SCHUBERT POLYNOMIALS, THETA AND ETA POLYNOMIALS, AND WEYL GROUP INVARIANTS

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In memory of Alain Lascoux

Abstract. We examine the relationship between the (double) Schubert polynomials of Billey-Haiman and Ikeda-Mihalcea-Naruse and the (double) theta and eta polynomials of Buch-Kresch-Tamvakis and Wilson from the perspective of Weyl group invariants. We obtain generators for the kernel of the natural map from the corresponding ring of Schubert polynomials to the (equivariant) cohomology ring of symplectic and orthogonal flag manifolds.

1. Introduction

The theory of Schubert polynomials due to Lascoux and Schützenberger [LS1] provides a canonical set of polynomial representatives for the Schubert classes on complete type A flag manifolds. The classical Schur polynomials are identified with the Schubert polynomials representing the classes which pull back from Grassmannians. There are natural analogues of these objects for the symplectic and orthogonal groups: the Schubert polynomials of Billey and Haiman [BH], and the theta and eta polynomials of Buch, Kresch, and the author [BKT1, BKT2], respectively. We also have 'double' versions of the aforementioned polynomials, which represent the Schubert classes in the torus-equivariant cohomology ring, and in the setting of degeneracy loci of vector bundles (see [L2, F1, IMN1] and [KL, L1, W, IM, TW, T6], respectively).

The goal of this work is to study the relation between these two families of polynomials from the point of view of Weyl group symmetries, following the program set out in [LS1, LS2, M2] in Lie type A. The key observation is that the theta and eta polynomials of a fixed level n form a basis of the Weyl group invariants in the associated ring of Schubert polynomials (Propositions 6 and 14). In this introduction, for simplicity, we review the story in type A, and describe its analogue in type C, in the case of 'single' polynomials, leaving the extensions to the 'double' case and the orthogonal Lie types B and D to the main body of the paper.

Let $S_\infty := \cup_k S_k$ be the group of permutations of the positive integers which leave all but a finite number of them fixed. For any $n \geq 1$, let $S^{(n)}$ denote the set of those permutations $\varpi = (\varpi_1, \varpi_2, \dots)$ in S_∞ such that $\varpi_{n+1} < \varpi_{n+2} < \cdots$. If $X_n := (x_1, \dots, x_n)$ is a family of n commuting independent variables, then the single Schubert polynomials $\mathfrak{S}_\varpi$ of Lascoux and Schützenberger [LS1], as ϖ ranges over

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$S^{(n)}$, form a $\mathbb{Z}$-basis of the polynomial ring $\mathbb{Z}[X_n]$. If $M_n := \mathrm{GL}_n/B$ denotes the complete type A flag manifold over $\mathbb{C}$, then there is a surjective ring homomorphism $\rho_n : \mathbb{Z}[X_n] \to \mathrm{H}^*(M_n)$ which maps the polynomial $\mathfrak{S}_\varpi$ to the cohomology class $[X_\varpi]$ of a codimension $\ell(\varpi)$ Schubert variety X_ϖ in M_n, if $\varpi \in S_n$, and to zero, otherwise.

The Weyl group S_n acts on $\mathbb{Z}[X_n]$ by permuting the variables, and the subring $\mathbb{Z}[X_n]^{S_n}$ of S_n-invariants is the ring Λ_n of symmetric polynomials in $x_1, \ldots, x_n$. We have $\Lambda_n = \mathbb{Z}[e_1(X_n), \ldots, e_n(X_n)]$, where $e_i(X_n)$ denotes the i-th elementary symmetric polynomial. The kernel of ρ_n is the ideal $\mathrm{I}\Lambda_n$ of $\mathbb{Z}[X_n]$ generated by the homogeneous elements of positive degree in Λ_n. We therefore have

$$\mathrm{I}\Lambda_n = \bigoplus_{\varpi \in S^{(n)} \smallsetminus S_n} \mathbb{Z}\, \mathfrak{S}_\varpi = \langle e_1(X_n), \ldots, e_n(X_n) \rangle$$

and recover the Borel presentation [Bo] of the cohomology ring

$$\mathrm{H}^*(\mathrm{GL}_n/B) \cong \mathbb{Z}[X_n]/\mathrm{I}\Lambda_n.$$

Any Schubert polynomial $\mathfrak{S}_\varpi$ which lies in Λ_n is equal to a Schur polynomial $s_\lambda(X_n)$ indexed by a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ associated to the Grassmannian permutation ϖ, and these elements form a $\mathbb{Z}$-basis of Λ_n. One knows that

$$s_\lambda(X_n) = A(x^{\lambda+\delta_{n-1}}) \big/ A(x^{\delta_{n-1}}), \tag{1}$$

where $A := \sum_{\varpi \in S_n} (-1)^{\ell(\varpi)} \varpi$ is the alternating operator on $\mathbb{Z}[X_n]$, x^α denotes $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for any integer vector exponent α, and $\delta_k := (k, \ldots, 1, 0)$ for every $k \geq 0$. Equation (1) may be identified with the Weyl character formula for GL_n. Alternatively, one has the (dual) Jacobi-Trudi identity

$$s_\lambda(X_n) = \det(e_{\lambda'_i+j-i}(X_n))_{i,j} = \prod_{i<j}(1 - R_{ij})\, e_{\lambda'}(X_n), \tag{2}$$

where λ' is the conjugate partition of λ, $e_\nu := e_{\nu_1} e_{\nu_2} \cdots$ for any integer vector ν, and the R_{ij} are Young's raising operators, with $R_{ij} e_\nu := e_{R_{ij}\nu}$ (see [M1, I.3]).

If ϖ_0 denotes the longest permutation in S_n, then the divided difference operator ∂_{ϖ_0} gives a Λ_n-linear map $\mathbb{Z}[X_n] \to \Lambda_n$, and the equation $\langle f, g \rangle = \partial_{\varpi_0}(fg)$ defines a scalar product on $\mathbb{Z}[X_n]$, with values in Λ_n. The Schubert polynomials $\{\mathfrak{S}_\varpi\}_{\varpi \in S_n}$ form a basis for $\mathbb{Z}[X_n]$ as a Λ_n-module, and satisfy an orthogonality property under this product, which corresponds to the natural duality pairing on $\mathrm{H}^*(M_n)$.

The above narrative admits an exact analogue for the symplectic group. Let $c := (c_1, c_2, \ldots)$ be a sequence of commuting variables, and set $c_0 := 1$ and $c_p = 0$ for $p < 0$. Consider the graded ring Γ which is the quotient of the polynomial ring $\mathbb{Z}[c]$ modulo the ideal generated by the relations

$$c_p^2 + 2\sum_{i=1}^{p} (-1)^i c_{p+i} c_{p-i} = 0, \quad \text{for all } p \geq 1. \tag{3}$$

The ring Γ is isomorphic to the ring of Schur Q-functions [M1, III.8] and to the stable cohomology ring of the Lagrangian Grassmannian, following [P, J].

Let W_k denote the hyperoctahedral group of signed permutations on the set $\{1, \ldots, k\}$. For each $k \geq 1$, we embed W_k in W_{k+1} by adjoining the fixed point $k+1$, and set $W_\infty := \cup_k W_k$. For any $n \geq 0$, let $W^{(n)}$ denote the set of those elements $w = (w_1, w_2, \ldots)$ in W_∞ such that $w_{n+1} < w_{n+2} < \cdots$. The type C single Schubert polynomials $\mathfrak{C}_w$ of Billey and Haiman [BH], as w ranges over $W^{(n)}$, form a

$\mathbb{Z}$-basis of the ring $\Gamma[X_n]$ (Proposition 1)[1]. If $\mathcal{M}_n := \mathrm{Sp}_{2n}/B$ denotes the complete symplectic flag manifold over $\mathbb{C}$, then there is a surjective ring homomorphism $\pi_n : \Gamma[X_n] \to \mathrm{H}^*(\mathcal{M}_n)$ which maps the polynomial $\mathfrak{C}_w$ to the class $[X_w]$ of a codimension $\ell(w)$ Schubert variety X_w in $\mathcal{M}_n$, if $w \in W_n$, and to zero, otherwise.

There is a natural action of the Weyl group W_n on $\Gamma[X_n]$ which extends the S_n action on $\mathbb{Z}[X_n]$ (see Section 2.1). The subring $\Gamma[X_n]^{W_n}$ of W_n-invariants is the ring $\Gamma^{(n)}$ of *theta polynomials of level* n (Proposition 6). The ring $\Gamma^{(n)}$ was defined in [BKT1, Section 5.1] as $\Gamma^{(n)} := \mathbb{Z}[{}^nc_1, {}^nc_2, \ldots]$, where

$$ {}^nc_p := \sum_{j=0}^{p} c_{p-j} e_j(X_n), \quad \text{for } p \geq 1. $$

The kernel of π_n is the ideal $\mathrm{I}\Gamma^{(n)}$ of $\Gamma[X_n]$ generated by the homogeneous elements of positive degree in $\Gamma^{(n)}$. We therefore have

$$ \mathrm{I}\Gamma^{(n)} = \bigoplus_{w \in W^{(n)} \smallsetminus W_n} \mathbb{Z}\,\mathfrak{C}_w = \langle {}^nc_1, {}^nc_2, \ldots \rangle \tag{4} $$

and obtain (Corollary 3) a canonical isomorphism

$$ \mathrm{H}^*(\mathrm{Sp}_{2n}/B) \cong \Gamma[X_n]/\mathrm{I}\Gamma^{(n)}. $$

Following [BKT1], any Schubert polynomial $\mathfrak{C}_w$ which lies in $\Gamma^{(n)}$ is equal to a *theta polynomial* ${}^n\Theta_\lambda$ indexed by an n-strict partition $\lambda = \lambda(w)$ associated to the n-Grassmannian element w, and these polynomials form a $\mathbb{Z}$-basis of $\Gamma^{(n)}$. The polynomial ${}^n\Theta_\lambda$ was defined in [BKT1] using the raising operator formula

$$ {}^n\Theta_\lambda := \prod_{i<j}(1 - R_{ij}) \prod_{(i,j)\in\mathcal{C}(\lambda)} (1 + R_{ij})^{-1}\, ({}^nc)_\lambda, \tag{5} $$

where $({}^nc)_\nu := {}^nc_{\nu_1}{}^nc_{\nu_2}\cdots$ and $\mathcal{C}(\lambda)$ denotes the set of pairs (i,j) with $i < j$ and $\lambda_i + \lambda_j > 2n + j - i$. This is the symplectic version of equation (2).

There is also a symplectic analogue of formula (1). Let w_0 denote the longest element of W_n, define $\widehat{w} := ww_0$ and consider the *multi-Schur Pfaffian*

$$ {}^{\nu(\widehat{w})}Q_{\lambda(\widehat{w})} := \prod_{i<j} \frac{1 - R_{ij}}{1 + R_{ij}}\, {}^{\nu(\widehat{w})}c_{\lambda(\widehat{w})}, \tag{6} $$

where $\nu(\widehat{w})$ and $\lambda(\widehat{w})$ are certain partitions associated to $\widehat{w}$ (see (13) and Definition 2). We then have (Theorem 2)

$$ {}^n\Theta_{\lambda(w)} = (-1)^{n(n+1)/2}\, \mathcal{A}\left({}^{\nu(\widehat{w})}Q_{\lambda(\widehat{w})}\right) \Big/ \mathcal{A}\left(x^{\delta_n + \delta_{n-1}}\right), \tag{7} $$

where $\mathcal{A} := \sum_{w \in W_n} (-1)^{\ell(w)} w$ is the alternating operator on $\Gamma[X_n]$. In the special case when $w \in S_\infty$, with $\lambda = \lambda(w)$, equation (7) becomes

$$ {}^n\Theta_\lambda = (-1)^{n(n+1)/2}\, \mathcal{A}\left({}^{\delta_{n-1}}Q_{\delta_n+\delta_{n-1}+\lambda'}\right) \big/ \mathcal{A}\left(x^{\delta_n+\delta_{n-1}}\right). $$

The maximal divided difference operator ∂_{w_0} gives a $\Gamma^{(n)}$-linear map $\Gamma[X_n] \to \Gamma^{(n)}$, and the equation $\langle f, g\rangle = \partial_{w_0}(fg)$ defines a scalar product on $\Gamma[X_n]$, with values in $\Gamma^{(n)}$. The Schubert polynomials $\{\mathfrak{C}_w\}_{w\in W_n}$ form a basis for $\Gamma[X_n]$ as a $\Gamma^{(n)}$-module (Corollary 4), and satisfy an orthogonality property under this product (Proposition 9), which corresponds to the duality pairing on $\mathrm{H}^*(\mathcal{M}_n)$. A similar scalar product in the finite case was introduced and studied in [LP1].

[1] The Billey-Haiman Schubert polynomials are actually power series; see Section 2.1

As mentioned earlier, we provide analogues of most of the above facts for the double Schubert, theta, and eta polynomials. Our main new results (Theorems 1 and 3) are the double versions of equation (4), which exhibit natural generators for the kernel of the *geometrization map* of [IMN1, Section 10] from the (stable) ring of double Schubert polynomials to the equivariant cohomology ring of the corresponding (finite dimensional) symplectic or orthogonal flag manifold. This is done by using an idea from [T1, Lemma 1] together with the transition equations of [B, IMN1] to write the Schubert polynomials in this kernel as an explicit linear combination of these generators, which is important in applications. The double versions of formula (7) rely on the equality of the multi-Schur Pfaffian (6) – and its orthogonal analogue – with certain double Schubert polynomials (Propositions 4 and 12). This latter fact is an extension of [IMN1, Thm. 1.2], which may be deduced from the (even more general) Pfaffian formulas of Anderson and Fulton [AF]. We give an independent treatment here, using the right divided difference operators.

This paper is organized as follows. In Section 2 we recall the type C double Schubert polynomials and the geometrization map π_n from $\Gamma[X_n, Y_n]$ to the equivariant cohomology ring of Sp_{2n}/B, and obtain canonical generators for the kernel of π_n. In Section 3 we define the statistics $\nu(w)$ and $\lambda(w)$ of a signed permutation w and prove the analogue of formula (7) for double theta polynomials. Section 4 examines some related facts about single type C Schubert polynomials, including the scalar product with values in the ring $\Gamma^{(n)}$ of W_n-invariants. Sections 5, 6 and 7 study the corresponding questions in the orthogonal Lie types B and D.

I dedicate this article to the memory of Alain Lascoux, whose warm personality and vision about symmetric functions and Schubert polynomials initially assisted, and subsequently inspired my research, from its beginning to the present day.

2. Double Schubert polynomials of type C

2.1. **Preliminaries.** We recall the type C double Schubert polynomials of Ikeda, Mihalcea, and Naruse [IMN1], employing the notational conventions of the introduction, which are similar to those used in [AF]. These differ from the Schubert polynomials found in [BH, IMN1] and our papers [T4, T5, T7], in that the ring Γ is realized using the generators c_p and relations (3) among them, instead of the formal power series known as Schur Q-functions, which are not required in the present work. The connection between these power series and the Schubert polynomials used here was first explained in [T2, T3] (in the case of single polynomials) and [IMN1, T4] (for their double versions). We refer to [T5, Section 7.3] and [T7, Section 5] for a detailed account of this history.

Let $X := (x_1, x_2, \ldots)$ and $Y := (y_1, y_2, \ldots)$ be two lists of commuting independent variables, and set $X_n := (x_1, \ldots, x_n)$ and $Y_n := (y_1, \ldots, y_n)$ for each $n \geq 1$. The Weyl group for the root system of type C_n is the group of signed permutations on the set $\{1, \ldots, n\}$, denoted W_n. We write the elements of W_n as n-tuples $(w_1, \ldots, w_n)$, where $w_i := w(i)$ for $1 \leq i \leq n$. The group $W_\infty = \cup_k W_k$ is generated by the simple transpositions $s_i = (i, i+1)$ for $i \geq 1$ together with the sign change s_0, which fixes all $j \geq 2$ and sends 1 to $\overline{1}$ (a bar over an integer here means a negative sign). For $w \in W_\infty$, we denote by $\ell(w)$ the *length* of w, which is the least integer ℓ such that we can write $w = s_{i_1} \cdots s_{i_\ell}$ for some indices $i_j \geq 0$.

There is an action of W_∞ on $\Gamma[X,Y]$ by ring automorphisms, defined as follows. The simple reflections s_i for $i \geq 1$ act by interchanging x_i and x_{i+1} while leaving all the remaining variables fixed. The reflection s_0 maps x_1 to $-x_1$, fixes the x_j for $j \geq 2$ and all the y_j, and satisfies

$$s_0(c_p) := c_p + 2\sum_{j=1}^{p} x_1^j c_{p-j} \quad \text{for all } p \geq 1. \tag{8}$$

For each $i \geq 0$, define the *divided difference operator* ∂_i^x on $\Gamma[X,Y]$ by

$$\partial_0^x f := \frac{f - s_0 f}{-2x_1}, \qquad \partial_i^x f := \frac{f - s_i f}{x_i - x_{i+1}} \quad \text{for } i \geq 1.$$

Consider the ring involution $\omega : \Gamma[X,Y] \to \Gamma[X,Y]$ determined by

$$\omega(x_j) = -y_j, \qquad \omega(y_j) = -x_j, \qquad \omega(c_p) = c_p$$

and set $\partial_i^y := \omega\partial_i^x\omega$ for each $i \geq 0$.

The double Schubert polynomials $\mathfrak{C}_w = \mathfrak{C}_w(X,Y)$ for $w \in W_\infty$ are the unique family of elements of $\Gamma[X,Y]$ such that

$$\partial_i^x \mathfrak{C}_w = \begin{cases} \mathfrak{C}_{ws_i} & \text{if } \ell(ws_i) < \ell(w), \\ 0 & \text{otherwise,} \end{cases} \qquad \partial_i^y \mathfrak{C}_w = \begin{cases} \mathfrak{C}_{s_iw} & \text{if } \ell(s_iw) < \ell(w), \\ 0 & \text{otherwise,} \end{cases} \tag{9}$$

for all $i \geq 0$, together with the condition that the constant term of $\mathfrak{C}_w$ is 1 if $w = 1$, and 0 otherwise. For any $w \in W_\infty$, the corresponding (single) Billey-Haiman Schubert polynomial of type C is $\mathfrak{C}_w(X) := \mathfrak{C}_w(X,0)$. It is known that the $\mathfrak{C}_w(X)$ for $w \in W_\infty$ form a $\mathbb{Z}$-basis of $\Gamma[X] = \Gamma[x_1, x_2, \ldots]$, and the $\mathfrak{C}_w(X,Y)$ for $w \in W_\infty$ form a $\mathbb{Z}[Y]$-basis of $\Gamma[X,Y] = \Gamma[x_1, y_1, x_2, y_2, \ldots]$. See [IMN1] for further details, noting that the polynomial called $\mathfrak{C}_w(z,t;x)$ in op. cit., which is a formal power series in the x variables, would be the polynomial denoted by $\mathfrak{C}_w(z,t)$ here.

In the sequel, for every $i \geq 0$, we set $\partial_i := \partial_i^x$. For any $w \in W_\infty$, we define a divided difference operator $\partial_w := \partial_{i_1} \circ \cdots \circ \partial_{i_\ell}$, for any choice of indices $(i_1, \ldots, i_\ell)$ such that $w = s_{i_1} \cdots s_{i_\ell}$ and $\ell = \ell(w)$. According to [IMN1, Prop. 8.4], for any $u, w \in W_\infty$, we have

$$\partial_u \mathfrak{C}_w(X,Y) = \begin{cases} \mathfrak{C}_{wu^{-1}}(X,Y) & \text{if } \ell(wu^{-1}) = \ell(w) - \ell(u), \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

2.2. **The set $W^{(n)}$ and the ring $\Gamma[X_n, Y_n]$.** For every $n \geq 1$, let

$$W^{(n)} := \{w \in W_\infty \mid w_{n+1} < w_{n+2} < \cdots\}.$$

Proposition 1. *The $\mathfrak{C}_w(X)$ for $w \in W^{(n)}$ form a $\mathbb{Z}$-basis of $\Gamma[X_n]$.*

Proof. We have that $\mathfrak{C}_w(X) \in \Gamma[X_n]$ if and only if $\partial_m \mathfrak{C}_w(X) = 0$ for all $m > n$ if and only if $w \in W^{(n)}$. Suppose that $f \in \Gamma[X_n]$ is a polynomial which is not in the $\mathbb{Z}$-span of the $\mathfrak{C}_w(X)$, $w \in W^{(n)}$. Then f can be written as an integer linear combination of Schubert polynomials

$$f(X) = \sum_w e_w \mathfrak{C}_w(X) \tag{11}$$

where there is at least one w with $e_w \neq 0$ and $w \notin W^{(n)}$. Hence for some $m > n$ we have $\partial_m \mathfrak{C}_w = \mathfrak{C}_{ws_m}$, and since $\partial_m f = 0$, we obtain from (11) a nontrivial linear dependence relation among the Schubert polynomials, which is a contradiction. This proves that the $\mathfrak{C}_w(X)$ for $w \in W^{(n)}$ span $\Gamma[X_n]$, and therefore the result. $\square$

Proposition 2. *The $\mathfrak{C}_w(X,Y)$ for $w \in W^{(n)}$ form a $\mathbb{Z}[Y]$-basis of $\Gamma[X_n, Y]$.*

Proof. The $\mathfrak{C}_w(X,Y)$ for $w \in W_\infty$ are linearly independent over $\mathbb{Z}[Y]$. By Proposition 1 we know that the $\mathfrak{C}_w(X)$ for $w \in W^{(n)}$ form a $\mathbb{Z}$-basis of $\Gamma[X_n]$. According to [IMN1, Cor. 8.10], we have

$$\mathfrak{C}_w(X,Y) = \sum_{uv=w} \mathfrak{S}_{u^{-1}}(-Y)\mathfrak{C}_v(X)$$

summed over all factorizations $uv = w$ with $\ell(u) + \ell(v) = \ell(w)$ and $u \in S_\infty$. Since the term of lowest y-degree in the sum is $\mathfrak{C}_w(X)$, the proposition follows. $\square$

Let $\mathfrak{C}_w^{(n)} = \mathfrak{C}_w^{(n)}(X_n, Y_n)$ be the polynomial obtained from $\mathfrak{C}_w(X,Y)$ by setting $x_j = y_j = 0$ for all $j > n$.

Corollary 1. *The $\mathfrak{C}_w^{(n)}$ for $w \in W^{(n)}$ form a $\mathbb{Z}[Y_n]$-basis of $\Gamma[X_n, Y_n]$.*

2.3. **The geometrization map π_n.** The double Schubert polynomials $\mathfrak{C}_w^{(n)}(X,Y)$ for $w \in W_n$ represent the equivariant Schubert classes on the symplectic flag manifold. Let $\{e_1, \ldots, e_{2n}\}$ denote the standard symplectic basis of $E := \mathbb{C}^{2n}$ and let F_i be the subspace spanned by the first i vectors of this basis, so that $F_{n-i}^{\perp} = F_{n+i}$ for $0 \leq i \leq n$. Let B denote the stabilizer of the flag $F_\bullet$ in the symplectic group $\mathrm{Sp}_{2n} = \mathrm{Sp}_{2n}(\mathbb{C})$, and let T be the associated maximal torus in the Borel subgroup B. The symplectic flag manifold given by $\mathcal{M}_n := \mathrm{Sp}_{2n}/B$ parametrizes complete flags $E_\bullet$ in E with $E_{n-i}^{\perp} = E_{n+i}$ for $0 \leq i \leq n$. The T-equivariant cohomology ring $\mathrm{H}_T^*(\mathcal{M}_n)$ is defined as the cohomology ring of the Borel mixing space $ET \times^T \mathcal{M}_n$. The ring $\mathrm{H}_T^*(\mathcal{M}_n)$ is a $\mathbb{Z}[Y_n]$-algebra, where y_i is identified with the equivariant Chern class $-c_1^T(F_{n+1-i}/F_{n-i})$, for $1 \leq i \leq n$.

The Schubert varieties in $\mathcal{M}_n$ are the closures of the B-orbits, and are indexed by the elements of W_n. Concretely, any $w \in W_n$ corresponds to a Schubert variety $X_w = X_w(F_\bullet)$ of codimension $\ell(w)$, defined by

$$X_w := \{E_\bullet \in \mathcal{M}_n \mid \dim(E_r \cap F_s) \geq d_w(r,s) \text{ for } 1 \leq r \leq n,\, 1 \leq s \leq 2n\},$$

where $d_w(r,s)$ is the rank function specified as follows. Consider the group monomorphism $\zeta : W_n \hookrightarrow S_{2n}$ with image

$$\zeta(W_n) = \{\, \varpi \in S_{2n} \mid \varpi_i + \varpi_{2n+1-i} = 2n+1, \quad \text{for all } i \,\},$$

and determined by setting, for each $w = (w_1, \ldots, w_n) \in W_n$ and $1 \leq i \leq n$,

$$\zeta(w)_i := \begin{cases} n+1-w_{n+1-i} & \text{if } w_{n+1-i} \text{ is unbarred,} \\ n+\overline{w}_{n+1-i} & \text{otherwise.} \end{cases}$$

Then $d_w(r,s)$ equals the number of $i \leq r$ such that $\zeta(w)_i > 2n - s$. Since X_w is stable under the action of T, we obtain an *equivariant Schubert class* $[X_w]^T := [ET \times^T X_w]$ in $\mathrm{H}_T^*(\mathcal{M}_n)$.

Following [IMN1], there is a surjective homomorphism of graded $\mathbb{Z}[Y_n]$-algebras

$$\pi_n : \Gamma[X_n, Y_n] \to \mathrm{H}_T^*(\mathcal{M}_n)$$

such that

$$\pi_n(\mathfrak{C}_w^{(n)}) = \begin{cases} [X_w]^T & \text{if } w \in W_n, \\ 0 & \text{if } w \in W^{(n)} \smallsetminus W_n. \end{cases}$$

We let E_i denote the i-th tautological vector vector bundle over $\mathcal{M}_n$, for $0 \leq i \leq 2n$. The geometrization map π_n is defined by the equations

$$\pi_n(x_i) = c_1^T(E_{n+1-i}/E_{n-i}) \quad \text{and} \quad \pi_n(c_p) = c_p^T(E - E_n - F_n)$$

for $1 \leq i \leq n$ and $p \geq 1$. Here $c_p^T(E - E_n - F_n)$ denotes the degree p component of the total Chern class $c^T(E - E_n - F_n) := c^T(E)c^T(E_n)^{-1}c^T(F_n)^{-1}$.

2.4. **The kernel of the map π_n.** For any integer $j \geq 0$ and sequence of variables $Z = (z_1, z_2, \ldots)$, define the elementary and complete symmetric functions $e_j(Z)$ and $h_j(Z)$ by the generating series

$$\prod_{i=1}^{\infty}(1 + z_i t) = \sum_{j=0}^{\infty} e_j(Z)t^j \quad \text{and} \quad \prod_{i=1}^{\infty}(1 - z_i t)^{-1} = \sum_{j=0}^{\infty} h_j(Z)t^j,$$

respectively. If $r \geq 1$ then we let $e_j^r(Z) := e_j(z_1, \ldots, z_r)$ and $h_j^r(Z) := h_j(z_1, \ldots, z_r)$ denote the polynomials obtained from $e_j(Z)$ and $h_j(Z)$ by setting $z_j = 0$ for all $j > r$. Let $e_j^0(Z) = h_j^0(Z) = \delta_{0j}$, where δ_{0j} denotes the Kronecker delta, and for $r < 0$, define $h_j^r(Z) := e_j^{-r}(Z)$ and $e_j^r(Z) := h_j^{-r}(Z)$.

For any $k, k' \in \mathbb{Z}$, define the polynomial ${}^k c_p^{k'} = {}^k c_p^{k'}(X, Y)$ by

$${}^k c_p^{k'} := \sum_{i=0}^{p} \sum_{j=0}^{p} c_{p-j-i} h_i^{-k}(X) h_j^{k'}(-Y).$$

Definition 1. Let

$$\widehat{\Gamma}^{(n)} := \mathbb{Z}[{}^n c_1^n, {}^n c_2^n, \ldots]$$

and let $\widehat{\mathrm{I}\Gamma}^{(n)}$ be the ideal of $\Gamma[X_n, Y_n]$ generated by the homogeneous elements in $\widehat{\Gamma}^{(n)}$ of positive degree.

For any $p \in \mathbb{Z}$, define $\widehat{e}_p \in \mathbb{Z}[X_n, Y_n]$ by

$$\widehat{e}_p = \widehat{e}_p(X_n/Y_n) := \sum_{i+j=p} e_i(X_n) h_j(-Y_n).$$

We then have the generating function equation

$$\sum_{p=0}^{\infty} {}^n c_p^n \, t^p = \left(\sum_{p=0}^{\infty} c_p t^p\right)\left(\sum_{j=0}^{n} \widehat{e}_j t^j\right) = \left(\sum_{p=0}^{\infty} c_p t^p\right) \prod_{j=1}^{n} \frac{1 + x_j t}{1 + y_j t}. \tag{12}$$

Lemma 1. *We have* $\widehat{\mathrm{I}\Gamma}^{(n)} \subset \operatorname{Ker} \pi_n$.

Proof. It suffices to show that ${}^n c_p^n \in \operatorname{Ker} \pi_n$ for each $p \geq 1$. We give two proofs of this result. A straightforward calculation using Chern roots shows that

$$\pi_n({}^k c_p^{k'}) = c_p^T(E - E_{n-k} - F_{n+k'})$$

for all $p, k, k' \in \mathbb{Z}$. Since $E = F_{2n}$, we deduce the lemma from this and the properties of Chern classes.

Our second proof proceeds as follows. There is a canonical isomorphism of $\mathbb{Z}[Y_n]$-algebras

$$\mathrm{H}_T^*(\mathcal{M}_n) \cong \mathbb{Z}[A_n, Y_n]/K_n,$$

where $A_n := (a_1, \ldots, a_n)$ and K_n is the ideal of $\mathbb{Z}[A_n, Y_n]$ generated by the differences $e_i(A_n^2) - e_i(Y_n^2)$ for $1 \leq i \leq n$ (see for example [F2, Section 3]). The geometrization map π_n satisfies $\pi_n(x_j) = -a_j$ for $1 \leq j \leq n$, while

$$\pi_n(c_p) := \sum_{i+j=p} e_i(A_n) h_j(Y_n), \quad p \geq 0.$$

A straightforward calculation using (12) gives

$$\pi_n\left(\sum_{p=0}^{\infty} {}^n c_p^n \, t^p\right) = \prod_{j=1}^{n} \frac{1-a_j^2t^2}{1-y_j^2t^2}.$$

On the other hand, we have

$$\begin{aligned}\prod_{j=1}^{n} \frac{1-a_j^2t^2}{1-y_j^2t^2} &= 1 + \left(\prod_{j=1}^{n}(1-a_j^2t^2) - \prod_{j=1}^{n}(1-y_j^2t^2)\right) \cdot \sum_{p=0}^{\infty} h_p(Y_n^2)t^{2p} \\ &= 1 + \left(\sum_{r=0}^{n}(-1)^r(e_r(A_n^2) - e_r(Y_n^2))t^{2r}\right) \cdot \sum_{p=0}^{\infty} h_p(Y_n^2)t^{2p}.\end{aligned}$$

The result follows immediately. □

For any three integer vectors $\alpha, \beta, \rho \in \mathbb{Z}^\ell$, which we view as integer sequences with finite support, define ${}^\rho c_\alpha^\beta := {}^{\rho_1} c_{\alpha_1}^{\beta_1} \, {}^{\rho_2} c_{\alpha_2}^{\beta_2} \cdots$. Recall that for each $i < j$, the operator R_{ij} acts on integer sequences $\alpha = (\alpha_1, \alpha_2, \ldots)$ by

$$R_{ij}(\alpha) := (\alpha_1, \ldots, \alpha_i + 1, \ldots, \alpha_j - 1, \ldots).$$

A *raising operator* R is any monomial in these R_{ij}'s. Given any raising operator $R = \prod_{i<j} R_{ij}^{n_{ij}}$, let $R \, {}^\rho c_\alpha^\beta := {}^\rho c_{R\alpha}^\beta$. Finally, define the *multi-Schur Pfaffian* ${}^\rho Q_\alpha^\beta$ by

$$(13) \qquad {}^\rho Q_\alpha^\beta := R^\infty \, {}^\rho c_\alpha^\beta,$$

where the raising operator expression R^∞ is given by

$$R^\infty := \prod_{i<j} \frac{1-R_{ij}}{1+R_{ij}}\,.$$

The name 'multi-Schur Pfaffian' is justified because ${}^\rho Q_\alpha^\beta$ is equal to the Pfaffian of the $r \times r$ skew-symmetric matrix with

$$\left\{ {}^{\rho_i,\rho_j} Q_{\alpha_i,\alpha_j}^{\beta_i,\beta_j} \right\}_{1 \leq i<j \leq r} = \left\{ \frac{1-R_{12}}{1+R_{12}} \, {}^{\rho_i,\rho_j} c_{\alpha_i,\alpha_j}^{\beta_i,\beta_j} \right\}_{1 \leq i<j \leq r}$$

above the main diagonal, following Kazarian [K]; here $r = 2\lfloor \ell/2 \rfloor$. We adopt the convention that when some superscript(s) are omitted, the corresponding indices are equal to zero. Thus ${}^k c_p := {}^k c_p^0$, $c_p^{k'} := {}^0 c_p^{k'}$, ${}^\rho c_\alpha := \prod_i {}^{\rho_i} c_{\alpha_i}^0$, ${}^\rho Q_\alpha := R^\infty \, {}^\rho c_\alpha$, $Q_\alpha := R^\infty \, c_\alpha$, etc.

If $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_\ell)$ is a strict partition of length ℓ, let w_λ be the corresponding increasing Weyl group element, so that the negative components of w_λ are exactly $(-\lambda_1, \ldots, -\lambda_\ell)$.

Lemma 2. *If λ is a strict partition with $\lambda_1 > n$, then $\mathfrak{C}_{w_\lambda}^{(n)}(X_n, Y_n) \in \widehat{\Gamma}^{(n)}$.*

Proof. For $p \geq 0$, recall that $c_p^{-n} := {}^0c_p^{-n} \in \Gamma[Y_n]$, so that we have

$$\sum_{p=0}^{\infty} c_p^{-n}\, t^p = \left(\sum_{p=0}^{\infty} c_p t^p\right) \prod_{j=1}^{n} (1-y_j t).$$

One has the generating function equation

$$\left(\sum_{p=0}^{\infty} {}^nc_p^n\, t^p\right)\left(\sum_{p=0}^{\infty} c_p^{-n}(-t)^p\right) = \sum_{j=0}^{n} e_j(X_n)t^j. \tag{14}$$

It follows from (14) that

$${}^nc_p^n - {}^nc_{p-1}^n c_1^{-n} + \cdots + (-1)^p(c_p^{-n}) = e_p(X_n)$$

for each $p \geq 0$. We deduce that $c_p^{-n} \in \widehat{\mathrm{I}\Gamma}^{(n)}$ when $p \geq n+1$.

According to [IMN1, Thm. 6.6], we have

$$\mathfrak{C}_{w_\lambda}(X,Y) = Q_\lambda^{\beta(\lambda)} = R^\infty\, c_\lambda^{\beta(\lambda)} \tag{15}$$

in $\Gamma[X,Y]$, where $\beta(\lambda)$ is equal to the integer vector $(1-\lambda_1,\ldots,1-\lambda_\ell)$. It follows that

$$\mathfrak{C}_{w_\lambda}^{(n)}(X_n,Y_n) = \overline{Q}_\lambda^{\beta(\lambda)}, \tag{16}$$

where $\overline{Q}_\lambda^{\beta(\lambda)}$ is obtained from $Q_\lambda^{\beta(\lambda)}$ by setting $y_j = 0$ for all $j > n$. The conclusion of the lemma now follows immediately by expanding the raising operator formula (15) for the double Schur Q-polynomial $\overline{Q}_\lambda^{\beta(\lambda)}$ and noting that in each monomial of the result, the first factor is equal to c_p^{-n} for some $p > n$. $\square$

Lemma 3. *For any $w \in W_\infty \smallsetminus W_n$, we have $\mathfrak{C}_w^{(n)} \in \widehat{\mathrm{I}\Gamma}^{(n)}$.*

Proof. For any positive integers $i < j$ we define reflections $t_{ij} \in S_\infty$ and $\bar{t}_{ij}, \bar{t}_{ii} \in W_\infty$ by their right actions

$$\begin{aligned}
(\ldots,w_i,\ldots,w_j,\ldots)\,t_{ij} &= (\ldots,w_j,\ldots,w_i,\ldots),\\
(\ldots,w_i,\ldots,w_j,\ldots)\,\bar{t}_{ij} &= (\ldots,\overline{w}_j,\ldots,\overline{w}_i,\ldots), \quad \text{and}\\
(\ldots,w_i,\ldots)\,\bar{t}_{ii} &= (\ldots,\overline{w}_i,\ldots),
\end{aligned}$$

and let $\bar{t}_{ji} := \bar{t}_{ij}$.

Let w be an element of W_∞. According to [B, Lemma 2], if $i \leq j$, then $\ell(w\bar{t}_{ij}) = \ell(w)+1$ if and only if (i) $-w_i < w_j$, (ii) in case $i < j$, either $w_i < 0$ or $w_j < 0$, and (iii) there is no $p < i$ such that $-w_j < w_p < w_i$, and no $p < j$ such that $-w_i < w_p < w_j$.

The group W_∞ acts on the polynomial ring $\mathbb{Z}[y_1,y_2,\ldots]$ in the usual way, with s_i for $i \geq 1$ interchanging y_i and y_{i+1} and leaving all the remaining variables fixed, and s_0 mapping y_1 to $-y_1$ and fixing the y_j with $j \geq 2$. Let $w \in W_\infty$ be non-increasing, let r be the last positive descent of w, let $s := \max(i > r \mid w_i < w_r)$, and let $v := wt_{rs}$. Following [IMN1, Prop. 6.12], the double Schubert polynomials $\mathfrak{C}_u = \mathfrak{C}_u(X,Y)$ obey the *transition equations*

$$\mathfrak{C}_w = (x_r - v(y_r))\mathfrak{C}_v + \sum_{\substack{1\leq i<r\\ \ell(vt_{ir})=\ell(w)}} \mathfrak{C}_{vt_{ir}} + \sum_{\substack{i\geq 1\\ \ell(v\bar{t}_{ir})=\ell(w)}} \mathfrak{C}_{v\bar{t}_{ir}} \tag{17}$$

in $\Gamma[X,Y]$. The recursion (17) terminates in a $\mathbb{Z}[X,Y]$-linear combination of elements $\mathfrak{C}_{w_\nu}(X,Y)$ for strict partitions ν.

For any $w \in W_\infty$, let $\mu(w)$ denote the strict partition whose parts are the elements of the set $\{|w_i| \ : \ w_i < 0\}$. Clearly we have $\mu(w) = \mu(wu)$ for any $u \in S_\infty$. In equation (17), we therefore have $\mu(v) = \mu(vt_{ir}) = \mu(w)$. Moreover, condition (i) above shows that the parts of $\mu(v\bar{t}_{ir})$ are greater than or equal to the parts of $\mu(w)$. In particular, if $\mu(w)_1 > n$, then $\mu(v\bar{t}_{ir})_1 > n$.

Assume first that $w \in W_{n+1} \smallsetminus W_n$. If $w_i = -n-1$ for some $i \leq n+1$, we use the transition recursion (17) to write $\mathfrak{C}_w^{(n)}$ as a $\mathbb{Z}[X_n,Y_n]$-linear combination of elements $\mathfrak{C}_{w_\nu}^{(n)}$ for strict partitions ν with $\nu_1 > n$. Lemma 2 now implies that $\mathfrak{C}_w^{(n)} \in \widehat{\mathrm{I}\Gamma}^{(n)}$.

Next, we consider the case when $w_i = n+1$ for some $i \leq n$. Let

$$\{v_2, \ldots, v_n\} := \{w_1, \ldots, \widehat{w_i}, \ldots, w_n\}$$

with $v_2 > \cdots > v_n$, and define

$$u := (n+1, v_2, \ldots, v_n, w_{n+1}) \in W_{n+1}$$

and

$$\overline{u} := us_0 = (\overline{n+1}, v_2, \ldots, v_n, w_{n+1}).$$

We have $\mathfrak{C}_{\overline{u}}^{(n)} \in \widehat{\mathrm{I}\Gamma}^{(n)}$ from the previous case, and, using (9), that $\partial_0(\mathfrak{C}_{\overline{u}}^{(n)}) = \mathfrak{C}_u^{(n)}$.

For any integer $i \in [0, n-1]$, it is easy to check that $s_i({}^n c_p^n) = {}^n c_p^n$, and therefore that $\partial_i({}^n c_p^n) = 0$. It follows that $\partial_i(\widehat{\mathrm{I}\Gamma}^{(n)}) \subset \widehat{\mathrm{I}\Gamma}^{(n)}$ for all indices $i \in [0, n-1]$. Since $\mathfrak{C}_{\overline{u}}^{(n)} \in \widehat{\mathrm{I}\Gamma}^{(n)}$, we deduce that $\mathfrak{C}_u^{(n)} \in \widehat{\mathrm{I}\Gamma}^{(n)}$. There exists a permutation $\sigma \in S_n$ such that $u = w\sigma$ and $\ell(\sigma) = \ell(u) - \ell(w)$. Using (10), we have $\mathfrak{C}_w^{(n)} = \partial_\sigma(\mathfrak{C}_u^{(n)})$, and hence conclude that $\mathfrak{C}_w^{(n)}$ lies in $\widehat{\mathrm{I}\Gamma}^{(n)}$.

Finally assume $w \notin W_{n+1}$ and let m be minimal such that $w \in W_m$. Then $w \in W_m \smallsetminus W_{m-1}$, so the above argument applies with $m-1$ in place of n. The result now follows by setting $x_j = y_j = 0$ for all $j > n$. □

Theorem 1. *Let* $J_n := \bigoplus_{w \in W^{(n)} \smallsetminus W_n} \mathbb{Z}[Y_n]\mathfrak{C}_w^{(n)}$. *Then we have*

$$\widehat{\mathrm{I}\Gamma}^{(n)} = J_n = \sum_{w \in W_\infty \smallsetminus W_n} \mathbb{Z}[Y_n]\mathfrak{C}_w^{(n)} = \operatorname{Ker}\pi_n. \tag{18}$$

We have a canonical isomorphism of $\mathbb{Z}[Y_n]$*-algebras*

$$\mathrm{H}_T^*(\mathrm{Sp}_{2n}/B) \cong \Gamma[X_n, Y_n]/\widehat{\mathrm{I}\Gamma}^{(n)}.$$

Proof. Lemmas 1 and 3 imply that

$$J_n \subset \sum_{w \in W_\infty \smallsetminus W_n} \mathbb{Z}[Y_n]\mathfrak{C}_w^{(n)} \subset \widehat{\mathrm{I}\Gamma}^{(n)} \subset \operatorname{Ker}\pi_n. \tag{19}$$

We claim that $\operatorname{Ker}\pi_n \subset J_n$. Indeed, if $f \in \operatorname{Ker}\pi_n$ then by Corollary 1 we have a unique expression

$$f = \sum_{w \in W^{(n)}} f_w \mathfrak{C}_w^{(n)} \tag{20}$$

for some coefficients $f_w \in \mathbb{Z}[Y_n]$. Applying the map π_n to (20) and using (19) gives

$$\sum_{w \in W_n} f_w [X_w]^T = 0.$$

Since the equivariant Schubert classes $[X_w]^T$ are a $\mathbb{Z}[Y_n]$-basis of $\mathrm{H}_T^*(\mathrm{Sp}_{2n}/B)$, we deduce that $f_w = 0$ for all $w \in W_n$. It follows that $f \in J_n$. $\square$

Remark 1. (a) It is easy to show that ${}^n c_p^n$ lies in $\sum_{w\in W_\infty \smallsetminus W_n} \mathbb{Z}[Y_n]\mathfrak{C}_w^{(n)}$ for all $n, p \geq 1$. This follows from the fact that ${}^n c_p^n = {}^{n+p-1}c_p^{(n+p-1)+1-p}(X_n, Y_n)$ is equal to the (restricted) double Schubert polynomial $\mathfrak{C}_{w_{(p)}}^{(n)}$, where

$$w_{(p)} := s_n s_{n+1} \cdots s_{n+p-1}.$$

In fact, $\mathfrak{C}_{w_{(p)}}(X, Y)$ is equal to the double theta polynomial ${}^{n+p-1}\Theta_p(X, Y)$ of level $n+p-1$, for every $p \geq 1$ (the definition of ${}^{n+p-1}\Theta_p(X, Y)$ is recalled in (30)).

(b) The equality $\sum_{w\in W_\infty \smallsetminus W_n} \mathbb{Z}[Y_n]\mathfrak{C}_w^{(n)} = \mathrm{Ker}\,\pi_n$ in (18) was proved earlier in [IMN1, Prop. 7.7] using different methods.

For any elements $f, g \in \Gamma[X_n, Y_n]$, we define the congruence $f \equiv g$ to mean $f - g \in \widehat{\mathrm{I}\Gamma}^{(n)}$. We claim that any element of $\Gamma[X_n, Y_n]$ is equivalent under $\equiv$ to a polynomial in $\mathbb{Z}[X_n, Y_n]$. Indeed, we have that

$$\left(\sum_{p=0}^{\infty} {}^n c_p^n\, t^p\right)\left(\sum_{p=0}^{\infty} c_p(-t)^p\right) = \sum_{j=0}^{\infty} \widehat{e}_j t^j. \tag{21}$$

It follows from (21) that

$$ {}^n c_p^n - {}^n c_{p-1}^n c_1 + \cdots + (-1)^r c_p = \widehat{e}_p \tag{22}$$

for each $p \geq 0$. The relation (22) implies that $c_p \equiv (-1)^p \widehat{e}_p(X_n/Y_n)$, for all $p \geq 0$, proving the claim.

We deduce that $c_\alpha \equiv (-1)^{|\alpha|} e_\alpha(X_n/Y_n)$ for each integer sequence α, and that

$$Q_\lambda = Q_\lambda(c) \equiv (-1)^{|\lambda|} \widetilde{Q}_\lambda(X_n/Y_n)$$

for any partition λ. Here $\widetilde{Q}_\lambda(X_n/Y_n)$ denotes a supersymmetric $\widetilde{Q}$-polynomial, namely

$$\widetilde{Q}_\lambda(X_n/Y_n) := R^\infty\, \widehat{e}_\lambda(X_n/Y_n).$$

The reader can compare this with the remarks in [T5, Section 7.3].

2.5. **Partial symplectic flag manifolds.** Following [Bo, KK], there is a standard way to generalize the presentation in Theorem 1 to partial flag manifolds Sp_{2n}/P, where P is a parabolic subgroup of Sp_{2n}. The parabolic subgroups P containing B correspond to sequences $\mathfrak{a}\ :\ a_1 < \cdots < a_p$ of nonnegative integers with $a_p < n$. The manifold Sp_{2n}/P parametrizes partial flags of subspaces

$$0 \subset E_1 \subset \cdots \subset E_p \subset E = \mathbb{C}^{2n}$$

with $\dim(E_j) = n - a_{p+1-j}$ for each $j \in [1, p]$ and E_p isotropic.

A sequence $\mathfrak{a}$ as above also parametrizes the parabolic subgroup W_P of W_n, which is generated by the simple reflections s_i for $i \notin \{a_1, \ldots, a_p\}$. Let $\Gamma[X_n, Y_n]^{W_P}$ be the subring of elements in $\Gamma[X_n, Y_n]$ which are fixed by the action of W_P, that is,

$$\Gamma[X_n, Y_n]^{W_P} = \{f \in \Gamma[X_n, Y_n] \mid s_i(f) = f,\ \forall\, i \notin \{a_1, \ldots, a_p\},\ i < n\}.$$

Since the action of W_n on $\Gamma[X_n, Y_n]$ is $\mathbb{Z}[Y_n]$-linear, we see that $\Gamma[X_n, Y_n]^{W_P}$ is a $\mathbb{Z}[Y_n]$-subalgebra of $\Gamma[X_n, Y_n]$. Let $W^P \subset W^{(n)}$ denote the set

$$W^P := \{w \in W^{(n)} \mid \ell(ws_i) = \ell(w) + 1,\ \forall\, i \notin \{a_1, \ldots, a_p\},\ i < n\}.$$

Proposition 3. *We have*

$$\Gamma[X_n,Y_n]^{W_P} = \bigoplus_{w\in W^P} \mathbb{Z}[Y_n]\mathfrak{C}_w^{(n)}. \tag{23}$$

Proof. If f is any element in $\Gamma[X_n,Y_n]^{W_P}$, Corollary 1 implies that we have an expansion $f=\sum_{w\in W^{(n)}} d_w\mathfrak{C}_w^{(n)}$ for some coefficients d_w in $\mathbb{Z}[Y_n]$. If $u\notin W^P$, there is an index $i<n$ with $i\notin\{a_1,\ldots,a_p\}$ and $\ell(us_i)=\ell(u)-1$. We have $\partial_i f=0$, and on the other hand, using (9), we see that

$$\partial_i f = \sum_u d_u \mathfrak{C}_{us_i}^{(n)}$$

summed over all u such that $\ell(us_i)=\ell(u)-1$. It follows that $d_u=0$ for all such u, and thus that $\Gamma[X_n,Y_n]^{W_P}$ is contained in the sum on the right hand side of (23).

For the reverse inclusion, is suffices to show that $\mathfrak{C}_w^{(n)}\in\Gamma[X_n,Y_n]^{W_P}$ for all $w\in W^P$. The definition of W^P implies that we have $\partial_i\mathfrak{C}_w^{(n)}=0$, or equivalently $s_i\mathfrak{C}_w^{(n)}=\mathfrak{C}_w^{(n)}$, for all $i<n$ with $i\notin\{a_1,\ldots,a_p\}$. The result follows. □

Corollary 2. *There is a canonical isomorphism of* $\mathbb{Z}[Y_n]$*-algebras*

$$\mathrm{H}_T^*(\mathrm{Sp}_{2n}/P)\cong\Gamma[X_n,Y_n]^{W_P}/\widehat{\mathrm{I}\Gamma}_P^{(n)},$$

where $\widehat{\mathrm{I}\Gamma}_P^{(n)}$ *is the ideal of* $\Gamma[X_n,Y_n]^{W_P}$ *generated by the homogeneous elements in* $\widehat{\Gamma}^{(n)}$ *of positive degree.*

Proof. It is well known that the canonical projection map $h:G/B\to G/P$ induces an injection $h^*:\mathrm{H}_T^*(\mathrm{Sp}_{2n}/P)\hookrightarrow\mathrm{H}_T^*(\mathrm{Sp}_{2n}/B)$ on equivariant cohomology rings, with the image of h^* equal to the W_P-invariants in $\mathrm{H}_T^*(\mathrm{Sp}_{2n}/B)$ (see for example [KK, Cor. (3.20)]). In fact, since the $\mathfrak{C}_w^{(n)}$ for $w\in W^P\cap W_n$ represent the equivariant Schubert classes coming from $\mathrm{H}_T^*(\mathrm{Sp}_{2n}/P)$, we deduce from Proposition 3 that the restriction of the geometrization map $\pi_n:\Gamma[X_n,Y_n]\to\mathrm{H}_T^*(\mathrm{Sp}_{2n}/B)$ to the W_P-invariants induces a surjection

$$\Gamma[X_n,Y_n]^{W_P}\to\mathrm{H}_T^*(\mathrm{Sp}_{2n}/P).$$

The result follows easily from this and Theorem 1. □

3. Divided differences and double theta polynomials

3.1. **Preliminaries.** For every $i\geq 0$, the divided difference operator $\partial_i=\partial_i^x$ on $\Gamma[X,Y]$ satisfies the Leibnitz rule

$$\partial_i(fg)=(\partial_i f)g+(s_i f)\partial_i g. \tag{24}$$

Observe that $\omega({}^kc_p^r)={}^{-r}c_p^{-k}$, for all $k,r,p\in\mathbb{Z}$. By applying this, it is easy to prove the following dual versions of [IM, Lemmas 5.4 and 8.2].

Lemma 4. *Suppose that* $k,p,r\in\mathbb{Z}$*. For all* $i\geq 0$*, we have*

$$\partial_i({}^kc_p^r)=\begin{cases}{}^{k-1}c_{p-1}^r & \text{if } k=\pm i,\\ 0 & \text{otherwise.}\end{cases}$$

Lemma 5. *Suppose that* $k\geq 0$ *and* $r\geq 1$*. Then we have*

$${}^kc_p^{-r}={}^{k+1}c_p^{-r+1}-(x_{k+1}+y_r)\,{}^kc_{p-1}^{-r+1}.$$

We also require the following lemma.

Lemma 6 ([IM], Prop. 5.4)**.** *Suppose that $k, r \geq 0$ and $p > k + r$. Then we have*

$$ {}^{(k_1,\ldots,k,k,\ldots,k_\ell)}Q^{(r_1,\ldots,-r,-r,\ldots,r_\ell)}_{(p_1,\ldots,p,p,\ldots,p_\ell)} = 0. $$

3.2. **The shape of a signed permutation.** We proceed to define certain statistics of an element of W_∞.

Definition 2. Let $w \in W_\infty$ be a signed permutation. The strict partition $\mu = \mu(w)$ is the one whose parts are the absolute values of the negative entries of w, arranged in decreasing order. The *A-code* of w is the sequence $\gamma = \gamma(w)$ with $\gamma_i := \#\{j > i \mid w_j < w_i\}$. We define a partition $\delta = \delta(w)$ whose parts are the non-zero entries γ_i arranged in weakly decreasing order, and let $\nu(w) := \delta(w)'$ be the conjugate of δ. Finally, the *shape* of w is the partition $\lambda(w) := \mu(w) + \nu(w)$.

It is easy to see that w is uniquely determined by $\mu(w)$ and $\gamma(w)$, and that $|\lambda(w)| = \ell(w)$. The shape $\lambda(w)$ of an element $w \in W_\infty$ is a natural generalization of the shape of a permutation, as defined in [M2, Chp. 1].

Example 1. (a) For the signed permutation $w := (\overline{3}, 2, \overline{7}, \overline{1}, 5, 4, \overline{6})$ in W_7, we obtain $\mu = (7, 6, 3, 1)$, $\gamma = (2, 3, 0, 1, 2, 1, 0)$, $\delta = (3, 2, 2, 1, 1)$, $\nu = (5, 3, 1)$, and $\lambda = (12, 9, 4, 1)$.
(b) An element $w \in W_\infty$ is n-Grassmannian if $\ell(ws_i) > \ell(w)$ for all $i \neq n$, while a partition λ is called n-strict if all its parts λ_i greater than n are distinct. Following [BKT1, Section 6.1], these two objects are in one-to-one correspondence with each other. If w is an n-Grassmannian element of W_∞, then $\lambda(w)$ is the n-strict partition associated to w, in the sense of op. cit.

Lemma 7. *If $i \geq 1$, $w \in W_\infty$, and $\gamma = \gamma(w)$, then*

$$ \gamma_i > \gamma_{i+1} \Leftrightarrow w_i > w_{i+1} \Leftrightarrow \ell(ws_i) = \ell(w) - 1. $$

If any of the above conditions hold, then

$$ \gamma(ws_i) = (\gamma_1, \ldots, \gamma_{i-1}, \gamma_{i+1}, \gamma_i - 1, \gamma_{i+2}, \gamma_{i+3}, \ldots). $$

Proof. These follow immediately from [M2, (1.23) and (1.24)]. $\square$

Let $\beta(w)$ be the sequence defined by $\beta(w)_i = \min(1 - \mu(w)_i, 0)$ for each $i \geq 1$. For each $n \geq 1$, let $w_0^{(n)} := (\overline{1}, \ldots, \overline{n})$ denote the longest element in W_n.

Proposition 4. *Suppose that $m > n \geq 0$ and $w \in W_m$ is an n-Grassmannian element. Set $\widehat{w} := ww_0^{(n)}$. Then we have*

$$ \mathfrak{C}_{\widehat{w}}(X, Y) = {}^{\nu(\widehat{w})}Q^{\beta(\widehat{w})}_{\lambda(\widehat{w})} $$

in the ring $\Gamma[X_n, Y_{m-1}]$. In particular, if $w \in S_m$, then we have

$$ \mathfrak{C}_{\widehat{w}}(X, Y) = {}^{\delta_{n-1}}Q^{(1-w_n,\ldots,1-w_1)}_{\delta_n+\delta_{n-1}+\lambda(w)'}. $$

Proof. We first consider the case when $w \in S_m$. We have

$$ w = (a_1, \ldots, a_n, d_1, \ldots, d_r) $$

where $r = m - n$, $0 < a_1 < \cdots < a_n$ and $0 < d_1 < \cdots < d_r$. If $\lambda := \lambda(w)$ then

$$ \lambda_j = n + j - d_j = m - d_j - (r - j) \quad \text{for } 1 \leq j \leq r. $$

Let $w_0^{(m)} := (\overline{1},\ldots,\overline{m})$ be the longest element in W_m. Then we have $w_0^{(m)} = \widehat{w}v_1\cdots v_r$, where $\ell(w_0^{(m)}) = \ell(\widehat{w}) + \sum_{j=1}^r \ell(v_j)$ and

$$v_j = s_{n+j-1}\cdots s_1 s_0 s_1 \cdots s_{d_j-1}, \quad 1\leq j\leq r.$$

One knows from [IMN1, Thm. 1.2] and [T7, Prop. 3.2] that the equation

$$\mathfrak{C}_{w_0^{(m)}}(X,Y) = {}^{\delta_{m-1}}Q^{-\delta_{m-1}}_{\delta_m+\delta_{m-1}}$$

holds in $\Gamma[X,Y]$. It follows from this and (10) that

$$\mathfrak{C}_{\widehat{w}} = \partial_{v_1}\cdots\partial_{v_r}\left(\mathfrak{C}_{w_0^{(m)}}\right) = \partial_{v_1}\cdots\partial_{v_r}\left({}^{\delta_{m-1}}Q^{-\delta_{m-1}}_{\delta_m+\delta_{m-1}}\right). \tag{25}$$

Using Lemmas 4 and 5, for any $p,q\in\mathbb{Z}$ with $p\geq 1$, we obtain

$$\partial_p({}^pc_q^{-p}) = {}^{p-1}c^{-p}_{q-1} = {}^pc^{1-p}_{q-1} - (x_p+y_p)\,{}^{p-1}c^{1-p}_{q-2}. \tag{26}$$

Let ϵ_j denote the j-th standard basis vector in $\mathbb{Z}^m$. The Leibnitz rule and (26) imply that for any integer vector $\alpha=(\alpha_1,\ldots,\alpha_m)$, we have

$$\partial_p\left({}^{\delta_{m-1}}c_\alpha^{-\delta_{m-1}}\right) = {}^{\delta_{m-1}}c^{-\delta_{m-1}+\epsilon_{m-p}}_{\alpha-\epsilon_{m-p}} - (x_p+y_p)\,{}^{\delta_{m-1}-\epsilon_{m-p}}c^{-\delta_{m-1}+\epsilon_{m-p}}_{\alpha-2\epsilon_{m-p}}.$$

We deduce from this and Lemma 6 that

$$\begin{aligned}\partial_p{}^{\delta_{m-1}}Q^{-\delta_{m-1}}_{\delta_m+\delta_{m-1}} &= {}^{\delta_{m-1}}Q^{-\delta_{m-1}+\epsilon_{m-p}}_{\delta_m+\delta_{m-1}-\epsilon_{m-p}} \\ &= {}^{(m-1,\ldots,1,0)}Q^{(1-m,\ldots,-1-p,1-p,1-p,2-p,\ldots,-1,0)}_{(2m-1,\ldots,2p+3,2p,2p-1,2p-3,\ldots,1)}.\end{aligned}$$

Iterating this calculation for $p=d_r-1,\ldots,1$ gives

$$(\partial_1\cdots\partial_{d_r-1})\mathfrak{C}_{w_0^{(m)}} = {}^{(m-1,\ldots,1,0)}Q^{(1-m,\ldots,-d_r,2-d_r,3-d_r,\ldots,-1,0,0)}_{(2m-1,\ldots,2d_r+1,2d_r-2,2d_r-4,\ldots,2,1)}.$$

Since $\partial_0({}^0c^0_1) = {}^{-1}c^0_0 = 1$, it follows that

$$(\partial_0\partial_1\cdots\partial_{d_r-1})\mathfrak{C}_{w_0^{(m)}} = {}^{(m-1,\ldots,1)}Q^{(1-m,\ldots,-d_r,2-d_r,3-d_r,\ldots,-1,0)}_{(2m-1,\ldots,2d_r+1,2d_r-2,2d_r-4,\ldots,2)}.$$

Applying Lemma 4 alone $m-1$ times now gives

$$\partial_{v_r}(\mathfrak{C}_{w_0^{(m)}}) = {}^{\delta_{m-2}}Q^{(1-m,\ldots,-d_r,2-d_r,3-d_r,\ldots,-1,0)}_{(2m-2,\ldots,2d_r,2d_r-3,\ldots,1)} = {}^{\delta_{m-2}}Q^{(1-m,\ldots,\widehat{1-d_r},\ldots,0)}_{\delta_{m-1}+\delta_{m-2}+1^{m-d_r}}.$$

Finally, we use (25) and repeat the above calculation $r-1$ more times to get

$$\mathfrak{C}_{\widehat{w}} = {}^{\delta_{n-1}}Q^{\rho}_{\delta_n+\delta_{n-1}+\xi},$$

where

$$\rho = (1-m,\ldots,\widehat{1-d_r},\ldots,\widehat{1-d_1},\ldots,-1,0) = (1-w_n,\ldots,1-w_1)$$

and

$$\xi = \sum_{j=1}^r 1^{m-d_j-(r-j)} = \sum_{j=1}^r 1^{n+j-d_j} = \sum_{j=1}^r 1^{\lambda_j} = \lambda(w)'.$$

Next consider the general case. Let $p>n$ and suppose that

$$w = (a_1,\ldots,\widehat{a}_{i_1},\ldots,\widehat{a}_{i_{p-n}},\ldots,a_p,-a_{i_{p-n}},\ldots,-a_{i_1},d_1,\ldots,d_r)$$

where $r=m-p$, $0<a_1<\cdots<a_p$ and $0<d_1<\cdots<d_r$. If

$$u := (a_1,\ldots,a_p,d_1,\ldots,d_r)$$

and $\widehat{u} := uw_0^{(p)}$, then

$$\widehat{u} = \widehat{w}v'_{p-n}\cdots v'_1, \tag{27}$$

where $v'_j = s_{p-j}\cdots s_{i_j-j+2}s_{i_j-j+1}$ for $1\leq j\leq p-n$. Now $\mathfrak{C}_{\widehat{u}}$ is known by the previous case, and

$$\mathfrak{C}_{\widehat{w}} = \partial_{v'_{p-n}}\cdots\partial_{v'_1}(\mathfrak{C}_{\widehat{u}}).$$

The proof is now completed by induction, using Lemma 7. The key observation is the following: Suppose that

$$u_0 = \widehat{u} > u_1 > \cdots > u_d = \widehat{w}$$

is the sequence of coverings in the right weak Bruhat order corresponding to the factorization (27), so that $u_{i+1} = u_i s_{r_i}$ with $\ell(u_{i+1}) = \ell(u_i)-1$ for each $i\in[0,d-1]$. Then if $\gamma := \gamma(u_i)$, we have $\gamma_{r_i+1} = \gamma_{r_i}-1$. Therefore Lemma 7 implies that $\gamma(u_{i+1})$ has two equal entries in positions r_i and r_i+1. Moreover, $\gamma(u_j)$ is a partition for all $j\in[0,d]$, and hence $\nu(u_j)$ is the conjugate of $\gamma(u_j)$. □

Remark 2. The work of Anderson and Fulton [AF] associates a partition λ to certain triples of ℓ-tuples of integers which define a class of symplectic degeneracy loci. The shape $\lambda(w)$ of an element $w\in W_\infty$ in Definition 2 (and its even orthogonal counterpart in Definition 5) is consistent with op. cit. In particular, Propositions 4 and 12 follow from the more general formulas for double Schubert polynomials which are established in [AF]. We give here an alternative proof, using [IMN1, Thm. 1.2] and the right divided difference operators.

3.3. **Double theta polynomials and alternating sums.** Let $n\geq 0$ and $w\in W_\infty$ be an n-Grassmannian element. Let $\lambda=\lambda(w)$ be the n-strict partition which corresponds to w, define a sequence $\beta(\lambda) = \{\beta_i(\lambda)\}_{i\geq 1}$ by

$$\beta_i(\lambda) := \begin{cases} w_{n+i}+1 & \text{if } w_{n+i}<0,\\ w_{n+i} & \text{if } w_{n+i}>0,\end{cases} \tag{28}$$

and a set of pairs $\mathcal{C}(\lambda)$ by

$$\mathcal{C}(\lambda) := \{(i,j)\in\mathbb{N}\times\mathbb{N} \mid 1\leq i<j \ \text{ and } \ w_{n+i}+w_{n+j}<0\} \tag{29}$$

(this agrees with the set $\mathcal{C}(\lambda)$ given in the introduction). The *double theta polynomial* ${}^n\Theta_\lambda(X,Y)$ of [TW, W] is defined by

$$ {}^n\Theta_\lambda(X,Y) := \prod_{i<j}(1-R_{ij})\prod_{(i,j)\in\mathcal{C}(\lambda)}(1+R_{ij})^{-1}\,({}^nc)^{\beta(\lambda)}_\lambda. \tag{30}$$

In the above formula, for any integer sequence $\alpha=(\alpha_1,\alpha_2,\ldots)$, we let $({}^nc)^{\beta(\lambda)}_\alpha := \prod_i {}^nc^{\beta_i(\lambda)}_{\alpha_i}$, and the raising operators R_{ij} act by $R_{ij}({}^nc)^{\beta(\lambda)}_\alpha := ({}^nc)^{\beta(\lambda)}_{R_{ij}\alpha}$. Note that ${}^n\Theta_\lambda(X,Y)$ lies in $\Gamma[X_n,Y]$ for any n-strict partition λ. To be precise, the polynomial ${}^n\Theta_\lambda(X,Y)$ is the image of the double theta polynomial $\Theta_\lambda(c\,|\,t)$ of [TW] (with $k=n$) in the ring $\Gamma[X_n,Y]$.

Let $\mathcal{A}:\Gamma[X_n,Y]\to\Gamma[X_n,Y]$ be the operator given by

$$\mathcal{A}(f) := \sum_{w\in W_n}(-1)^{\ell(w)}w(f).$$

Let $w_0 = w_0^{(n)}$ denote the longest element in W_n and set $\widehat{w} := ww_0$.

Theorem 2. *Let λ be an n-strict partition and w be the corresponding n-Grassmannian element of W_∞. Then we have*

$$ {}^n\Theta_\lambda(X,Y) = \partial_{w_0}\left({}^{\nu(\widehat{w})}Q^{\beta(\widehat{w})}_{\lambda(\widehat{w})}\right) \tag{31}$$

$$ = (-1)^{n(n+1)/2}\,\mathcal{A}\left({}^{\nu(\widehat{w})}Q^{\beta(\widehat{w})}_{\lambda(\widehat{w})}\right)\Big/\,\mathcal{A}\left(x^{\delta_n+\delta_{n-1}}\right). \tag{32}$$

Proof. We deduce from (9) that the double Schubert polynomial $\mathfrak{C}_w(X,Y)$ satisfies

$$\mathfrak{C}_w(X,Y) = \partial_{w_0}\left(\mathfrak{C}_{\widehat{w}}(X,Y)\right). \tag{33}$$

The equality (31) follows from (33), Proposition 4, and the fact, proved in [IM, Thm. 1.2], that $\mathfrak{C}_w(X,Y) = {}^n\Theta_\lambda(X,Y)$ in the ring $\Gamma[X_n,Y]$.

To establish the equality (32), recall from [D, Lemma 4] and [PR, Prop. 5.5] that we have

$$\partial_{w_0}(f) = (-1)^{n(n+1)/2}\left(2^n x_1\cdots x_n\prod_{1\le i<j\le n}(x_i^2-x_j^2)\right)^{-1}\cdot\mathcal{A}(f).$$

On the other hand, it follows from [PR, Cor. 5.6(ii)] that

$$\partial_{w_0}(x^{\delta_n+\delta_{n-1}}) = (-1)^{n(n+1)/2}$$

and hence that

$$\mathcal{A}(x^{\delta_n+\delta_{n-1}}) = 2^n x_1\cdots x_n\prod_{1\le i<j\le n}(x_i^2-x_j^2).$$

The proof of (32) is completed by using these two equations in (31). □

4. Single Schubert polynomials of type C

In this section, we work with the single type C Schubert polynomials $\mathfrak{C}_w(X)$. The entire section is inspired by [LS1, LS2, M2] and [PR, LP1].

4.1. **Theta polynomials as Weyl group invariants.** Let $\chi:\Gamma[X_n]\to\mathbb{Z}$ be the homomorphism defined by $\chi(c_p)=\chi(x_j)=0$ for all p,j. In other words, $\chi(f)$ is the constant term of f, for each polynomial $f\in\Gamma[X_n]$.

Proposition 5. *For any $f\in\Gamma[X_n]$, we have $f=\sum_{w\in W^{(n)}}\chi(\partial_w f)\mathfrak{C}_w(X_n)$.*

Proof. By Proposition 1 and linearity, it is only necessary to verify this when f is a Schubert polynomial $\mathfrak{C}_v(X_n)$, $v\in W^{(n)}$. In this case, it follows from the properties of Schubert polynomials in Section 2.1 that $\chi(\partial_w(\mathfrak{C}_v(X_n)))$ is equal to 1 when $w=v$ and equal to zero, otherwise. □

Following [BKT1, Section 5.1], let

$$\Gamma^{(n)} := \mathbb{Z}[{}^nc_1, {}^nc_2, \ldots]$$

be the ring of theta polynomials of level n. Notice that the elements denoted by $\vartheta_r(x\,;y)$ in loc. cit. correspond to the generators ${}^nc_r$ here. According to [BKT1, Thm. 2], the single theta polynomials ${}^n\Theta_\lambda = {}^n\Theta_\lambda(X)$ for all n-strict partitions λ form a $\mathbb{Z}$-basis of $\Gamma^{(n)}$. In the next result, the Weyl group W_n acts on the ring $\Gamma[X_n]$ in the usual way.

Proposition 6. *The ring $\Gamma^{(n)}$ is equal to the subring $\Gamma[X_n]^{W_n}$ of W_n-invariants in $\Gamma[X_n]$.*

Proof. We have $g \in \Gamma[X_n]^{W_n}$ if and only if $s_i g = g$ for all $i \in [0, n-1]$ if and only if $\partial_i g = 0$ for $0 \le i \le n-1$. Suppose that $f \in \Gamma[X_n]^{W_n}$ and employ Proposition 1 to write

$$f(X_n) = \sum_{w \in W^{(n)}} a_w \mathfrak{C}_w(X_n). \tag{34}$$

Applying the divided differences ∂_i for $i \in [0, n-1]$ to (34) and using (9), we deduce that $a_w = 0$ for all $w \in W^{(n)}$ such that $\ell(ws_i) < \ell(w)$ for some $i \in [0, n-1]$. Therefore, f is in the $\mathbb{Z}$-span of those $\mathfrak{C}_w(X_n)$ for $w \in W^{(n)}$ with $\ell(ws_i) > \ell(w)$ for all $i \in [0, n-1]$. These are exactly the n-Grassmannian elements w in W_∞. According to [BKT1, Prop. 6.2], for any such w, we have $\mathfrak{C}_w(X_n) = {}^n\Theta_{\lambda(w)}(X)$ in $\Gamma[X_n]$. It follows that f is a $\mathbb{Z}$-linear combination of theta polynomials of level n, and hence that $f \in \Gamma^{(n)}$. The converse is clear, since $\partial_i h = 0$ for all $i \in [0, n-1]$ and $h \in \Gamma^{(n)}$. □

Example 2. It follows from Proposition 6 that

$$\Gamma^{(n)} \cap \mathbb{Z}[X_n] = \mathbb{Z}[X_n]^{W_n} = \mathbb{Z}[e_1(X_n^2), \ldots, e_n(X_n^2)],$$

where $X_n^2 := (x_1^2, \ldots, x_n^2)$. This can also be seen directly, using the identities

$$({}^n c_p)^2 + 2\sum_{i=1}^{p} (-1)^i \, ({}^n c_{p+i})({}^n c_{p-i}) = e_p(X_n^2)$$

for all $p \ge 0$ (compare with [BKT1, Eqn. (19)]).

Let $I\Gamma^{(n)} = \langle {}^n c_1, {}^n c_2, \ldots \rangle$ be the ideal of $\Gamma[X_n]$ generated by the homogeneous elements in $\Gamma^{(n)}$ of positive degree. For any parabolic subgroup P of Sp_{2n}, let $I\Gamma_P^{(n)}$ be the corresponding ideal of $\Gamma[X_n]^{W_P}$, and set $W_n^P := W^P \cap W_n$. The following result about the cohomology ring of Sp_{2n}/P is an immediate consequence of Theorem 1, Corollary 2 and the discussion in Section 2.

Corollary 3. *There is a canonical ring isomorphism*

$$\mathrm{H}^*(\mathrm{Sp}_{2n}/B) \cong \Gamma[X_n]/I\Gamma^{(n)}$$

which maps the cohomology class of the codimension $\ell(w)$ Schubert variety X_w to the class of the Schubert polynomial $\mathfrak{C}_w(X)$, for any $w \in W_n$. Moreover, for any parabolic subgroup P of Sp_{2n}, there is a canonical ring isomorphism

$$\mathrm{H}^*(\mathrm{Sp}_{2n}/P) \cong \Gamma[X_n]^{W_P}/I\Gamma_P^{(n)}$$

which maps the cohomology class of the codimension $\ell(w)$ Schubert variety X_w to the class of the Schubert polynomial $\mathfrak{C}_w(X)$, for any $w \in W_n^P$.

Example 3. The version of Lemma 2 for single polynomials states that if λ is a strict partition of length ℓ and $p > \max(n, \lambda_1)$, then $Q_{(p,\lambda)} \in I\Gamma^{(n)}$. We can exhibit this containment more explicitly as follows. For any integer $m > n$, we have

$$c_m = \sum_{j=1}^{\infty} (-1)^{j-1} \, c_{m-j} \, {}^n c_j.$$

This implies that for any integer vector $\alpha = (\alpha_1, \ldots, \alpha_\ell)$, the equality

$$c_{(m,\alpha)} = \sum_{j=1}^{\infty} (-1)^{j-1} \, c_{(m-j,\alpha)} \, {}^n c_j$$

holds, and therefore, by applying the Pfaffian operator R^∞, that

$$Q_{(p,\lambda)} = \sum_{j=1}^{\infty} (-1)^{j-1} Q_{(p-j,\lambda)} {}^n c_j. \tag{35}$$

It is important to notice that the terms $Q_{(p-j,\lambda)}$ in (35) can be non-zero even when $p - j < 0$. The 'straightening law' for such terms was found by Hoffman and Humphreys. For any integer k, let $n(k) := \#\{i \mid \lambda_i > |k|\}$, and define the sets

$$A_\lambda := \{r \in [0, p-1] \mid r \neq \lambda_i \text{ for all } i \leq \ell\} \quad \text{and} \quad B_\lambda := \{\lambda_1, \ldots, \lambda_\ell\}.$$

It follows from [HH, Thm. 9.2] that for any integer $k < p$, we have

$$Q_{(k,\lambda)} = \begin{cases} (-1)^{n(k)}\, Q_{\lambda\cup k} & \text{if } k \in A_\lambda, \\ (-1)^{k+n(k)}\, 2\, Q_{\lambda \smallsetminus |k|} & \text{if } |k| \in B_\lambda, \\ 0 & \text{otherwise,} \end{cases}$$

where $\lambda \cup k$ and $\lambda \smallsetminus |k|$ denote the partitions obtained by adding (resp. removing) a part equal to k (resp. $|k|$) from λ. Applying this in (35), we obtain

$$Q_{(p,\lambda)} = \sum_{r\in A_\lambda} (-1)^{p-1-r+n(r)}\, Q_{\lambda\cup r} {}^n c_{p-r} + 2 \sum_{r\in B_\lambda} (-1)^{p-1+n(r)}\, Q_{\lambda\smallsetminus r} {}^n c_{p+r}. \tag{36}$$

The particular case of (36) when $(p,\lambda) = \delta_{n+1}$ reads

$$Q_{\delta_{n+1}} = Q_{\delta_n} {}^n c_{n+1} + 2 \sum_{r=1}^{n} (-1)^r\, Q_{\delta_n \smallsetminus r} {}^n c_{n+1+r}.$$

4.2. **The ring $\Gamma[X_n]$ as a $\Gamma^{(n)}$-module.** Set $e_p := e_p(X_n)$ for each $p \in \mathbb{Z}$, and recall that $e_\alpha := \prod_i e_{\alpha_i}$ for any integer sequence α. Let $\mathcal{P}_n$ denote the set of all strict partitions λ with $\lambda_1 \leq n$.

Proposition 7. *$\Gamma[X_n]$ is a free $\Gamma^{(n)}$-module of rank $2^n n!$ with basis*

$$\{e_\lambda(-X_n) x^\alpha \mid \lambda \in \mathcal{P}_n, \quad 0 \leq \alpha_i \leq n-i, \ i \in [1,n]\}.$$

Proof. It is well known (see e.g [M2, (5.1′)]) that $\Gamma[X_n]$ is a free $\Gamma[e_1,\ldots,e_n]$-module with basis given by the monomials x^α with $0 \leq \alpha_i \leq n-i$ for $i \in [1,n]$. It will therefore suffice to show that $\Gamma[e_1,\ldots,e_n]$ is a free $\Gamma^{(n)}$-module with basis $e_\lambda(-X_n)$ for $\lambda \in \mathcal{P}_n$. Setting $y_j = 0$ for $1 \leq j \leq n$ in equation (21) gives

$$E(X_n,t) := \sum_{p=0}^{\infty} e_p t^p = \left(\sum_{p=0}^{\infty} {}^n c_p\, t^p\right)\left(\sum_{p=0}^{\infty} c_p (-t)^p\right).$$

Using this and the relations (3), we obtain

$$E(X_n,t)E(X_n,-t) = \left(\sum_{p=0}^{\infty} {}^n c_p t^r\right)\left(\sum_{p=0}^{\infty} {}^n c_p (-t)^p\right)$$

and therefore that

$$e_p^2(-X_n) + 2\sum_{i=1}^{p} (-1)^i e_{p+i}(-X_n) e_{p-i}(-X_n) \in \Gamma^{(n)}$$

for each $p \geq 1$. It follows that the monomials $e_\lambda(-X_n)$ for $\lambda \in \mathcal{P}_n$ generate $\Gamma[e_1,\ldots,e_n]$ as a $\Gamma^{(n)}$-module. It remains to prove that these monomials $e_\lambda(-X_n)$ are linearly independent over $\Gamma^{(n)}$.

We claim that the Schubert polynomials $\mathfrak{C}_w(X_n)$ for $w \in W_n$ are linearly independent over $\Gamma^{(n)}$. Indeed, suppose that

$$\sum_{w\in W_n} f_w \mathfrak{C}_w(X_n) = 0$$

for some coefficients $f_w \in \Gamma^{(n)}$, and that $v \in W_n$ is an element of maximal length such that $f_v \neq 0$. Then, by applying (10), we have

$$0 = \partial_v \left(\sum_{w\in W_n} f_w \mathfrak{C}_w(X_n) \right) = f_v \partial_v(\mathfrak{C}_v(X_n)) = f_v,$$

which is a contradiction, proving the claim. We have used here the fact that the divided differences ∂_i are $\Gamma^{(n)}$-linear for each $i \in [0, n-1]$.

It follows that the Schur Q-polynomials $Q_\lambda = Q_\lambda(c)$ for $\lambda \in \mathcal{P}_n$ are linearly independent over $\Gamma^{(n)}$ (since these are exactly the Schubert polynomials $\mathfrak{C}_w(X_n)$ which lie in Γ, with $w = w_\lambda \in W_n$). But the elements $\{Q_\lambda\}$ and $\{c_\lambda\}$ for $\lambda \in \mathcal{P}_n$ are related by an unitriangular change of basis matrix, and so are the elements $\{c_\lambda\}$ and $\{e_\lambda(-X_n)\}$. It follows that the Q_λ for $\lambda \in \mathcal{P}_n$ generate $\Gamma[e_1, \ldots, e_n]$ as a $\Gamma^{(n)}$-module, and hence that the three aforementioned sets each form a basis. $\square$

Following [PR], for any partition λ, the $\widetilde{Q}$-polynomial is defined by

$$\widetilde{Q}_\lambda(X_n) := R^\infty \, e_\lambda(X_n). \tag{37}$$

Corollary 4. *The ring $\Gamma[X_n]$ is a free $\Gamma[X_n]^{S_n}$-module with basis $\{\mathfrak{S}_\varpi(X)\}$ for $\varpi \in S_n$. The ring $\Gamma[X_n]^{S_n}$ is a free $\Gamma^{(n)}$-module with basis $\{\widetilde{Q}_\lambda(-X_n)\}$ for $\lambda \in \mathcal{P}_n$. The ring $\Gamma[X_n]$ is a free $\Gamma^{(n)}$-module on the basis $\{\mathfrak{C}_w(X_n)\}$ of single type C Schubert polynomials for $w \in W_n$, and is also free on the product basis $\{\widetilde{Q}_\lambda(-X_n)\mathfrak{S}_\varpi(X)\}$ for $\lambda \in \mathcal{P}_n$ and $\varpi \in S_n$.*

Proof. Since $\Gamma[X_n]^{S_n} = \Gamma[e_1, \ldots, e_n]$, the first statement follows from Proposition 7 and [M2, (4.11)]. The assertions involving the polynomials $\widetilde{Q}_\lambda(-X_n)$ are justified using Proposition 7 and equation (37), and the fact that the Schubert polynomials $\{\mathfrak{C}_w(X_n)\}$ for $w \in W_n$ form a basis is also clear. $\square$

4.3. **A scalar product on $\Gamma[X_n]$.** Recall that $w_0 = (\overline{1}, \ldots, \overline{n})$ denotes the element of longest length in W_n. If $f \in \Gamma[X_n]$, then $\partial_i(\partial_{w_0} f) = 0$ for all i with $0 \le i \le n-1$. Proposition 6 implies that $\partial_{w_0}(f) \in \Gamma^{(n)}$, for each $f \in \Gamma[X_n]$.

Definition 3. We define a scalar product $\langle\ ,\ \rangle$ on $\Gamma[X_n]$, with values in $\Gamma^{(n)}$, by the rule

$$\langle f, g\rangle := \partial_{w_0}(fg), \quad f, g \in \Gamma[X_n].$$

Proposition 8. *The scalar product $\langle\ ,\ \rangle : \Gamma[X_n] \times \Gamma[X_n] \to \Gamma^{(n)}$ is $\Gamma^{(n)}$-linear. For any $f, g \in \Gamma[X_n]$ and $w \in W_n$, we have*

$$\langle \partial_w f, g\rangle = \langle f, \partial_{w^{-1}} g\rangle.$$

Proof. The scalar product is $\Gamma^{(n)}$-linear, since the same is true for the operator ∂_{w_0}. For the second statement, given $f, g \in \Gamma[X_n]$, it suffices to show that $\langle \partial_i f, g\rangle = \langle f, \partial_i g\rangle$ for $0 \le i \le n-1$. We have

$$\langle \partial_i f, g\rangle = \partial_{w_0}((\partial_i f)g) = \partial_{w_0 s_i}\partial_i((\partial_i f)g) = \partial_{w_0 s_i}((\partial_i f)(\partial_i g))$$

because $s_i(\partial_i f) = \partial_i f$. The expression on the right is symmetric in f and g, hence

$$\langle \partial_i f, g\rangle = \langle \partial_i g, f\rangle = \langle f, \partial_i g\rangle,$$

as required.

□

Proposition 9. *Let $u, v \in W_n$ be such that $\ell(u)+\ell(v) = n^2$. Then we have*

$$\langle \mathfrak{C}_u(X_n), \mathfrak{C}_v(X_n)\rangle = \begin{cases} 1 & \textit{if } v = w_0u, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Using (10) and Proposition 8, we obtain

$$\langle \mathfrak{C}_u(X_n), \mathfrak{C}_v(X_n)\rangle = \langle \partial_{u^{-1}w_0}\mathfrak{C}_{w_0}(X_n), \mathfrak{C}_v(X_n)\rangle = \langle \mathfrak{C}_{w_0}(X_n), \partial_{w_0u}\mathfrak{C}_v(X_n)\rangle.$$

Also $\ell(w_0u) = \ell(w_0) - \ell(u) = \ell(v)$, and we deduce that

$$\partial_{w_0u}\mathfrak{C}_v(X_n) = \begin{cases} 1 & \text{if } v = w_0u, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\langle \mathfrak{C}_{w_0}(X_n), 1\rangle = \partial_{w_0}(\mathfrak{C}_{w_0}(X_n)) = 1$, the result follows. □

Although the elements of the $\Gamma^{(n)}$-basis $\{\widetilde{Q}_\lambda(-X_n)\mathfrak{S}_\varpi(X)\}$ of $\Gamma[X_n]$ do not represent the Schubert classes on the symplectic flag manifold, this product basis is convenient for computational purposes. Indeed, following Lascoux and Pragacz [LP1] (in the finite case), one can identify the dual $\Gamma^{(n)}$-basis of $\Gamma[X_n]$ relative to the scalar product $\langle\ ,\ \rangle$, by working as shown below.

Let $\varpi_0 = (n, n-1, \ldots, 1)$ denote the permutation of longest length in S_n, and define $v_0 := w_0\varpi_0 = \varpi_0 w_0$. We have

$$\partial_{w_0} = \partial_{v_0}\partial_{\varpi_0} = \partial_{\varpi_0}\partial_{v_0}. \tag{38}$$

We define a $\Gamma[X_n]^{S_n}$-valued scalar product $(\ ,\ )$ on $\Gamma[X_n]$ by the rule

$$(f, g) := \partial_{\varpi_0}(fg), \quad f, g \in \Gamma[X_n].$$

According to [M2, (5.12)], the Schubert polynomials $\mathfrak{S}_u(X)$ for $u \in S_n$ satisfy the orthogonality relation

$$(\,\mathfrak{S}_u(X),\, \varpi_0\mathfrak{S}_{u'\varpi_0}(-X)\,) = \delta_{u,u'}$$

for any $u, u' \in S_n$.

Furthermore, define a $\Gamma^{(n)}$-valued scalar product $\{\ ,\ \}$ on $\Gamma[X_n]^{S_n}$ by the rule

$$\{f, g\} := \partial_{v_0}(fg), \quad f, g \in \Gamma[X_n]^{S_n}.$$

According to [PR, Thm. 5.23], for any two partitions $\lambda, \mu \in \mathcal{P}_n$, we have

$$\left\{\widetilde{Q}_\lambda(-X_n), \widetilde{Q}_{\delta_n \smallsetminus \mu}(-X_n)\right\} = \delta_{\lambda,\mu},$$

where $\delta_n \smallsetminus \mu$ is the strict partition whose parts complement the parts of μ in the set $\{n, n-1, \ldots, 1\}$, and $\delta_{\lambda,\mu}$ denotes the Kronecker delta.

Observe that $(\ ,\ )$ is $\Gamma[X_n]^{S_n}$-linear and $\{\ ,\ \}$ is $\Gamma^{(n)}$-linear. Then (38) gives

$$\langle f, g\rangle = \{(f, g)\}, \quad \text{for any } f, g \in \Gamma[X_n],$$

and moreover the orthogonality relation

$$\left\langle \widetilde{Q}_\lambda(-X_n)\mathfrak{S}_u(X), \widetilde{Q}_{\delta_n\smallsetminus\mu}(-X_n)(\varpi_0\mathfrak{S}_{u'\varpi_0}(-X))\right\rangle = \delta_{u,u'}\delta_{\lambda,\mu}$$

holds, for any $u, u' \in S_n$ and $\lambda, \mu \in \mathcal{P}_n$. The reader should compare this to the discussion in [LP1, Section 1].

5. Double Schubert polynomials of types B and D

5.1. **Preliminaries.** Let $b := (b_1, b_2, \ldots)$ be a sequence of commuting variables, and set $b_0 := 1$ and $b_p = 0$ for $p < 0$. Consider the graded ring Γ' which is the quotient of the polynomial ring $\mathbb{Z}[b]$ modulo the ideal generated by the relations

$$b_p^2 + 2\sum_{i=1}^{p-1}(-1)^i b_{p+i}b_{p-i} + (-1)^p b_{2p} = 0, \quad \text{for all } p \geq 1.$$

The ring Γ' is isomorphic to the ring of Schur P-functions. Following [P], the P-functions map naturally to the Schubert classes on maximal (odd or even) orthogonal Grassmannians. We regard Γ as a subring of Γ' via the injective ring homomorphism which sends c_p to $2b_p$ for every $p \geq 1$.

The Weyl group for the root system of type B_n is the same group W_n as the one for type C_n. The Ikeda-Mihalcea-Naruse type B double Schubert polynomials $\mathfrak{B}_w(X,Y)$ for $w \in W_\infty$ form a natural $\mathbb{Z}[Y]$-basis of $\Gamma'[X,Y]$. For any Weyl group element w, the polynomial $\mathfrak{B}_w(X,Y)$ satisfies

$$\mathfrak{B}_w(X,Y) = 2^{-s(w)}\mathfrak{C}_w(X,Y),$$

where $s(w)$ denotes the number of indices i such that $w_i < 0$. The algebraic theory of these polynomials is thus nearly identical to that in type C, provided that one uses coefficients in the ring $\mathbb{Z}[\frac{1}{2}]$.

If $\mathfrak{B}'_w = \mathfrak{B}'_w(X,Y)$ is the polynomial obtained from $\mathfrak{B}_w(X,Y)$ by setting $x_j = y_j = 0$ for all $j > n$, then the $\mathfrak{B}'_w$ for $w \in W^{(n)}$ form a $\mathbb{Z}[Y_n]$-basis of $\Gamma'[X_n, Y_n]$. The polynomials $\mathfrak{B}'_w$ for $w \in W_n$ represent the equivariant Schubert classes on the odd orthogonal flag manifold SO_{2n+1}/B, whose equivariant cohomology ring (with $\mathbb{Z}[\frac{1}{2}]$-coefficients) is isomorphic to that of Sp_{2n}/B. For further details, the reader may consult the references [IMN1] and [T5, Section 6.3.1].

In the rest of this paper we discuss the corresponding theory for the even orthogonal group, that is, in Lie type D, and assume that $n \geq 2$. The Weyl group $\widetilde{W}_n$ for the root system D_n is the subgroup of W_n consisting of all signed permutations with an even number of sign changes. The group $\widetilde{W}_n$ is an extension of S_n by the element $s_\square = s_0 s_1 s_0$, which acts on the right by

$$(w_1, w_2, \ldots, w_n)s_\square = (\overline{w}_2, \overline{w}_1, w_3, \ldots, w_n).$$

There is a natural embedding $\widetilde{W}_k \hookrightarrow \widetilde{W}_{k+1}$ of Weyl groups defined by adjoining the fixed point $k+1$, and we let $\widetilde{W}_\infty := \cup_k \widetilde{W}_k$. The elements of the set $\mathbb{N}_\square := \{\square, 1, \ldots\}$ index the simple reflections in $\widetilde{W}_\infty$. The *length* $\ell(w)$ of an element $w \in \widetilde{W}_\infty$ is defined as in type C. The element of longest length $\widetilde{w}_0 = \widetilde{w}_0^{(n)}$ in $\widetilde{W}_n$ satisfies

$$\widetilde{w}_0 = \begin{cases} (\overline{1}, \ldots, \overline{n}) & \text{if } n \text{ is even,} \\ (1, \overline{2}, \ldots, \overline{n}) & \text{if } n \text{ is odd.} \end{cases}$$

We define an action of $\widetilde{W}_\infty$ on $\Gamma'[X,Y]$ by ring automorphisms as follows. The simple reflections s_i for $i > 0$ act by interchanging x_i and x_{i+1} and leaving all the remaining variables fixed. The reflection $s_\Box$ maps (x_1,x_2) to $(-x_2,-x_1)$, fixes the x_j for $j\geq 3$ and all the y_j, and satisfies, for any $p\geq 1$,

$$s_\Box(b_p) := b_p + (x_1+x_2)\sum_{j=0}^{p-1}\left(\sum_{a+b=j} x_1^a x_2^b\right) c_{p-1-j}.$$

For each $i\in\mathbb{N}_\Box$, define the divided difference operator ∂_i^x on $\Gamma'[X,Y]$ by

$$\partial_\Box^x f := \frac{f-s_\Box f}{-x_1-x_2}, \qquad \partial_i^x f := \frac{f-s_i f}{x_i-x_{i+1}} \quad \text{for } i\geq 1.$$

Consider the ring involution $\omega:\Gamma'[X,Y]\to\Gamma'[X,Y]$ determined by

$$\omega(x_j)=-y_j, \qquad \omega(y_j)=-x_j, \qquad \omega(b_p)=b_p$$

and set $\partial_i^y := \omega\partial_i^x\omega$ for each $i\in\mathbb{N}_\Box$.

The Ikeda-Mihalcea-Naruse double Schubert polynomials $\mathfrak{D}_w=\mathfrak{D}_w(X,Y)$ for $w\in\widetilde{W}_\infty$ are the unique family of elements of $\Gamma'[X,Y]$ satisfying the equations

$$\partial_i^x\mathfrak{D}_w = \begin{cases}\mathfrak{D}_{ws_i} & \text{if } \ell(ws_i)<\ell(w),\\ 0 & \text{otherwise,}\end{cases} \qquad \partial_i^y\mathfrak{D}_w = \begin{cases}\mathfrak{D}_{s_iw} & \text{if } \ell(s_iw)<\ell(w),\\ 0 & \text{otherwise,}\end{cases} \tag{39}$$

for all $i\in\mathbb{N}_\Box$, together with the condition that the constant term of $\mathfrak{D}_w$ is 1 if $w=1$, and 0 otherwise.

The operators $\partial_i := \partial_i^x$ for $i\in\mathbb{N}_\Box$ satisfy the same Leibnitz rule (24) as in the type C case, and for any $w\in\widetilde{W}_\infty$, the divided difference operator ∂_w is defined as before. For any $u,w\in\widetilde{W}_\infty$, we have

$$\partial_u\mathfrak{D}_w(X,Y) = \begin{cases}\mathfrak{D}_{wu^{-1}}(X,Y) & \text{if } \ell(wu^{-1})=\ell(w)-\ell(u),\\ 0 & \text{otherwise.}\end{cases}$$

5.2. **The set $\widetilde{W}^{(n)}$ and the ring $\Gamma'[X_n,Y_n]$.** It is known that the $\mathfrak{D}_w$ for $w\in\widetilde{W}_\infty$ form a $\mathbb{Z}[Y]$-basis of $\Gamma'[X,Y]$. Let $\mathfrak{D}_w^{(n)}=\mathfrak{D}_w^{(n)}(X_n,Y_n)$ be the polynomial obtained from $\mathfrak{D}_w(X,Y)$ by setting $x_j=y_j=0$ for all $j>n$. For every $n\geq 1$, let

$$\widetilde{W}^{(n)} := \{w\in\widetilde{W}_\infty \mid w_{n+1}<w_{n+2}<\cdots\}.$$

Let $\mathfrak{D}_w(X):=\mathfrak{D}_w(X,0)$ denote the single Schubert polynomial.

Proposition 10. *The $\mathfrak{D}_w(X)$ for $w\in\widetilde{W}^{(n)}$ form a $\mathbb{Z}$-basis of $\Gamma'[X_n]$, and a $\mathbb{Z}[Y]$-basis of $\Gamma'[X_n,Y]$. The $\mathfrak{D}_w^{(n)}$ for $w\in\widetilde{W}^{(n)}$ form a $\mathbb{Z}[Y_n]$-basis of $\Gamma'[X_n,Y_n]$.*

Proof. The argument is the same as for the proofs of Propositions 1, 2, and Corollary 1 in Section 2. □

5.3. **The geometrization map π_n'.** The double Schubert polynomials $\mathfrak{D}_w^{(n)}(X,Y)$ for $w\in\widetilde{W}_n$ represent the equivariant Schubert classes on the even orthogonal flag manifold. Let $\{e_1,\ldots,e_{2n}\}$ denote the standard orthogonal basis of $E:=\mathbb{C}^{2n}$ and let F_i be the subspace spanned by the first i vectors of this basis, so that $F_{n-i}^\perp=F_{n+i}$ for $0\leq i\leq n$. We say that two maximal isotropic subspaces L and L' of E are *in the same family* if $\dim(L\cap L')\equiv n \pmod 2$. The orthogonal flag manifold $\mathcal{M}_n'$ parametrizes complete flags $E_\bullet$ in E with $E_{n-i}^\perp=E_{n+i}$ for $0\leq i\leq n$, and E_n in the same family as $\langle e_{n+1},\ldots,e_{2n}\rangle$. Equivalently, E_n is in the same family as F_n, if n is even, and in the opposite family, if n is odd. We have that

$\mathcal{M}'_n = \mathrm{SO}_{2n}/B$ for a Borel subgroup B of the orthogonal group $\mathrm{SO}_{2n} = \mathrm{SO}_{2n}(\mathbb{C})$. If T denotes the associated maximal torus in B, then the T-equivariant cohomology ring $\mathrm{H}^*_T(\mathcal{M}'_n)$ is a $\mathbb{Z}[Y_n]$-algebra, where y_i is identified with the equivariant Chern class $-c_1^T(F_{n+1-i}/F_{n-i})$, for $1 \leq i \leq n$.

The Schubert varieties in $\mathcal{M}'_n$ are the closures of the B-orbits, and are indexed by the elements of $\widetilde{W}_n$. Concretely, any $w \in \widetilde{W}_n$ corresponds to a Schubert variety $X_w = X_w(F_\bullet)$ of codimension $\ell(w)$, which is the closure of the B-orbit

$$X_w^\circ := \{E_\bullet \in \mathcal{M}'_n \mid \dim(E_r \cap F_s) = d'_w(r,s) \text{ for } 1 \leq r \leq n-1,\, 1 \leq s \leq 2n\},$$

where $d'_w(r,s)$ denotes the rank function defined as follows. There is a group monomorphism $\zeta : \widetilde{W}_n \hookrightarrow S_{2n}$, defined by restricting the map ζ of Section 2.3 to $\widetilde{W}_n$. Then $d'_w(r,s)$ equals the number of $i \leq r$ such that $\zeta(\widetilde{w}_0 w \widetilde{w}_0)_i > 2n - s$. Since X_w is stable under the action of T, we obtain an *equivariant Schubert class* $[X_w]^T := [ET \times^T X_w]$ in $\mathrm{H}^*_T(\mathcal{M}'_n)$.

Following [IMN1], there is a surjective homomorphism of graded $\mathbb{Z}[Y_n]$-algebras

$$\pi'_n : \Gamma'[X_n, Y_n] \to \mathrm{H}^*_T(\mathcal{M}'_n)$$

such that

$$\pi'_n(\mathfrak{D}_w^{(n)}) = \begin{cases} [X_w]^T & \text{if } w \in \widetilde{W}_n, \\ 0 & \text{if } w \in \widetilde{W}^{(n)} \smallsetminus \widetilde{W}_n. \end{cases} \tag{40}$$

We let E_i denote the i-th tautological vector vector bundle over $\mathcal{M}'_n$, for $0 \leq i \leq 2n$. The map π'_n is defined by the equations

$$\pi'_n(x_i) = c_1^T(E_{n+1-i}/E_{n-i}) \ \text{ and } \ \pi'_n(b_p) = \frac{1}{2}\, c_p^T(E - E_n - F_n) \tag{41}$$

for $1 \leq i \leq n$ and $p \geq 1$.

Remark 3. The above convention on the family of E_n in the definition of $\mathcal{M}'_n$ differs from that stated in [T5, Section 6.3.2] and [T6, §4.1], and corrects these latter two references. This is necessary in order for the formulas (40) and (41) to hold, which are directly analogous to the ones for the Lie types B and C.

5.4. **The kernel of the map π'_n.** In the following discussion, it suffices to work with coefficients in the ring $\mathbb{Z}[\frac{1}{2}]$, but for ease of notation we will employ the rational numbers $\mathbb{Q}$ instead. For any abelian group A, let $A_\mathbb{Q} := A \otimes_\mathbb{Z} \mathbb{Q}$, and use the tensor product to extend π'_n to a homomorphism of $\mathbb{Q}[Y_n]$-algebras

$$\pi'_n : \Gamma'[X_n, Y_n]_\mathbb{Q} \to \mathrm{H}^*_T(\mathcal{M}'_n)_\mathbb{Q}.$$

Definition 4. Define

$$\widetilde{b}_n := \sum_{j=0}^{n-1} b_{n-j} e_j(-Y_n),$$

let

$$\widehat{B}^{(n)} := \mathbb{Z}[\widetilde{b}_n, {}^n c_1^n, {}^n c_2^n, \ldots],$$

and let $\widehat{\mathrm{I}B}^{(n)}$ be the ideal of $\Gamma'[X_n, Y_n]_\mathbb{Q}$ generated by the homogeneous elements in $\widehat{B}^{(n)}$ of positive degree.

Lemma 8. *We have* $\widehat{\mathrm{I}B}^{(n)} \subset \operatorname{Ker} \pi'_n$.

Proof. Let $A_n := (a_1, \ldots, a_n)$ and $\mathrm{H}_n := \mathbb{Q}[A_n, Y_n]/L_n$, where L_n is the ideal of $\mathbb{Q}[A_n, Y_n]$ generated by the differences $e_i(A_n^2) - e_i(Y_n^2)$ for $1 \leq i \leq n-1$ and the difference $e_n(A_n) - e_n(-Y_n)$. It is known that the equivariant cohomology ring $\mathrm{H}_T^*(\mathcal{M}'_n)_{\mathbb{Q}}$ is canonically isomorphic to H_n as a $\mathbb{Q}[Y_n]$-algebra (compare with [F2, Section 3]). The geometrization map $\pi'_n : \Gamma'[X_n, Y_n]_{\mathbb{Q}} \to \mathrm{H}_n$ satisfies $\pi'_n(x_j) = -a_j$ for $1 \leq j \leq n$, while

$$\pi'_n(b_p) := \frac{1}{2} \sum_{i+j=p} e_i(A_n) h_j(Y_n), \quad p \geq 1.$$

The element $e_n(A_n) - e_n(-Y_n)$ is thus identified with the difference $(-1)^n c_n^T(E_n) - c_n^T(F_n)$. Our conventions on the families of E_n and F_n imply that the latter class vanishes in $\mathrm{H}_T^*(\mathcal{M}'_n)_{\mathbb{Q}}$, by a result of Edidin and Graham [EG, Thm. 1].

We deduce that ${}^n c_p^n \in \mathrm{Ker}\, \pi'_n$ for each $p \geq 1$ as in the proof of Lemma 1, so it suffices to check that $\widetilde{b}_n \in \mathrm{Ker}\, \pi'_n$. Indeed, we have

$$\pi'_n(2\widetilde{b}_n) = \sum_{j=0}^{n-1} e_j(-Y_n) \sum_{\alpha+\beta=n-j} e_\alpha(A_n) h_\beta(Y_n)$$
$$= \sum_{\alpha=1}^{n} e_\alpha(A_n) \sum_{j=0}^{n-\alpha} e_j(-Y_n) h_{n-\alpha-j}(Y_n) + \sum_{j=0}^{n-1} e_j(-Y_n) h_{n-j}(Y_n)$$
$$= e_n(A_n) - e_n(-Y_n).$$

□

If $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_\ell)$ is a strict partition, let $\widetilde{w}_\lambda$ be the corresponding increasing element of $\widetilde{W}_\infty$, so that the negative components of $\widetilde{w}_\lambda$ are exactly $-\lambda_1 - 1, \ldots, -\lambda_\ell - 1$ and possibly also -1, depending on the parity of $\ell = \ell(\lambda)$.

Lemma 9. *If λ is a strict partition with $\lambda_1 \geq n$, then $\mathfrak{D}_{\widetilde{w}_\lambda}^{(n)}(X_n, Y_n) \in \widehat{\mathrm{I}B}^{(n)}$.*

Proof. For each strict partition μ of length ℓ, let $P_\mu := \mathfrak{D}_{\widetilde{w}_\mu}^{(n)}(X_n, Y_n)$. According to [IMN1, Thm. 6.6] and [IMN2, (2.11)], we have the Pfaffian recursion

$$P_\mu = \sum_{j=2}^{\ell} (-1)^j P_{\mu_1, \mu_j} P_{\mu_2, \ldots, \widehat{\mu}_j, \ldots, \mu_\ell}. \tag{42}$$

Moreover, it follows from [IMN2, Prop. 2.1] that every factor P_{μ_1,μ_j} in (42) is a $\mathbb{Z}[Y_n]$-linear combination of products $P_r P_s$ with $r \geq \mu_1$.

It is easy to show that for any integer $r \geq 1$,

$$P_r = \begin{cases} \sum_{j=0}^{r-1} b_{r-j} e_j(-Y_r) & \text{if } r \leq n, \\ \sum_{j=0}^{n} b_{r-j} e_j(-Y_n) & \text{if } r > n. \end{cases}$$

It follows that $P_r = \frac{1}{2}\overline{Q}_r^{1-r}$ for all $r > n$, while $P_n = \widetilde{b}_n$. We now deduce from equation (16) and Lemma 2 that $P_r \in \widehat{\mathrm{I}B}^{(n)}$ for every $r \geq n$. The proof is finished by combining this fact with (42). □

Lemma 10. *For any $w \in \widetilde{W}_\infty \smallsetminus \widetilde{W}_n$, we have $\mathfrak{D}_w^{(n)} \in \widehat{\mathrm{I}B}^{(n)}$.*

Proof. Let w be an element of $\widetilde{W}_\infty$ and $i<j$. Following [B, Lemma 2], we have $\ell(w\bar{t}_{ij})=\ell(w)+1$ if and only if (i) $-w_i<w_j$, and (ii) there is no $p<i$ such that $-w_j<w_p<w_i$, and no $p<j$ such that $-w_i<w_p<w_j$.

The group $\widetilde{W}_\infty$ acts on the polynomial ring $\mathbb{Z}[y_1,y_2,\ldots]$, with s_i for $i\geq 1$ interchanging y_i and y_{i+1} and leaving all the remaining variables fixed, and $s_\Box$ mapping (y_1,y_2) to $(-y_2,-y_1)$ and fixing the y_j with $j\geq 3$. Let $w\in\widetilde{W}_\infty$ be non-increasing, let r be the last positive descent of w, let $s:=\max(i>r\mid w_i<w_r)$, and let $v:=wt_{rs}$. According to [IMN1, Prop. 6.12], the double Schubert polynomials $\mathfrak{D}_u=\mathfrak{D}_u(X,Y)$ satisfy the *transition equations*

$$\mathfrak{D}_w=(x_r-v(y_r))\mathfrak{D}_v+\sum_{\substack{1\leq i<r\\ \ell(vt_{ir})=\ell(w)}}\mathfrak{D}_{vt_{ir}}+\sum_{\substack{i\geq 1,i\neq r\\ \ell(v\bar{t}_{ir})=\ell(w)}}\mathfrak{D}_{v\bar{t}_{ir}} \tag{43}$$

in $\Gamma'[X,Y]$. The recursion (43) terminates in a $\mathbb{Z}[X,Y]$-linear combination of elements $\mathfrak{D}_{\widetilde{w}_\nu}(X,Y)$ for strict partitions ν.

For any $w\in\widetilde{W}_\infty$, let $\mu(w)$ denote the strict partition whose parts are the elements of the set $\{|w_i|-1\ :\ w_i<0\}$. Clearly we have $\mu(w)=\mu(wu)$ for any $u\in S_\infty$. In equation (43), we therefore have $\mu(v)=\mu(vt_{ir})=\mu(w)$. Moreover, condition (i) above shows that the parts of $\mu(v\bar{t}_{ir})$ are greater than or equal to the parts of $\mu(w)$. In particular, if $\mu(w)_1\geq n$, then $\mu(v\bar{t}_{ir})_1\geq n$.

Assume first that $w\in\widetilde{W}_{n+1}\smallsetminus\widetilde{W}_n$. If $w_i=-n-1$ for some $i\leq n+1$, we use the transition recursion (43) to write $\mathfrak{D}_w^{(n)}$ as a $\mathbb{Z}[X_n,Y_n]$-linear combination of elements $\mathfrak{D}^{(n)}_{\widetilde{w}_\nu}$ for strict partitions ν with $\nu_1\geq n$. Lemma 9 now implies that $\mathfrak{D}_w^{(n)}\in\widehat{\mathrm{I}B}^{(n)}$.

We next suppose that $w_i=n+1$ for some $i\leq n$. Let

$$\{v_2,\ldots,v_n\}:=\{w_1,\ldots,\widehat{w_i},\ldots,w_n\}$$

with $v_2>\cdots>v_n$, and define

$$u:=(\overline{v_2},n+1,v_3,\ldots,v_n,w_{n+1})\in\widetilde{W}_{n+1}$$

and

$$\overline{u}:=us_\Box=(\overline{n+1},v_2,v_3,\ldots,v_n,w_{n+1}).$$

Then we have $\mathfrak{D}^{(n)}_{\overline{u}}\in\widehat{\mathrm{I}B}^{(n)}$ from the previous case, and $\partial_\Box(\mathfrak{D}^{(n)}_{\overline{u}})=\mathfrak{D}^{(n)}_u$.

For any i such that $\Box\leq i\leq n-1$, it is easy to verify that $s_i({}^nc^n_p)={}^nc^n_p$ and $s_i(\widetilde{b}_n)=\widetilde{b}_n$, and hence that $\partial_i({}^nc_p^n)=\partial_i(\widetilde{b}_n)=0$. We therefore obtain that $\partial_i(\widehat{\mathrm{I}B}^{(n)})\subset\widehat{\mathrm{I}B}^{(n)}$ for all $i\in[\Box,n-1]$. It follows that $\mathfrak{D}^{(n)}_u\in\widehat{\mathrm{I}B}^{(n)}$, and the proof is now concluded in the same way as in Lemma 3. $\Box$

Theorem 3. *Let* $J'_n:=\bigoplus_{w\in\widetilde{W}^{(n)}\smallsetminus\widetilde{W}_n}\mathbb{Q}[Y_n]\mathfrak{D}^{(n)}_w$. *Then we have*

$$\widehat{\mathrm{I}B}^{(n)}=J'_n=\sum_{w\in\widetilde{W}_\infty\smallsetminus\widetilde{W}_n}\mathbb{Q}[Y_n]\mathfrak{D}^{(n)}_w=\operatorname{Ker}\pi'_n.$$

We have a canonical isomorphism of $\mathbb{Q}[Y_n]$*-algebras*

$$\mathrm{H}^*_T(\mathrm{SO}_{2n}/B,\mathbb{Q})\cong\Gamma'[X_n,Y_n]_{\mathbb{Q}}/\widehat{\mathrm{I}B}^{(n)}.$$

Proof. The argument is the same as the proof of Theorem 1, this time using Lemma 8, Lemma 10, and Proposition 10. $\Box$

5.5. **Partial even orthogonal flag manifolds.** We can generalize the presentation in Theorem 3 to the partial flag manifolds SO_{2n}/P, where P is a parabolic subgroup of SO_{2n}. The parabolic subgroups P containing B correspond to sequences $\mathfrak{a} \,:\, a_1 < \cdots < a_p$ of elements of $\mathbb{N}_\Box$ with $a_p < n$. The manifold SO_{2n}/P parametrizes partial flags of subspaces

$$0 \subset E_1 \subset \cdots \subset E_p \subset E = \mathbb{C}^{2n}$$

with $\dim(E_j) = n - a_{p+1-j}$ for each $j \in [1,p]$ and E_p isotropic. If $a_1 = \Box$, so that $\dim(E_p) = n$, then we insist that the family of E_p obeys the same convention as in Section 5.3.

A sequence $\mathfrak{a}$ as above parametrizes the parabolic subgroup $\widetilde{W}_P$ of $\widetilde{W}_n$, which is generated by the simple reflections s_i for $i \notin \{a_1,\ldots,a_p\}$. Let $\Gamma'[X_n,Y_n]_{\mathbb{Q}}^{\widetilde{W}_P}$ be the subring of elements in $\Gamma'[X_n,Y_n]_{\mathbb{Q}}$ which are fixed by the action of $\widetilde{W}_P$, i.e.,

$$\Gamma'[X_n,Y_n]_{\mathbb{Q}}^{\widetilde{W}_P} = \{f \in \Gamma'[X_n,Y_n]_{\mathbb{Q}} \mid s_i(f) = f,\ \forall\, i \notin \{a_1,\ldots,a_p\},\ i<n\}.$$

Then $\Gamma'[X_n,Y_n]_{\mathbb{Q}}^{\widetilde{W}_P}$ is a $\mathbb{Q}[Y_n]$-subalgebra of $\Gamma'[X_n,Y_n]_{\mathbb{Q}}$. Let $\widetilde{W}^P \subset \widetilde{W}^{(n)}$ denote the set

$$\widetilde{W}^P := \{w \in \widetilde{W}^{(n)} \mid \ell(ws_i) = \ell(w)+1,\ \forall\, i \notin \{a_1,\ldots,a_p\},\ i<n\}.$$

Then by arguing as in Section 2.5, we obtain the following two results.

Proposition 11. *We have*

$$\Gamma'[X_n,Y_n]_{\mathbb{Q}}^{\widetilde{W}_P} = \bigoplus_{w\in\widetilde{W}^P} \mathbb{Q}[Y_n]\mathfrak{D}_w^{(n)}. \tag{44}$$

Corollary 5. *There is a canonical isomorphism of $\mathbb{Q}[Y_n]$-algebras*

$$\mathrm{H}_T^*(\mathrm{SO}_{2n}/P,\mathbb{Q}) \cong \Gamma'[X_n,Y_n]_{\mathbb{Q}}^{\widetilde{W}_P}/\mathrm{I}\widehat{B}_P^{(n)},$$

where $\mathrm{I}\widehat{B}_P^{(n)}$ is the ideal of $\Gamma'[X_n,Y_n]_{\mathbb{Q}}^{W_P}$ generated by the homogeneous elements in $\widehat{B}^{(n)}$ of positive degree.

6. Divided differences and double eta polynomials

6.1. **Preliminaries.** Fix $k \geq 0$, and set ${}^k c_p = {}^k c_p(X) := \sum_{i=0}^p c_{p-i} h_i^{-k}(X)$. Define ${}^k b_p := {}^k c_p$ for $p<k$, ${}^k b_p := \frac{1}{2}{}^k c_p$ for $p>k$, and set

$${}^k b_k := \frac{1}{2}{}^k c_k + \frac{1}{2}e_k^k(X) \quad \text{and} \quad {}^k\widetilde{b}_k := \frac{1}{2}{}^k c_k - \frac{1}{2}e_k^k(X).$$

Let f_k be an indeterminate of degree k, which will equal ${}^k b_k$, ${}^k\widetilde{b}_k$, or $\frac{1}{2}\,{}^k c_k$ in the sequel. We also let $f_0 \in \{0,1\}$. For any $p,r \in \mathbb{Z}$, define ${}^k\widehat{c}_p^{\,r}$ by

$${}^k\widehat{c}_p^{\,r} := {}^k c_p^r + \begin{cases} (2f_k - {}^k c_k)e_{p-k}^{p-k}(-Y) & \text{if } r = k-p < 0, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $\omega({}^k c_p^r) = {}^{-r} c_p^{-k}$ for any $k,r\in\mathbb{Z}$, and if $r \leq 0 \leq k$, then

$$\omega({}^k\widehat{c}_p^{\,r}) = {}^{-r}\widehat{c}_p^{\,-k}.$$

We now have the following even orthogonal analogues of Lemmas 4 and 5, which are dual versions of results from [T6, Prop. 1 and Prop. 2] and [T7].

Lemma 11. *Suppose that $k,p,r\in\mathbb{Z}$ and $i\geq 1$.*

(a) *We have*

$$\partial_i({}^k c_p^r) = \begin{cases} {}^{k-1}c_{p-1}^r & \text{if } k=\pm i, \\ 0 & \text{otherwise.} \end{cases}$$

(b) *If $p>k\geq 0$, we have*

$$\partial_i({}^k\widehat{c}_p^{\,k-p}) = \begin{cases} {}^{k-1}\widehat{c}_{p-1}^{\,k-p} & \text{if } i=p-k\geq 2, \\ 2\omega(f_k) & \text{if } i=p-k=1, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 12. *Suppose that $k,p,r\in\mathbb{Z}$ and $r\leq 0$. We have*

$$\partial_\square\left({}^k c_p^r\right) = \begin{cases} {}^{-2}c_{p-1}^r & \text{if } k=-1, \\ 2\left({}^{-2}c_{p-1}^r\right) & \text{if } k=0, \\ 2\left({}^{-1}c_{p-1}^r\right) - {}^0c_{p-1}^r & \text{if } k=1, \\ 0 & \text{if } |k|\geq 2. \end{cases}$$

For $s\in\{0,1\}$, define

$$f_k^s := f_k + \sum_{j=1}^{k} {}^k c_{k-j} h_j^s(-Y),$$

set $\widetilde{f}_k := {}^k c_k - f_k$ and $\widetilde{f}_k^s := {}^k c_k - 2f_k + f_k^s$.

Lemma 13. *Suppose that $k,p\in\mathbb{Z}$ with $p>k$. Then we have*

$$\partial_\square\left({}^k\widehat{c}_p^{\,k-p}\right) = \begin{cases} 2\omega(\widetilde{f}_k^1) & \text{if } k-p=-1, \\ 0 & \text{if } k-p<-1. \end{cases}$$

Lemma 14. *Suppose that $k\geq 0$ and $r\geq 1$. Then we have*

$${}^k\widehat{c}_p^{\,-r} = {}^{k+1}\widehat{c}_p^{\,-r+1} - (x_{k+1}+y_r)\,{}^k\widehat{c}_{p-1}^{\,-r+1}.$$

Let ρ be a composition and let $\alpha=(\alpha_1,\ldots,\alpha_\ell)$ and $\beta=(\beta_1,\ldots,\beta_\ell)$ be two integer vectors. Define

$${}^\rho\widehat{c}_\alpha^\beta := {}^{\rho_1}\widehat{c}_{\alpha_1}^{\beta_1}\,{}^{\rho_2}\widehat{c}_{\alpha_2}^{\beta_2}\cdots,$$

where, for each $i\geq 1$,

$${}^{\rho_i}\widehat{c}_{\alpha_i}^{\beta_i} := {}^{\rho_i}c_{\alpha_i}^{\beta_i} + \begin{cases} (-1)^i e_{\rho_i}^{\rho_i}(X)e_{\alpha_i-\rho_i}^{\alpha_i-\rho_i}(-Y) & \text{if } \beta_i=\rho_i-\alpha_i<0, \\ 0 & \text{otherwise.} \end{cases}$$

If $R:=\prod_{i<j}R_{ij}^{n_{ij}}$ is any raising operator, denote by $\mathrm{supp}(R)$ the set of all indices i and j such that $n_{ij}>0$. Set $\nu:=R\alpha$, and define

$$R\star{}^\rho\widehat{c}_\alpha^\beta = {}^\rho\overline{c}_\nu^\beta := {}^{\rho_1}\overline{c}_{\nu_1}^{\beta_1}\cdots{}^{\rho_\ell}\overline{c}_{\nu_\ell}^{\beta_\ell},$$

where for each $i\geq 1$,

$${}^{\rho_i}\overline{c}_{\nu_i}^{\beta_i} := \begin{cases} {}^{\rho_i}c_{\nu_i}^{\beta_i} & \text{if } i\in\mathrm{supp}(R), \\ {}^{\rho_i}\widehat{c}_{\nu_i}^{\beta_i} & \text{otherwise.} \end{cases}$$

If α is a partition of length ℓ, then we set

$${}^\rho\widehat{P}_\alpha^\beta := 2^{-\ell}\,R^\infty\star{}^\rho\widehat{c}_\alpha^\beta.$$

Lemma 15 ([T7], Lemma 4.9)**.** *Suppose that $\beta_i = \rho_i - \alpha_i < 0$ for every $i \in [1,\ell]$, and $\alpha_j = \alpha_{j+1}$ and $\beta_j = \beta_{j+1}$ for some $j \in [1, \ell-1]$. Then we have ${}^{\rho}\widehat{P}^{\beta}_{\alpha} = 0$.*

6.2. **The shape of an element of $\widetilde{W}_\infty$.** We next define certain statistics of a signed permutation in $\widetilde{W}_\infty$, analogous to the ones given in Section 3.2.

Definition 5. Let $w \in \widetilde{W}_\infty$. The strict partition $\mu(w)$ is the one whose parts are the absolute values of the negative entries of w minus one, arranged in decreasing order. The *A-code* of w is the sequence $\gamma = \gamma(w)$ with $\gamma_i := \#\{j > i \mid w_j < w_i\}$. The parts of the partition $\delta(w)$ are the non-zero entries γ_i arranged in weakly decreasing order, and $\nu(w) := \delta(w)'$. The *shape* of w is the partition $\lambda(w) := \mu(w) + \nu(w)$.

Note that w is uniquely determined by $\mu(w)$ and $\gamma(w)$, and that $|\lambda(w)| = \ell(w)$.

Example 4. (a) For the signed permutation $w := (\overline{3}, 2, \overline{7}, \overline{1}, 5, 4, \overline{6})$ in $\widetilde{W}_7$, we obtain $\mu = (6,5,2)$, $\gamma = (2,3,0,1,2,1,0)$, $\delta = (3,2,2,1,1)$, $\nu = (5,3,1)$, and $\lambda = (11,8,3)$.
(b) Recall from [T5, Section 4.2] that an element w of $\widetilde{W}_\infty$ is n-Grassmannian if $\ell(ws_i) > \ell(w)$ for all $i \neq n$. The type of an n-Grassmannian element w is 0 if $|w_1| = 1$, and 1 (respectively, 2) if $w_1 > 1$ (respectively, if $w_1 < -1$). According to [BKT1, Section 6.1], there is a type preserving bijection between the n-Grassmannian elements of $\widetilde{W}_\infty$ and typed n-strict partitions. If w is an n-Grassmannian element of $\widetilde{W}_\infty$ of type 0 or 1, then $\lambda(w)$ is the (typed) n-strict partition associated to w, in the sense of op. cit. However, this latter property can fail if $w_1 < -1$, for example the 2-Grassmannian element $v := (\overline{3}, 4, \overline{1}, 2)$ is associated to the typed partition of shape $(2,2)$, while $\lambda(v) = (3,1)$.

Let $\beta(w)$ denote the sequence defined by $\beta(w)_i = -\mu(w)_i$ for each $i \geq 1$. Recall that $\widetilde{w}_0^{(n)}$ denotes the longest element in $\widetilde{W}_n$.

Proposition 12. *Suppose that $m > n \geq 0$ and $w \in \widetilde{W}_m$ is an n-Grassmannian element. Set $\widehat{w} := w\widetilde{w}_0^{(n)}$. Then we have*

$$\mathfrak{D}_{\widehat{w}}(X,Y) = {}^{\nu(\widehat{w})}\widehat{P}^{\beta(\widehat{w})}_{\lambda(\widehat{w})}$$

in the ring $\Gamma[X_n, Y_{m-1}]$. In particular, if $w \in S_m$, then we have

$$\mathfrak{D}_{\widehat{w}}(X,Y) = {}^{\delta_{n-1}}\widehat{P}^{(1-w_n,\ldots,1-w_1)}_{2\delta_{n-1}+\lambda(w)'}.$$

Proof. We first consider the case where $w \in S_m$. We have

$$w = (a_1, \ldots, a_n, d_1, \ldots, d_r),$$

where $r = m - n$, $0 < a_1 < \cdots < a_n$ and $0 < d_1 < \cdots < d_r$. If $\lambda := \lambda(w)$ then

$$\lambda_j = n + j - d_j = m - d_j - (r-j) \quad \text{for } 1 \leq j \leq r.$$

We have $\widetilde{w}_0^{(m)} = \widehat{w}v_1 \cdots v_r$, where $\ell(\widetilde{w}_0^{(m)}) = \ell(\widehat{w}) + \sum_{j=1}^r \ell(v_j)$ and

$$v_j := s_{n+j-1} \cdots s_3 s_2 s_\Box s_1 s_2 \cdots s_{d_j-1}, \quad 2 \leq j \leq r,$$

while

$$v_1 := \begin{cases} s_n \cdots s_3 s_2 s_\Box s_1 s_2 \cdots s_{d_1-1} & \text{if } d_1 > 1, \\ s_n \cdots s_2 s_1 & \text{if } d_1 = 1. \end{cases}$$

Using [IMN1, Thm. 1.2] and [T7, Prop. 4.10], it follows that

$$\mathfrak{D}_{\widehat{w}} = \partial_{v_1} \cdots \partial_{v_r}\left(\mathfrak{D}_{\widetilde{w}_0^{(m)}}\right) = \partial_{v_1} \cdots \partial_{v_r}\left({}^{\delta_{m-1}}\widehat{P}^{-\delta_{m-1}}_{2\delta_{m-1}}\right). \tag{45}$$

According to Lemmas 11 and 14, for any $p,q\in\mathbb{Z}$ with $p\geq 2$, we have

$$\partial_p({}^p\widehat{c}_q^{\,-p}) = {}^{p-1}\widehat{c}_{q-1}^{\,-p} = {}^p\widehat{c}_{q-1}^{\,1-p} - (x_p+y_p)\,{}^{p-1}\widehat{c}_{q-2}^{\,1-p}. \tag{46}$$

Let ϵ_j denote the j-th standard basis vector in $\mathbb{Z}^m$. The Leibnitz rule and (46) imply that for any integer vector $\alpha=(\alpha_1,\ldots,\alpha_m)$, we have

$$\partial_p\left({}^{\delta_{m-1}}\widehat{c}_{\alpha}^{\,-\delta_{m-1}}\right) = {}^{\delta_{m-1}}\widehat{c}_{\alpha-\epsilon_{m-p}}^{\,-\delta_{m-1}+\epsilon_{m-p}} - (x_p+y_p)\,{}^{\delta_{m-1}-\epsilon_{m-p}}\widehat{c}_{\alpha-2\epsilon_{m-p}}^{\,-\delta_{m-1}+\epsilon_{m-p}}.$$

We deduce from this and Lemma 15 that

$$\begin{aligned}\partial_p\,{}^{\delta_{m-1}}\widehat{P}_{2\delta_{m-1}}^{-\delta_{m-1}} &= {}^{\delta_{m-1}}\widehat{P}_{2\delta_{m-1}-\epsilon_{m-p}}^{-\delta_{m-1}+\epsilon_{m-p}} \\ &= {}^{(m-1,\ldots,1)}\widehat{P}_{(2m-2,\ldots,2p+2,2p-1,2p-2,2p-4,\ldots,2)}^{(1-m,\ldots,-1-p,1-p,1-p,2-p,\ldots,-1)}.\end{aligned}$$

Iterating this calculation for $p=d_r-1,\ldots,2$ gives

$$\begin{aligned}(\partial_2\cdots\partial_{d_r-1})\mathfrak{D}_{\widetilde{w}_0^{(m)}} &= {}^{(m-1,\ldots,1)}\widehat{P}_{(2m-2,\ldots,2d_r,2d_r-3,2d_r-5,\ldots,3,2)}^{(1-m,\ldots,-d_r,2-d_r,3-d_r,\ldots,-1,-1)} \\ &= {}^{(m-1,\ldots,1)}\widehat{P}_{(2m-2,\ldots,2d_r,2d_r-3,2d_r-5,\ldots,3,2)}^{(1-m,\ldots,-d_r,2-d_r,3-d_r,\ldots,-1,-1)}.\end{aligned}$$

We next compute that

$$\partial_\Box\partial_1({}^1\widehat{c}_p^{\,-1}) = \begin{cases} 2 & \text{if } p=2, \\ 0 & \text{otherwise.}\end{cases}$$

By arguing as in [T7, Prop. 4.10], it follows that

$$(\partial_\Box\partial_1\cdots\partial_{d_r-1})\mathfrak{D}_{\widetilde{w}_0^{(m)}} = {}^{(m-1,\ldots,2)}\widehat{P}_{(2m-2,\ldots,2d_r,2d_r-3,2d_r-5,\ldots,3)}^{(1-m,\ldots,-d_r,2-d_r,3-d_r,\ldots,-1)}.$$

Applying Lemma 11 alone $m-2$ times to the last equality gives

$$\partial_{v_r}(\mathfrak{D}_{\widetilde{w}_0^{(m)}}) = {}^{\delta_{m-2}}\widehat{P}_{(2m-3,\ldots,2d_r-1,2d_r-4,\ldots,2)}^{(1-m,\ldots,-d_r,2-d_r,3-d_r,\ldots,-1)} = {}^{\delta_{m-2}}\widehat{P}_{2\delta_{m-2}+1^{m-d_r}}^{(1-m,\ldots,\widehat{1-d_r},\ldots,-1)}.$$

We now use (45) and repeat the above computation $r-1$ more times to get

$$\mathfrak{D}_{\widehat{w}} = {}^{\delta_{n-1}}\widehat{P}_{2\delta_{n-1}+\xi}^{\rho},$$

where

$$\rho = (1-m,\ldots,\widehat{1-d_r},\ldots,\widehat{1-d_1},\ldots,-1) = (1-w_n,\ldots,1-w_1)$$

and

$$\xi = \sum_{j=1}^r 1^{m-d_j-(r-j)} = \sum_{j=1}^r 1^{\lambda_j} = \lambda(w)'.$$

(Note that in the case when $d_1=1$, the last stage of the calculation is simpler).

The general case now follows as in the proof of Proposition 4. $\square$

6.3. **Double eta polynomials and alternating sums.** Let λ be a typed n-strict partition which corresponds to the n-Grassmannian element $w\in\widetilde{W}_\infty$, and define a sequence $\beta(\lambda)$ and a set $\mathcal{C}(\lambda)$ using the same formulas (28) and (29) as in Lie type C. The *double eta polynomial* ${}^nH_\lambda(X,Y)$ of [T6] is defined by

$${}^nH_\lambda(X,Y) := 2^{-\ell_n(\lambda)}\prod_{i<j}(1-R_{ij})\prod_{(i,j)\in\mathcal{C}(\lambda)}(1+R_{ij})^{-1}\star({}^n\widehat{c})_\lambda^{\beta(\lambda)},$$

where $\ell_n(\lambda)$ denotes the number of parts λ_i which are greater than n (see op. cit. for the precise definitions of typed n-strict partitions and $\star$).

Let $\mathcal{A}' : \Gamma'[X_n, Y] \to \Gamma'[X_n, Y]$ be the operator defined by

$$\mathcal{A}'(f) := \sum_{w \in \widetilde{W}_n} (-1)^{\ell(w)} w(f).$$

Let $\widetilde{w}_0$ denote the longest element in $\widetilde{W}_n$ and set $\widehat{w} := w\widetilde{w}_0$.

Theorem 4. *Let λ be a typed n-strict partition and w be the corresponding n-Grassmannian element of $\widetilde{W}_\infty$. Then we have*

$$ {}^nH_\lambda(X,Y) = \partial_{\widetilde{w}_0}\left({}^{\nu(\widehat{w})}\widehat{P}^{\beta(\widehat{w})}_{\lambda(\widehat{w})}\right) \tag{47}$$

$$ = (-1)^{n(n-1)/2} \cdot 2^{n-1}\ \mathcal{A}'\left({}^{\nu(\widehat{w})}\widehat{P}^{\beta(\widehat{w})}_{\lambda(\widehat{w})}\right)\Big/\ \mathcal{A}'\left(x^{2\delta_{n-1}}\right). \tag{48}$$

Proof. We deduce from (39) that the double Schubert polynomial $\mathfrak{D}_w(X,Y)$ satisfies

$$\mathfrak{D}_w(X,Y) = \partial_{\widetilde{w}_0}\left(\mathfrak{D}_{\widehat{w}}(X,Y)\right). \tag{49}$$

The equality (47) follows from (49), Proposition 12, and the fact, proved in [T6], that $\mathfrak{D}_w(X,Y) = {}^nH_\lambda(X,Y)$ in the ring $\Gamma'[X_n, Y]$.

For the second equality, recall from [D, Lemma 4] and [PR] that we have

$$\partial_{\widetilde{w}_0}(f) = (-1)^{n(n-1)/2} \prod_{1\leq i<j\leq n}(x_i^2 - x_j^2)^{-1} \cdot \mathcal{A}'(f).$$

On the other hand, it follows from [PR, Lemma 5.16(ii)] that

$$\partial_{\widetilde{w}_0}(x^{2\delta_{n-1}}) = (-1)^{n(n-1)/2} \cdot 2^{n-1}$$

and hence that

$$\mathcal{A}'(x^{2\delta_{n-1}}) = 2^{n-1} \prod_{1\leq i<j\leq n}(x_i^2 - x_j^2).$$

The proof of (48) is completed by using these two equations in (47). □

7. Single Schubert polynomials of type D

7.1. **Eta polynomials as Weyl group invariants.** In this section, we work with the single type D Schubert polynomials $\mathfrak{D}_w(X)$. Let $\chi' : \Gamma'[X_n] \to \mathbb{Z}$ be the homomorphism defined by $\chi'(b_p) = \chi'(x_j) = 0$ for all p, j.

Proposition 13. *For any $f \in \Gamma'[X_n]$, we have $f = \sum_{w\in \widetilde{W}^{(n)}} \chi'(\partial_w f)\mathfrak{D}_w(X_n)$.*

Proof. The argument is the same as for the proof of Proposition 5. □

We next define a ring $B^{(n)}$, following [BKT2, Section 5.2]. For each integer $p \geq 1$, let

$${}^nb_p := \begin{cases} e_p(X_n) + 2\sum_{j=0}^{p-1} b_{p-j}e_j(X_n) & \text{if } p < n, \\ \sum_{j=0}^{p} b_{p-j}e_j(X_n) & \text{if } p \geq n \end{cases}$$

and

$${}^nb'_n := \sum_{j=0}^{n-1} b_{n-j}e_j(X_n).$$

Observe that the elements denoted by $\eta_r(x\,;y)$ and $\eta'_k(x\,;y)$ in loc. cit. correspond to the elements ${}^nb_r$ and ${}^nb'_n$ here. Let

$$B^{(n)} := \mathbb{Z}[{}^nb_1, \ldots, {}^nb_{n-1}, {}^nb_n, {}^nb'_n, {}^nb_{n+1}, \ldots]$$

be the ring of eta polynomials of level n. We have

$$ {}^n c_p = \begin{cases} {}^n b_p & \text{if } p < n, \\ {}^n b_n + {}^n b'_n & \text{if } p = n, \\ 2 \cdot {}^n b_p & \text{if } p > n \end{cases} $$

and thus $\Gamma^{(n)}$ is a subring of $B^{(n)}$.

According to [BKT2, Thm. 4], the single eta polynomials ${}^n H_\lambda$ for all typed n-strict partitions λ form a $\mathbb{Z}$-basis of $B^{(n)}$. In the next result, the Weyl group $\widetilde{W}_n$ acts on the ring $\Gamma'[X_n]$ in the manner described in §5.1.

Proposition 14. *The ring $B^{(n)}$ is equal to the subring $\Gamma'[X_n]^{\widetilde{W}_n}$ of $\widetilde{W}_n$-invariants in $\Gamma'[X_n]$.*

Proof. The proof is identical to that of Proposition 6, using [BKT2, Prop. 6.3] for the fact that the Schubert polynomials $\mathfrak{D}_w(X_n)$ for those $w \in \widetilde{W}_\infty$ with $|w_1| < w_2 < \cdots < w_n$ and $w_{n+1} < w_{n+2} < \cdots$ coincide with the (single) eta polynomials of level $n \geq 2$. □

Example 5. It follows from Proposition 14 that we have

$$ B^{(n)} \cap \mathbb{Z}[X_n] = \mathbb{Z}[X_n]^{\widetilde{W}_n} = \mathbb{Z}[e_1(X_n^2), \ldots, e_{n-1}(X_n^2), e_n(X_n)]. $$

This can also be shown as in Example 2, using the fact that ${}^n b_n - {}^n b'_n = e_n(X_n)$.

For any parabolic subgroup P of SO_{2n}, let $\widetilde{W}_n^P := \widetilde{W}^P \cap \widetilde{W}_n$. Let $\mathrm{I}B^{(n)}$ (respectively $\mathrm{I}B_P^{(n)}$) be the ideal of $\Gamma'[X_n]_{\mathbb{Q}}$ (respectively $\Gamma'[X_n]_{\mathbb{Q}}^{\widetilde{W}_P}$) generated by the homogeneous elements in $B^{(n)}$ of positive degree. We then have the following immediate consequence of Theorem 3 and the discussion in §5.3.

Corollary 6. *There is a canonical ring isomorphism*

$$ \mathrm{H}^*(\mathrm{SO}_{2n}/B, \mathbb{Q}) \cong \Gamma'[X_n]_{\mathbb{Q}}/\mathrm{I}B^{(n)} $$

which maps the cohomology class of the codimension $\ell(w)$ Schubert variety X_w to the class of the Schubert polynomial $\mathfrak{D}_w(X)$, for any $w \in \widetilde{W}_n$. Moreover, for any parabolic subgroup P of SO_{2n}, there is a canonical ring isomorphism

$$ \mathrm{H}^*(\mathrm{SO}_{2n}/P, \mathbb{Q}) \cong \Gamma'[X_n]_{\mathbb{Q}}^{\widetilde{W}_P}/\mathrm{I}B_P^{(n)} $$

which maps the cohomology class of the codimension $\ell(w)$ Schubert variety X_w to the class of the Schubert polynomial $\mathfrak{D}_w(X)$, for any $w \in \widetilde{W}_n^P$.

7.2. **The ring $\Gamma'[X_n]_{\mathbb{Q}}$ as a $B_{\mathbb{Q}}^{(n)}$-module.** Let $e_p := e_p(X_n)$ for each $p \in \mathbb{Z}$, and recall that $\mathcal{P}_n$ denotes the set of strict partitions λ with $\lambda_1 \leq n$.

Proposition 15. *The $\mathbb{Q}$-algebra $\Gamma'[X_n]_{\mathbb{Q}}$ is a free $B_{\mathbb{Q}}^{(n)}$-module of rank $2^{n-1}n!$ with basis*

$$ \{e_\lambda(-X_n)x^\alpha \mid \lambda \in \mathcal{P}_{n-1}, \ \ 0 \leq \alpha_i \leq n-i, \ i \in [1,n]\}. $$

Proof. We have that $\Gamma'[X_n]$ is a free $\Gamma'[e_1, \ldots, e_n]$-module with basis given by the monomials x^α with $0 \leq \alpha_i \leq n-i$ for $i \in [1,n]$. It will therefore suffice to show that $\Gamma'[e_1, \ldots, e_n]_{\mathbb{Q}}$ is a free $B_{\mathbb{Q}}^{(n)}$-module with basis $e_\lambda(-X_n)$ for $\lambda \in \mathcal{P}_{n-1}$.

As in the proof of Proposition 7, we see that the monomials $e_\lambda(-X_n)$ for $\lambda \in \mathcal{P}_n$ generate $\Gamma'[e_1, \ldots, e_n]_{\mathbb{Q}}$ as a $B_{\mathbb{Q}}^{(n)}$-module. Furthermore, since $e_n = {}^n b_n - {}^n b'_n$, it follows that the monomials $e_\lambda(-X_n)$ for $\lambda \in \mathcal{P}_{n-1}$ also generate this module. The rest of the argument is the same as in type C. $\square$

For any strict partition λ, define the $\widetilde{P}$-polynomial of [PR] by

$$\widetilde{P}_\lambda(X_n) := 2^{-\ell(\lambda)}\,\widetilde{Q}_\lambda(X_n).$$

Corollary 7. *The ring $\Gamma'[X_n]$ is a free $\Gamma'[X_n]^{S_n}$-module with basis $\{\mathfrak{S}_\varpi(X)\}$ for $\varpi \in S_n$. The $\mathbb{Q}$-algebra $\Gamma'[X_n]^{S_n}_{\mathbb{Q}}$ is a free $B^{(n)}_{\mathbb{Q}}$-module with basis $\{P_\lambda(-X_n)\}$ for $\lambda \in \mathcal{P}_{n-1}$. The $\mathbb{Q}$-algebra $\Gamma'[X_n]_{\mathbb{Q}}$ is a free $B^{(n)}_{\mathbb{Q}}$-module on the basis $\{\mathfrak{D}_w(X_n)\}$ of single type D Schubert polynomials for $w \in \widetilde{W}_n$, and is also free on the product basis $\{P_\lambda(-X_n)\mathfrak{S}_\varpi(X)\}$ for $\lambda \in \mathcal{P}_{n-1}$ and $\varpi \in S_n$.*

7.3. **A scalar product on $\Gamma'[X_n]$.** Let $\widetilde{w}_0$ be the element of longest length in $\widetilde{W}_n$. If $f \in \Gamma'[X_n]$, then $\partial_i(\partial_{\widetilde{w}_0} f) = 0$ for all i with $\square \leq i \leq n-1$. Hence Proposition 14 implies that $\partial_{\widetilde{w}_0}(f) \in B^{(n)}$, for each $f \in \Gamma'[X_n]$.

Definition 6. We define a scalar product $\langle\ ,\ \rangle$ on $\Gamma'[X_n]$, with values in $B^{(n)}$, by the rule

$$\langle f, g\rangle := \partial_{\widetilde{w}_0}(fg), \quad f, g \in \Gamma'[X_n].$$

Proposition 16. (a) *The scalar product $\langle\ ,\ \rangle : \Gamma'[X_n] \times \Gamma'[X_n] \to B^{(n)}$ is $B^{(n)}$-linear. For any $f, g \in \Gamma'[X_n]$ and $w \in \widetilde{W}_n$, we have*

$$\langle \partial_w f, g\rangle = \langle f, \partial_{w^{-1}} g\rangle.$$

(b) *Let $u, v \in \widetilde{W}_n$ be such that $\ell(u) + \ell(v) = n^2 - n$. Then we have*

$$\langle \mathfrak{D}_u(X_n), \mathfrak{D}_v(X_n)\rangle = \begin{cases} 1 & \text{if } v = \widetilde{w}_0 u, \\ 0 & \text{otherwise.}\end{cases}$$

Proof. The argument is identical to the proofs of Propositions 8 and 9. $\square$

Let ϖ_0 denote the longest permutation in S_n, and define $\widetilde{v}_0 := \widetilde{w}_0\varpi_0$. We define a $\Gamma'[X_n]^{S_n}$-valued scalar product $(\ ,\ )$ on $\Gamma'[X_n]$ by the rule

$$(f, g) := \partial_{\varpi_0}(fg), \quad f, g \in \Gamma'[X_n],$$

and a $B^{(n)}$-valued scalar product $\{\ ,\ \}$ on $\Gamma'[X_n]^{S_n}$ by the rule

$$\{f, g\} := \partial_{\widetilde{v}_0}(fg), \quad f, g \in \Gamma'[X_n]^{S_n}.$$

Following [PR, Thm. 5.23], for any two partitions $\lambda, \mu \in \mathcal{P}_{n-1}$, we have

$$\left\{\widetilde{P}_\lambda(-X_n), \widetilde{P}_{\delta_{n-1}\smallsetminus\mu}(-X_n)\right\} = \delta_{\lambda,\mu},$$

where $\delta_{n-1} \smallsetminus \mu$ is the strict partition whose parts complement the parts of μ in the set $\{n-1, n-2, \ldots, 1\}$

Observe that the scalar product $(\ ,\ )$ is $\Gamma'[X_n]^{S_n}$-linear and $\{\ ,\ \}$ is $B^{(n)}$-linear. Since $\partial_{\widetilde{w}_0} = \partial_{\widetilde{v}_0}\partial_{\varpi_0}$, we deduce that

$$\langle f, g\rangle = \{(f, g)\}, \quad \text{for any } f, g \in \Gamma'[X_n].$$

Furthermore, according to [LP2, (2.20)], the orthogonality relation

$$\left\langle \widetilde{P}_\lambda(-X_n)\mathfrak{S}_u(X), \widetilde{P}_{\delta_{n-1}\smallsetminus\mu}(-X_n)(\varpi_0\mathfrak{S}_{u'\varpi_0}(-X))\right\rangle = \delta_{u,u'}\delta_{\lambda,\mu}$$

holds, for any $u, u' \in S_n$ and $\lambda, \mu \in \mathcal{P}_{n-1}$. We have therefore identified the dual $B_{\mathbb{Q}}^{(n)}$-basis of the product basis $\{\widetilde{P}_\lambda(-X_n)\mathfrak{S}_u(X)\}$ of $\Gamma'[X_n]_{\mathbb{Q}}$, relative to the scalar product $\langle\ ,\ \rangle$.

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