THE THEORY OF SCHUR POLYNOMIALS REVISITED

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Abstract. We use Young's raising operators to give short and uniform proofs of several well known results about Schur polynomials and symmetric functions, starting from the Jacobi-Trudi identity.

1. Introduction

One of the earliest papers to study the symmetric functions later known as the Schur polynomials s_λ is that of Jacobi [J], where the following two formulas are found. The first is Cauchy's definition of s_λ as a quotient of determinants:

$$s_\lambda(x_1, \ldots, x_n) = \det(x_i^{\lambda_i+n-j})_{i,j} \Big/ \det(x_i^{n-j})_{i,j} \tag{1}$$

where $\lambda = (\lambda_1, \ldots, \lambda_n)$ is an integer partition with at most n non-zero parts. The second is the Jacobi-Trudi identity

$$s_\lambda = \det(h_{\lambda_i+j-i})_{1\leq i,j\leq n} \tag{2}$$

which expresses s_λ as a polynomial in the complete symmetric functions h_r, $r \geq 0$. Nearly a century later, Littlewood [L] obtained the positive combinatorial expansion

$$s_\lambda(x) = \sum_T x^{c(T)} \tag{3}$$

where the sum is over all semistandard Young tableaux T of shape λ, and $c(T)$ denotes the content vector of T.

The traditional approach to the theory of Schur polynomials begins with the classical definition (1); see for example [FH, M, Ma]. Since equation (1) is a special case of the Weyl character formula, this method is particularly suitable for applications to representation theory. The more combinatorial treatments [Sa, Sta] use (3) as the definition of $s_\lambda(x)$, and proceed from there. It is not hard to relate formulas (1) and (3) to each other directly; see e.g. [Pr, Ste].

In this article, we take the Jacobi-Trudi formula (2) as the starting point, where the h_r represent algebraically independent variables. We avoid the use of the x variables or 'alphabets' and try to prove as much as we can without them. For this purpose, it turns out to be very useful to express (2) in the alternative form

$$s_\lambda = \prod_{i<j}(1 - R_{ij})\, h_\lambda \tag{4}$$

where the R_{ij} are Young's raising operators [Y] and $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_n}$. The equivalence of (2) and (4) follows immediately from the Vandermonde identity.

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The motivation for this approach to the subject comes from Schubert calculus. It is well known that the algebra of Schur polynomials agrees with that of the Schubert classes in the cohomology ring of the complex Grassmannian $G(k, r)$, when k and r are sufficiently large. Giambelli [G] showed that the Schubert classes on $G(k, r)$ satisfy the determinantal formula (2); the closely related Pieri rule [P] had been obtained geometrically a few years earlier. Recently, with Buch and Kresch [BKT1, BKT2], we proved analogues of the Pieri and Giambelli formulas for the isotropic Grassmannians which are quotients of the symplectic and orthogonal groups. Our Giambelli formulas for the Schubert classes on these spaces are not determinantal, but rather are stated in terms of raising operators. In [T], we used raising operators to obtain a tableau formula for the corresponding theta polynomials, which is an analogue of Littlewood's equation (3) in this context. Moreover, the same methods were applied in loc. cit. to provide new proofs of similar facts about the Hall-Littlewood functions.

Our aim here is to give a self-contained treatment of those aspects of the theory of Schur polynomials and symmetric functions which follow naturally from the above raising operator approach. Using (4) as the definition of Schur polynomials, we give short proofs of the Pieri and Littlewood-Richardson rules, and follow this with a discussion – in this setting – of the duality involution, Cauchy identities, and skew Schur polynomials. We next introduce the variables $x = (x_1, x_2, \ldots)$ and study the ring Λ of symmetric functions in x from scratch. In particular, we derive the bialternant and tableau formulas (1) and (3) for $s_\lambda(x)$. See [La] for an approach to these topics which begins with (2) but is based on alphabets and properties of determinants such as the Binet-Cauchy formula, and [vL, Ste] for a different treatment which employs alternating sums stemming from (1).

Most of the proofs in this article are streamlined versions of more involved arguments contained in [BKT2], [M], and [T]. The proof we give of the Littlewood-Richardson rule from the Pieri rule is essentially that of Remmel-Shimozono [RS] and Gasharov [G], but expressed in the concise form adapted by Stembridge [Ste]. Each of these proofs employs the same sign reversing involution on a certain set of Young tableaux, which originates in the work of Berenstein-Zelevinsky [BZ]. The version given here does not use formulas (1) and (3) at all, but relies on the alternating property of the determinant (2), which serves the same purpose.

The reduction formula (22) for the number of variables in $s_\lambda(x_1, \ldots, x_n)$ is classically known as a 'branching rule' for the characters of the general linear group [Pr, W]. Our terminology differs because there are similar results in situations where the connection with representation theory is not available (see [T]). We use the reduction formula to derive (3) from (4); a different cancellation argument relating formulas (2) and (3) to each other is due to Gessel-Viennot [GV, Sa].

We find that the short arguments in this article are quite uniform, especially when compared to other treatments of the same material. On the other hand, much of the theory of Schur polynomials does not readily fit into the present framework. Missing from the discussion are the Hall inner product, the Hopf algebra structure on Λ, the basis of power sums, the character theory of the symmetric and general linear groups, Young tableau algorithms such as jeu de taquin, the plactic algebra, and noncommutative symmetric functions. These topics and many more can be added following standard references such as [F, La, M, Ma, Sa, Sta, Z], but are not as natural from the point of view adopted here, which stems from Grassmannian

Schubert calculus. A similar approach may be used to study the theory of Schur Q-polynomials and more generally of Hall-Littlewood functions; some of this story may be found in [T].

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2. The algebra of Schur polynomials

2.1. **Preliminaries.** An *integer sequence* or *integer vector* is a sequence of integers $\alpha = (\alpha_1, \alpha_2, \ldots)$ with only finitely α_i non-zero. The *length* of α, denoted $\ell(\alpha)$, is largest integer $\ell \geq 0$ such that $\alpha_\ell \neq 0$. We identify an integer sequence of length ℓ with the vector consisting of its first ℓ terms. We let $|\alpha| = \sum \alpha_i$ and write $\alpha \geq \beta$ if $\alpha_i \geq \beta_i$ for each i. An integer sequence α is a *composition* if $\alpha_i \geq 0$ for all i and a *partition* if $\alpha_i \geq \alpha_{i+1} \geq 0$ for all i.

Consider the polynomial ring $\mathbb{A} = \mathbb{Z}[u_1, u_2, \ldots]$ where the u_i are countably infinite commuting independent variables. We regard $\mathbb{A}$ as a graded ring with each u_i having graded degree i, and adopt the convention here and throughout the paper that $u_0 = 1$ while $u_r = 0$ for $r < 0$. For each integer vector α, set $u_\alpha = \prod_i u_{\alpha_i}$; then $\mathbb{A}$ has a free $\mathbb{Z}$-basis consisting of the monomials u_λ for all partitions λ.

For two integer sequences α, β such that $|\alpha| = |\beta|$, we say that α *dominates* β and write $\alpha \succeq \beta$ if $\alpha_1 + \cdots + \alpha_i \geq \beta_1 + \cdots + \beta_i$ for each i. Given any integer sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ and $i < j$, we define

$$R_{ij}(\alpha) = (\alpha_1, \ldots, \alpha_i + 1, \ldots, \alpha_j - 1, \ldots).$$

A *raising operator* R is any monomial in these R_{ij}'s. Note that we have $R\,\alpha \succeq \alpha$ for all integer sequences α. For any raising operator R, define $R\,u_\alpha = u_{R\alpha}$. Here the operator R acts on the index α, and not on the monomial u_α itself. Thus, if the components of α are a permutation of the components of β, then $u_\alpha = u_\beta$ as elements of $\mathbb{A}$, but it may happen that $R\,u_\alpha \neq R\,u_\beta$. Formal manipulations using these raising operators are justified carefully in the following section. Note that if $\alpha_\ell < 0$ for $\ell = \ell(\alpha)$, then $R\,u_\alpha = 0$ in $\mathbb{A}$ for any raising operator R.

2.2. **Schur polynomials.** For any integer vector α, define the *Schur polynomial* U_α by the formula

$$U_\alpha := \prod_{i<j} (1 - R_{ij})\, u_\alpha. \tag{5}$$

Although the product in (5) is infinite, if we expand it into a formal series we find that only finitely many of the summands are nonzero; hence, U_α is well defined. We will show that equation (5) may be written in the determinantal form

$$U_\alpha = \det(u_{\alpha_i + j - i})_{1 \leq i,j \leq \ell} = \sum_{w \in S_\ell} (-1)^w u_{w(\alpha + \rho_\ell) - \rho_\ell} \tag{6}$$

where ℓ denotes the length of α and $\rho_\ell = (\ell - 1, \ell - 2, \ldots, 1, 0)$.

Algebraic expressions and identities involving raising operators like the above can be justified by viewing them as the image of a $\mathbb{Z}$-linear map $\mathbb{Z}[\mathbb{Z}^\ell] \to \mathbb{A}$, where $\mathbb{Z}[\mathbb{Z}^\ell]$ denotes the group algebra of $(\mathbb{Z}^\ell, +)$. We let $x_1, \ldots, x_\ell$ be independent variables and identify $\mathbb{Z}[\mathbb{Z}^\ell]$ with $\mathbb{Z}[x_1, x_1^{-1}, \ldots, x_\ell, x_\ell^{-1}]$. For any integer vector $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ and raising operator R, set $x^\alpha = x_1^{\alpha_1} \cdots x_\ell^{\alpha_\ell}$ and $R\,x^\alpha = x^{R\alpha}$. Then if $\psi : \mathbb{Z}[\mathbb{Z}^\ell] \to \mathbb{A}$

is the $\mathbb{Z}$-linear map determined by $\psi(x^\alpha) = u_\alpha$ for each α, we have $R\, u_\alpha = \psi(x^{R\alpha})$. It follows from the Vandermonde identity

$$\prod_{1\leq i<j\leq \ell}(x_j - x_i) = \det(x_i^{j-1})_{1\leq i,j\leq \ell}$$

that

$$\prod_{1\leq i<j\leq \ell}(1-R_{ij})\, x^\alpha = \prod_{1\leq i<j\leq \ell}(1-x_ix_j^{-1})\, x^\alpha = \det(x_i^{\alpha_i+j-i})_{1\leq i,j\leq \ell}.$$

Now apply the map ψ to both ends of the above equation to obtain (6).

Example 1. We have

$$\begin{aligned} U_{(5,4,2)} &= (1-R_{12})(1-R_{13})(1-R_{23})\, u_{(5,4,2)} \\ &= (1-R_{12}-R_{13}-R_{23}+R_{12}R_{13}+R_{12}R_{23}+R_{13}R_{23}-R_{12}R_{13}R_{23})\, u_{(5,4,2)} \\ &= u_{(5,4,2)} - u_{(6,3,2)} - u_{(6,4,1)} - u_{(5,5,1)} + u_{(7,3,1)} + u_{(6,4,1)} + u_{(6,5,0)} - u_{(7,4,0)} \\ &= u_5u_4u_2 - u_6u_3u_2 - u_5^2u_1 + u_7u_3u_1 + u_6u_5 - u_7u_4 = \begin{vmatrix} u_5 & u_6 & u_7 \\ u_3 & u_4 & u_5 \\ 1 & u_1 & u_2 \end{vmatrix}. \end{aligned}$$

If $\alpha = (\alpha_1,\ldots,\alpha_\ell)$ and $\beta = (\beta_1,\ldots,\beta_m)$ are two integer vectors and $r,s\in\mathbb{Z}$, we let (α,r,s,β) denote the integer vector $(\alpha_1,\ldots,\alpha_\ell,r,s,\beta_1,\ldots,\beta_m)$. The next lemma is known as a 'straightening law' for the U_α.

Lemma 1. (a) *Let α and β be integer vectors. Then for any $r,s\in\mathbb{Z}$ we have*

$$U_{(\alpha,r,s,\beta)} = -U_{(\alpha,s-1,r+1,\beta)}\,.$$

(b) *Let $\alpha = (\alpha_1,\ldots,\alpha_\ell)$ be any integer vector. Then $U_\alpha = 0$ unless $\alpha+\rho_\ell = w(\mu+\rho_\ell)$ for a (unique) permutation $w\in S_\ell$ and partition μ. In the latter case, we have $U_\alpha = (-1)^w U_\mu$.*

Proof. Both parts follow immediately from (6) and the alternating property of the determinant. □

If λ is any partition, clearly (5) implies that $U_\lambda = u_\lambda + \sum_{\mu\succ\lambda} a_{\lambda\mu}u_\mu$ where $a_{\lambda\mu}\in\mathbb{Z}$ and the sum is over partitions μ which strictly dominate λ. We deduce that the U_λ for λ a partition form another $\mathbb{Z}$-basis of $\mathbb{A}$.

2.3. **Mirror identities.** We will represent a partition λ by its Young diagram of boxes, arranged in left-justified rows, with λ_i boxes in row i. We write $\lambda\subset\mu$ instead of $\lambda\leq\mu$ for the containment relation between two Young diagrams; in this case the set-theoretic difference $\mu\smallsetminus\lambda$ is the skew diagram μ/λ. A skew diagram is a *horizontal* (resp. *vertical*) *strip* if it does not contain two boxes in the same column (resp. row). We write $\lambda\xrightarrow{p}\mu$ if μ/λ is a horizontal strip with p boxes.

Lemma 2. *Let λ be a partition and $p\geq 0$ be an integer. Then we have*

$$\sum_{\alpha\geq 0,\ |\alpha|=p} U_{\lambda+\alpha} = \sum_{\lambda\xrightarrow{p}\mu} U_\mu \quad \textit{and} \quad \sum_{\alpha\geq 0,\ |\alpha|=p} U_{\lambda-\alpha} = \sum_{\mu\xrightarrow{p}\lambda} U_\mu \tag{7}$$

where the sums are over compositions $\alpha\geq 0$ with $|\alpha|=p$ and partitions $\mu\supset\lambda$ (respectively $\mu\subset\lambda$) such that $\lambda\xrightarrow{p}\mu$ (respectively, $\mu\xrightarrow{p}\lambda$). Moreover, for every $n\geq\ell(\lambda)$, the identities (7) remain true if the sums are taken over α and μ of length at most n.

Proof. The proofs of the two identities are very similar, so we will only discuss the second. Let us rewrite the sum $\sum_{\alpha\geq 0} U_{\lambda-\alpha}$ as $\sum_{\nu\leq\lambda} U_\nu$, where the latter sum is over integer sequences ν such that $\nu_i \leq \lambda_i$ for each i and $|\nu| = |\lambda| - p$. Call any such sequence ν *bad* if there exists a $j \geq 1$ such that $\nu_j < \lambda_{j+1}$, and let X be the set of all bad sequences. Define an involution $\iota : X \to X$ as follows: for $\nu \in X$, choose j minimal such that $\nu_j < \lambda_{j+1}$, and set

$$\iota(\nu) = (\nu_1, \ldots, \nu_{j-1}, \nu_{j+1} - 1, \nu_j + 1, \nu_{j+2}, \ldots).$$

Lemma 1(a) implies that $U_\nu + U_{\iota(\nu)} = 0$ for every $\nu \in X$. Therefore all bad indices may be omitted from the sum $\sum_{\nu\leq\lambda} U_\nu$, and this completes the proof. Moreover, to evaluate $\sum_{\nu\leq\lambda} U_\nu$ in the situation where $\nu_j = 0$ for all $j > n$, notice that if the minimal j such that $\nu_j < \lambda_{j+1}$ is $j = n$, then $\nu_n < 0$ and therefore $U_\nu = 0$. $\square$

2.4. **The Pieri rule.** For any $d \geq 1$ define the operator R^d by

$$R^d = \prod_{1\leq i<j\leq d} (1 - R_{ij}).$$

For $p > 0$ and any partition λ of length ℓ, we compute

$$u_p \cdot U_\lambda = u_p \cdot R^\ell\, u_\lambda = R^\ell\, u_{(\lambda,p)} = R^{\ell+1} \cdot \prod_{i=1}^{\ell} (1 - R_{i,\ell+1})^{-1}\, u_{(\lambda,p)}$$

$$= R^{\ell+1} \cdot \prod_{i=1}^{\ell} (1 + R_{i,\ell+1} + R_{i,\ell+1}^2 + \cdots)\, u_{(\lambda,p)} = \sum_{\alpha\geq 0} U_{\lambda+\alpha},$$

where the sum is over all compositions α such that $|\alpha| = p$ and $\alpha_j = 0$ for $j > \ell+1$. Applying Lemma 2, we arrive at the *Pieri rule*

$$u_p \cdot U_\lambda = \sum_{\lambda \xrightarrow{p} \mu} U_\mu. \tag{8}$$

Conversely, suppose that we are given a family $\{X_\lambda\}$ of elements of $\mathbb{A}$, one for each partition λ, such that $X_p = u_p$ for every integer $p \geq 0$ and the X_λ satisfy the Pieri rule $X_p \cdot X_\lambda = \sum_{\lambda \xrightarrow{p} \mu} X_\mu$. We claim then that

$$X_\lambda = U_\lambda = \prod_{i<j} (1 - R_{ij})\, u_\lambda$$

for every partition λ. To see this, note that the Pieri rule implies that

$$U_\lambda + \sum_{\mu \succ \lambda} a_{\lambda\mu}\, U_\mu = u_{\lambda_1} \cdots u_{\lambda_\ell} = X_\lambda + \sum_{\mu \succ \lambda} a_{\lambda\mu}\, X_\mu \tag{9}$$

for some constants $a_{\lambda\mu} \in \mathbb{Z}$. The claim now follows by induction on λ.

Example 2. We have

$$u_2 \cdot U_{(3,3,1)} = U_{(5,3,1)} + U_{(4,3,2)} + U_{(4,3,1,1)} + U_{(3,3,3)} + U_{(3,3,2,1)}.$$

2.5. **Kostka numbers.** A *(semistandard) tableau* T on the skew shape λ/μ is a filling of the boxes of λ/μ with positive integers, so that the entries are weakly increasing along each row from left to right and strictly increasing down each column. We can identify such a tableau T with a sequence of partitions

$$\mu = \lambda^0 \xrightarrow{c_1} \lambda^1 \xrightarrow{c_2} \cdots \xrightarrow{c_r} \lambda^r = \lambda$$

such that for $1 \leq i \leq r$ the horizontal strip λ^i/λ^{i-1} consists of the c_i boxes in T with entry i. The composition $c(T) = (c_1, \ldots, c_r)$ is called the *content* of T.

Let μ be a partition and α any integer vector. The equation

$$u_\alpha \, U_\mu = \sum_\lambda K_{\lambda/\mu,\alpha} \, U_\lambda$$

summed over partitions λ such that $\lambda \supset \mu$ defines the *Kostka numbers* $K_{\lambda/\mu,\alpha}$. If α is not a composition such that $|\alpha| = |\lambda/\mu|$ then we have $K_{\lambda/\mu,\alpha} = 0$. Otherwise, iteration of the Pieri rule shows that $K_{\lambda/\mu,\alpha}$ equals the number of tableaux T of shape λ/μ and content vector $c(T) = \alpha$. We deduce from equation (9) that the *Kostka matrix* $K = \{K_{\lambda,\mu}\}$, whose rows and columns are indexed by partitions, is lower unitriangular with respect to the dominance order.

2.6. **The Littlewood-Richardson rule.** Define the *Littlewood-Richardson coefficients* to be the structure constants $c_{\mu\nu}^\lambda$ in the equation

$$U_\mu \cdot U_\nu = \sum_\lambda c_{\mu\nu}^\lambda \, U_\lambda. \tag{10}$$

If $\ell = \ell(\nu)$, we compute that

$$\begin{aligned} U_\mu \cdot U_\nu &= \sum_{w \in S_\ell} (-1)^w u_{w(\nu+\rho_\ell)-\rho_\ell} \, U_\mu \\ &= \sum_\lambda \sum_{w \in S_\ell} (-1)^w K_{\lambda/\mu, w(\nu+\rho_\ell)-\rho_\ell} \, U_\lambda \end{aligned}$$

from which we deduce that

$$c_{\mu\nu}^\lambda = \sum_{(w,T)} (-1)^w \tag{11}$$

where the sum is over all pairs (w,T) such that $w \in S_\ell$ and T is a tableau on λ/μ with $c(T) + \rho_\ell = w(\nu+\rho_\ell)$. Observe that $c(T)$ is a partition if and only if $c(T)+\rho_\ell$ is a strict partition, in which case $c(T)+\rho_\ell = w(\nu+\rho_\ell)$ implies that $w = 1$.

For any tableau T, let $T_{\geq r}$ denote the subtableau of T formed by the entries in columns r and higher, and define $T_{>r}$ and $T_{<r}$ similarly. We say that a pair (w,T) is *bad* if $c(T_{\geq r})$ is not a partition for some r. Let Y denote the set of bad pairs indexing the sum (11), and define a sign reversing involution $\iota : Y \to Y$ as follows. Given $(w,T) \in Y$, choose r maximal such that $c(T_{\geq r})$ is not a partition, and let j be minimal such that $c_j(T_{\geq r}) < c_{j+1}(T_{\geq r})$. Call an entry j (resp. $j+1$) in T *free* if there is no $j+1$ (resp. j) in its column. Let T' denote the filling of λ/μ obtained from T by replacing all free j's (resp. $(j+1)$'s) that lie in $T_{<r}$ with $(j+1)$'s (resp. j's), and then arranging the entries of each row in weakly increasing order. Since $c(T_{>r})$ is a partition, we deduce that T contains a single entry $j+1$ in column r, and no j in column r, while $c_j(T_{\geq r})+1 = c_{j+1}(T_{\geq r})$. It follows easily from this that T' is a tableau. We define $\iota(w,T) = (\epsilon_j w, T')$, where ϵ_j denotes the transposition $(j, j+1)$. Since $\epsilon_j c(T_{<r}) = c(T'_{<r})$ and $\epsilon_j(c(T_{\geq r})+\rho_\ell) = c(T_{\geq r})+\rho_\ell$, while $T_{\geq r}$

coincides with $T'_{\geq r}$, it follows that $\epsilon_j(c(T)+\rho_\ell) = c(T')+\rho_\ell$ and $\iota(w,T) \in Y$. We conclude that the bad pairs can be cancelled from the sum (11).

The above argument proves that $c^\lambda_{\mu\nu}$ is equal to the number of tableaux T of shape λ/μ and content ν such that $T_{\geq r}$ is a partition for each r. This is one among many equivalent forms of the *Littlewood-Richardson rule*.

2.7. **Duality involution.** Let $v_r = U_{(1^r)}$ for $r \geq 1$, $v_0 = 1$, and $v_r = 0$ for $r < 0$. By expanding the determinant $U_{(1^r)} = \det(u_{1+j-i})_{1\leq i,j\leq r}$ along the first row, we obtain the identity

$$v_r - u_1 v_{r-1} + u_2 v_{r-2} - \cdots + (-1)^r u_r = 0. \tag{12}$$

Define a ring homomorphism $\omega : \mathbb{A} \to \mathbb{A}$ by setting $\omega(u_r) = v_r$ for every integer r. For any integer sequence α, let $v_\alpha = \prod_i v_{\alpha_i}$, and for any partition λ, set

$$V_\lambda = \omega(U_\lambda) = \prod_{i<j}(1-R_{ij})\, v_\lambda.$$

We deduce from (8) that the V_λ satisfy the Pieri rule

$$v_p \cdot V_\lambda = \sum_{\lambda \xrightarrow{p} \mu} V_\mu. \tag{13}$$

On the other hand, the Littlewood-Richardson rule easily implies that

$$U_{(1^p)} \cdot U_\lambda = \sum_\mu U_\mu \tag{14}$$

summed over all partitions $\mu \supset \lambda$ such that μ/λ is a *vertical p-strip*. It follows from (13), (14), and induction on λ that $V_\lambda = U_{\lambda'}$ for each λ. Here λ' denotes the partition which is conjugate to λ, i.e. such that $\lambda'_i = \#\{j \mid \lambda_j \geq i\}$ for all i. In particular, the equality $\omega(U_\lambda) = U_{\lambda'}$ proves that ω is an involution of $\mathbb{A}$, a fact that can also be deduced from (12).

2.8. **Cauchy identities and skew Schur polynomials.** Define a new $\mathbb{Z}$-basis t_λ of $\mathbb{A}$ by the transition equations

$$U_\lambda = \sum_\mu K_{\lambda,\mu}\, t_\mu. \tag{15}$$

In other words, the transition matrix $M(U,t)$ between the bases U_λ and t_λ of $\mathbb{A}$ is defined to be the lower unitriangular Kostka matrix K. Then $A := M(t,U) = K^{-1}$ and $B := M(u,U) = K^t$. We have

$$\begin{aligned}\sum_\lambda t_\lambda \otimes u_\lambda &= \sum_{\lambda,\mu,\nu} A_{\lambda\mu} B_{\lambda\nu}\, U_\mu \otimes U_\nu \\ &= \sum_{\lambda,\mu,\nu} A^t_{\mu\lambda} B_{\lambda\nu}\, U_\mu \otimes U_\nu = \sum_\mu U_\mu \otimes U_\mu\end{aligned}$$

in $\mathbb{A}\otimes_{\mathbb{Z}}\mathbb{A}$, where the above sums are either formal or restricted to run over partitions of a fixed integer n. We deduce the Cauchy identity

$$\sum_\lambda U_\lambda \otimes U_\lambda = \sum_\lambda t_\lambda \otimes u_\lambda \tag{16}$$

and, by applying the automorphism $1\otimes\omega$ to (16), the dual Cauchy identity

$$\sum_\lambda U_\lambda\otimes V_\lambda=\sum_\lambda t_\lambda\otimes v_\lambda. \tag{17}$$

For any skew diagram λ/μ, define the *skew Schur polynomial* $U_{\lambda/\mu}$ by generalizing equation (15):

$$U_{\lambda/\mu}:=\sum_\nu K_{\lambda/\mu,\nu}\,t_\nu.$$

We have the following computation in the ring $\mathbb{A}\otimes_{\mathbb{Z}}\mathbb{A}\otimes_{\mathbb{Z}}\mathbb{A}$.

$$\begin{aligned}\sum_{\mu,\nu}U_\mu\otimes U_\nu\otimes U_\mu U_\nu&=\sum_{\mu,\nu}U_\mu\otimes t_\nu\otimes U_\mu u_\nu=\sum_{\lambda,\mu,\nu}U_\mu\otimes t_\nu\otimes K_{\lambda/\mu,\nu}U_\lambda\\&=\sum_{\lambda,\mu}U_\mu\otimes U_{\lambda/\mu}\otimes U_\lambda.\end{aligned}$$

By comparing the coefficient of $U_\mu\otimes U_\nu\otimes U_\lambda$ on either end of the previous equation, we obtain

$$U_{\lambda/\mu}=\sum_\nu c_{\mu\nu}^\lambda\,U_\nu \tag{18}$$

where the coefficients $c_{\mu\nu}^\lambda$ are the same as the ones in (10). Since $\omega(U_\lambda)=U_{\lambda'}$ implies the identity $c_{\mu\nu}^\lambda=c_{\mu'\nu'}^{\lambda'}$, we deduce from (18) that

$$\omega(U_{\lambda/\mu})=U_{\lambda'/\mu'}. \tag{19}$$

3. Symmetric functions

3.1. **Initial definitions.** Let $x=(x_1,x_2,\ldots)$ be an infinite sequence of commuting variables. For any composition α we set $x^\alpha=\prod_i x_i^{\alpha_i}$. Given $k\geq 0$, let Λ^k denote the abelian group of all formal power series $\sum_{|\alpha|=k}c_\alpha x^\alpha\in\mathbb{Z}[[x_1,x_2,\ldots]]$ which are invariant under any permutation of the variables x_i. The elements of Λ^k are called homogeneous symmetric functions of degree k, and the graded ring $\Lambda=\bigoplus_{k\geq 0}\Lambda^k$ is the ring of symmetric functions.

For each partition λ of k, we obtain an element $m_\lambda\in\Lambda^k$ by symmetrizing the monomial x^λ. In other words, $m_\lambda(x)=\sum_\alpha x^\alpha$ where the sum is over all distinct permutations $\alpha=(\alpha_1,\alpha_2,\ldots)$ of $\lambda=(\lambda_1,\lambda_2,\ldots)$. We call m_λ a *monomial symmetric function*. The definition implies that if $f=\sum_\alpha c_\alpha x^\alpha\in\Lambda^k$, then $f=\sum_\lambda c_\lambda m_\lambda$. It follows that the m_λ for all partitions λ of k (respectively, for all partitions λ) form a $\mathbb{Z}$-basis of Λ^k (respectively, of Λ).

Let $h_r=h_r(x)$ denote the r-th *complete symmetric function*, defined by

$$h_r(x)=\sum_{\lambda\,:\,|\lambda|=r}m_\lambda(x)=\sum_{i_1\leq\cdots\leq i_r}x_{i_1}\cdots x_{i_r}.$$

We have the generating function equation

$$H(t)=\sum_{r=0}^\infty h_r(x)t^r=\prod_{i=1}^\infty(1-x_it)^{-1}. \tag{20}$$

Let $h_\alpha=\prod_i h_{\alpha_i}$ for any integer sequence α.

There is a unique ring homomorphism $\phi : \mathbb{A} \to \Lambda$ defined by setting $\phi(u_r) = h_r$ for every $r \geq 0$. For any integer sequence α, the *Schur function* s_α is defined by $s_\alpha = \phi(U_\alpha)$. We have

$$s_\alpha = \prod_{i<j}(1 - R_{ij})\, h_\alpha = \det(h_{\alpha_i+j-i})_{i,j}.$$

3.2. **Reduction and tableau formulas.** Let $y = (y_1, y_2, \ldots)$ be a second sequence of variables, choose $n \geq 1$, and set $x^{(n)} = (x_1, \ldots, x_n)$. It follows easily from equation (20) that for any integer p,

$$h_p(x^{(n)}, y) = \sum_{i=0}^{p} h_i(x_n)\, h_{p-i}(x^{(n-1)}, y).$$

Therefore, for any integer vector ν, we have

$$h_\nu(x^{(n)}, y) = \sum_{\alpha \geq 0} h_\alpha(x_n)\, h_{\nu-\alpha}(x^{(n-1)}, y) = \sum_{\alpha \geq 0} x_n^{|\alpha|}\, h_{\nu-\alpha}(x^{(n-1)}, y)$$

summed over all compositions α. If R denotes any raising operator and λ is any partition, we obtain

$$R\, h_\lambda(x^{(n)}, y) = \sum_{\alpha \geq 0} x_n^{|\alpha|}\, h_{R\lambda-\alpha}(x^{(n-1)}, y) = \sum_{\alpha \geq 0} x_n^{|\alpha|}\, R\, h_{\lambda-\alpha}(x^{(n-1)}, y). \tag{21}$$

Since $s_\lambda = \prod_{i<j}(1 - R_{ij})\, h_\lambda$, we deduce from (21) that

$$s_\lambda(x^{(n)}, y) = \sum_{\alpha \geq 0} x_n^{|\alpha|} s_{\lambda-\alpha}(x^{(n-1)}, y) = \sum_{p=0}^{\infty} x_n^p \sum_{|\alpha|=p} s_{\lambda-\alpha}(x^{(n-1)}, y).$$

Applying Lemma 2, we obtain the reduction formula

$$s_\lambda(x^{(n)}, y) = \sum_{p=0}^{\infty} x_n^p \sum_{\mu \xrightarrow{p} \lambda} s_\mu(x^{(n-1)}, y). \tag{22}$$

Repeated application of the reduction equation (22) results in

$$s_\lambda(x^{(n)}, y) = \sum_{\mu \subset \lambda} s_\mu(y) \sum_{T \text{ on } \lambda/\mu} x^{c(T)} \tag{23}$$

where the first sum is over partitions $\mu \subset \lambda$ and the second over all tableau T of shape λ/μ with entries at most n. As n is arbitrary, equation (23) holds with $x = (x_1, x_2, \ldots)$ in place of $x^{(n)}$. It follows that

$$s_\lambda(x, y) = \sum_{\mu \subset \lambda} s_\mu(y) \sum_{T \text{ on } \lambda/\mu} x^{c(T)}$$

where the second sum is over all tableau T of shape λ/μ. Substituting $y = 0$ proves Littlewood's tableau formula

$$s_\lambda(x) = \sum_{T \text{ on } \lambda} x^{c(T)} = \sum_{\mu} K_{\lambda,\mu}\, m_\mu(x). \tag{24}$$

From (24) we deduce immediately that the s_λ for λ a partition form a $\mathbb{Z}$-basis of Λ, and comparing with (15) shows that $\phi(t_\lambda) = m_\lambda$. It follows that the functions h_λ for λ a partition also form a $\mathbb{Z}$-basis of Λ.

3.3. **Duality and Cauchy identities.** Let $e_r = e_r(x)$ denote the r-th *elementary symmetric function* in the variables x, so that

$$e_r(x) = m_{(1^r)}(x) = \sum_{i_1<\cdots<i_r} x_{i_1}\cdots x_{i_r}.$$

The generating function $E(t)$ for the e_r satisfies

$$E(t) = \sum_{r=0}^{\infty} e_r(x)t^r = \prod_{i=1}^{\infty}(1+x_it).$$

Since $E(t)H(-t)=1$, we obtain

$$e_r - h_1e_{r-1} + h_2e_{r-2} - \cdots + (-1)^r h_r = 0 \tag{25}$$

for each $r \geq 1$. For any integer sequence α, we set $e_\alpha = \prod_i e_{\alpha_i}$.

By comparing equations (12) and (25), we deduce that $\phi(v_r) = e_r$ for each r, and hence $\phi(v_\lambda) = e_\lambda$ and $\phi(V_\lambda) = s_{\lambda'}$. The duality involution on $\mathbb{A}$ transfers to an automorphism $\omega : \Lambda \to \Lambda$ which sends h_λ to e_λ and s_λ to $s_{\lambda'}$, for each partition λ. We deduce that the e_λ form another $\mathbb{Z}$-basis of Λ. Moreover, by applying ϕ to (16) and (17), we obtain the usual form of the Cauchy identities

$$\sum_\lambda s_\lambda(x)s_\lambda(y) = \sum_\lambda m_\lambda(x)h_\lambda(y) = \prod_{i,j}\frac{1}{1-x_iy_j}$$

and

$$\sum_\lambda s_\lambda(x)s_{\lambda'}(y) = \sum_\lambda m_\lambda(x)e_\lambda(y) = \prod_{i,j}(1+x_iy_j)$$

where the sums are taken over all partitions λ.

3.4. **Skew Schur functions.** Define the *skew Schur functions* $s_{\lambda/\mu}$ by

$$s_{\lambda/\mu}(x) = \phi(U_{\lambda/\mu}) = \sum_\nu K_{\lambda/\mu,\nu}\, m_\nu(x) = \sum_{T \text{ on } \lambda/\mu} x^{c(T)}.$$

Equation (23) then implies that

$$s_\lambda(x,y) = \sum_{\mu\subset\lambda} s_{\lambda/\mu}(x)s_\mu(y) = \sum_{\mu\subset\lambda} s_\mu(x)s_{\lambda/\mu}(y). \tag{26}$$

Applying the operator $\prod_{i<j}(1-R_{ij})$ to both sides of the equation

$$h_\lambda(x,y) = \sum_{\alpha\geq 0} h_\alpha(x)h_{\lambda-\alpha}(y)$$

gives

$$s_\lambda(x,y) = \sum_{\alpha\geq 0} h_\alpha(x)s_{\lambda-\alpha}(y). \tag{27}$$

Since $h_\alpha = \sum_\mu K_{\mu,\alpha}s_\mu$, comparing (26) with (27) proves that

$$s_{\lambda/\mu} = \sum_{\alpha\geq 0} K_{\mu,\alpha}s_{\lambda-\alpha}. \tag{28}$$

Observe that (28) is a generalization of the second identity in Lemma 2.

Using Lemma 1(b) in (27), we obtain that

$$s_\lambda(x,y) = \sum_\mu s_\mu(y) \sum_{w\in S_\ell} (-1)^w h_{\lambda+\rho_\ell - w(\mu+\rho_\ell)}(x) \tag{29}$$

where the first sum is over all partitions μ and $\ell = \ell(\lambda)$. Equating the coefficients of $s_\mu(y)$ in (26) and (29) proves the following generalization of the Jacobi-Trudi identity (2):

$$s_{\lambda/\mu} = \sum_{w \in S_\ell} (-1)^w h_{\lambda+\rho_\ell - w(\mu+\rho_\ell)} = \det(h_{\lambda_i - \mu_j + j - i})_{i,j}. \tag{30}$$

By applying the involution ω to (30) and using (19), we derive the dual equation

$$s_{\lambda'/\mu'} = \det(e_{\lambda_i - \mu_j + j - i})_{i,j}.$$

3.5. **The classical definition of Schur polynomials.** In this section we fix n, the number of variables, and work with integer vectors and partitions in $\mathbb{Z}^n$. Let $x = (x_1, \ldots, x_n)$ and set $\rho = \rho_n = (n-1, \ldots, 1, 0)$. For each $\alpha \in \mathbb{Z}^n$, define

$$A_\alpha = \sum_{w \in S_n} (-1)^w x^{w(\alpha)} = \det(x_i^{\alpha_j})_{1 \leq i,j \leq n}$$

and set $\tilde{s}_\alpha(x) = A_{\alpha+\rho}/A_\rho$. Consider the $\mathbb{Z}$-linear map $\mathbb{A} \to \mathbb{Z}[x_1, \ldots, x_n]$ sending U_λ to $A_{\lambda+\rho}$ for any partition λ with $\ell(\lambda) \leq n$, and to zero, if $\ell(\lambda) > n$. It follows from Lemma 1(b) that this map sends U_α to $A_{\alpha+\rho}$ for any composition $\alpha \in \mathbb{Z}^n$. Lemma 2 therefore implies that for any partition $\lambda \in \mathbb{Z}^n$ and integer $r \geq 0$, we have

$$\sum_{\alpha \geq 0} A_{\lambda+\alpha+\rho} = \sum_{\lambda \xrightarrow{r} \mu} A_{\mu+\rho} \tag{31}$$

where the sums are over compositions $\alpha \geq 0$ with $|\alpha| = r$ and $\ell(\alpha) \leq n$ and partitions μ with $\lambda \xrightarrow{r} \mu$ and $\ell(\mu) \leq n$. Furthermore, we have

$$\begin{aligned} A_{\lambda+\rho}\, h_r(x) &= \sum_{w \in S_n} (-1)^w \sum_{\alpha \geq 0\, :\, |\alpha| = r} x^{w(\lambda+\rho)+\alpha} \\ &= \sum_{w \in S_n} (-1)^w \sum_{\alpha \geq 0\, :\, |\alpha| = r} x^{w(\lambda+\rho)+w(\alpha)} \\ &= \sum_{\alpha \geq 0\, :\, |\alpha| = r} A_{\lambda+\alpha+\rho} = \sum_{\lambda \xrightarrow{r} \mu} A_{\mu+\rho}, \end{aligned}$$

by (31). Now divide by A_ρ to deduce that

$$\tilde{s}_\lambda(x)\, h_r(x) = \sum_{\lambda \xrightarrow{r} \mu} \tilde{s}_\mu(x). \tag{32}$$

Applying (32) with $\lambda = 0$ gives $\tilde{s}_r(x) = h_r(x)$, for every $r \geq 1$. Since the $\tilde{s}_\lambda(x)$ satisfy the Pieri rule, it follows by induction on λ as in §2.4 that

$$\tilde{s}_\lambda(x) = \prod_{i<j} (1 - R_{ij})\, h_\lambda(x) = s_\lambda(x)$$

for each partition λ of length at most n. We have thus proved equation (1).

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