_\lambda^{(1)}\, \sigma_\mu^{(1)}\, \sigma_\nu^{(1)} = \frac{1}{2} \int_{\mathrm{LG}(n+1,2n+2)} \sigma_\lambda\, \sigma_\mu\, \sigma_\nu .$$

The first equality is an example of a "quantum = classical" result; here the isotropic Grassmannian $\mathrm{IG}(n-1,2n) = Sp_{2n}/P_{n-1}$ is the parameter space of lines on $\mathrm{LG}(n,2n)$, and $\sigma_\lambda^{(1)}$, $\sigma_\mu^{(1)}$, $\sigma_\nu^{(1)}$ are certain Schubert classes in $\mathrm{H}^*(\mathrm{IG})$.

5.4.2. *Symmetries of Gromov-Witten invariants.* Kresch and the author [14] also studied the quantum cohomology of the maximal orthogonal Grassmannians $\mathrm{OG} = \mathrm{OG}(n,2n+1) = SO_{2n+1}/P_n$. There are quantum Pieri rules for LG and OG which extend the known ones in classical cohomology. Using them, one shows that the Gromov-Witten invariants on these spaces enjoy a $(\mathbb{Z}/2\mathbb{Z})^3$-symmetry, which implies that the tables of Gromov-Witten invariants for $\mathrm{LG}(n-1,2n-2)$ and $\mathrm{OG}(n,2n+1)$ coincide, after applying an involution. Similar symmetries were observed by Postnikov [15] and others for type A Grassmannians; recently, Chaput, Manivel, and Perrin have extended them to all hermitian symmetric spaces.

The original proofs of all the above results relied on intersection theory on $\overline{M}_{0,3}(X,d)$ or Quot schemes. A technical breakthrough was found by Buch [3]; his 'Ker/Span' ideas greatly simplified most of the arguments involved. Using this approach, Buch, Kresch, and the author have made a corresponding analysis of $\mathrm{QH}^*(G/P)$ when $G$ is a classical group and $P$ any maximal parabolic subgroup.

## 6. "Quantum = Classical" results

The title refers to theorems which equate any Gromov-Witten invariant on a hermitian symmetric Grassmannian with a classical triple intersection number on a related homogeneous space. These results were discovered in joint work of the author with Buch and Kresch [4]. More recently, Chaput, Manivel, and Perrin [5] have presented the theory in a uniform framework which includes the exceptional symmetric spaces $E_6/P_6$ and $E_7/P_7$. There follows a summary of this story.

Assume that $X = G/P$ is a hermitian symmetric space. For $x, y \in X$, let $\delta(x,y)$ be the minimum $d \geq 0$ such that there exists a rational curve of degree $d$ passing through the points $x$ and $y$. The invariant $\delta(x,y)$ parametrizes the $G$ orbits in $X \times X$. If $\delta(x,y) = d$, then define $Z(x,y) = \overline{\bigcup C_{x,y}}$, where the union is over all rational curves $C_{x,y}$ of degree $d$ through the points $x$ and $y$. Then $Z(x,y)$ is a homogeneous Schubert variety $X_{w_d}$ in $X$. Now $G$ acts transitively on the set of translates $\{gX_{w_d} \mid g \in G\}$; therefore the variety $Y_d$ parametrizing all such $X_{w_d}$ in $X$ is a homogeneous space $G/P_d$ for some (generally non maximal) parabolic subgroup $P_d$ of $G$. To each Schubert class $\sigma_\lambda$ in $\mathrm{H}^*(X)$ there corresponds naturally a Schubert class $\sigma_\lambda^{(d)}$ in $\mathrm{H}^*(Y_d)$. Then we have

$$\langle \sigma_\lambda, \sigma_\mu, \sigma_\nu \rangle_d = \int_{Y_d} \sigma_\lambda^{(d)}\, \sigma_\mu^{(d)}\, \sigma_\nu^{(d)} .$$

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