SCHUBERT POLYNOMIALS AND DEGENERACY LOCUS FORMULAS

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Abstract. In previous work [T6], we employed the approach to Schubert polynomials by Fomin, Stanley, and Kirillov to obtain simple, uniform proofs that the double Schubert polynomials of Lascoux and Schützenberger and Ikeda, Mihalcea, and Naruse represent degeneracy loci for the classical groups in the sense of Fulton. Using this as our starting point, and purely combinatorial methods, we obtain a new proof of the general formulas of [T5], which represent the degeneracy loci coming from any isotropic partial flag variety. Along the way, we also find several new formulas and elucidate the connections between some earlier ones.

Contents

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0. Introduction

In the 1990s, Fulton [Fu1, Fu2] introduced a notion of degeneracy loci determined by flags of vector bundles associated to any classical Lie group G. For any two (isotropic) flags $E_\bullet$ and $F_\bullet$ of subbundles of a (symplectic or orthogonal) vector bundle E over a base variety M, and an element w in the Weyl group of G, there is a locus $\mathfrak{X}_w \subset M$ defined by incidence relations between the flags. The *degeneracy locus problem* is to find a universal polynomial P_w in the Chern classes of the vector bundles involved such that $[\mathfrak{X}_w] = P_w \cap [M]$. We ask that P_w should be combinatorially explicit and manifestly respect the symmetries (that is, the descent sets) of both w and w^{-1}, whenever possible.

When G is the general linear group, the degeneracy locus problem was solved by Buch, Kresch, Yong, and the author [BKTY]. This answer was extended in a type uniform way to the symplectic and orthogonal Lie groups in [T5]. The formulas of [BKTY, T5] rely in part on a theory of *Schubert polynomials*, which express the classes of the degeneracy loci in terms of the Chern *roots* of the vector bundles $E_\bullet$ and $F_\bullet$. The desired combinatorial theory of Schubert polynomials, together with its connection to geometry, was established in the papers [LS, L, Fu1] (for Lie type A) and [BH, T2, T3, IMN1, T5] (for Lie types B, C, and D).

In previous work [T6, §7.3], we employed Fomin, Stanley, and Kirillov's nil-Coxeter algebra approach to Schubert polynomials [FS, FK] to give simple, uniform proofs that the double Schubert polynomials of Lascoux and Schützenberger [LS, L] and Ikeda, Mihalcea, and Naruse [IMN1] represent degeneracy loci of vector bundles, in the above sense. Our main goal in this paper is to begin with the same definition of Schubert polynomials from [T5, T6] and, by purely combinatorial methods, derive the splitting formulas for these polynomials found in [T5, §3 and §6]. The latter results then imply the general degeneracy locus formulas of [T5]. The proof of the corresponding type A splitting formula from [BKTY] is essentially combinatorial; this is clarified in [T5, §1.4] and §1 of the present paper.

As in [T5], our arguments depend on two key results from [BKT2, BKT3], which state that the single Schubert polynomials indexed by Grassmannian elements of the Weyl groups are represented by (single) *theta* and *eta polynomials*. The original proofs of these theorems used the classical Pieri rules from [BKT1], which were derived geometrically in op. cit. by intersecting Schubert cells. More recent proofs by Ikeda and Matsumura [IM], the author and Wilson [TW, T7], and Anderson and Fulton [AF2] use localization in equivariant cohomology (following [Ar, KK]) or employ other geometric arguments stemming from Kazarian's work [Ka]. However, the statements of the aforementioned theorems from [BKT2, BKT3] are entirely combinatorial, and it is natural to seek proofs of them within the same framework. The corresponding result in type A is the elementary fact that the symmetric Schubert polynomials are equal to Schur polynomials.

The approach we take here begins by extending Billey and Haiman's formula [BH, Prop. 4.15] for the single Schubert polynomials indexed by the longest element in the Weyl group of G to the double Schubert polynomials. From this, we deduce corresponding Pfaffian formulas equivalent to those in [IMN1, Thm. 1.2] and [AF1, AF2] for the (equivariant) Schubert class of a point on the complete flag variety G/B. Following the method of [IM, §8] in the symplectic case, by employing the left divided difference operators, we then derive analogous formulas for the class of a point on symplectic and even orthogonal Grassmannians, which first appeared in Kazarian's paper [Ka]. The proof continues by using the arguments found in [TW, T7] to arrive at *double theta* and *double eta polynomials*, and then specializing to obtain their single versions in [BKT2, BKT3]. Finally, we establish more general versions of the Schubert splitting formulas of [T5] with the help of the double mixed Stanley functions and k-transition trees introduced in op. cit.

The result of our efforts is a straightforward and type uniform combinatorial proof of the equivariant Giambelli and degeneracy locus formulas of [Ka, I, IN, IMN1, IM, T5, T7, TW]. Along the way, we obtain several new formulas, and illuminate the connections between some earlier ones. In particular, we define in §2.3 the *reverse double Schubert polynomials* $\widetilde{\mathfrak{S}}_{\varpi}(Y,Z)$, which provide a bridge between the type A and type C theories. Furthermore, in type D, inspired by [I, IN, IMN1], we use Ivanov's double (or factorial) Schur P-functions [Iv1, Iv2, Iv3], and obtain in §4.2 new results about them, which are suitable for our purposes.

Our earlier paper [T6] contained a detailed exposition of the various ingredients that went into [T5], revealing the author's perspective on the subject of degeneracy loci, as it stood in 2009. We revisit some of that material here, for completeness, but focus in §1 – §4 on what is required for our new combinatorial proofs of the main results. In §5, we discuss how to use *geometrization* to translate the theorems of the previous sections into Chern class formulas, and provide some detailed remarks on the history of this problem, which supplement the ones contained in [T6].

In a recent preprint, Anderson and Fulton [AF2] use Young's raising operators and algebro-geometric arguments to define *multi-theta* and *multi-eta polynomials*, which extend the double theta and eta polynomials of [TW, T7, W] even further. The resulting degeneracy locus formulas and their proofs are an important contribution to the theory of theta and eta polynomials, but hold only for certain special elements w of the Weyl group of G. We leave the task of including them within the present algebraic and combinatorial framework to future research.

As we mentioned above, besides new and uniform proofs of earlier theorems, the present paper also contains many original results, which appear in the text without attribution. For the reader's convenience, we provide a list of the main ones here. To the best of our knowledge, Definition 2.4, Propositions 2.5, 3.1, 3.8, 3.9, 4.2, 4.4, 4.5, and 4.13, and Corollaries 2.6, 4.3, and 4.11 are new.

This article is organized as follows. In §1 we review the type A theory, culminating in the relevant splitting formulas for type A double Schubert polynomials from [BKTY]. In §2, we discuss flagged Schur polynomials, type A duality, and introduce the reverse double Schubert polynomials $\widetilde{\mathfrak{S}}_{\varpi}(Y,Z)$. These objects are used in §3 and §4, which provide the corresponding theory in Lie types C and D. Finally, in §5 we discuss the history of the geometrization of the single and double Schubert polynomials, focusing on the symplectic case.

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1. The type A theory

1.1. **Schubert polynomials and divided differences.** The Weyl group S_n of permutations of $\{1,\ldots,n\}$ is generated by the $n-1$ simple transpositions $s_i =$

$(i, i+1)$. A *reduced word* of a permutation ϖ in S_∞ is a sequence $a_1 \cdots a_\ell$ of positive integers such that $\varpi = s_{a_1} \cdots s_{a_\ell}$ and ℓ is minimal, so (by definition) equal to the length $\ell(\varpi)$ of ϖ. We say that ϖ has a *descent* at position r if $\ell(\varpi s_r) < \ell(\varpi)$, where s_r is the simple reflection indexed by r.

Our main references for type A Schubert polynomials are [LS, M2, FS]. We recall from [FS] their construction using the *nilCoxeter algebra* $\mathcal{N}_n$ of the symmetric group S_n. By definition, $\mathcal{N}_n$ is the free associative algebra with unit generated by the elements $u_1, \ldots, u_{n-1}$ modulo the relations

$$\begin{array}{rcll} u_i^2 & = & 0 & i \geq 1\ ; \\ u_i u_j & = & u_j u_i & |i-j| \geq 2\ ; \\ u_i u_{i+1} u_i & = & u_{i+1} u_i u_{i+1} & i \geq 1. \end{array}$$

For any $\varpi \in S_n$, choose a reduced word $a_1 \cdots a_\ell$ for ϖ and define $u_\varpi := u_{a_1} \ldots u_{a_\ell}$. Then the u_ϖ for $\varpi \in S_n$ are well defined and form a free $\mathbb{Z}$-basis of $\mathcal{N}_n$. We denote the coefficient of $u_\varpi \in \mathcal{N}_n$ in the expansion of the element $\xi \in \mathcal{N}_n$ by $\langle \xi, \varpi \rangle$; in other words, $\xi = \sum_{\varpi \in S_n} \langle \xi, \varpi \rangle \, u_\varpi$, for all $\xi \in \mathcal{N}_n$.

Let t be an indeterminate and define

$$A_i(t) := (1 + t u_{n-1})(1 + t u_{n-2}) \cdots (1 + t u_i)\ ;$$
$$\tilde{A}_i(t) := (1 - t u_i)(1 - t u_{i+1}) \cdots (1 - t u_{n-1}).$$

Suppose that $Y = (y_1, y_2, \ldots)$ and $Z = (z_1, z_2, \ldots)$ are two infinite sequences of commuting independent variables. For any $\varpi \in S_n$, the *double Schubert polynomial* $\mathfrak{S}_\varpi(Y, Z)$ of Lascoux and Schützenberger [LS, L] is given by the prescription

$$\mathfrak{S}_\varpi(Y, Z) := \left\langle \tilde{A}_{n-1}(z_{n-1}) \cdots \tilde{A}_1(z_1) A_1(y_1) \cdots A_{n-1}(y_{n-1}), \varpi \right\rangle. \tag{1.1}$$

The polynomial $\mathfrak{S}_\varpi(Y) := \mathfrak{S}_\varpi(Y, 0)$ is the single Schubert polynomial. The definition (1.1) implies that $\mathfrak{S}_\varpi(Y)$ has nonnegative integer coefficients, which admit a combinatorial interpretation (compare with [BJS, Thm. 1.1]).

For each $n \geq 1$, there is an injective group homomorphism $i_n : S_n \hookrightarrow S_{n+1}$, defined by adjoining the fixed point $n+1$, and we let $S_\infty := \cup_n S_n$. The polynomials $\mathfrak{S}_\varpi(Y, Z)$ have an important *stability property* under the inclusions i_n of the Weyl groups, namely, if $\varpi \in S_n$, then we have

$$\mathfrak{S}_{i_n(\varpi)}(Y, Z) = \mathfrak{S}_\varpi(Y, Z).$$

The stability property implies that $\mathfrak{S}_\varpi(Y, Z)$ is well defined for all $\varpi \in S_\infty$.

For any $i \geq 1$, there are *right* and *left* divided difference operators ∂_i^y and ∂_i^z which act on the polynomial ring $\mathbb{Z}[Y, Z]$. We define an action of S_∞ on $\mathbb{Z}[Y, Z]$ by ring automorphisms by letting the simple transpositions s_i act by interchanging y_i and y_{i+1} and leaving all the remaining variables fixed. Define ∂_i^y on $\mathbb{Z}[Y, Z]$ by

$$\partial_i^y f := \frac{f - s_i f}{y_i - y_{i+1}}.$$

Consider the ring involution $\omega : \mathbb{Z}[Y, Z] \to \mathbb{Z}[Y, Z]$ determined by $\omega(y_j) = -z_j$ and $\omega(z_j) = -y_j$ for each j, and set $\partial_i^z := \omega \partial_i^y \omega$ for each $i \geq 1$.

Both the single and double Schubert polynomials may be characterized by their compatibility with the divided difference operators. In fact, the polynomials $\mathfrak{S}_\varpi$

for $\varpi \in S_\infty$ are the unique family of elements of $\mathbb{Z}[Y,Z]$ satisfying the equations

$$\partial_i^y \mathfrak{S}_\varpi = \begin{cases} \mathfrak{S}_{\varpi s_i} & \text{if } \ell(\varpi s_i) < \ell(\varpi), \\ 0 & \text{otherwise,} \end{cases} \qquad \partial_i^z \mathfrak{S}_\varpi = \begin{cases} \mathfrak{S}_{s_i \varpi} & \text{if } \ell(s_i \varpi) < \ell(\varpi), \\ 0 & \text{otherwise,} \end{cases} \tag{1.2}$$

for all $i \geq 1$, together with the condition that the constant term of $\mathfrak{S}_\varpi$ is 1 if $\varpi = 1$, and 0 otherwise. This characterization theorem is straightforward to prove directly from the definition (1.1), following [FS, Thm. 2.2].

The above result has two important consequences. The first is that $\mathfrak{S}_\varpi(Y,Z)$ is symmetric in y_i, y_{i+1} (respectively in z_j, z_{j+1}) if and only if $\varpi_i < \varpi_{i+1}$ (respectively $\varpi_j^{-1} < \varpi_{j+1}^{-1}$). In other words, the descents of ϖ and ϖ^{-1} determine the symmetries of the polynomial $\mathfrak{S}_\varpi(Y,Z)$. A second consequence is that the double Schubert polynomials $\mathfrak{S}_\varpi(Y,Z)$ represent the universal Schubert classes in type A flag bundles, and therefore degeneracy loci of vector bundles, in the sense of [Fu1].

Let $\varpi_0 = (n, n-1, \ldots, 1)$ be the longest element of S_n. According to [FS, Lemma 2.1], for all commuting variables s, t and indices i, we have $A_i(s)A_i(t) = A_i(t)A_i(s)$. Since $\tilde{A}_i(t) = A_i(t)^{-1}$, we also have $A_i(s)\tilde{A}_i(t) = \tilde{A}_i(t)A_i(s)$. Fomin and Stanley [FS, Cor. 4.4] use this fact and the definition (1.1) to show that

$$\mathfrak{S}_{\varpi_0}(Y,Z) = \prod_{i+j \leq n} (y_i - z_j). \tag{1.3}$$

1.2. **Schur polynomials.** Following [LS, M2], the product in (1.3) may be written in the form of a multi-Schur determinant. Furthermore, by applying divided differences to $\mathfrak{S}_{\varpi_0}$ and using the equations (1.2), one can express more general Schubert polynomials $\mathfrak{S}_\varpi$ as Schur type determinants. We will not reprove these formulas here, but we do need some more notation to recall the ones that we will require.

For any integer $j \geq 0$, define the elementary and complete symmetric functions $e_j(Y)$ and $h_j(Y)$ by the generating series

$$\prod_{i=1}^{\infty}(1 + y_i t) = \sum_{j=0}^{\infty} e_j(Y) t^j \quad \text{and} \quad \prod_{i=1}^{\infty}(1 - y_i t)^{-1} = \sum_{j=0}^{\infty} h_j(Y) t^j,$$

respectively. We define the supersymmetric functions $h_p(Y/Z)$ for $p \in \mathbb{Z}$ by the generating function equation

$$\sum_{p=0}^{\infty} h_p(Y/Z) t^p = \left(\sum_{j=0}^{\infty} h_j(Y) t^j\right) \left(\sum_{j=0}^{\infty} e_j(Z)(-t)^j\right).$$

If $r \geq 1$ then we let $e_j^r(Y) := e_j(y_1, \ldots, y_r)$ and $h_j^r(Y) := h_j(y_1, \ldots, y_r)$ denote the polynomials obtained from $e_j(Y)$ and $h_j(Y)$ by setting $y_j = 0$ for all $j > r$. Let $e_j^0(Y) = h_j^0(Y) = \delta_{0j}$, where δ_{0j} denotes the Kronecker delta, and for $r < 0$, define $h_j^r(Y) := e_j^{-r}(Y)$ and $e_j^r(Y) := h_j^{-r}(Y)$.

We will work with integer sequences $\alpha = (\alpha_1, \alpha_2, \ldots)$ which are assumed to have finite support when they appear as subscripts. The sequence α is a *composition* if $\alpha_i \geq 0$ for all i, and a *partition* if $\alpha_i \geq \alpha_{i+1} \geq 0$ for all $i \geq 1$. We set $|\alpha| := \sum_i \alpha_i$. Partitions are traditionally identified with their Young diagram of boxes, and this is used to define the inclusion relation $\mu \subset \lambda$ between two partitions μ and λ.

Given an integer sequence α, we define $s_\alpha(Y/Z)$ by the determinantal equation

$$s_\alpha(Y/Z) := \det(h_{\alpha_i + j - i}(Y/Z))_{i,j}. \tag{1.4}$$

Notice that the matrix $\{h_{\alpha_i+j-i}(Y/Z)\}_{i,j}$ is upper unitriangular for i and j sufficiently large, so the determinant in (1.4) is well defined. When $\alpha = \lambda$ is a partition, then $s_\lambda(Y/Z)$ is called a *supersymmetric Schur function*. The usual Schur S-function $s_\lambda(Y)$ satisfies $s_\lambda(Y) := s_\lambda(Y/Z)|_{Z=0}$. Moreover, we have $s_\lambda(0/Z) = s_\lambda(Y/Z)|_{Y=0} = (-1)^{|\lambda|}s_{\lambda'}(Z)$, where λ' denotes the conjugate (or transpose) partition of λ. Observe that $s_\lambda(y_1,\ldots,y_m)=0$ if the *length* $\ell(\lambda)$, that is, the number of non-zero parts λ_i, is greater than m.

For any positive integers $i<j$ and integer sequence α, define the Young raising operator R_{ij} by $R_{ij}(\alpha) := (\alpha_1,\ldots,\alpha_i+1,\ldots,\alpha_j-1,\ldots)$. Using these operators, the equation (1.4) may be rewritten as

$$s_\alpha(Y/Z) = \prod_{i<j}(1-R_{ij})\, h_\alpha(Y/Z),$$

where $h_\alpha := \prod_i h_{\alpha_i}$ and each operator R_{ij} acts on the expression h_α (regarded as a noncommutative monomial) by the prescription $R_{ij}h_\alpha := h_{R_{ij}\alpha}$. See [T4] for more information on raising operators.

A permutation $\varpi \in S_\infty$ is *Grassmannian* if there exists an $m \geq 1$ such that $\varpi_i < \varpi_{i+1}$ for all $i \neq m$. The *shape* of such a Grassmannian permutation ϖ is the partition $\lambda = (\lambda_1,\ldots,\lambda_m)$ with $\lambda_{m+1-j} = \varpi_j - j$ for $1 \leq j \leq m$. If $\varpi \in S_\infty$ is a Grassmannian permutation with a unique descent at m and shape λ, then we have

$$\mathfrak{S}_\varpi(Y) = s_\lambda(y_1,\ldots,y_m). \tag{1.5}$$

A short proof of (1.5) starting from the formula (1.3) for $\mathfrak{S}_{\varpi_0}(Y)$ is in [M2, (4.8)].

1.3. **Stanley symmetric functions and splitting formulas.** Given any permutations $u_1,\ldots,u_p,\varpi$, we will write $u_1\cdots u_p = \varpi$ if $\ell(u_1)+\cdots+\ell(u_p) = \ell(\varpi)$ and the product of $u_1,\ldots,u_p$ is equal to ϖ. In this case we say that $u_1\cdots u_p$ is a *reduced factorization* of ϖ. Equation (1.1) impies the relation

$$\mathfrak{S}_\varpi(Y,Z) = \sum_{uv=\varpi} \mathfrak{S}_{u^{-1}}(-Z)\mathfrak{S}_v(Y) \tag{1.6}$$

summed over all reduced factorizations $uv=\varpi$ in S_∞.

If $A(Y) := A_1(y_1)A_1(y_2)\cdots$, then the function $G_\varpi(Y)$ defined for $\varpi \in S_n$ by

$$G_\varpi(Y) := \langle A(Y), \varpi\rangle$$

is symmetric in Y. G_ϖ is the type A *Stanley symmetric function*, which was introduced in [S].[1] If $\tilde{A}(Z) := \tilde{A}_1(z_1)\tilde{A}_1(z_2)\cdots$, then we define the *double Stanley symmetric function* $G_\varpi(Y/Z)$ by

$$G_\varpi(Y/Z) := \langle \tilde{A}(Z)A(Y), \varpi\rangle = \sum_{uv=\varpi} G_{u^{-1}}(-Z)G_v(Y)$$

with the sum over all reduced factorizations $uv=\varpi$ in S_∞.

Given $m \geq 1$ and any $\varpi \in S_n$, the permutation $1_m \times \varpi \in S_{m+n}$ is defined by $(1_m\times\varpi)(j) = j$ for $1 \leq j \leq m$ and $(1_m\times\varpi)(j) = m+\varpi(j-m)$ for $j>m$. We say that a permutation ϖ is *increasing up to m* if $\varpi(1)<\varpi(2)<\cdots<\varpi(m)$. If ϖ is increasing up to m, then we have the following key identity:

$$\mathfrak{S}_\varpi(Y) = \sum_{v(1_m\times u)=\varpi} G_v(y_1,\ldots,y_m)\mathfrak{S}_u(y_{m+1},y_{m+2},\ldots) \tag{1.7}$$

[1]In Stanley's paper, the function $G_{\varpi^{-1}}$ is assigned to ϖ.

where the sum is over all reduced factorizations $v(1_m \times u) = \varpi$ in S_∞. Equation (1.7) admits a double version: let $G_v(Y_{(m)}/Z_{(\ell)})$ denote the polynomial obtained from $G_v(Y/Z)$ by setting $y_i = z_j = 0$ for all $i > m$ and $j > \ell$. Then if ϖ is increasing up to m and ϖ^{-1} is increasing up to ℓ, we have

$$\mathfrak{S}_\varpi(Y,Z) = \sum \mathfrak{S}_{u^{-1}}(-Z_{>\ell})G_v(Y_{(m)}/Z_{(\ell)})\mathfrak{S}_{u'}(Y_{>m}), \tag{1.8}$$

where $Y_{>m} := (y_{m+1}, y_{m+2}, \ldots)$, $-Z_{>\ell} := (-z_{\ell+1}, -z_{\ell+2}, \ldots)$, and the sum is over all reduced factorizations $(1_\ell \times u)v(1_m \times u') = \varpi$ in S_∞. Equations (1.7) and (1.8) are easy to show directly from the definitions of $\mathfrak{S}_\varpi(Y,Z)$ and $G_v(Y/Z)$ (see [T5, §1.4] for a detailed proof of (1.7); the proof of (1.8) is similar).

We say that a permutation $\varpi \in S_\infty$ is *compatible* with the sequence $\mathfrak{a} : a_1 < \cdots < a_p$ of positive integers if all descent positions of ϖ are contained in $\mathfrak{a}$. Let $\mathfrak{b} : b_1 < \cdots < b_q$ be a second sequence of positive integers and assume that ϖ is compatible with $\mathfrak{a}$ and ϖ^{-1} is compatible with $\mathfrak{b}$. We say that a reduced factorization $u_1 \cdots u_{p+q-1} = \varpi$ is *compatible* with $\mathfrak{a}$, $\mathfrak{b}$ if $u_j(i) = i$ whenever $j < q$ and $i \leq b_{q-j}$ or whenever $j > q$ and $i \leq a_{j-q}$ (and where we set $u_j(0) = 0$). Set $Y_i := \{y_{a_{i-1}+1}, \ldots, y_{a_i}\}$ for each $i \geq 1$ and $Z_j := \{z_{b_{j-1}+1}, \ldots, z_{b_j}\}$ for each $j \geq 1$.

Proposition 1.1 ([BKTY]). *Suppose that ϖ and ϖ^{-1} are compatible with $\mathfrak{a}$ and $\mathfrak{b}$, respectively. Then the Schubert polynomial $\mathfrak{S}_\varpi(Y,Z)$ satisfies*

$$\mathfrak{S}_\varpi = \sum G_{u_1}(0/Z_q) \cdots G_{u_{q-1}}(0/Z_2)G_{u_q}(Y_1/Z_1)G_{u_{q+1}}(Y_2) \cdots G_{u_{p+q-1}}(Y_p)$$

summed over all reduced factorizations $u_1 \cdots u_{p+q-1} = \varpi$ compatible with $\mathfrak{a}$, $\mathfrak{b}$.

Proof. The result follows easily by using (1.8) and iterating the identity (1.7). □

When the Stanley symmetric function G_ϖ is expanded in the basis of Schur functions, one obtains a formula

$$G_\varpi(Y) = \sum_{\lambda\,:\,|\lambda| = \ell(\varpi)} a_\lambda^\varpi s_\lambda(Y) \tag{1.9}$$

for some nonnegative integers a_λ^ϖ. There exist several different combinatorial interpretations of these coefficients, for instance using the *transition trees* of Lascoux and Schützenberger (see for example [M2, (4.37)]). One also knows that $a_\lambda^\varpi = a_{\lambda'}^{\varpi^{-1}}$.

Theorem 1.2 ([BKTY], Thm. 4). *Suppose that ϖ is compatible with $\mathfrak{a}$ and ϖ^{-1} is compatible with $\mathfrak{b}$. Then we have*

$$\mathfrak{S}_\varpi = \sum_{\underline{\lambda}} a_{\underline{\lambda}}^\varpi\, s_{\lambda^1}(0/Z_q) \cdots s_{\lambda^{q-1}}(0/Z_2)s_{\lambda^q}(Y_1/Z_1)s_{\lambda^{q+1}}(Y_2) \cdots s_{\lambda^{p+q-1}}(Y_p) \tag{1.10}$$

summed over all sequences of partitions $\underline{\lambda} = (\lambda^1, \ldots, \lambda^{p+q-1})$, where

$$a_{\underline{\lambda}}^\varpi := \sum_{u_1 \cdots u_{p+q-1} = \varpi} a_{\lambda^1}^{u_1} \cdots a_{\lambda^{p+q-1}}^{u_{p+q-1}},$$

summed over all reduced factorizations $u_1 \cdots u_{p+q-1} = \varpi$ compatible with $\mathfrak{a}$, $\mathfrak{b}$.

Proof. The result follows from Proposition 1.1 by using the equation (1.9). □

Equation (1.10) generalizes the monomial positivity of the Schubert polynomial $\mathfrak{S}_\varpi(Y,Z)$ from [BJS, FS] to a combinatorial formula which manifestly respects the descent sets of ϖ and ϖ^{-1}, and therefore exhibits the symmetries of $\mathfrak{S}_\varpi$. Moreover, the splitting formula is uniquely determined once ϖ and the compatible sequences

$\mathfrak{a}$ and $\mathfrak{b}$ are specified. The main geometric application of equation (1.10) is that it directly implies a corresponding Chern class formula for the type A degeneracy locus indexed by ϖ, with the symmetries native to the partial flag variety associated to $\mathfrak{a}$. For more details on this, as well as examples of explicit computations of the *splitting coefficients* $a^{\varpi}_{\underline{\lambda}}$, see [BKTY, §4] and [T6, §4 and §6].

2. Flagged Schur polynomials and duality

2.1. **Flagged Schur polynomials.** Let $\alpha = \{\alpha_j\}_{1\leq j\leq \ell}$, $\beta = \{\beta_j\}_{1\leq j\leq \ell}$, and $\rho = \{\rho_j\}_{1\leq j\leq \ell}$ be three integer sequences, and let $t = (t_1, t_2, \ldots)$ be a sequence of independent variables. The Schur type determinant

$$S^{\rho}_{\alpha/\beta}(h(t)) := \det\left(h^{\rho_i}_{\alpha_i-\beta_j+j-i}(t)\right)_{1\leq i,j\leq \ell}$$

is called a *flagged Schur polynomial*. We define the polynomial $S^{\rho}_{\alpha/\beta}(e(t))$ in a similar way. The method of Gessel and Viennot [GV] or Wachs [Wa] shows that for any partition λ and increasing composition ρ, we have

$$S^{\rho}_{\lambda}(h(t)) = \sum t^U \tag{2.1}$$

where the sum is over all column strict Young tableaux U of shape λ whose entries in the i-th row are $\leq \rho_i$ for all $i\geq 1$.

The following well known result will be used in §3.2.

Lemma 2.1. *Let λ and μ be two partitions of length at most ℓ with $\max(\lambda_1,\mu_1)\leq k$. Then we have*

$$\det\left(h^{k+i-\lambda_i}_{\lambda_i-\mu_j+j-i}(t)\right)_{1\leq i,j\leq \ell} = \det\left(e^{k+\lambda'_i-i}_{\lambda'_i-\mu'_j+j-i}(t)\right)_{1\leq i,j\leq k}. \tag{2.2}$$

Proof. The argument follows the one in [M1, I.2, eq. (2.9)]. Let $N := k+\ell$ and define the matrices

$$A := \left(h^{-i}_{i-j}(-t)\right)_{0\leq i,j\leq N-1} \quad \text{and} \quad B := \left(h^{j+1}_{i-j}(t)\right)_{0\leq i,j\leq N-1}.$$

It is well known that A and B are inverse to each other (see for example [M1, I.3, Ex. 21]). Therefore each minor of A is equal to the complementary cofactor of B^t, the transpose of B. For the minor of $A = (e^i_{i-j}(-t))$ with row indices λ'_i+k-i $(1\leq i\leq k)$ and column indices μ'_j+k-j $(1\leq j\leq k)$, the complementary cofactor of $B^t = (h^{i+1}_{j-i}(t))$ has row indices $k-1+i-\lambda_i$ $(1\leq i\leq \ell)$ and column indices $k-1+j-\mu_j$ $(1\leq j\leq \ell)$. The equality (2.2) follows by taking determinants. □

2.2. **The duality involution.** For any $r\geq 1$, let δ_r denote the partition $(r, r-1,\ldots,1)$. Let $Y_{(n)} := (y_1,\ldots,y_n)$ and $I_n \subset \mathbb{Z}[Y_{(n)}]$ be the ideal generated by the elementary symmetric polynomials $e_i(Y_{(n)})$ for $1\leq i\leq n$. Set $H_n := \mathbb{Z}[Y_{(n)}]/I_n$ and let $\mathcal{H}_n$ be the $\mathbb{Z}$-linear subspace of $\mathbb{Z}[Y_{(n)}]$ spanned by the monomials $y^{\alpha} := y_1^{\alpha_1}\cdots y_{n-1}^{\alpha_{n-1}}$ for all compositions $\alpha\leq \delta_{n-1}$.

Let $\mathfrak{Y} = \mathrm{GL}_n/B$ be the variety which parametrizes complete flags

$$E_\bullet \;:\; 0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_n = \mathbb{C}^n$$

of subspaces of $\mathbb{C}^n$. It is well known that the inclusion $\mathcal{H}_n \subset \mathbb{Z}[Y_{(n)}]$ induces an isomorphism of abelian groups $\mathcal{H}_n \xrightarrow{\sim} H_n$, and that H_n is isomorphic to the cohomology ring of the flag variety $\mathfrak{Y}$. Moreover, the Schubert polynomials $\mathfrak{S}_{\varpi}(Y)$

for $\varpi \in S_n$ form a $\mathbb{Z}$-basis of $\mathcal{H}_n$, and represent the Schubert classes in $\mathfrak{Y}$ under the above isomorphism (see for example [Fu3, §10]).

Define an involution $\varpi \mapsto \varpi^*$ of S_n by setting $\varpi^* := \varpi_0\varpi\varpi_0$, and define the *duality involution* $* : \mathcal{H}_n \to \mathcal{H}_n$ to be the $\mathbb{Z}$-linear map determined by $*\,\mathfrak{S}_\varpi(Y) := \mathfrak{S}_{\varpi^*}(Y)$, for each $\varpi \in S_n$. We let $\mathbb{Z}[Y_{(n)}] \to \mathbb{Z}[Y_{(n)}]$ be the ring involution given by $(y_1, \ldots, y_n) \mapsto (-y_n, \ldots, -y_1)$, which descends to a ring involution $D : H_n \to H_n$. The following result states that $*$ and D are mutually compatible.

Lemma 2.2. *For any permutation $\varpi \in S_n$, we have*

$$D(\mathfrak{S}_\varpi(Y) + I_n) = \mathfrak{S}_{\varpi^*}(Y) + I_n \tag{2.3}$$

in the quotient ring H_n.

Proof. Modulo the defining relations for the ideal I_n of $\mathbb{Z}[Y_{(n)}]$, we have

$$A_1(y_1)A_1(y_2)\cdots A_1(y_n) \equiv 1. \tag{2.4}$$

Set

$$B_i(t) := (1 + tu_{n-i})(1 + tu_{n-i-1}t)\cdots(1 + tu_1)$$

so that

$$B_i(t)^{-1} = (1 - tu_1)(1 - tu_2)\cdots(1 - tu_{n-i}).$$

Observe that (2.4) may be written as

$$A_1(y_1)A_2(y_2)\cdots A_{n-1}(y_{n-1}) \cdot B_{n-1}(y_2)B_{n-2}(y_3)\cdots B_1(y_n) \equiv 1$$

or equivalently

$$A_1(y_1)A_2(y_2)\cdots A_{n-1}(y_{n-1}) \equiv B_1(y_n)^{-1}B_2(y_{n-1})^{-1}\cdots B_{n-1}(y_2)^{-1}.$$

Notice that $(s_i)^* = s_{n-i}$ for each simple transposition s_i in S_n. It follows that for any $\varpi \in S_n$, we have

$$\begin{aligned} \mathfrak{S}_\varpi(Y) &= \langle A_1(y_1)A_2(y_2)\cdots A_{n-1}(y_{n-1}), \varpi\rangle \\ &\equiv \langle B_1(y_n)^{-1}B_2(y_{n-1})^{-1}\cdots B_{n-1}(y_2)^{-1}, \varpi\rangle = D\,\mathfrak{S}_{\varpi^*}(Y), \end{aligned}$$

as required. □

Geometrically, the above involutions correspond to the duality isomorphism $\mathfrak{Y} \to \mathfrak{Y}$ which sends each complete flag $E_\bullet$ in $\mathbb{C}^n$ to the dual flag $E'_\bullet$ in $(\mathbb{C}^n)^*$. Here the subspace E'_i is defined as the kernel of the canonical linear map $(\mathbb{C}^n)^* \to E^*_{n-i}$, for $1 \leq i \leq n$. For more details on this, see [Fu3, §10, Exercise 13].

Example 2.3. One can prove the equality (2.3) directly in the case when $\mathfrak{S}_\varpi(Y)$ is an elementary symmetric polynomial $e_i(y_1, \ldots, y_r)$. We have $\prod_{i=1}^n(1 + y_it) = 1$ in $H_n[t]$, and hence

$$\prod_{i=1}^{r}(1 + y_it) \equiv \prod_{j=r+1}^{n} \frac{1}{1 + y_jt}$$

which implies that $e_i(y_1, \ldots, y_r) = (-1)^i h_i(y_{r+1}, \ldots, y_n)$ in H_n, for any $i, r \in [1, n]$. Applying the automorphism D gives

$$D(e_i(y_1, \ldots, y_r) + I_n) = h_i(y_1, \ldots, y_{n-r}) + I_n. \tag{2.5}$$

2.3. **Reverse double Schubert polynomials.**

Definition 2.4. For any $\varpi \in S_\infty$, define the *reverse double Schubert polynomial*

$$\widetilde{\mathfrak{S}}_\varpi(Y,Z) := \sum_{uv=\varpi} \mathfrak{S}_u(Y)\mathfrak{S}_{v^{-1}}(-Z), \tag{2.6}$$

where the sum is over all reduced factorizations $uv = \varpi$.

The adjective 'reverse' in Definition 2.4 is justified by comparing (2.6) with formula (1.6). Observe also that if $\varpi \in S_n$, then $\widetilde{\mathfrak{S}}_\varpi(Y,Z) = \langle \widetilde{\mathfrak{S}}(Y,Z), \varpi\rangle$, where

$$\widetilde{\mathfrak{S}}(Y,Z) := A_1(y_1)\cdots A_{n-1}(y_{n-1})\widetilde{A}_{n-1}(z_{n-1})\cdots \widetilde{A}_1(z_1)$$

and, for each variable t, the factors $A_i(t)$ and $\widetilde{A}_i(t)$ are defined as in §1.1.

Choose any integer $m \geq 0$, let $\delta^\vee_{n-1}(m) := (m+1,\ldots,m+n-1)$, and $\delta^\vee_{n-1} := (1,2,\ldots,n-1)$. Let $\Omega := (\omega_1,\ldots,\omega_m)$ be an m-tuple of independent variables and $\Omega + Y = (\omega_1,\ldots,\omega_m,y_1,y_2,\ldots)$ denote the concatenation of the alphabets Ω and Y. For every $i \in [1, m+n-1]$, define $\mathcal{A}_i(t) := (1+tu_{m+n-1})\cdots(1+tu_i)$ and $\widetilde{\mathcal{A}}_i(t) := \mathcal{A}_i(t)^{-1}$. Furthermore, set $\mathcal{A}(\Omega) := \mathcal{A}_1(\omega_1)\cdots\mathcal{A}_m(\omega_m)$ and

$$\widetilde{\mathfrak{S}}(\Omega+Y,Z) := \mathcal{A}(\Omega)\mathcal{A}_{m+1}(y_1)\cdots\mathcal{A}_{m+n-1}(y_{n-1})\widetilde{\mathcal{A}}_{m+n-1}(z_{n-1})\cdots\widetilde{\mathcal{A}}_{m+1}(z_1).$$

Let ϖ_0 denote the longest element of S_n and consider the polynomial

$$\widetilde{\mathfrak{S}}_{1_m\times\varpi_0}(\Omega+Y,Z) := \langle \widetilde{\mathfrak{S}}(\Omega+Y,Z), 1_m\times\varpi_0\rangle. \tag{2.7}$$

Proposition 2.5. *For every integer $m \geq 0$, we have*

$$\widetilde{\mathfrak{S}}_{1_m\times\varpi_0}(\Omega+Y,Z) = S^{(\delta^\vee_{n-1}(m),\delta^\vee_{n-1})}_{\delta_{n-1}}(h(\Omega+Y,-Z)), \tag{2.8}$$

where the superscript $(\delta^\vee_{n-1}(m),\delta^\vee_{n-1})$ indicates the number of ω, y, and z variables used in each row of the flagged Schur polynomial.

Proof. Let $\Omega' := (\omega'_1,\ldots,\omega'_m)$. Then

$$\mathfrak{S}_{1_m\times\varpi_0}(\Omega+Y,\Omega'+Z) = \sum_{v_2^{-1}uv_1=\varpi_0} \mathfrak{S}_{v_2}(-Z)G_u(\Omega/\Omega')\mathfrak{S}_{v_1}(Y),$$

so setting $\omega'_j = 0$ for each $j \in [1,m]$ gives

$$\mathfrak{S}_{1_m\times\varpi_0}(\Omega+Y,Z) = \sum_{u,v_1\,:\,\ell(u)+\ell(v_1)=\ell(uv_1)} G_u(\Omega)\mathfrak{S}_{v_1}(Y)\mathfrak{S}_{uv_1\varpi_0}(-Z). \tag{2.9}$$

Moreover, it follows from equations (1.7) and (2.7) that

$$\widetilde{\mathfrak{S}}_{1_m\times\varpi_0}(\Omega+Y,Z) = \sum_{u,v_1\,:\,\ell(u)+\ell(v_1)=\ell(uv_1)} G_u(\Omega)\mathfrak{S}_{v_1}(Y)\mathfrak{S}_{\varpi_0uv_1}(-Z). \tag{2.10}$$

Suppose that $Y_1,\ldots,Y_\ell$ and $Z_1\ldots,Z_\ell$ denote finite sets of independent variables. For any integer vector $\alpha = (\alpha_1,\ldots,\alpha_\ell)$, we introduce the *multi-Schur polynomial*

$$S_\alpha(Y_1-Z_1\,;\,\cdots\,;\,Y_\ell-Z_\ell) := \det(h_{\alpha_i+j-i}(Y_i/Z_i))_{1\leq i,j\leq \ell}.$$

These generalize the supersymmetric Schur polynomials defined in §1.2.

A permutation ϖ is called *vexillary* if it is 2143-avoiding, that is, there is no sequence $i<j<k<r$ such that $\varpi_j < \varpi_i < \varpi_r < \varpi_k$. The Schubert polynomials which are indexed by vexillary permutations may be expressed as multi-Schur polynomials (see [M2, (6.16)]). Let $Y_{(r)} := (y_1,\ldots,y_r)$ and $Z_{(r)} := (z_1,\ldots,z_r)$ for each $r \geq 0$. Since the permutation $1_m \times \varpi_0$ is vexillary, we deduce from loc. cit. that

$$\mathfrak{S}_{1_m\times\varpi_0}(\Omega+Y,\Omega'+Z) = S_{\delta_{n-1}}(\Omega+Y_{(1)}-\Omega'-Z_{(n-1)}\,;\,\cdots\,;\,\Omega+Y_{(n-1)}-\Omega'-Z_{(1)})$$

and therefore, setting $\omega'_j = 0$ for each $j \in [1,m]$, that

$$\mathfrak{S}_{1_m\times\varpi_0}(\Omega+Y,Z) = S_{\delta_{n-1}}(\Omega+Y_{(1)}-Z_{(n-1)}\,;\,\cdots\,;\,\Omega+Y_{(n-1)}-Z_{(1)}). \tag{2.11}$$

Consider the duality isomorphism with respect to the Z-variables

$$D_z : \mathbb{Z}[\Omega, Y_{(n)}, Z_{(n)}] \to \mathbb{Z}[\Omega, Y_{(n)}, Z_{(n)}]$$

which sends $(z_1,\ldots,z_n)$ to $(-z_n,\ldots,-z_1)$ and leaves all the remaining variables fixed. Let $I^z_n \subset \mathbb{Z}[\Omega, Y_{(n)}, Z_{(n)}]$ be the ideal generated by the elementary symmetric polynomials $e_i(Z_{(n)})$ for $1 \leq i \leq n$. It follows from (2.9), (2.10), and Lemma 2.2 that

$$D_z(\mathfrak{S}_{1_m\times\varpi_0}(\Omega+Y,Z)+I^z_n) = \widetilde{\mathfrak{S}}_{1_m\times\varpi_0}(\Omega+Y,Z)+I^z_n.$$

Furthermore, for $r, s \in [1, n-1]$, equation (2.5) implies that

$$D_z(h_j(Y_{(r)}-Z_{(s)})) \equiv h_j(Y_{(r)}, -Z_{(n-s)})$$

modulo the ideal I^z_n. It follows from (2.11) that

$$\begin{aligned} D_z(\mathfrak{S}_{1_m\times\varpi_0}(\Omega+Y,Z)) &\equiv S_{\delta_{n-1}}(h(\Omega+Y_1,-Z_1),\ldots,h(\Omega+Y_{n-1},-Z_{n-1})) \\ &= S^{(\delta^\vee_{n-1}(m),\delta^\vee_{n-1})}_{\delta_{n-1}}(h(\Omega+Y,-Z)). \end{aligned}$$

We deduce that equation (2.8) holds modulo I^z_n.

Let $\mathcal{H}^z_n$ be the $\mathbb{Z}[\Omega, Y_{(n)}]$-linear subspace of $\mathbb{Z}[\Omega, Y_{(n)}, Z_{(n)}]$ spanned by the monomials z^α for $0 \leq \alpha \leq \delta_{n-1}$. Then the monomial expression (2.1) for the flagged Schur polynomial $S^{(\delta^\vee_{n-1}(m),\delta^\vee_{n-1})}_{\delta_{n-1}}(h(\Omega+Y,-Z))$ in (2.8) proves that the latter lies in $\mathcal{H}^z_n$. Since the same is clearly true of $\widetilde{\mathfrak{S}}_{1_m\times\varpi_0}(\Omega+Y,Z)$, the proposition follows. □

The next result is obtained by setting $m = 0$ in Proposition 2.5.

Corollary 2.6. *We have* $\widetilde{\mathfrak{S}}_{\varpi_0}(Y,Z) = S^{(\delta^\vee_{n-1},\delta^\vee_{n-1})}_{\delta_{n-1}}(h(Y,-Z))$.

3. The type C theory

3.1. **Schubert polynomials and divided differences.** The Weyl group for the root system of type B_n or C_n is the *hyperoctahedral group* W_n, which consists of signed permutations on the set $\{1,\ldots,n\}$. The group W_n is generated by the simple transpositions $s_i = (i, i+1)$ for $1 \leq i \leq n-1$ and the sign change $s_0(1) = \overline{1}$ (as is customary, we use a bar to denote an entry with a negative sign). There is a natural embedding $W_n \hookrightarrow W_{n+1}$ defined by adding the fixed point $n+1$, and we let $W_\infty := \cup_n W_n$. The notions of length, reduced words, and descents of elements of W_∞ are defined as in the case of the symmetric group S_∞, only now the simple reflections are indexed by the integers in the set $\mathbb{N}_0 := \{0, 1, \ldots\}$.

The *nilCoxeter algebra* $\mathcal{W}_n$ of the hyperoctahedral group W_n is the free associative algebra with unit generated by the elements $u_0, u_1, \ldots, u_{n-1}$ modulo the relations

$$\begin{array}{rcll} u_i^2 &=& 0 & i \in \mathbb{N}_0\ ; \\ u_iu_j &=& u_ju_i & |i-j| \geq 2\ ; \\ u_iu_{i+1}u_i &=& u_{i+1}u_iu_{i+1} & i > 0\ ; \\ u_0u_1u_0u_1 &=& u_1u_0u_1u_0. & \end{array}$$

For any $w \in W_n$, define $u_w := u_{a_1} \ldots u_{a_\ell}$, where $a_1 \cdots a_\ell$ is any reduced word for w. Then the u_w for $w \in W_n$ form a free $\mathbb{Z}$-basis of $\mathcal{W}_n$. As in §1.1, we denote the coefficient of $u_w \in \mathcal{W}_n$ in the expansion of the element $\xi \in \mathcal{W}_n$ by $\langle \xi, w \rangle$.

Let t be a variable and define

$$C(t) := (1 + tu_{n-1}) \cdots (1 + tu_1)(1 + tu_0)(1 + tu_0)(1 + tu_1) \cdots (1 + tu_{n-1}).$$

Suppose that $X = (x_1, x_2, \ldots)$ is another infinite sequence of commuting variables, let $C(X) := C(x_1)C(x_2) \cdots$, and for $w \in W_n$, define

$$\mathfrak{C}_w(X\,; Y, Z) := \left\langle \tilde{A}_{n-1}(z_{n-1}) \cdots \tilde{A}_1(z_1)C(X)A_1(y_1) \cdots A_{n-1}(y_{n-1}), w \right\rangle . \tag{3.1}$$

Set $\mathfrak{C}_w(X\,; Y) := \mathfrak{C}_w(X\,; Y, 0)$. The polynomials $\mathfrak{C}_w(X\,; Y)$ are the type C Schubert polynomials of Billey and Haiman [BH] and the $\mathfrak{C}_w(X\,; Y, Z)$ are their double versions introduced by Ikeda, Mihalcea, and Naruse [IMN1].

Note that $\mathfrak{C}_w$ is really a polynomial in the Y and Z variables, with coefficients which are formal power series in X, with integer coefficients. These power series are symmetric in the X variables, since $C(s)C(t) = C(t)C(s)$, for any two commuting variables s and t (see [FK, Prop. 4.2]). We set

$$F_w(X) := \mathfrak{C}_w(X\,; 0, 0) = \langle C(X), w \rangle \tag{3.2}$$

and call F_w the *type C Stanley symmetric function* indexed by $w \in W_n$. We deduce from (3.2) that the coefficients of $F_w(X)$ have a combinatorial interpretation (compare with [BH, (3.5)]). Observe also that we have $F_w = F_{w^{-1}}$.

The above definition of $\mathfrak{C}_w(X\,; Y, Z)$ implies that it is stable under the natural inclusions $W_n \hookrightarrow W_{n+1}$ of the Weyl groups, and hence is well defined for $w \in W_\infty$. Equation (3.1) implies the relation

$$\mathfrak{C}_w(X\,; Y, Z) = \sum_{uv\varpi = w} \mathfrak{S}_{u^{-1}}(-Z)F_v(X)\mathfrak{S}_\varpi(Y) \tag{3.3}$$

summed over all reduced factorizations $uv\varpi = w$ with $u, \varpi \in S_\infty$.

The type C Stanley symmetric functions $F_w(X)$ lie in the ring Γ of Schur Q-functions $Q_\lambda(X)$. For each $r \in \mathbb{Z}$, define the basic function $q_r(X)$ by the equation

$$\prod_{i=1}^{\infty} \frac{1 + x_i t}{1 - x_i t} = \sum_{r=0}^{\infty} q_r(X) t^r.$$

For any integer vector α, let $q_\alpha := \prod_i q_{\alpha_i}$, and define $Q_\alpha := Q_\alpha(X)$ by

$$Q_\alpha := R^\infty \, q_\alpha \tag{3.4}$$

where the raising operator expression R^∞ is given by

$$R^\infty := \prod_{i<j} \frac{1 - R_{ij}}{1 + R_{ij}} \,.$$

Following [M1, III.8], we can equivalently write formula (3.4) using a Schur Pfaffian [Sc]. More precisely, for integer vectors $\alpha = (\alpha_1, \alpha_2)$ with only two parts, we have

$$Q_{(\alpha_1,\alpha_2)} = \frac{1 - R_{12}}{1 + R_{12}} \, q_{(\alpha_1,\alpha_2)} = q_{\alpha_1} q_{\alpha_2} + 2 \sum_{j \geq 1} (-1)^j q_{\alpha_1 + j} q_{\alpha_2 - j},$$

while for an integer vector $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ with three or more components,

$$Q_\alpha = \text{Pfaffian}(Q_{(\alpha_i,\alpha_j)})_{1 \leq i < j \leq 2\ell'} \tag{3.5}$$

where ℓ' is the least positive integer such that $2\ell' \geq \ell$.

A partition λ is strict if all its (non-zero) parts λ_i are distinct. It is known that the $Q_\lambda(X)$ for λ a strict partition form a free $\mathbb{Z}$-basis of the ring $\Gamma := \mathbb{Z}[q_1(X), q_2(X), \ldots]$. For any $w \in W_\infty$, we have an identity

$$F_w(X) = \sum_{\lambda\,:\,|\lambda|=\ell(w)} e^w_\lambda\, Q_\lambda(X), \tag{3.6}$$

summed over all strict partitions λ with $|\lambda| = \ell(w)$. At this stage we only need to know that an equation (3.6) exists with $e^w_\lambda \in \mathbb{Q}$. This latter fact follows immediately from the cancellation rule $C(t)C(-t) = 1$ and a result of Pragacz [P, Thm. 2.11] (see also [FK, §4]). We refer to [B, BH, La] for three different combinatorial proofs that the coefficients e^w_λ, when non-zero, are positive integers. Equations (3.3) and (3.6) show that the Schubert polynomials $\mathfrak{C}_w(X\,;Y,Z)$ lie in the ring $\Gamma[Y,Z]$.

We define an action of W_∞ on $\Gamma[Y,Z]$ by ring automorphisms as follows. The simple reflections s_i for $i > 0$ act by interchanging y_i and y_{i+1} and leaving all the remaining variables fixed, as in §1.1. The reflection s_0 maps y_1 to $-y_1$, fixes the y_j for $j \geq 2$ and all the z_j, and satisfies

$$s_0(q_r(X)) := q_r(y_1, x_1, x_2, \ldots) = q_r(X) + 2\sum_{j=1}^{r} y_1^j q_{r-j}(X).$$

For each $i \geq 0$, define the *divided difference operator* ∂^y_i on $\Gamma[Y,Z]$ by

$$\partial^y_0 f := \frac{f - s_0 f}{-2y_1}, \qquad \partial^y_i f := \frac{f - s_i f}{y_i - y_{i+1}} \quad \text{for } i > 0.$$

Consider the ring involution $\omega : \Gamma[Y,Z] \to \Gamma[Y,Z]$ determined by

$$\omega(y_j) = -z_j, \qquad \omega(z_j) = -y_j, \qquad \omega(q_r(X)) = q_r(X)$$

and set $\partial^z_i := \omega\partial^y_i\omega$ for each $i \geq 0$.

The polynomials $\mathfrak{C}_w(X\,;Y,Z)$ for $w \in W_\infty$ are the unique family of elements of $\Gamma[Y,Z]$ satisfying the equations

$$\partial^y_i \mathfrak{C}_w = \begin{cases} \mathfrak{C}_{ws_i} & \text{if } \ell(ws_i) < \ell(w), \\ 0 & \text{otherwise,} \end{cases} \qquad \partial^z_i \mathfrak{C}_w = \begin{cases} \mathfrak{C}_{s_iw} & \text{if } \ell(s_iw) < \ell(w), \\ 0 & \text{otherwise,} \end{cases} \tag{3.7}$$

for all $i \geq 0$, together with the condition that the constant term of $\mathfrak{C}_w$ is 1 if $w = 1$, and 0 otherwise. As a consequence, the the descents of w and w^{-1} determine the symmetries of the polynomial $\mathfrak{C}_w(X\,;Y,Z)$. Note however the special role that descents at *zero* play here. Furthermore, as in type A, one can show that the double Schubert polynomials $\mathfrak{C}_w(X\,;Y,Z)$ represent the universal Schubert classes in type C flag bundles, and therefore degeneracy loci of symplectic vector bundles, in the sense of [Fu2]. For a simple proof of these assertions, see [T6, §7.3]. The *geometrization* of the Schubert polynomials $\mathfrak{C}_w$ will be discussed in §5.

3.2. The Schubert polynomial indexed by the longest element.

Let w_0 denote the longest element in W_n. A formula for the top single Schubert polynomial $\mathfrak{C}_{w_0}(X\,;Y)$ was given by Billey and Haiman [BH, Prop. 4.15]. In this section, we derive the analogue of their result for the double Schubert polynomial $\mathfrak{C}_{w_0}(X\,;Y,Z)$, and use it to give a new proof of a Pfaffian formula for $\mathfrak{C}_{w_0}$ due to Ikeda, Mihalcea, and Naruse [IMN1, Thm. 1.2].

Observe first that $w_0u = uw_0$ for any $u \in S_n$. Using this and equation (3.3) gives

$$\mathfrak{C}_{w_0}(X\,;Y,Z) = \sum_{wvu=w_0} F_w(X)\mathfrak{S}_{u^{-1}}(-Z)\mathfrak{S}_v(Y)$$

where the u, v in the sum lie in S_n, and the factorization $wvu = w_0$ is reduced. It follows that

$$\mathfrak{C}_{w_0}(X\,;Y,Z) = \sum_{\sigma\in S_n} F_{w_0\sigma^{-1}}(X)\widetilde{\mathfrak{S}}_\sigma(Y,Z). \tag{3.8}$$

Proposition 3.1. *The equation*

$$\mathfrak{C}_{w_0}(X\,;Y,Z) = \sum_{\lambda\subset\delta_{n-1}} Q_{\delta_n+\lambda}(X)S^{(\delta^\vee_{n-1},\delta^\vee_{n-1})}_{\delta_{n-1}/\lambda'}(h(Y,-Z)) \tag{3.9}$$

holds in $\Gamma[Y,Z]$.

Proof. Recall that ϖ_0 denotes the longest permutation in S_n. According to [BH, Thm. 3.16] and [La, Thm. 3.15], we have, for every $u \in S_n$,

$$F_{w_0u}(X) = \sum_\lambda a_\lambda^{u^{-1}\varpi_0} Q_{\delta_n+\lambda}(X), \tag{3.10}$$

summed over partitions λ. We deduce from (3.8) and (3.10) that

$$\mathfrak{C}_{w_0}(X\,;Y,Z) = \sum_\lambda Q_{\delta_n+\lambda}(X)\sum_{\sigma\in S_n} a_\lambda^{\sigma\varpi_0}\widetilde{\mathfrak{S}}_\sigma(Y,Z).$$

It therefore suffices to show that for every partition λ, we have

$$\sum_{\sigma\in S_n} a_\lambda^{\sigma\varpi_0}\widetilde{\mathfrak{S}}_\sigma(Y,Z) = S^{(\delta^\vee_{n-1},\delta^\vee_{n-1})}_{\delta_{n-1}/\lambda'}(h(Y,-Z)). \tag{3.11}$$

This is a generalization of [BH, Eqn. (4.63)], and its proof is similar.

Choose any integer $m \geq 0$ and define $\Omega := (\omega_1,\ldots,\omega_m)$ and $\delta^\vee_{n-1}(m) := (m+1,\ldots,m+n-1)$ as in §2.3. For any integer vector $\gamma = (\gamma_1,\ldots,\gamma_{n-1})$, we have

$$h_\gamma^{(\delta^\vee_{n-1}(m),\delta^\vee_{n-1})}(\Omega+Y,-Z) = \sum_{\alpha\geq 0} h_{\gamma-\alpha}(\Omega)h_\alpha^{(\delta^\vee_{n-1},\delta^\vee_{n-1})}(Y,-Z) \tag{3.12}$$

summed over all compositions $\alpha = (\alpha_1,\ldots,\alpha_{n-1})$. Moreover, for any such composition α, we have $s_{\delta_{n-1}-\alpha}(\Omega) = 0$ unless $\delta_{n-1}-\alpha+\delta_{n-2} = \sigma(\lambda+\delta_{n-2})$ for some partition λ and permutation $\sigma \in S_{n-1}$, in which case $s_{\delta_{n-1}-\alpha}(\Omega) = (-1)^\sigma s_\lambda(\Omega)$. Using this and equation (3.12), we compute that

$$\begin{aligned} S_{\delta_{n-1}}^{(\delta^\vee_{n-1}(m),\delta^\vee_{n-1})}(h(\Omega+Y,-Z)) &= \prod_{i<j}(1-R_{ij})\,h_{\delta_{n-1}}^{(\delta^\vee_{n-1}(m),\delta^\vee_{n-1})}(\Omega+Y,Z) \\ &= \sum_{\alpha\geq 0} s_{\delta_{n-1}-\alpha}(\Omega)\,h_\alpha^{(\delta^\vee_{n-1},\delta^\vee_{n-1})}(Y,-Z) \\ &= \sum_\lambda s_\lambda(\Omega)\sum_{\sigma\in S_{n-1}}(-1)^\sigma\,h^{(\delta^\vee_{n-1},\delta^\vee_{n-1})}_{\delta_{n-1}+\delta_{n-2}-\sigma(\lambda+\delta_{n-2})}(Y,-Z). \end{aligned}$$

We deduce that

$$S_{\delta_{n-1}}^{(\delta^\vee_{n-1}(m),\delta^\vee_{n-1})}(\Omega+Y,-Z) = \sum_{\lambda\subset\delta_{n-1}} s_\lambda(\Omega)\,S^{(\delta^\vee_{n-1},\delta^\vee_{n-1})}_{\delta_{n-1}/\lambda}(h(Y,-Z)). \tag{3.13}$$

On the other hand, it follows from (1.7), (1.9), and the definition (2.7) that

$$\widetilde{\mathfrak{S}}_{1_m\times\varpi_0}(\Omega+Y,Z) = \sum_{uv=\varpi_0} G_u(\Omega)\widetilde{\mathfrak{S}}_v(Y,Z) = \sum_\lambda\sum_{v\in S_n} a_\lambda^{\varpi_0 v^{-1}} s_\lambda(\Omega)\widetilde{\mathfrak{S}}_v(Y,Z)$$

and hence

$$\widetilde{\mathfrak{S}}_{1_m\times\varpi_0}(\Omega+Y,Z) = \sum_\lambda s_\lambda(\Omega)\sum_{\sigma\in S_n} a_{\lambda'}^{\sigma\varpi_0}\widetilde{\mathfrak{S}}_\sigma(Y,Z). \tag{3.14}$$

By Proposition 2.5, the left hand sides of equations (3.13) and (3.14) coincide. Comparing the coefficients of $s_\lambda(\Omega)$ on the right hand sides of the same equations completes the proof of (3.11), and hence of the proposition. □

Our next goal is to express the top polynomial $\mathfrak{C}_{w_0}(X\,;Y,Z)$ as a multi-Schur Pfaffian analogous to equations (3.4) and (3.5). For any $k,r\in\mathbb{Z}$, we define the polynomial ${}^k c_p^r = {}^k c_p^r(X\,;Y,Z)$ by

$${}^k c_p^r := \sum_{i=0}^p\sum_{j=0}^p q_{p-j-i}(X)h_i^{-k}(Y)h_j^r(-Z). \tag{3.15}$$

The polynomials ${}^k c_p^r$ were first studied by Wilson in [W, Def. 6 and Prop. 6]. For any integer sequences α,β,ρ, define ${}^\rho c_\alpha^\beta := \prod_i {}^{\rho_i}c_{\alpha_i}^{\beta_i}$. Given any raising operator R, let $R\,{}^\rho c_\alpha^\beta := {}^\rho c_{R\alpha}^\beta$. Finally, define the *multi-Schur Pfaffian* ${}^\rho Q_\alpha^\beta(c)$ by

$${}^\rho Q_\alpha^\beta(c) := R^\infty\,{}^\rho c_\alpha^\beta.$$

Proposition 3.2. *The equation*

$$\mathfrak{C}_{w_0}(X\,;Y,Z) = {}^{\delta_{n-1}}Q_{\delta_n+\delta_{n-1}}^{-\delta_{n-1}}(c) \tag{3.16}$$

holds in $\Gamma[Y,Z]$.

Proof. If we set ${}^k h_m^r(Y,-Z) := \sum_{j=0}^m h_j^{-k}(Y)h_{m-j}^r(-Z)$, then we have

$${}^k c_p^r = \sum_{j=0}^p q_{p-j}(X)\,{}^k h_j^r(Y,-Z).$$

In particular, if $k,r\geq 0$, then we have

$${}^k h_m^{-r}(Y,-Z) = e_m(y_1,\ldots,y_k,-z_1,\ldots,-z_r) = e_m(Y_{(k)},-Z_{(r)}) \tag{3.17}$$

so that ${}^k h_m^{-r}(Y,-Z)=0$ whenever $m>k+r$.

For any integer sequences α,β,ρ, define ${}^\rho h_\alpha^\beta := \prod_i {}^{\rho_i}h_{\alpha_i}^{\beta_i}$. Notice, using (3.17), that for any integer vector $\gamma=(\gamma_1,\ldots,\gamma_n)$, we have

$${}^{\delta_{n-1}}c_\gamma^{-\delta_{n-1}} = \sum_{0\leq\alpha\leq 2\delta_{n-1}} q_{\gamma-\alpha}(X)\,{}^{\delta_{n-1}}h_\alpha^{-\delta_{n-1}}(Y,-Z),$$

and hence, by the definition (3.4),

$${}^{\delta_{n-1}}Q_{\delta_n+\delta_{n-1}}^{-\delta_{n-1}}(c) = \sum_{0\leq\alpha\leq 2\delta_{n-1}} Q_{\delta_n+\delta_{n-1}-\alpha}(X)\,{}^{\delta_{n-1}}h_\alpha^{-\delta_{n-1}}(Y,-Z).$$

Recall for example from [BKT2, Lemma 1.3] that the Schur Q-functions $Q_\gamma(X)$ are alternating in the components (γ_i,γ_j) of the index γ, provided that $\gamma_i+\gamma_j>0$. Therefore, we have $Q_{\delta_n+\delta_{n-1}-\alpha}(X)=0$ in the above sum unless

$$\delta_n+\delta_{n-1}-\alpha = 1^n+\sigma(\delta_{n-1}+\lambda)$$

for some partition $\lambda \subset \delta_{n-1}$ and permutation $\sigma \in S_{n-1}$. Observe that $\alpha = 2\delta_{n-1} - \sigma(\delta_{n-1}+\lambda)$ is uniquely determined from λ and σ. It follows that

$$\begin{aligned} {}^{\delta_{n-1}}Q^{-\delta_{n-1}}_{\delta_n+\delta_{n-1}}(c) &= \sum_{\lambda\subset\delta_{n-1}}\sum_{\sigma\in S_{n-1}} Q_{1^n+\sigma(\delta_{n-1}+\lambda)}(X)\,{}^{\delta_{n-1}}h^{-\delta_{n-1}}_{2\delta_{n-1}-\sigma(\delta_{n-1}+\lambda)}(Y,-Z) \\ &= \sum_{\lambda\subset\delta_{n-1}} Q_{\delta_n+\lambda}(X)\sum_{\sigma\in S_{n-1}}(-1)^{\sigma}\,{}^{\delta_{n-1}}h^{-\delta_{n-1}}_{2\delta_{n-1}-\sigma(\lambda+\delta_{n-1})}(Y,-Z) \\ &= \sum_{\lambda\subset\delta_{n-1}} Q_{\delta_n+\lambda}(X)\,S^{(\delta_{n-1},\delta_{n-1})}_{\delta_{n-1}/\lambda}(e(Y,-Z)). \end{aligned}$$

Taking $\lambda = \delta_{n-1}$, $\mu = \lambda'$, $k = n$, $\ell = n-1$, and $t = (y_1, -z_1, \ldots, y_{n-1}, -z_{n-1})$ in Lemma 2.1 gives

$$S^{(\delta_{n-1},\delta_{n-1})}_{\delta_{n-1}/\lambda}(e(Y,-Z)) = S^{(\delta^\vee_{n-1},\delta^\vee_{n-1})}_{\delta_{n-1}/\lambda'}(h(Y,-Z)). \tag{3.18}$$

The result now follows by using equations (3.9) and (3.18). □

It is easy to show (see for example [IM, §8.2]) that equation (3.16) is equivalent to the Pfaffian formula for $\mathfrak{C}_{w_0}$ found in [IMN1, Thm. 1.2].

3.3. **The Schubert polynomials indexed by maximal elements.** Consider a sequence $\mathfrak{a} : a_1 < \cdots < a_p$ of nonnegative integers with $a_p < n$. The sequence $\mathfrak{a}$ parametrizes a parabolic subgroup $W_\mathfrak{a}$ of W_n, generated by the simple reflections s_i for $i \notin \{a_1, \ldots, a_p\}$. We let $W_n^\mathfrak{a}$ denote the set of minimal length left $W_\mathfrak{a}$-coset representatives. Recall that

$$W_n^\mathfrak{a} = \{w \in W_n \mid \ell(ws_i) = \ell(w)+1,\ \forall\, i \notin \{a_1,\ldots,a_p\}\}.$$

Let $w_0(\mathfrak{a})$ denote the longest element in $W_n^\mathfrak{a}$; we have

$$w_0(\mathfrak{a}) = \begin{cases} \overline{a_2}\cdots\overline{1}\,\overline{a_3}\cdots\overline{a_2+1}\cdots\overline{n}\cdots\overline{a_p+1} & \text{if } a_1 = 0, \\ 1\cdots a_1\overline{a_2}\cdots\overline{a_1+1}\cdots\overline{n}\cdots\overline{a_p+1} & \text{if } a_1 > 0. \end{cases}$$

It is known (see for example [St, §2]) that $W_n^\mathfrak{a}$ is an order ideal of the left weak Bruhat order of W_n, and that $w_0(\mathfrak{a})$ is the unique maximal element of $W_n^\mathfrak{a}$ under this ordering.

Fix an integer k with $0 \leq k < n$. The elements of the set $W_n^{(k)}$ are the k-*Grassmannian* elements of $\widetilde{W}_n$. Let $w^{(k,n)} = 1\cdots k\,\overline{n}\cdots\overline{k+1}$ denote the longest element of $W_n^{(k)}$. Following [TW, §6.2], we will require a formula analogous to (3.16) for the Schubert polynomial $\mathfrak{C}_{w^{(k,n)}}(X\,;Y,Z)$, which maps to Kazarian's multi-Schur Pfaffian formula from [Ka, Thm. 1.1]. Similar Pfaffian formulas for the Schubert polynomials $\mathfrak{C}_{w_0(\mathfrak{a})}(X\,;Y,Z)$ were obtained by Anderson and Fulton in [AF1]. Ikeda and Matsumura [IM, §8.2] gave proofs of these formulas by applying left difference operators to the top Schubert polynomial $\mathfrak{C}_{w_0}$, and we will follow that approach here.

For every $i \geq 0$, the operator $\partial_i := \partial_i^z$ on $\Gamma[Y,Z]$ satisfies the Leibniz rule

$$\partial_i(fg) = (\partial_i f)g + (s_i f)\partial_i g. \tag{3.19}$$

The argument depends on the following basic lemmas.

Lemma 3.3 ([IM], Lemma 5.4)**.** *Suppose that $p, r \in \mathbb{Z}$ and let $k \geq 0$. For all $i \geq 0$, we have*
$$\partial_i({}^k c^r_p) = \begin{cases} {}^k c^{r+1}_{p-1} & \text{if } r = \pm i, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 3.4 ([IM], Lemma 8.2)**.** *Suppose that $i \geq 0$ and $k > 0$. Then we have*
$${}^k c^{-i}_p = {}^{k-1} c^{-i-1}_p + (z_{i+1} + y_k)\, {}^{k-1} c^{-i}_{p-1}.$$

Lemma 3.5 ([IM], Prop. 5.4)**.** *Suppose that $k, r \geq 0$ and $p > k + r$. Then we have*
$${}^{(k_1,\ldots,k,k,\ldots,k_\ell)} Q^{(i_1,\ldots,-r,-r,\ldots,i_\ell)}_{(p_1,\ldots,p,p,\ldots,p_\ell)}(c) = 0.$$

Example 3.6. Let $\delta^*_n := (n, n-1, \ldots, 2) \in \mathbb{Z}^{n-1}$. For any integer sequence $\alpha = (\alpha_1, \ldots, \alpha_n)$, we have
$$\partial_0({}^{n-1} c^{1-n}_{\alpha_1} \cdots {}^1 c^{-1}_{\alpha_{n-1}}\, {}^0 c^0_{\alpha_n}) = {}^{n-1} c^{1-n}_{\alpha_1} \cdots {}^1 c^{-1}_{\alpha_{n-1}}\, {}^0 c^1_{\alpha_n - 1}.$$
It follows from this and equations (3.7) and (3.16) that
$$\mathfrak{C}_{s_0 w_0}(X\,; Y, Z) = \partial_0\, \mathfrak{C}_{w_0}(X\,; Y, Z) = {}^{\delta_{n-1}} Q^{-\delta_{n-1}}_{\delta^*_n + \delta_{n-1}}(c).$$
Arguing as in §3.2, we can show that
$$\mathfrak{C}_{s_0 w_0}(X\,; Y, Z) = \sum_{\lambda \subset \delta^*_n} Q_{\delta_{n-1}+\lambda}(X) S^{(\delta_{n-1},\delta_{n-1})}_{\delta^*_n/\lambda}(e(Y, -Z)). \tag{3.20}$$

Proposition 3.7. *We have*
$$\mathfrak{C}_{w^{(k,n)}}(X\,; Y, Z) = {}^{(k,k,\ldots,k)} Q^{(1-n,2-n,\ldots,-k)}_{(n+k,n+k-1,\ldots,2k+1)}(c). \tag{3.21}$$
in $\Gamma[Y, Z]$.

Proof. If $v^{(k,n)} = \overline{k} \cdots \overline{1}\, \overline{n} \cdots \overline{k+1}$ is the longest element in $W^{(0,k)}_n$, then we have a reduced factorization $w_0 = v_1 v_2 v^{(k,n)}$, where
$$v_1 := (s_{k-1} \cdots s_1)(s_{k-1} \cdots s_2) \cdots (s_{k-1} s_{k-2}) s_{k-1} \tag{3.22}$$
if $k \geq 2$, and $v_1 := 1$, otherwise, while
$$v_2 := (s_{n-1} \cdots s_{k+1})(s_{n-1} \cdots s_{k+2}) \cdots (s_{n-1} s_{n-2}) s_{n-1}. \tag{3.23}$$
Using (3.7), this implies the equation
$$\mathfrak{C}_{v^{(k,n)}} = \partial_{n-1}(\partial_{n-2}\partial_{n-1}) \cdots (\partial_{k+1} \cdots \partial_{n-1}) \cdot \partial_{k-1}(\partial_{k-2}\partial_{k-1}) \cdots (\partial_1 \cdots \partial_{k-1}) \mathfrak{C}_{w_0}.$$

Assume that $k \geq 2$, as the proof when $k \in \{0, 1\}$ is easier. Using Lemmas 3.3 and 3.4, for any $p \in \mathbb{Z}$ we have
$$\partial_{k-1}\, {}^{k-1} c^{1-k}_p = {}^{k-1} c^{2-k}_{p-1} = {}^{k-2} c^{1-k}_{p-1} + (z_{k-1} + y_{k-1})\, {}^{k-2} c^{2-k}_{p-2}. \tag{3.24}$$
Let ϵ_j denote the j-th standard basis vector in $\mathbb{Z}^n$. The Leibnitz rule and (3.24) imply that for any integer vector $\alpha = (\alpha_1, \ldots, \alpha_n)$, we have
$$\partial_{k-1}\, {}^{\delta_{n-1}} c^{-\delta_{n-1}}_{\alpha} = {}^{\delta_{n-1} - \epsilon_{n+1-k}} c^{-\delta_{n-1}}_{\alpha - \epsilon_{n+1-k}} + (z_{k-1} + y_{k-1})^{\delta_{n-1} - \epsilon_{n+1-k}} c^{-\delta_{n-1} + \epsilon_{n+1-k}}_{\alpha - 2\epsilon_{n+1-k}}.$$
We deduce from this and Lemma 3.5 that
$$\begin{aligned} \partial_{k-1}\, {}^{\delta_{n-1}} Q^{-\delta_{n-1}}_{\delta_n + \delta_{n-1}}(c) &= {}^{\delta_{n-1} - \epsilon_{n+1-k}} Q^{-\delta_{n-1}}_{\delta_n + \delta_{n-1} - \epsilon_{n+1-k}}(c) \\ &= {}^{(n-1,\ldots,k,k-2,k-2,\ldots,1)} Q^{(1-n,\ldots,-1,0)}_{(2n-1,\ldots,2k+1,2k-2,2k-3,\ldots,1)}(c). \end{aligned}$$

Iterating this calculation gives

$$(\partial_1\cdots\partial_{k-1})\mathfrak{C}_{w_0} = {}^{(n-1,\ldots,k,k-2,k-3,\ldots,0)}Q^{(1-n,\ldots,-1,0)}_{(2n-1,\ldots,2k+1,2k-2,2k-4,\ldots,2,1)}(c)$$

and furthermore

$$\partial_{k-1}(\partial_{k-2}\partial_{k-1})\cdots(\partial_1\cdots\partial_{k-1})\mathfrak{C}_{w_0} = {}^{(n-1,\ldots,k,0,0,\ldots,0)}Q^{(1-n,\ldots,-1,0)}_{(2n-1,\ldots,2k+1,k,k-1,\ldots,1)}(c).$$

Applying the operator $\partial_{n-1}(\partial_{n-2}\partial_{n-1})\cdots(\partial_{k+1}\cdots\partial_{n-1})$ to both sides of the above equation, we similarly obtain

$$\mathfrak{C}_{v^{(k,n)}} = {}^{(k,k,\ldots,k,0,0,\ldots,0)}Q^{(1-n,\ldots,-1,0)}_{(n+k,n+k-1,\ldots,2k+1,k,k-1,\ldots,1)}(c).$$

Since $v^{(k,n)} = (s_0\cdots s_{k-1})\cdots(s_0s_1)s_0w^{(k,n)}$, equation (3.7) gives

$$\mathfrak{C}_{w^{(k,n)}} = \partial_0(\partial_1\partial_0)\cdots(\partial_{k-1}\cdots\partial_0)\mathfrak{C}_{v^{(k,n)}}.$$

Finally, since ${}^{(\rho,r)}Q^{(\beta,b)}_{(\alpha,0)}(c) = {}^{\rho}Q^{\beta}_{\alpha}(c)$ for any integers r and b, it follows that

$$\begin{aligned}
\mathfrak{C}_{w^{(k,n)}} &= \partial_0(\partial_1\partial_0)\cdots(\partial_{k-1}\cdots\partial_0)\mathfrak{C}_{v^{(k,n)}} \\
&= \partial_0(\partial_1\partial_0)\cdots(\partial_{k-2}\cdots\partial_0)^{(k,\ldots,k,0,\ldots,0)}Q^{(1-n,\ldots,-k,2-k\ldots,0)}_{(n+k,\ldots,2k+1,k-1,\ldots,1)}(c) \\
&= {}^{(k,\ldots,k)}Q^{(1-n,\ldots,-k)}_{(n+k,\ldots,2k+1)}(c).
\end{aligned}$$

□

Since $w^{(k,n)} = (w^{(k,n)})^{-1}$, the polynomial $\mathfrak{C}_{w^{(k,n)}}(X\,;Y,Z)$ is symmetric in the Z variables as well as in the Y variables, however this is not reflected in equation (3.21). The next proposition makes this symmetry apparent. Recall (for example from [Fu3, §6.2]) that for any three partitions λ,μ, and ν, the *Littlewood-Richardson number* $N^{\lambda}_{\mu\nu}$ is the nonnegative integer defined by the equation of Schur S-functions

$$s_\mu(t)s_\nu(t) = \sum_\lambda N^\lambda_{\mu\nu}s_\lambda(t).$$

Let $\mu_0 := (2k)^{n-k} = (2k,\ldots,2k)$, and for every $\mu\subset\mu_0$, define $\mu^\vee := (2k-\mu_{n-k},\ldots,2k-\mu_1)$. Note that $(n+k,n+k-1,\ldots,2k+1) = \delta_{n-k}+\mu_0$.

Proposition 3.8. *We have*

$$\begin{aligned}
\mathfrak{C}_{w^{(k,n)}}(X\,;Y,Z) &= {}^{(k,\ldots,k)}Q^{(-k,\ldots,-k)}_{\delta_{n-k}+\mu_0}(c) \\
&= \sum_{\nu_1,\nu_2\subset\mu\subset\mu_0} N^\mu_{\nu_1\nu_2}Q_{\delta_{n-k}+\mu^\vee}(X)s_{\nu_1'}(Y_{(k)})s_{\nu_2'}(-Z_{(k)}),
\end{aligned}$$

in $\Gamma[Y,Z]$*, where* $N^\mu_{\nu_1\nu_2}$ *denotes a Littlewood-Richardson number.*

Proof. For any integer vector $\gamma = (\gamma_1,\ldots,\gamma_{n-k})$, we have

$$(3.25)\qquad {}^{(k,\ldots,k)}c^{(1-n,\ldots,-k)}_\gamma = \sum_{0\le\alpha\le\delta_{n-k-1}+\mu_0} q_{\gamma-\alpha}(X)e^{(n+k-1,\ldots,2k)}_\alpha(Y_{(k)},-Z)$$

where the alphabet $Y_{(k)}$ in the factor $e^{(n+k-1,\ldots,2k)}_\alpha(Y_{(k)},-Z)$ is constant, while Z varies down from $Z_{(n-1)}$ to $Z_{(k)}$. Now using (3.21) and (3.25) while applying the raising operator R^∞, along with the alternating property of Schur Q-functions, gives

$$\begin{aligned}\mathfrak{C}_{w^{(k,n)}}(X\,;Y,Z) &= \sum_{0\leq\alpha\leq\delta_{n-k-1}+\mu_0} Q_{\delta_{n-k}+\mu_0-\alpha}(X)e_\alpha^{(n+k-1,\ldots,2k)}(Y_{(k)},-Z)\\ &= \sum_{\mu\subset\mu_0} Q_{\delta_{n-k}+\mu}(X)\det(e^{(n+k-1,\ldots,2k)}_{2k-\mu_j+j-i}(Y_{(k)},-Z))_{1\leq i,j\leq n-k}\\ &= \sum_{\mu\subset\mu_0} Q_{\delta_{n-k}+\mu}(X)S^{(n+k-1,\ldots,2k)}_{\mu_0/\mu}(e(Y_{(k)},-Z)).\end{aligned}$$

We claim that for each partition $\mu\subset\mu_0$, we have

$$\begin{aligned}\det(e^{(n+k-1,\ldots,2k)}_{2k-\mu_j+j-i}(Y_{(k)},-Z)) &= \det(e^{(2k,\ldots,2k)}_{2k-\mu_j+j-i}(Y_{(k)},-Z_{(k)}))\\ &= S_{\mu_0/\mu}(e(Y_{(k)},-Z_{(k)})).\end{aligned}$$

The proof of this follows [M2, (3.4)]. For each i,j with $1\leq i,j\leq n-k$,

$$\begin{aligned}e^{n+k-i}_{2k-\mu_j-i+j}(Y_{(k)},-Z) &= e_{2k-\mu_j-i+j}(Y_{(k)},-Z_{n-i})\\ &= \sum_{p=1}^{n-k} e_{p-i}(-B_i)e_{2k-\mu_j+j-p}(Y_{(k)},-Z_{(k)}),\end{aligned}$$

where $B_i=(z_{k+1},\ldots,z_{n-i})$ (in particular, $B_{n-k}=\emptyset$). Therefore the matrix

$$\{e^{n+k-i}_{2k-\mu_j-i+j}(Y_{(k)},-Z)\}_{1\leq i,j\leq n-k}$$

is the product of the matrix

$$\{e_{p-i}(-B_i)\}_{1\leq i,p\leq n-k},$$

which is unitriangular, and the matrix

$$\{e_{2k-\mu_j+j-p}(Y_{(k)},-Z_{(k)})\}_{1\leq p,j\leq n-k}.$$

Taking determinants completes the proof of the claim.

Since $S_{\mu_0/\mu}(e(Y_{(k)},-Z_{(k)}))=S_{\mu^\vee}(e(Y_{(k)},-Z_{(k)}))$, we deduce that

$$\begin{aligned}\mathfrak{C}_{w^{(k,n)}}(X\,;Y,Z) &= \sum_{\mu\subset\mu_0} Q_{\delta_{n-k}+\mu}(X)S_{\mu^\vee}(e(Y_{(k)},-Z_{(k)}))\\ &= {}^{(k,\ldots,k)}Q^{(-k,\ldots,-k)}_{\delta_{n-k}+\mu_0}(c).\end{aligned}$$

Furthermore, using [M1, I.(5.9)], we compute that

$$\begin{aligned}\mathfrak{C}_{w^{(k,n)}}(X\,;Y,Z) &= \sum_{\mu\subset\mu_0} Q_{\delta_{n-k}+\mu^\vee}(X)S_\mu(e(Y_{(k)},-Z_{(k)}))\\ &= \sum_{\mu\subset\mu_0} Q_{\delta_{n-k}+\mu^\vee}(X)s_{\mu'}(Y_{(k)},-Z_{(k)})\\ &= \sum_{\nu\subset\mu\subset\mu_0} Q_{\delta_{n-k}+\mu^\vee}(X)s_{\mu'/\nu'}(Y_{(k)})s_{\nu'}(-Z_{(k)})\\ &= \sum_{\nu_1,\nu_2\subset\mu\subset\mu_0} N^{\mu'}_{\nu_1'\nu_2'}Q_{\delta_{n-k}+\mu^\vee}(X)s_{\nu_1'}(Y_{(k)})s_{\nu_2'}(-Z_{(k)}).\end{aligned}$$

Since we have $N^{\mu'}_{\nu_1'\nu_2'}=N^\mu_{\nu_1\nu_2}$, the result follows. □

Although we only require the formula for $\mathfrak{C}_{w^{(k,n)}}(X\,;Y,Z)$, we will record the general result here for comparison with the orthogonal case, which is discussed in §4.4. According to [AF1] and [IM, Thm. 8.2], we have

$$\mathfrak{C}_{w_0(\mathfrak{a})}(X\,;Y,Z) = {}^{\rho(\mathfrak{a})}Q^{\beta(\mathfrak{a})}_{\lambda(\mathfrak{a})}(c),$$

where $\lambda(\mathfrak{a})$, $\beta(\mathfrak{a})$, and $\rho(\mathfrak{a})$ denote the sequences

$$\lambda(\mathfrak{a}) = (n+a_p,\ldots,2a_p+1,\ldots,a_i+a_{i+1},\ldots,2a_i+1,\ldots,a_1+a_2,\ldots,2a_1+1)\,;$$

$$\beta(\mathfrak{a}) = (1-n,\ldots,-a_p,\ldots,1-a_{i+1},\ldots,-a_i,\ldots,1-a_2,\ldots,-a_1)\,;$$

and

$$\rho(\mathfrak{a}) = (a_p^{n-a_p},\ldots,a_i^{a_{i+1}-a_i},\ldots,a_1^{a_2-a_1}).$$

3.4. **Theta polynomials.** The next step in this program is to prove formulas for the Schubert polynomials indexed by the k-Grassmannian elements of W_∞. We will see that they may be expressed using *theta polynomials*.

We say that a partition λ is *k-strict* if no part greater than k is repeated, that is, $\lambda_j > k$ implies $\lambda_{j+1} < \lambda_j$ for each $j \geq 1$. There is an explicit bijection between k-Grassmannian elements w of W_∞ and k-strict partitions λ, such that the elements in W_n correspond to those partitions whose diagram fits inside an $(n-k)\times(n+k)$ rectangle. According to [BKT1, §4.1 and §4.4], if the element w corresponds to the k-strict partition λ, then the bijection is given by the equations

$$\lambda_i = \begin{cases} |w_{k+i}| + k & \text{if } w_{k+i} < 0, \\ \#\{p \leq k \,:\, w_p > w_{k+i}\} & \text{if } w_{k+i} > 0. \end{cases}$$

Using the above bijection, we attach to any k-strict partition λ a finite set of pairs

$$\mathcal{C}(\lambda) := \{(i,j) \in \mathbb{N}\times\mathbb{N} \mid 1 \leq i < j \ \text{ and } \ w_{k+i} + w_{k+j} < 0\} \tag{3.26}$$

and a sequence $\beta(\lambda) = \{\beta_j(\lambda)\}_{j\geq 1}$ defined by

$$\beta_j(\lambda) := \begin{cases} w_{k+j}+1 & \text{if } w_{k+j} < 0, \\ w_{k+j} & \text{if } w_{k+j} > 0. \end{cases} \tag{3.27}$$

Following [BKT2], let λ be any k-strict partition, and consider the raising operator expression R^λ given by

$$R^\lambda := \prod_{i<j}(1-R_{ij}) \prod_{(i,j)\in\mathcal{C}(\lambda)} (1+R_{ij})^{-1}. \tag{3.28}$$

For any integer sequences α and β, define $c_\alpha^\beta := \prod_i {}^k c_{\alpha_i}^{\beta_i}$. According to [TW, W], the *double theta polynomial* $\Theta_\lambda(X\,;Y_{(k)},Z)$ is defined by

$$\Theta_\lambda(X\,;Y_{(k)},Z) := R^\lambda\, c_\lambda^{\beta(\lambda)}. \tag{3.29}$$

The single theta polynomial $\Theta_\lambda(X\,;Y_{(k)})$ of [BKT2] is given by

$$\Theta_\lambda(X\,;Y_{(k)}) := \Theta_\lambda(X\,;Y_{(k)},0).$$

Note that we are working here with the images of the theta polynomials $\Theta_\lambda(c)$ and $\Theta_\lambda(c\,|\,t)$ from [BKT2, TW] in the ring $\Gamma[Y,Z]$ of double Schubert polynomials, following [T5, T6].

Fix a rank n and let

$$\lambda_0 := (n+k, n+k-1, \ldots, 2k+1)$$

be the k-strict partition associated to the k-Grassmannian element $w^{(k,n)}$ of maximal length in W_n. We deduce from Proposition 3.7 and the definition (3.29) that

$$\mathfrak{C}_{w^{(k,n)}}(X\,;Y,Z) = \Theta_{\lambda_0}(X\,;Y_{(k)},Z) \tag{3.30}$$

in $\Gamma[Y,Z]$.

It follows from [IM, Lemma 5.5] that for all $i \geq 1$ and indices p and q, we have

$$\partial_i\, c^{(-i,i)}_{(p,q)} = c^{(-i+1,i+1)}_{(p-1,q)} + c^{(-i+1,i+1)}_{(p,q-1)} = (1+R_{12})\, c^{(-i+1,i+1)}_{(p-1,q)}.$$

Using this identity and Lemma 3.3, it is shown in [TW, Prop. 5] that if λ and μ are k-strict partitions such that $|\lambda| = |\mu|+1$ and $w_\lambda = s_i w_\mu$ for some simple reflection $s_i \in W_\infty$, then we have

$$\partial_i \Theta_\lambda(X\,;Y_{(k)},Z) = \Theta_\mu(X\,;Y_{(k)},Z) \tag{3.31}$$

in $\Gamma[Y,Z]$. Now (3.30) and (3.31) imply that for any k-strict partition λ with associated k-Grassmannian element w_λ, we have

$$\mathfrak{C}_{w_\lambda}(X\,;Y,Z) = \Theta_\lambda(X\,;Y_{(k)},Z)$$

in $\Gamma[Y,Z]$. In particular, we recover the equality

$$\mathfrak{C}_{w_\lambda}(X\,;Y) = \Theta_\lambda(X\,;Y_{(k)}) \tag{3.32}$$

in $\Gamma[Y]$ from [BKT2, Prop. 6.2] for the single polynomials.

3.5. **Mixed Stanley functions and splitting formulas.** Following [T5, §2], for any $w \in W_\infty$, the *double mixed Stanley function* $J_w(X\,;Y/Z)$ is defined by the equation

$$J_w(X\,;Y/Z) := \langle \tilde{A}(Z)C(X)A(Y), w\rangle = \sum_{uv\varpi=w} G_{u^{-1}}(-Z)F_v(X)G_\varpi(Y),$$

where the sum is over all reduced factorizations $uv\varpi = w$ with $u,\varpi \in S_\infty$. The single mixed Stanley function $J_w(X\,;Y)$ is given by setting $Z=0$ in $J_w(X\,;Y/Z)$. Observe that $J_w(X\,;Y/Z)$ is separately symmetric in the three sets of variables X, Y, and Z, and that we have $J_w(X\,;0) = F_w(X)$.

Fix an integer $k \geq 0$. We say that an element $w \in W_\infty$ is *increasing up to* k if $0 < w_1 < w_2 < \cdots < w_k$ (this condition is automatically true if $k=0$). If w is increasing up to k, then [BH, Eqn. (2.5)] and equation (1.7) have a natural analogue for the *restricted mixed Stanley function* $J_w(X\,;Y_{(k)})$, which is obtained from $J_w(X;Y)$ after setting $y_i = 0$ for $i > k$. In this case, according to [T5, Prop. 5], we have

$$\mathfrak{C}_w(X\,;Y) = \sum_{v(1_k\times\varpi)=w} J_v(X\,;Y_{(k)})\mathfrak{S}_\varpi(y_{k+1},y_{k+2},\ldots), \tag{3.33}$$

where the sum is over all reduced factorizations $v(1_k\times\varpi) = w$ in W_∞ with $\varpi \in S_\infty$. Moreover, there is a double version of equation (3.33) which is parallel to (1.8). Let $J_v(X\,;Y_{(k)}/Z_{(\ell)})$ denote the power series obtained from $J_v(X\,;Y/Z)$ by setting $y_i = z_j = 0$ for all $i > k$ and $j > \ell$. Then if w is increasing up to k and w^{-1} is increasing up to ℓ, we have

$$\mathfrak{C}_w(X\,;Y,Z) = \sum \mathfrak{S}_{u^{-1}}(-Z_{>\ell})J_v(X\,;Y_{(k)}/Z_{(\ell)})\mathfrak{S}_\varpi(Y_{>k}), \tag{3.34}$$

where $Y_{>k} := (y_{k+1},y_{k+2},\ldots)$, $-Z_{>\ell} := (-z_{\ell+1},-z_{\ell+2},\ldots)$, and the sum is over all reduced factorizations $(1_\ell\times u)v(1_k\times\varpi) = w$ in W_∞ with $u,\varpi \in S_\infty$.

We say that an element $w \in W_\infty$ is *compatible* with the sequence $\mathfrak{a} : a_1 < \cdots < a_p$ of elements of $\mathbb{N}_0$ if all descent positions of w are contained in $\mathfrak{a}$. Let $\mathfrak{b} : b_1 < \cdots < b_q$ be a second sequence of elements of $\mathbb{N}_0$ and assume that w is compatible with $\mathfrak{a}$ and w^{-1} is compatible with $\mathfrak{b}$. We say that a reduced factorization $u_1 \cdots u_{p+q-1} = w$ is *compatible* with $\mathfrak{a}$, $\mathfrak{b}$ if $u_i \in S_\infty$ for all $i \neq q$, $u_j(i) = i$ whenever $j < q$ and $i \leq b_{q-j}$ or whenever $j > q$ and $i \leq a_{j-q}$. Set $Y_i := \{y_{a_{i-1}+1}, \ldots, y_{a_i}\}$ for each $i \geq 1$ and $Z_j := \{z_{b_{j-1}+1}, \ldots, z_{b_j}\}$ for each $j \geq 1$.

Proposition 3.9. *Suppose that w and w^{-1} are compatible with $\mathfrak{a}$ and $\mathfrak{b}$, respectively. Then the Schubert polynomial $\mathfrak{C}_w(X\,;Y,Z)$ satisfies*

$$\mathfrak{C}_w = \sum G_{u_1}(0/Z_q) \cdots G_{u_{q-1}}(0/Z_2) J_{u_q}(X\,;Y_1/Z_1) G_{u_{q+1}}(Y_2) \cdots G_{u_{p+q-1}}(Y_p)$$

summed over all reduced factorizations $u_1 \cdots u_{p+q-1} = w$ compatible with $\mathfrak{a}$, $\mathfrak{b}$.

Proof. The result is established by combining the identity (1.7) with (3.34). □

If w is increasing up to k, then the following generalization of equation (3.6) holds (see [T5, Thm. 1]):

$$J_w(X\,;Y_{(k)}) = \sum_{\lambda\,:\,|\lambda|=\ell(w)} e_\lambda^w\, \Theta_\lambda(X;Y_{(k)}), \tag{3.35}$$

where the sum is over k-strict partitions λ with $|\lambda| = \ell(w)$. The *mixed Stanley coefficients* e_λ^w in (3.35) are nonnegative integers. In fact, to any $w \in W_\infty$ increasing up to k we associate a *k-transition tree* $T^k(w)$ whose leaves are k-Grassmannian elements, and e_λ^w is equal to the number of leaves of the tree $T^k(w)$ which have shape λ. The proof of (3.35) in [T5] is a straightforward application of Billey's transition equations for symplectic flag varieties [B] combined with equation (3.32).

Assume that w is increasing up to k and w^{-1} is increasing up to ℓ. At present there is no clear analogue of equation (3.35) for the (restricted) double mixed Stanley function $J_w(X\,;Y_{(k)}/Z_{(\ell)})$. However, we have

$$J_w(X\,;Y_{(k)}/Z_{(\ell)}) = \sum_{uv=w} G_{u^{-1}}(-Z_{(\ell)}) J_v(X\,;Y_{(k)}) \tag{3.36}$$

$$= \sum_{uv=w^{-1}} G_{u^{-1}}(Y_{(k)}) J_v(X\,;-Z_{(\ell)}), \tag{3.37}$$

where the factorizations under the sum signs are reduced with $u \in S_\infty$. We can now use equations (1.9) and (3.35) in (3.36) and (3.37) to obtain two dual expansions of $J_w(X\,;Y_{(k)}/Z_{(\ell)})$ as a positive sum of products of Schur S-polynomials with theta polynomials.

Example 3.10. Let $w = 231 = s_1 s_2 \in W_3$ and take $k = \ell = 1$. We have

$$\mathfrak{C}_{231}(X\,;Y,Z) = q_2(X) + q_1(X)(y_1 + y_2 - z_1) + (y_1 - z_1)(y_2 - z_1)$$

and hence $J_{231}(X\,;Y_{(1)}/Z_{(1)}) = q_2(X) + q_1(X)(y_1 - z_1) - (y_1 - z_1)z_1$. Equality (3.36) gives

$$\begin{aligned} J_{231}(X\,;Y_{(1)}/Z_{(1)}) &= J_{231}(X\,;Y_{(1)}) + G_{213}(-Z_{(1)})J_{132}(X\,;Y_{(1)}) + G_{312}(-Z_{(1)}) \\ &= \Theta_2(X\,;y_1) + s_1(-z_1)\Theta_1(X\,;y_1) + s_2(-z_1) \\ &= (q_2(X) + q_1(X)y_1) + (-z_1)(q_1(X) + y_1) + z_1^2, \end{aligned}$$

while equality (3.37) gives

$$\begin{aligned} J_{231}(X\,;Y_{(1)}/Z_{(1)}) &= J_{312}(X\,;-Z_{(1)}) + G_{132}(Y_{(1)})J_{213}(X\,;-Z_{(1)}) + G_{231}(Y_{(1)}) \\ &= \Theta_{(1,1)}(X\,;-z_1) + s_1(y_1)\Theta_1(X\,;-z_1) + s_{(1,1)}(y_1) \\ &= (q_2(X) - q_1(X)z_1 + z_1^2) + y_1(q_1(X) - z_1). \end{aligned}$$

Theorem 3.11 ([T5], Cor. 1)**.** *Suppose that w is compatible with $\mathfrak{a}$ and w^{-1} is compatible with $\mathfrak{b}$, where $b_1 = 0$. Then we have*

$$\mathfrak{C}_w = \sum_{\underline{\lambda}} f^w_{\underline{\lambda}}\, s_{\lambda^1}(0/Z_q)\cdots s_{\lambda^{q-1}}(0/Z_2)\Theta_{\lambda^q}(X\,;Y_1)s_{\lambda^{q+1}}(Y_2)\cdots s_{\lambda^{p+q-1}}(Y_p) \tag{3.38}$$

summed over all sequences of partitions $\underline{\lambda} = (\lambda^1,\ldots,\lambda^{p+q-1})$ with λ^q a_1-strict, where

$$f^w_{\underline{\lambda}} := \sum_{u_1\cdots u_{p+q-1}=w} a^{u_1}_{\lambda^1}\cdots a^{u_{q-1}}_{\lambda^{q-1}} e^{u_q}_{\lambda^q} a^{u_{q+1}}_{\lambda^{q+1}}\cdots a^{u_{p+q-1}}_{\lambda^{p+q-1}} \tag{3.39}$$

summed over all reduced factorizations $u_1\cdots u_{p+q-1} = w$ compatible with $\mathfrak{a}$, $\mathfrak{b}$.

Proof. The result follows from Proposition 3.9 by using equations (1.9) and (3.35). □

Proposition 3.9 and Theorem 3.11 are symplectic analogues of Proposition 1.1 and Theorem 1.2, and similar remarks about their algebraic, combinatorial, and geometric significance apply. We refer the reader to [T5], [T6, §4 and §6], and §5 of the present paper for further details and for examples which illustrate computations of the mixed Stanley coefficients $e^w_{\underline{\lambda}}$.

4. The type D theory

For the orthogonal Lie types B and D we work with coefficients in the ring $\mathbb{Z}[\frac{1}{2}]$. For $w \in W_\infty$, the type B double Schubert polynomial $\mathfrak{B}_w$ of [IMN1] is related to the type C Schubert polynomial by the equation $\mathfrak{B}_w = 2^{-s(w)}\mathfrak{C}_w$, where $s(w)$ denotes the number of indices i such that $w_i < 0$. We will therefore omit any further discussion of type B, and concentrate on the even orthogonal case. The exposition is parallel to that of §3, but there are some interesting variations in the results and in their proofs.

4.1. **Schubert polynomials and divided differences.** The Weyl group $\widetilde{W}_n$ for the root system D_n is the subgroup of W_n consisting of all signed permutations with an even number of sign changes. The group $\widetilde{W}_n$ is an extension of S_n by the element $s_\square = s_0s_1s_0$, which acts on the right by

$$(w_1, w_2, \ldots, w_n)s_\square = (\overline{w}_2, \overline{w}_1, w_3, \ldots, w_n).$$

There is a natural embedding $\widetilde{W}_n \hookrightarrow \widetilde{W}_{n+1}$ of Weyl groups defined by adjoining the fixed point $n+1$, and we let $\widetilde{W}_\infty := \cup_n \widetilde{W}_n$. The elements of the set $\mathbb{N}_\square := \{\square, 1, \ldots\}$ index the simple reflections in $\widetilde{W}_\infty$; these are used to define the reduced words and descents of elements in $\widetilde{W}_\infty$ as in the previous sections.

The nilCoxeter algebra $\widetilde{\mathcal{W}}_n$ of $\widetilde{W}_n$ is the free associative algebra with unit generated by the elements $u_\square, u_1, \ldots, u_{n-1}$ modulo the relations

$$\begin{aligned} u_i^2 &= 0 && i \in \mathbb{N}_\square ; \\ u_\square u_1 &= u_1 u_\square && \\ u_\square u_2 u_\square &= u_2 u_\square u_2 && \\ u_i u_{i+1} u_i &= u_{i+1} u_i u_{i+1} && i > 0 ; \\ u_i u_j &= u_j u_i && j > i+1, \text{ and } (i,j) \neq (\square, 2). \end{aligned}$$

As in §3.1, for any $w \in \widetilde{W}_n$, choose a reduced word $a_1 \cdots a_\ell$ for w, and define $u_w := u_{a_1} \ldots u_{a_\ell}$. Denote the coefficient of $u_w \in \widetilde{\mathcal{W}}_n$ in the expansion of the element $\xi \in \widetilde{\mathcal{W}}_n$ in the u_w basis by $\langle \xi, w \rangle$. Let t be a variable and, following Lam [La], define

$$D(t) := (1 + t u_{n-1}) \cdots (1 + t u_2)(1 + t u_1)(1 + t u_\square)(1 + t u_2) \cdots (1 + t u_{n-1}).$$

Let $D(X) := D(x_1) D(x_2) \cdots$, and for $w \in \widetilde{W}_n$, define

$$\mathfrak{D}_w(X\,; Y, Z) := \left\langle \tilde{A}_{n-1}(z_{n-1}) \cdots \tilde{A}_1(z_1) D(X) A_1(y_1) \cdots A_{n-1}(y_{n-1}), w \right\rangle . \tag{4.1}$$

The power series $\mathfrak{D}_w(X\,; Y) := \mathfrak{D}_w(X\,; Y, 0)$ are the type D Billey-Haiman Schubert polynomials, and the $\mathfrak{D}_w(X\,; Y, Z)$ are their double versions from [IMN1].

The double Schubert polynomial $\mathfrak{D}_w(X\,; Y, Z)$ is stable under the natural inclusions $\widetilde{W}_n \hookrightarrow \widetilde{W}_{n+1}$, and hence is well defined for $w \in \widetilde{W}_\infty$. We set

$$E_w(X) := \mathfrak{D}_w(X\,; 0, 0) = \langle D(X), w \rangle$$

and call E_w the *type D Stanley symmetric function* indexed by $w \in \widetilde{W}_n$. Observe that we have $E_w = E_{w^{-1}}$. Equation (4.1) implies the relation

$$\mathfrak{D}_w(X\,; Y, Z) = \sum_{uv\varpi = w} \mathfrak{S}_{u^{-1}}(-Z) E_v(X) \mathfrak{S}_\varpi(Y) \tag{4.2}$$

summed over all reduced factorizations $uv\varpi = w$ with $u, \varpi \in S_\infty$.

For each strict partition λ, the Schur P-function $P_\lambda(X)$ is defined by the equation $P_\lambda(X) := 2^{-\ell(\lambda)} Q_\lambda(X)$, where $\ell(\lambda)$ denotes the length of λ. The type D Stanley symmetric functions $E_w(X)$ lie in the ring $\Gamma' := \mathbb{Z}[P_1, P_2, \ldots]$ of Schur P-functions. In fact, for any $w \in \widetilde{W}_\infty$, we have an equation

$$E_w(X) = \sum_{\lambda \,:\, |\lambda| = \ell(w)} d_\lambda^w \, P_\lambda(X) \tag{4.3}$$

summed over all strict partitions λ with $|\lambda| = \ell(w)$. Since $D(t)D(-t) = 1$, it follows from [P, Thm. 2.11] that an identity (4.3) exists with coefficients $d_\lambda^w \in \mathbb{Z}$. Given equation (4.2), this implies that $\mathfrak{D}_w(X\,; Y, Z)$ is an element of $\Gamma'[Y, Z]$, for any $w \in \widetilde{W}_\infty$. For three different proofs that $d_\lambda^w \geq 0$, see [B, BH, La].

We define an action of $\widetilde{W}_\infty$ on $\Gamma'[Y, Z]$ by ring automorphisms as follows. The simple reflections s_i for $i > 0$ act by interchanging y_i and y_{i+1} and leaving all the remaining variables fixed, as in §1.1. The reflection $s_\square$ maps (y_1, y_2) to $(-y_2, -y_1)$, fixes the y_j for $j \geq 3$ and all the z_j, and satisfies, for any $r \geq 1$,

$$\begin{aligned} s_\square(P_r(X)) &:= P_r(y_1, y_2, x_1, x_2, \ldots) \\ &= P_r(X) + (y_1 + y_2) \sum_{j=0}^{r-1} \left(\sum_{a+b=j} y_1^a y_2^b \right) Q_{r-1-j}(X). \end{aligned}$$

For each $i \in \mathbb{N}_\Box$, define the divided difference operator ∂_i^y on $\Gamma'[Y,Z]$ by

$$\partial_\Box^y f := \frac{f - s_\Box f}{-y_1 - y_2}, \qquad \partial_i^y f := \frac{f - s_i f}{y_i - y_{i+1}} \quad \text{for } i > 0.$$

Consider the ring involution $\omega : \Gamma'[Y,Z] \to \Gamma'[Y,Z]$ determined by

$$\omega(y_j) = -z_j, \qquad \omega(z_j) = -y_j, \qquad \omega(P_r(X)) = P_r(X)$$

and set $\partial_i^z := \omega \partial_i^y \omega$ for each $i \in \mathbb{N}_\Box$.

The polynomials $\mathfrak{D}_w(X\,;Y,Z)$ for $w \in \widetilde{W}_\infty$ are the unique family of elements of $\Gamma'[Y,Z]$ satisfying the equations

$$\partial_i^y \mathfrak{D}_w = \begin{cases} \mathfrak{D}_{ws_i} & \text{if } \ell(ws_i) < \ell(w), \\ 0 & \text{otherwise}, \end{cases} \qquad \partial_i^z \mathfrak{D}_w = \begin{cases} \mathfrak{D}_{s_i w} & \text{if } \ell(s_i w) < \ell(w), \\ 0 & \text{otherwise}, \end{cases} \tag{4.4}$$

for all $i \in \mathbb{N}_\Box$, together with the condition that the constant term of $\mathfrak{D}_w$ is 1 if $w = 1$, and 0 otherwise. As in §1.1 and §3.1, it follows that descents of w and w^{-1} determine the symmetries of the double Schubert polynomial $\mathfrak{D}_w(X\,;Y,Z)$, and that the polynomials $\mathfrak{D}_w$ represent degeneracy loci of even orthogonal vector bundles, in the sense of [Fu2].

4.2. **Schur P-functions and their double analogues.** Let $n \geq 1$ be an integer and $\ell \in [1,n]$. Let $\alpha = (\alpha_1 \ldots, \alpha_\ell)$ be a composition, and define a polynomial $P_\alpha^{(\ell)}(x_1, \ldots, x_n)$ by the equation

$$P_\alpha^{(\ell)}(x_1, \ldots, x_n) := \frac{1}{(n-\ell)!} \sum_{\varpi \in S_n} \varpi \left(x_1^{\alpha_1} \cdots x_\ell^{\alpha_\ell} \prod_{i \leq \ell,\, i<j \leq n} \frac{x_i + x_j}{x_i - x_j} \right). \tag{4.5}$$

Let $S_{n-\ell}$ denote the subgroup of S_n consisting of permutations of $\{\ell+1, \ldots, n\}$. Since the expression $x^\alpha \displaystyle\prod_{i \leq \ell, i<j \leq n} \frac{x_i + x_j}{x_i - x_j}$ is symmetric in $(x_{\ell+1}, \ldots, x_n)$, we deduce that

$$P_\alpha^{(\ell)}(x_1, \ldots, x_n) = \sum_{\sigma \in S_n / S_{n-\ell}} \sigma \left(x_1^{\alpha_1} \cdots x_\ell^{\alpha_\ell} \prod_{i \leq \ell, i<j \leq n} \frac{x_i + x_j}{x_i - x_j} \right). \tag{4.6}$$

It follows from [Iv1, Prop. 1.1(c)] that $P_\alpha^{(\ell)}(x_1, \ldots, x_n) = 0$ if $\alpha_i = \alpha_j$ for some $i \neq j$. Hence, the polynomial $P_\alpha^{(\ell)}(x_1, \ldots, x_n)$ is alternating in the indices $(\alpha_1, \ldots, \alpha_\ell)$.

Lemma 4.1. *Assume that n is even. If $\alpha_\ell = 0$, then we have*

$$P_\alpha^{(\ell)}(x_1, \ldots, x_n) = \begin{cases} 0 & \text{if } \ell \text{ is odd}, \\ P_\alpha^{(\ell-1)}(x_1, \ldots, x_n) & \text{if } \ell \text{ is even}. \end{cases}$$

Proof. According to [Iv2, Prop. 2.4], for any $m \geq 1$, we have

$$\sum_{\varpi \in S_m} \varpi \left(\prod_{j=2}^m \frac{x_1 + x_j}{x_1 - x_j} \right) = \begin{cases} 0 & \text{if } m \text{ is even}, \\ (m-1)! & \text{if } m \text{ is odd}. \end{cases} \tag{4.7}$$

Let $H \cong S_{n+1-\ell}$ denote the subgroup of S_n consisting of permutations of $\{\ell, \ldots, n\}$, and set $P_\alpha^{(\ell,n)} := P_\alpha^{(\ell)}(x_1, \ldots, x_n)$. Using (4.6) and equation (4.7), we compute that

$$\begin{aligned}
(n-\ell)!P_\alpha^{(\ell,n)} &= \sum_{\sigma\in S_n/H}\sum_{\varpi\in H}\sigma\varpi\left(x^\alpha\prod_{i\leq\ell, i<j\leq n}\frac{x_i+x_j}{x_i-x_j}\right)\\
&= \sum_{\sigma\in S_n/H}\sum_{\varpi\in H}\sigma\left(x^\alpha\prod_{i<\ell, i<j\leq n}\frac{x_i+x_{\varpi(j)}}{x_i-x_{\varpi(j)}}\right)\prod_{j=\ell+1}^{n}\frac{x_{\sigma\varpi(\ell)}+x_{\sigma\varpi(j)}}{x_{\sigma\varpi(\ell)}-x_{\sigma\varpi(j)}}\\
&= \sum_{\sigma\in S_n/H}\sigma\left(x^\alpha\prod_{i<\ell, i<j\leq n}\frac{x_i+x_j}{x_i-x_j}\right)\cdot\sum_{\varpi\in H}\prod_{j=\ell+1}^{n}\frac{x_{\sigma\varpi(\ell)}+x_{\sigma\varpi(j)}}{x_{\sigma\varpi(\ell)}-x_{\sigma\varpi(j)}}\\
&= \begin{cases}0 & \text{if } \ell \text{ is odd,}\\ (n-\ell)!\,P_\alpha^{(\ell-1,n)} & \text{if } \ell \text{ is even.}\end{cases}
\end{aligned}$$

□

Let $t=(t_1,t_2\ldots)$ be a sequence of independent variables, as in §2.1, and define $(x\,|\,t)^r := (x-t_1)\cdots(x-t_r)$. Given a strict partition λ of length ℓ and $n\geq\ell$, Ivanov's double Schur P-function $P_\lambda(x_1,\ldots,x_n\,|\,t)$ is defined by

$$P_\lambda(x_1,\ldots,x_n\,|\,t) := \frac{1}{(n-\ell)!}\sum_{\varpi\in S_n}\varpi\left(\prod_{i=1}^{\ell}(x_i\,|\,t)^{\lambda_i}\prod_{i\leq\ell,\, i<j\leq n}\frac{x_i+x_j}{x_i-x_j}\right). \tag{4.8}$$

Following [IMN1, §4.2], we let $P_\lambda(X\,|\,t)$ denote the (even) projective limit of the functions $P_\lambda(x_1,\ldots,x_{2m}\,|\,t)$ as $m\to\infty$. We have that $P_\lambda(X\,|\,0)=P_\lambda(X)$ is the Schur P-function indexed by the partition λ.

Proposition 4.2. *Let λ be a strict partition of length ℓ.*

(a) *Suppose that ℓ is even and $\lambda=\delta_{\ell-1}+\mu$ for some partition μ. Then*

$$P_\lambda(X\,|\,t)=\sum_{\nu\subset\mu}P_{\delta_{\ell-1}+\nu}(X)S^\lambda_{\mu/\nu}(e(-t)).$$

(b) *Suppose that ℓ is odd and $\lambda=\delta_\ell+\mu$ for some partition μ. Then*

$$P_\lambda(X\,|\,t)=\sum_{\nu\subset\mu}P_{\delta_\ell+\nu}(X)S^\lambda_{\mu/\nu}(e(-t)).$$

Proof. For any $r\geq 1$, we have $(x\,|\,t)^r=\sum_{p=0}^r x^p e^r_{r-p}(-t)$. Therefore for any partition λ of length ℓ, we have

$$\prod_{i=1}^{\ell}(x_i\,|\,t)^{\lambda_i}=\sum_{\alpha\geq 0}x^\alpha e^\lambda_{\lambda-\alpha}(-t). \tag{4.9}$$

It follows from (4.5), (4.8), and (4.9) that for any $n\geq\ell$, we have

$$P_\lambda(x_1,\ldots,x_n\,|\,t)=\sum_{0\leq\alpha\leq\lambda}P_\alpha^{(\ell)}(x_1,\ldots,x_n)e^\lambda_{\lambda-\alpha}(-t).$$

Taking the even projective limit as $n\to\infty$ gives

$$P_\lambda(X\,|\,t)=\sum_{0\leq\alpha\leq\lambda}P_\alpha^{(\ell)}(X)e^\lambda_{\lambda-\alpha}(-t). \tag{4.10}$$

For part (a), using (4.10), Lemma 4.1, and the alternating property of the functions $P^{(\ell)}_\alpha(X)$ gives

$$\begin{aligned} P_\lambda(X\,|\,t) &= \sum_{\nu\subset\mu}\sum_{\sigma\in S_\ell}(-1)^\sigma P_{\delta_{\ell-1}+\nu}(X)e^\lambda_{\delta_{\ell-1}+\mu-\sigma(\delta_{\ell-1}+\nu)}(-t) \\ &= \sum_{\nu\subset\mu} P_{\delta_{\ell-1}+\nu}(X)S^\lambda_{\mu/\nu}(e(-t)). \end{aligned}$$

For part (b), we similarly obtain

$$\begin{aligned} P_\lambda(X\,|\,t) &= \sum_{\nu\subset\mu}\sum_{\sigma\in S_\ell}(-1)^\sigma P_{\delta_\ell+\nu}(X)e^\lambda_{\delta_\ell+\mu-\sigma(\delta_\ell+\nu)}(-t) \\ &= \sum_{\nu\subset\mu} P_{\delta_\ell+\nu}(X)S^\lambda_{\mu/\nu}(e(-t)). \end{aligned}$$

□

Corollary 4.3. (a) *If n is odd, then*

$$P_{2\delta_{n-1}}(X\,|\,t) = \sum_{\nu\subset\delta^*_n} P_{\delta_{n-2}+\nu}(X)S^{2\delta_{n-1}}_{\delta^*_n/\nu}(e(-t)).$$

(b) *If n is even, then*

$$P_{2\delta_{n-1}}(X\,|\,t) = \sum_{\nu\subset\delta_{n-1}} P_{\delta_{n-1}+\nu}(X)S^{2\delta_{n-1}}_{\delta_{n-1}/\nu}(e(-t)).$$

According to [Iv3, §9] and [IN, §8.3], we have a Pfaffian formula

$$P_\lambda(X\,|\,t) = \mathrm{Pfaffian}(P_{\lambda_i,\lambda_j}(X\,|\,t))_{1\le i<j\le 2\ell'}, \tag{4.11}$$

where $2\ell'$ is the least even integer which is greater than or equal to $\ell(\lambda)$. In equation (4.11), we use the conventions that $P_{a,b}(X\,|\,t) := -P_{b,a}(X\,|\,t)$ whenever $0\le a\le b$, and $P_{a,0}(X\,|\,t) := P_a(X\,|\,t)$. We will require a raising operator expression analogous to (3.4) for the functions $P_\lambda(X\,|\,t)$. This uses a more involved Pfaffian formalism which stems from the work of Knuth [Kn, §4] and Kazarian [Ka, App. C and D].

For any $r\in\mathbb{Z}$, we define the polynomial $\mathfrak{c}^r_p = \mathfrak{c}^r_p(X\,|\,t)$ by

$$\mathfrak{c}^r_p := \sum_{j=0}^p q_{p-j}(X)e^r_j(-t).$$

For any integer sequences α, β, let

$$\widehat{\mathfrak{c}}^\beta_\alpha := \widehat{\mathfrak{c}}^{\beta_1}_{\alpha_1}\widehat{\mathfrak{c}}^{\beta_2}_{\alpha_2}\cdots$$

where, for each $i\ge 1$,

$$\widehat{\mathfrak{c}}^{\beta_i}_{\alpha_i} := \mathfrak{c}^{\beta_i}_{\alpha_i} + \begin{cases} (-1)^i e^{\alpha_i}_{\alpha_i}(-t) & \text{if } \beta_i=\alpha_i>0, \\ 0 & \text{otherwise.}\end{cases}$$

If $R := \prod_{i<j} R_{ij}^{n_{ij}}$ is any raising operator, denote by $\mathrm{supp}(R)$ the set of all indices i and j such that $n_{ij}>0$. Let $\alpha=(\alpha_1,\ldots,\alpha_\ell)$ and $\beta=(\beta_1,\ldots,\beta_\ell)$ be integer vectors, set $\nu := R\alpha$, and define

$$R\star\widehat{\mathfrak{c}}^\beta_\alpha = \overline{\mathfrak{c}}^\beta_\nu := \overline{\mathfrak{c}}^{\beta_1}_{\nu_1}\cdots\overline{\mathfrak{c}}^{\beta_\ell}_{\nu_\ell}$$

where for each $i \geq 1$,

$$\bar{\mathfrak{c}}^{\beta_i}_{\nu_i} := \begin{cases} \mathfrak{c}^{\beta_i}_{\nu_i} & \text{if } i \in \mathrm{supp}(R), \\ \widehat{\mathfrak{c}}^{\beta_i}_{\nu_i} & \text{otherwise.} \end{cases}$$

Proposition 4.4. *For any strict partition λ, we have*

$$P_\lambda(X\,|\,t) = 2^{-\ell(\lambda)}\, R^\infty \star \widehat{\mathfrak{c}}^\lambda_\lambda. \tag{4.12}$$

Proof. It follows from [Kn, Ka] that the equation

$$R^\infty \star \widehat{\mathfrak{c}}^\lambda_\lambda = 2^{\ell(\lambda)}\, \mathrm{Pfaffian}(P_{\lambda_i,\lambda_j}(X\,|\,t))_{i<j} \tag{4.13}$$

holds if and only if it holds for all strict partitions λ of length ℓ at most 3. The latter is a formal identity which is straightforward to check from the definitions; compare with [AF2, App. A] and [IMN2, §2.3]. We conclude from (4.11) and (4.13) that (4.12) is also true. □

4.3. **The Schubert polynomial indexed by the longest element.** Let $\widetilde{w}_0$ denote the longest element in $\widetilde{W}_n$. We have

$$\widetilde{w}_0 = \begin{cases} (\overline{1},\ldots,\overline{n}) & \text{if } n \text{ is even,} \\ (1,\overline{2},\ldots,\overline{n}) & \text{if } n \text{ is odd.} \end{cases}$$

A formula for the top single Schubert polynomial $\mathfrak{D}_{\widetilde{w}_0}(X\,;Y)$ was given by Billey and Haiman [BH, Prop. 4.15]. In this section, we derive the analogue of their result for the double Schubert polynomial $\mathfrak{D}_{\widetilde{w}_0}(X\,;Y,Z)$, and use it to give a combinatorial proof of the Pfaffian formula for $\mathfrak{D}_{\widetilde{w}_0}$ from [IMN1, Thm. 1.2].

Proposition 4.5. *If n is even, then we have*

$$\mathfrak{D}_{\widetilde{w}_0}(X\,;Y,Z) = \sum_{\lambda\subset\delta_{n-1}} P_{\delta_{n-1}+\lambda}(X) S^{(\delta_{n-1},\delta_{n-1})}_{\delta_{n-1}/\lambda}(e(Y,-Z))$$

in $\Gamma'[Y,Z]$.

Proof. Using equation (4.2), we have

$$\mathfrak{D}_{\widetilde{w}_0}(X\,;Y,Z) = \sum_{uwv=\widetilde{w}_0} E_w(X)\mathfrak{S}_{u^{-1}}(-Z)\mathfrak{S}_v(Y).$$

summed over all reduced factorizations $uwv = \widetilde{w}_0$ in $\widetilde{W}_\infty$ with $u,v\in S_\infty$. If n is even, then every permutation in S_n commutes with $\widetilde{w}_0$, and it follows that

$$\mathfrak{D}_{\widetilde{w}_0}(X\,;Y,Z) = \sum_{\sigma\in S_n} E_{\widetilde{w}_0\sigma^{-1}}(X)\widetilde{\mathfrak{S}}_\sigma(Y,Z). \tag{4.14}$$

According to [BH, Thm. 3.16] and [La, Thm. 5.14], we have, for every $u\in S_n$,

$$E_{\widetilde{w}_0 u}(X) = \sum_\lambda a_\lambda^{u^{-1}\varpi_0} P_{\delta_{n-1}+\lambda}(X). \tag{4.15}$$

We deduce from (4.14) and (4.15) that

$$\mathfrak{D}_{\widetilde{w}_0}(X\,;Y,Z) = \sum_\lambda P_{\delta_{n-1}+\lambda}(X) \sum_{\sigma\in S_n} a_\lambda^{\sigma\varpi_0}\widetilde{\mathfrak{S}}_\sigma(Y,Z).$$

The result now follows by combining equations (3.11) and (3.18). □

Fix $k \geq 0$, and set ${}^k c_p(X\,;Y) := \sum_{i=0}^p q_{p-i}(X) h_i^{-k}(Y)$, so that we have

$$ {}^k c_p^r(X\,;Y,Z) = \sum_{j=0}^p {}^k c_{p-j}(X\,;Y) h_j^r(-Z). $$

Define ${}^k b_r := {}^k c_r$ for $r < k$, ${}^k b_r := \frac{1}{2}{}^k c_r$ for $r > k$, and set

$$ {}^k b_k := \frac{1}{2}{}^k c_k + \frac{1}{2} e_k^k(Y) \quad \text{and} \quad {}^k\widetilde{b}_k := \frac{1}{2}{}^k c_k - \frac{1}{2} e_k^k(Y). $$

Let f_k be an indeterminate of degree k, which will be equal to ${}^k b_k$, ${}^k\widetilde{b}_k$, or $\frac{1}{2}\,{}^k c_k$, depending on the context. We also let $f_0 \in \{0,1\}$. For any $p, r \in \mathbb{Z}$, define ${}^k\widehat{c}_p^r$ by

$$ {}^k\widehat{c}_p^r := {}^k c_p^r + \begin{cases} (2f_k - {}^k c_k) e_{p-k}^{p-k}(-Z) & \text{if } r = k-p < 0, \\ 0 & \text{otherwise.} \end{cases} $$

In particular, we have

$$ {}^0\widehat{c}_p^r = {}^0 c_p^r \pm \begin{cases} e_p^p(-Z) & \text{if } r = -p < 0, \\ 0 & \text{otherwise,} \end{cases} $$

and thus ${}^0\widehat{c}_p^{-p} = {}^0 c_p^{-p} \pm e_p^p(-Z)$ when $p > 0$, while ${}^0\widehat{c}_0^0 = 1$. It follows from [IMN2, Eq. (2.14)] that

$$ P_r(X \,|\, Z) = \frac{1}{2}({}^0 c_r^{-r} - e_r^r(-Z)), $$

that is, $P_r(X \,|\, Z) = \frac{1}{2}\,{}^0\widehat{c}_r^{-r}$ with the choice of $f_0 = 0$.

Recall that we have defined left divided differences $\partial_i = \partial_i^z$ for each $i \in \mathbb{N}_\Box$. These operators satisfy the same Leibnitz rule (3.19) as in the type C case. We now have the following even orthogonal analogues of Lemmas 3.3 and 3.4.

Lemma 4.6 ([T7], Prop. 2)**.** *Suppose that $p, r \in \mathbb{Z}$ and let $k \geq 0$ and $i \geq 1$.*

(a) *We have*

$$ \partial_i({}^k c_p^r) = \begin{cases} {}^k c_{p-1}^{r+1} & \text{if } r = \pm i, \\ 0 & \text{otherwise.} \end{cases} $$

(b) *If $p > k$, we have*

$$ \partial_i({}^k\widehat{c}_p^{k-p}) = \begin{cases} {}^k\widehat{c}_{p-1}^{k-p+1} & \text{if } i = p-k \geq 2, \\ 2f_k & \text{if } i = p-k = 1, \\ 0 & \text{otherwise.} \end{cases} $$

Lemma 4.7. *Suppose that $i \geq 0$ and $k > 0$. Then we have*

$$ {}^k\widehat{c}_p^{-i} = {}^{k-1}\widehat{c}_p^{-i-1} + (z_{i+1} + y_k)\,{}^{k-1}\widehat{c}_{p-1}^{-i}. $$

Proof. We know from Lemma 3.4 that

$$ {}^k c_p^{-i} = {}^{k-1} c_p^{-i-1} + (z_{i+1} + y_k)\,{}^{k-1} c_{p-1}^{-i}. \tag{4.16} $$

If $i \neq p-k$ or $i = 0$ there is nothing more to prove. If $i = p-k > 0$ the result follows from (4.16) and the fact that

$$ e_k^k(Y) e_{p-k}^{p-k}(-Z) = e_{k-1}^{k-1}(Y) e_{p-k+1}^{p-k+1}(-Z) + (z_{p-k+1} + y_k)\, e_{k-1}^{k-1}(Y) e_{p-k}^{p-k}(-Z). $$

□

For any integer sequences α, β and composition ρ, let

$$ {}^{\rho}\widehat{c}^{\beta}_{\alpha} := {}^{\rho_1}\widehat{c}^{\beta_1}_{\alpha_1}\,{}^{\rho_2}\widehat{c}^{\beta_2}_{\alpha_2}\cdots $$

where, for each $i \geq 1$,

$$ {}^{\rho_i}\widehat{c}^{\beta_i}_{\alpha_i} := {}^{\rho_i}c^{\beta_i}_{\alpha_i} + \begin{cases} (-1)^i e^{\rho_i}_{\rho_i}(Y) e^{\alpha_i-\rho_i}_{\alpha_i-\rho_i}(-Z) & \text{if } \beta_i = \rho_i - \alpha_i < 0, \\ 0 & \text{otherwise.} \end{cases} $$

Definition 4.8. Let ρ be a composition and $\alpha = (\alpha_1, \ldots, \alpha_\ell)$, $\beta = (\beta_1, \ldots, \beta_\ell)$ be two integer vectors. Let R be any raising operator, set $\nu := R\alpha$, and define

$$ R \star {}^{\rho}\widehat{c}^{\beta}_{\alpha} = {}^{\rho}\overline{c}^{\beta}_{\nu} := {}^{\rho_1}\overline{c}^{\beta_1}_{\nu_1}\cdots {}^{\rho_\ell}\overline{c}^{\beta_\ell}_{\nu_\ell} $$

where for each $i \geq 1$,

$$ {}^{\rho_i}\overline{c}^{\beta_i}_{\nu_i} := \begin{cases} {}^{\rho_i}c^{\beta_i}_{\nu_i} & \text{if } i \in \mathrm{supp}(R), \\ {}^{\rho_i}\widehat{c}^{\beta_i}_{\nu_i} & \text{otherwise.} \end{cases} $$

Set

$$ {}^{\rho}\widehat{P}^{\beta}_{\alpha}(c) := 2^{-\ell}\, R^{\infty} \star {}^{\rho}\widehat{c}^{\beta}_{\alpha}. \tag{4.17} $$

Lemma 4.9. *Suppose that $\beta_i = \rho_i - \alpha_i < 0$ for every $i \in [1, \ell]$. Then we have*

$$ {}^{\rho}\widehat{P}^{\beta}_{\alpha}(c) = 2^{-\ell}\ \mathrm{Pfaffian}({}^{\rho_i,\rho_j}\widehat{P}^{\beta_i,\beta_j}_{\alpha_i,\alpha_j}(c))_{i<j}. \tag{4.18} $$

In addition, if $\alpha_j = \alpha_{j+1}$ and $\beta_j = \beta_{j+1}$ for some $j \in [1, \ell-1]$, then

$$ {}^{\rho}\widehat{P}^{\beta}_{\alpha}(c) = 0. \tag{4.19} $$

Proof. Arguing as in §4.2 for the double Schur P-functions, one shows that the raising operator expression $R^{\infty} \star {}^{\rho}\widehat{c}^{\beta}_{\alpha}$ in (4.17) may be written formally as the Schur-type Pfaffian in (4.18). The proof of the vanishing statement (4.19) is similar to [IM, Prop. 5.4]. Suppose that $k, r \geq 0$, let ξ be a formal variable, and

$$ F(\xi) := \sum_{p=0}^{\infty} {}^{k}c^{-r}_{p}\xi^p = \prod_{i=1}^{\infty}\frac{1+x_i\xi}{1-x_i\xi}\prod_{j=1}^{k}(1+y_j\xi)\prod_{m=1}^{r}(1-z_m\xi) $$

be the generating function for the sequence $\{{}^{k}c^{-r}_{p}\}_{p\geq 0}$. Then we clearly have

$$ F(\xi)F(-\xi) = \prod_{j=1}^{k}(1-y_j^2\xi^2)\prod_{m=1}^{r}(1-z_m^2\xi^2). \tag{4.20} $$

Equating the like even powers of ξ on both sides of (4.20) gives

$$ \frac{1-R_{12}}{1+R_{12}}\,{}^{k,k}c^{-r,-r}_{p,p} = \begin{cases} e_k(y_1^2,\ldots,y_k^2)e_r(z_1^2,\ldots,z_r^2) & \text{if } p = k+r, \\ 0 & \text{if } p > k+r. \end{cases} $$

We deduce that if $p \geq k+r$, then

$$ \frac{1-R_{12}}{1+R_{12}} \star {}^{k,k}\widehat{c}^{-r,-r}_{p,p} = 0, $$

and therefore that ${}^{k,k}\widehat{P}^{-r,-r}_{p,p}(c) = 0$. Equation (4.19) now follows using (4.18) and the alternating properties of Pfaffians, as in [Ka, §1] and [IM, §4]. $\square$

The next result is equivalent to Ikeda, Mihalcea, and Naruse's Pfaffian formula for $\mathfrak{D}_{\widetilde{w}_0}$ from [IMN1, Thm. 1.2].

Proposition 4.10. *For any integer $n \geq 1$, we have*

$$\mathfrak{D}_{\widetilde{w}_0}(X\,;Y,Z) = {}^{\delta_{n-1}}\widehat{P}^{-\delta_{n-1}}_{2\delta_{n-1}}(c) \tag{4.21}$$

in $\Gamma'[Y,Z]$.

Proof. It follows from Propositions 4.4 and 4.5 and Corollary 4.3(b) that (4.21) holds when n is even. Assume that n is even for the rest of this proof. Since we have

$$s_{n-1}\cdots s_1 s_\Box s_2 \cdots s_{n-1}\widetilde{w}_0^{(n)} = \widetilde{w}_0^{(n-1)}$$

in $\widetilde{W}_n$, we get using (4.4) a corresponding equation of divided differences

$$\mathfrak{D}_{\widetilde{w}_0^{(n-1)}} = (\partial_{n-1}\cdots\partial_2)(\partial_\Box\partial_1)(\partial_2\cdots\partial_{n-1})\mathfrak{D}_{\widetilde{w}_0^{(n)}}$$

in $\Gamma'[Y,Z]$. We therefore obtain the equality

$$\mathfrak{D}_{\widetilde{w}_0^{(n-1)}} = 2^{1-n}(\partial_{n-1}\cdots\partial_2)(\partial_\Box\partial_1)(\partial_2\cdots\partial_{n-1})\left(R^\infty \star {}^{\delta_{n-1}}\widehat{c}^{\,-\delta_{n-1}}_{2\delta_{n-1}}\right). \tag{4.22}$$

We next compute the action of the divided differences on the right hand side of (4.22). By using Lemmas 4.7 and 4.9 and arguing as in §3.3, we see that

$$(\partial_2\cdots\partial_{n-1})\left(R^\infty \star {}^{\delta_{n-1}}\widehat{c}^{\,-\delta_{n-1}}_{2\delta_{n-1}}\right) = R^\infty \star {}^{(n-2,n-3,\ldots,1,1)}\widehat{c}^{\,-\delta_{n-1}}_{(2n-3,2n-5,\ldots,3,2)}.$$

For any integer vector $\alpha := (\alpha_1\ldots,\alpha_{n-1})$, we have

$$\partial_1({}^{(n-2,n-3,\ldots,1,1)}\widehat{c}^{\,-\delta_{n-1}}_\alpha) = {}^{\delta_{n-2}}\widehat{c}^{\,(1-n,2-n,\ldots,-2)}_{(\alpha_1,\ldots,\alpha_{n-2})}\,{}^1g^0_{\alpha_{n-1}-1},$$

where

$${}^1g^0_p = \begin{cases} 2\,{}^1f_1 & \text{if } p=1,\\ {}^1\widehat{c}^{\,0}_p & \text{otherwise.}\end{cases}$$

According to [T7, §1], for $k \geq 1$, we have

$$\partial_\Box({}^kc_p) = 2\,{}^kc^2_{p-1} \quad\text{and}\quad \partial_\Box({}^kb_k) = \partial_\Box({}^k\widetilde{b}_k) = {}^kc^2_{k-1}. \tag{4.23}$$

We therefore also have $\partial_\Box({}^1f_1) = 1$ by (4.23) and $\partial_\Box({}^1\widehat{c}^{\,0}_p) = 0$ for $p \leq 0$. It follows that

$$\partial_\Box\partial_1({}^{(n-2,n-3,\ldots,1,1)}\widehat{c}^{\,-\delta_{n-1}}_\alpha) = \begin{cases} 2\cdot{}^{\delta_{n-2}}\widehat{c}^{\,(1-n,2-n,\ldots,-2)}_{(\alpha_1,\ldots,\alpha_{n-2})} & \text{if } \alpha_{n-1}=2,\\ 0 & \text{if } \alpha_{n-1}<2.\end{cases} \tag{4.24}$$

We deduce from (4.24) that $\partial_\Box\partial_1$ commutes with the action of the raising operators R in the expansion of R^∞ in its $\star$-action on ${}^{(n-2,n-3,\ldots,1,1)}\widehat{c}^{\,-\delta_{n-1}}_{(2n-3,2n-5,\ldots,3,2)}$, and hence that

$$(\partial_\Box\partial_1\cdots\partial_{n-1})\left(R^\infty \star {}^{\delta_{n-1}}\widehat{c}^{\,-\delta_{n-1}}_{2\delta_{n-1}}\right) = 2\,R^\infty \star {}^{\delta_{n-2}}\widehat{c}^{\,(1-n,2-n,\ldots,-2)}_{(2n-3,2n-5,\ldots,3)}.$$

We continue applying Lemma 4.6(b) to compute the action of $\partial_{n-1}\cdots\partial_2$ on $R^\infty \star {}^{\delta_{n-2}}\widehat{c}^{\,(1-n,2-n,\ldots,-2)}_{(2n-3,2n-5,\ldots,3)}$, to conclude that

$$\mathfrak{D}_{\widetilde{w}_0^{(n-1)}} = 2^{2-n}\,(\partial_{n-1}\cdots\partial_2)\left(R^\infty \star {}^{\delta_{n-2}}\widehat{c}^{\,(1-n,2-n,\ldots,-2)}_{(2n-3,2n-5,\ldots,3)}\right) = 2^{2-n}\,R^\infty \star {}^{\delta_{n-2}}\widehat{c}^{\,-\delta_{n-2}}_{2\delta_{n-2}},$$

and hence that (4.21) holds for all $n \geq 1$, as required. □

Corollary 4.11. *If n is odd, then we have*

$$\mathfrak{D}_{\widetilde{w}_0}(X\,;Y,Z) = \sum_{\lambda\subset\delta_n^*} P_{\delta_{n-2}+\lambda}(X) S^{(\delta_{n-1},\delta_{n-1})}_{\delta_n^*/\lambda}(e(Y,-Z)) \tag{4.25}$$

in $\Gamma'[Y,Z]$.

Proof. This follows by combining Proposition 4.10 with (4.12) and Corollary 4.3(a). □

Note the similarity between formulas (3.20) and (4.25). It would be interesting to expose a more direct argument connecting the two to each other.

4.4. **The Schubert polynomials indexed by maximal elements.** Consider a sequence $\mathfrak{a} \ : \ a_1 < \cdots < a_p$ of elements of $\mathbb{N}_\Box$ with $a_p < n$. The sequence $\mathfrak{a}$ parametrizes a parabolic subgroup $\widetilde{W}_{\mathfrak{a}}$ of $\widetilde{W}_n$, which is generated by the simple reflections s_i for $i \notin \{a_1,\ldots,a_p\}$. In type D, we will only consider sequences $\mathfrak{a}$ with $a_1 \neq 1$, since these suffice to parametrize all the relevant homogeneous spaces and degeneracy loci, up to isomorphism.[2] This claim is due to the natural involution of the Dynkin diagram of type D_n. Geometrically, it is explained by the fact that any isotropic subspace E_{n-1} of $\mathbb{C}^{2n}$ (equipped with an orthogonal form) with $\dim(E_{n-1}) = n-1$ can be uniquely extended to a two-step flag $E_{n-1} \subset E_n$ with E_n maximal isotropic and in a given family (compare with [T6, §6.3.2]).

Define the set $\widetilde{W}_n^{\mathfrak{a}}$ by

$$\widetilde{W}_n^{\mathfrak{a}} := \{w \in \widetilde{W}_n \mid \ell(ws_i) = \ell(w)+1, \ \forall\, i \notin \{a_1,\ldots,a_p\}\}$$

and let $\widetilde{w}_0(\mathfrak{a})$ denote the longest element in $\widetilde{W}_n^{\mathfrak{a}}$. We have

$$\widetilde{w}_0(\mathfrak{a}) = \begin{cases} \overline{a_2}\cdots\widehat{\overline{21}}\overline{a_3}\cdots\overline{a_2+1}\cdots\overline{n}\cdots\overline{a_p+1} & \text{if } a_1 = \Box, \\ \widehat{1}2\cdots a_1\overline{a_2}\cdots\overline{a_1+1}\cdots\overline{n}\cdots\overline{a_p+1} & \text{if } a_1 \neq \Box, \end{cases}$$

where $\widetilde{1}$ is equal to either 1 or $\overline{1}$, specified so that $\widetilde{w}_0(\mathfrak{a})$ contains an even number of barred integers.

Fix an element $k \in \mathbb{N}_\Box$ with $\Box \leq k < n$, and set $\widetilde{W}_n^{(1)} := \widetilde{W}_n^{(\Box,1)}$. The elements of the set $\widetilde{W}_n^{(k)}$ are the k-*Grassmannian* elements of $\widetilde{W}_n$. Let $\widetilde{w}^{(k,n)} = \widehat{1}2\cdots k\,\overline{n}\cdots\overline{k+1}$ denote the longest element of $\widetilde{W}_n^{(k)}$. Following [T7, §3.2], we will require a formula analogous to (4.21) for the Schubert polynomial $\mathfrak{D}_{\widetilde{w}^{(k,n)}}(X\,;Y,Z)$, which maps to Kazarian's multi-Schur Pfaffian formula from [Ka, Thm. 1.1]. Corresponding Pfaffian formulas for the Schubert polynomials $\mathfrak{D}_{w_0(\mathfrak{a})}(X\,;Y,Z)$ were obtained in [AF1].

Proposition 4.12. *We have*

$$\mathfrak{D}_{\widetilde{w}^{(k,n)}}(X\,;Y,Z) = {}^{(k,\ldots,k)}\widehat{P}^{(1-n,\ldots,-k)}_{(n+k-1,\ldots,2k)}(c).$$

in $\Gamma'[Y,Z]$.

Proof. Let $\widetilde{v}^{(k,n)} = \overline{k}\cdots\widehat{\overline{21}}\,\overline{n}\cdots\overline{k+1}$ be the longest element in $\widetilde{W}_n^{(\Box,k)}$. Then we have a reduced factorization $\widetilde{w}_0 = v_1v_2\widetilde{v}^{(k,n)}$, where v_1 and v_2 are defined by (3.22) and (3.23), as in the type C case. Using the equations (4.4), we obtain the relation

$$\mathfrak{D}_{\widetilde{v}^{(k,n)}} = \partial_{n-1}(\partial_{n-2}\partial_{n-1})\cdots(\partial_{k+1}\cdots\partial_{n-1})\cdot\partial_{k-1}(\partial_{k-2}\partial_{k-1})\cdots(\partial_1\cdots\partial_{k-1})\mathfrak{D}_{\widetilde{w}_0}.$$

[2]This convention is simpler than the one used in [T5, §6] and [T6, §5.3].

Assume that $k \geq 2$, as the proof is easier if $k \in \{\Box, 1\}$. First, using Lemmas 4.6 and 4.7, for any $p \in \mathbb{Z}$ we have

$$\partial_{k-1} {}^{k-1}\widehat{c}_p^{1-k} = {}^{k-1}\widehat{c}_{p-1}^{2-k} = {}^{k-2}\widehat{c}_{p-1}^{1-k} + (z_{k-1} + y_{k-1})^{k-2}\widehat{c}_{p-2}^{2-k}. \tag{4.26}$$

The Leibnitz rule and (4.26) imply that for any integer vector $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$, we have

$$\partial_{k-1} {}^{\delta_{n-1}}\widehat{c}_\alpha^{-\delta_{n-1}} = {}^{\delta_{n-1}-\epsilon_{n+1-k}}\widehat{c}_{\alpha-\epsilon_{n+1-k}}^{-\delta_{n-1}} + (z_{k-1} + y_{k-1})^{\delta_{n-1}-\epsilon_{n+1-k}}\widehat{c}_{\alpha-2\epsilon_{n+1-k}}^{-\delta_{n-1}+\epsilon_{n+1-k}}.$$

The last equation implies that ∂_{k-1} commutes with $\star$ action of the raising operators R in the expansion of R^∞ on ${}^{\delta_{n-1}}\widehat{c}_{2\delta_{n-1}}^{-\delta_{n-1}}$. We deduce from this and Lemma 4.9 that

$$\begin{aligned}\partial_{k-1} {}^{\delta_{n-1}}\widehat{P}_{2\delta_{n-1}}^{-\delta_{n-1}}(c) &= {}^{\delta_{n-1}-\epsilon_{n+1-k}}\widehat{P}_{2\delta_{n-1}-\epsilon_{n+1-k}}^{-\delta_{n-1}}(c) \\ &= {}^{(n-1,\ldots,k,k-2,k-2,\ldots,1)}\widehat{P}_{(2n-2,\ldots,2k,2k-3,2k-4,\ldots,2)}^{-\delta_{n-1}}(c).\end{aligned}$$

Iterating this calculation gives

$$(\partial_1 \cdots \partial_{k-1})\mathfrak{D}_{\widetilde{w}_0} = {}^{(n-1,\ldots,k,k-2,k-3,\ldots,0)}\widehat{P}_{(2n-2,\ldots,2k,2k-3,2k-5,\ldots,3,1)}^{-\delta_{n-1}}(c)$$

and furthermore

$$\partial_{k-1}(\partial_{k-2}\partial_{k-1}) \cdots (\partial_1 \cdots \partial_{k-1})\mathfrak{D}_{\widetilde{w}_0} = {}^{(n-1,\ldots,k,0,0,\ldots,0)}\widehat{P}_{(2n-2,\ldots,2k,k-1,k-2,\ldots,1)}^{-\delta_{n-1}}(c).$$

Applying the operator $\partial_{n-1}(\partial_{n-2}\partial_{n-1}) \cdots (\partial_{k+1} \cdots \partial_{n-1})$ to the latter, we similarly get

$$\mathfrak{D}_{\widetilde{v}^{(k,n)}} = {}^{(k,k,\ldots,k,0,0,\ldots,0)}\widehat{P}_{(n+k-1,n+k-2,\ldots,2k,k-1,\ldots,1)}^{-\delta_{n-1}}(c).$$

We also have $\widetilde{v}^{(k,n)} = (s_\Box s_2 \cdots s_{k-1}) \cdots (s_\Box s_2)s_\Box \widetilde{w}^{(k,n)}$, and hence (4.4) gives

$$\mathfrak{D}_{\widetilde{w}^{(k,n)}} = \partial_\Box(\partial_2\partial_\Box) \cdots (\partial_{k-1} \cdots \partial_2\partial_\Box)\mathfrak{D}_{\widetilde{v}^{(k,n)}}.$$

Finally, using the fact that

$${}^{(\rho,r)}\widehat{P}_{(\alpha,0)}^{(\beta,b)}(c) = {}^{\rho}\widehat{P}_\alpha^\beta(c),$$

we compute that

$$\begin{aligned}\mathfrak{D}_{\widetilde{w}^{(k,n)}} &= \partial_\Box(\partial_2\partial_\Box) \cdots (\partial_{k-1} \cdots \partial_2\partial_\Box)\mathfrak{D}_{\widetilde{v}^{(k,n)}} \\ &= \partial_\Box(\partial_2\partial_\Box) \cdots (\partial_{k-2} \cdots \partial_2\partial_\Box)^{(k,\ldots,k,0,\ldots,0)}\widehat{P}_{(n+k-1,\ldots,2k,k-2,\ldots,1)}^{(1-n,\ldots,-k,2-k\ldots,0)}(c) \\ &= {}^{(k,\ldots,k)}\widehat{P}_{(n+k-1,\ldots,2k)}^{(1-n,\ldots,-k)}(c).\end{aligned}$$

□

More generally, using similar arguments to those above, we can prove that

$$\mathfrak{D}_{\widetilde{w}_0(\mathfrak{a})}(X\,; Y, Z) = {}^{\rho(\mathfrak{a})}\widehat{P}_{\lambda(\mathfrak{a})}^{\beta(\mathfrak{a})}(c),$$

where $\lambda(\mathfrak{a})$, $\beta(\mathfrak{a})$, and $\rho(\mathfrak{a})$ denote the sequences

$$\lambda(\mathfrak{a}) = (n + a_p - 1, \ldots, 2a_p, \ldots, a_i + a_{i+1} - 1, \ldots, 2a_i, \ldots, a_1 + a_2 - 1, \ldots, 2a_1)\,;$$

$$\beta(\mathfrak{a}) = (1 - n, \ldots, -a_p, \ldots, 1 - a_{i+1}, \ldots, -a_i, \ldots, 1 - a_2, \ldots, -a_1)\,;$$

and

$$\rho(\mathfrak{a}) = (a_p^{n-a_p}, \ldots, a_i^{a_{i+1}-a_i}, \ldots, a_1^{a_2-a_1}).$$

4.5. **Eta polynomials.** According to [BKT1, BKT3], a *typed k-strict partition* is a pair consisting of a k-strict partition λ together with an integer $\mathrm{type}(\lambda) \in \{0,1,2\}$, which is positive if and only if $\lambda_j = k$ for some index j. We assume that $k > 0$ here, although it is straightforward to include the case $k = 0$, where a typed 0-strict partition is simply a strict partition (of type zero). There is a bijection between the k-Grassmannian elements w of $\widetilde{W}_\infty$ and typed k-strict partitions λ, under which the elements in $\widetilde{W}_n$ correspond to typed partitions whose diagram fits inside an $(n-k)\times(n+k-1)$ rectangle, obtained as follows. If the element w corresponds to the typed partition λ, then for each $j \geq 1$,

$$\lambda_j = \begin{cases} |w_{k+j}| + k - 1 & \text{if } w_{k+j} < 0, \\ \#\{p \leq k \,:\, |w_p| > w_{k+j}\} & \text{if } w_{k+j} > 0 \end{cases}$$

while $\mathrm{type}(\lambda) > 0$ if and only if $|w_1| > 1$, and in this case $\mathrm{type}(\lambda)$ is equal to 1 or 2 depending on whether $w_1 > 0$ or $w_1 < 0$, respectively. To any typed k-strict partition λ, we associate a finite set of pairs $\mathcal{C}(\lambda)$ and a sequence $\beta(\lambda) = \{\beta_j(\lambda)\}_{j\geq 1}$ using this bijection and the same equations (3.26) and (3.27) as in the type C case.

The Schubert polynomials indexed by k-Grassmannian elements are represented by *eta polynomials*. For any typed k-strict partition λ, the raising operator expression R^λ is defined by equation (3.28), as before. Let ℓ denote the length of λ, let $\ell_k(\lambda)$ denote the number of parts λ_i which are strictly greater than k, let $m := \ell_k(\lambda) + 1$ and $\beta := \beta(\lambda)$. If $R := \prod_{i<j} R_{ij}^{n_{ij}}$ is any raising operator, denote by $\mathrm{supp}_m(R)$ the set of all indices i and j such that $n_{ij} > 0$ and $j < m$, and set $\nu := R\lambda$. If $\mathrm{type}(\lambda) = 0$, then define

$$R \star \widehat{c}_\lambda^{\beta} = \overline{c}_\nu^{\beta} := \overline{c}_{\nu_1}^{\beta_1} \cdots \overline{c}_{\nu_\ell}^{\beta_\ell}$$

where for each $i \geq 1$,

$$\overline{c}_{\nu_i}^{\beta_i} := \begin{cases} {}^k c_{\nu_i}^{\beta_i} & \text{if } i \in \mathrm{supp}_m(R), \\ {}^k \widehat{c}_{\nu_i}^{\beta_i} & \text{otherwise.} \end{cases}$$

For any $p, r \in \mathbb{Z}$ and $s \in \{0,1\}$, define

$$a_p^s := \frac{1}{2}\,{}^k c_p + \omega_p^s, \quad b_k^s := {}^k b_k + \omega_k^s, \quad \text{and} \quad \widetilde{b}_k^s := {}^k \widetilde{b}_k + \omega_k^s,$$

where $\omega_p^s = \omega_p^s(X\,;Y,Z) := \sum_{j=1}^{p} {}^k c_{p-j}\, h_j^s(-Z)$.

If $\mathrm{type}(\lambda) > 0$ and R involves any factors R_{ij} with $i = m$ or $j = m$, then define

$$R \star \widehat{c}_\lambda^{\beta} := \overline{c}_{\nu_1}^{\beta_1} \cdots \overline{c}_{\nu_{m-1}}^{\beta_{m-1}}\, a_{\nu_m}^{\beta_m}\, c_{\nu_{m+1}}^{\beta_{m+1}} \cdots c_{\nu_\ell}^{\beta_\ell},$$

where $c_{\nu_i}^{\beta_i} := {}^k c_{\nu_i}^{\beta_i}$ for each i. If R has no such factors, then define

$$R \star \widehat{c}_\lambda^{\beta} := \begin{cases} \overline{c}_{\nu_1}^{\beta_1} \cdots \overline{c}_{\nu_{m-1}}^{\beta_{m-1}}\, b_k^{\beta_m}\, c_{\nu_{m+1}}^{\beta_{m+1}} \cdots c_{\nu_\ell}^{\beta_\ell} & \text{if } \mathrm{type}(\lambda) = 1, \\ \overline{c}_{\nu_1}^{\beta_1} \cdots \overline{c}_{\nu_{m-1}}^{\beta_{m-1}}\, \widetilde{b}_k^{\beta_m}\, c_{\nu_{m+1}}^{\beta_{m+1}} \cdots c_{\nu_\ell}^{\beta_\ell} & \text{if } \mathrm{type}(\lambda) = 2. \end{cases}$$

Following [T7], define the *double eta polynomial* $H_\lambda(X\,;Y_{(k)},Z)$ by

$$H_\lambda(X\,;Y_{(k)},Z) := 2^{-\ell_k(\lambda)} R^\lambda \star \widehat{c}_\lambda^{\beta(\lambda)}. \tag{4.27}$$

The single eta polynomial $H_\lambda(X\,;Y_{(k)})$ of [BKT3] is given by

$$H_\lambda(X\,;Y_{(k)}) := H_\lambda(X\,;Y_{(k)},0).$$

As in §3.4, we note that we are working here with the images in the ring $\Gamma'[Y,Z]$ of the eta polynomials $H_\lambda(c)$ and $H_\lambda(c\,|\,t)$ from [T6, T7].

Fix a rank n and let

$$\widetilde{\lambda}_0 := (n+k-1, n+k-2, \ldots, 2k)$$

be the typed k-strict partition associated to the k-Grassmannian element $\widetilde{w}^{(k,n)}$ of maximal length in $\widetilde{W}_n$. We deduce from Proposition 4.12 and (4.27) that

$$\mathfrak{D}_{\widetilde{w}^{(k,n)}}(X\,;Y,Z) = H_{\widetilde{\lambda}_0}(X\,;Y_{(k)},Z). \tag{4.28}$$

Using raising operators, it is shown in [T7, Prop. 5] that if λ and μ are typed k-strict partitions such that $|\lambda| = |\mu| + 1$ and $w_\lambda = s_i w_\mu$ for some simple reflection $s_i \in \widetilde{W}_\infty$, then we have

$$\partial_i H_\lambda(X\,;Y_{(k)},Z) = H_\mu(X\,;Y_{(k)},Z) \tag{4.29}$$

in $\Gamma'[Y,Z]$.[3] It follows easily from (4.28) and (4.29) that for any typed k-strict partition λ with associated k-Grassmannian element w_λ, we have

$$\mathfrak{D}_{w_\lambda}(X\,;Y,Z) = H_\lambda(X\,;Y_{(k)},Z)$$

in $\Gamma'[Y,Z]$. In particular, we recover the equality

$$\mathfrak{D}_{w_\lambda}(X\,;Y) = H_\lambda(X\,;Y_{(k)}) \tag{4.30}$$

in $\Gamma'[Y]$ from [BKT3, Prop. 6.3] for the single Schubert and eta polynomials.

4.6. **Mixed Stanley functions and splitting formulas.** For any $w \in \widetilde{W}_\infty$, the *double mixed Stanley function* $I_w(X\,;Y/Z)$ is defined by the equation

$$I_w(X\,;Y/Z) := \langle \tilde{A}(Z)D(X)A(Y), w\rangle = \sum_{uv\varpi = w} G_{u^{-1}}(-Z)E_v(X)G_\varpi(Y),$$

where the sum is over all reduced factorizations $uv\varpi = w$ with $u, \varpi \in S_\infty$. The single mixed Stanley function $I_w(X\,;Y)$ from [T5, §6] is given by setting $Z=0$ in $I_w(X\,;Y/Z)$.

Fix an element $k \in \mathbb{N}_\Box$. If $k \geq 2$, we say that an element $w \in \widetilde{W}_\infty$ is *increasing up to k* if $|w_1| < w_2 < \cdots < w_k$. Furthermore, we adopt the convention that every element of $\widetilde{W}_\infty$ is increasing up to $\Box$ and increasing up to 1. If w is increasing up to k, then there is an analogue of (3.33) for the *restricted mixed Stanley function* $I_w(X\,;Y_{(k)})$, which is obtained from $I_w(X;Y)$ after setting $y_i = 0$ for $i > k$. In this case, according to [T5, Eqn. (33)], we have

$$\mathfrak{D}_w(X\,;Y) = \sum_{v(1_k\times\varpi)=w} I_v(X\,;Y_{(k)})\mathfrak{S}_\varpi(y_{k+1}, y_{k+2}, \ldots), \tag{4.31}$$

where the sum is over all reduced factorizations $v(1_k \times \varpi) = w$ in $\widetilde{W}_\infty$ with $\varpi \in S_\infty$. Akin to §1.3 and §3.5, equation (4.31) has a double version: let $I_v(X\,;Y_{(k)}/Z_{(\ell)})$ be the power series obtained from $I_v(X\,;Y/Z)$ by setting $y_i = z_j = 0$ for all $i > k$ and $j > \ell$. Then if w is increasing up to k and w^{-1} is increasing up to ℓ, we have

$$\mathfrak{D}_w(X\,;Y,Z) = \sum \mathfrak{S}_{u^{-1}}(-Z_{>\ell})I_v(X\,;Y_{(k)}/Z_{(\ell)})\mathfrak{S}_\varpi(Y_{>k}), \tag{4.32}$$

where the sum is over all reduced factorizations $(1_\ell \times u)v(1_k \times \varpi) = w$ in $\widetilde{W}_\infty$ with $u, \varpi \in S_\infty$.

[3] The paper [T7] assumes that $k > 0$, but the proofs also work (and are simpler) when $k = 0$.

We say that an element $w \in \widetilde{W}_\infty$ is *compatible* with the sequence $\mathfrak{a} : a_1 < \cdots < a_p$ of elements of $\mathbb{N}_\Box$ if all descent positions of w are contained in $\mathfrak{a}$ (following §4.4, we assume that $a_1 \neq 1$). Let $\mathfrak{b} : b_1 < \cdots < b_q$ be a second sequence of elements of $\mathbb{N}_\Box$, and suppose that w is compatible with $\mathfrak{a}$ and w^{-1} is compatible with $\mathfrak{b}$. The notion of a reduced factorization $u_1 \cdots u_{p+q-1} = w$ compatible with $\mathfrak{a}$, $\mathfrak{b}$ and the sets of variables Y_i and Z_j for $i, j \geq 1$ are defined exactly as in §3.5.

Proposition 4.13. *Suppose that w and w^{-1} are compatible with $\mathfrak{a}$ and $\mathfrak{b}$, respectively. Then the Schubert polynomial $\mathfrak{D}_w(X\,;Y,Z)$ satisfies*

$$\mathfrak{D}_w = \sum G_{u_1}(0/Z_q) \cdots G_{u_{q-1}}(0/Z_2) I_{u_q}(X\,;Y_1/Z_1) G_{u_{q+1}}(Y_2) \cdots G_{u_{p+q-1}}(Y_p)$$

summed over all reduced factorizations $u_1 \cdots u_{p+q-1} = w$ compatible with $\mathfrak{a}$, $\mathfrak{b}$.

Proof. The result is shown by using (4.32) and iterating the identity (1.7). $\Box$

If w is increasing up to k, then the following generalization of equation (4.3) holds (see [T5, Eqn. (32)]):

$$I_w(X\,;Y_{(k)}) = \sum_{\lambda\,:\,|\lambda|=\ell(w)} d_\lambda^w\, H_\lambda(X;Y_{(k)}), \tag{4.33}$$

where the sum is over typed k-strict partitions λ with $|\lambda| = \ell(w)$. The *mixed Stanley coefficients* d_λ^w in (4.33) are nonnegative integers which have a combinatorial interpretation, as in §3.5. The proof of (4.33) in [T5] uses (4.30), and is similar to the type C case.

Assume that w is increasing up to k and w^{-1} is increasing up to ℓ. Then the (restricted) double mixed Stanley function $I_w(X\,;Y_{(k)}/Z_{(\ell)})$ satisfies

$$I_w(X\,;Y_{(k)}/Z_{(\ell)}) = \sum_{uv=w} G_{u^{-1}}(-Z_{(\ell)}) I_v(X\,;Y_{(k)}) \tag{4.34}$$

$$= \sum_{uv=w^{-1}} G_{u^{-1}}(Y_{(k)}) I_v(X\,;-Z_{(\ell)}), \tag{4.35}$$

where the sums are over reduced factorizations as shown, with $u \in S_\infty$. We can now use equations (1.9) and (4.33) in (4.34) and (4.35) to obtain two expansions of $I_w(X\,;Y_{(k)}/Z_{(\ell)})$ as a positive sum of products of Schur S-polynomials with eta polynomials, as in Example 3.10.

Theorem 4.14 ([T5], Cor. 3)**.** *Suppose that w is compatible with $\mathfrak{a}$ and w^{-1} is compatible with $\mathfrak{b}$, where $b_1 = \Box$. Then we have*

$$\mathfrak{D}_w = \sum_{\underline{\lambda}} g_{\underline{\lambda}}^w\, s_{\lambda^1}(0/Z_q) \cdots s_{\lambda^{q-1}}(0/Z_2) H_{\lambda^q}(X\,;Y_1) s_{\lambda^{q+1}}(Y_2) \cdots s_{\lambda^{p+q-1}}(Y_p)$$

summed over all sequences of partitions $\underline{\lambda} = (\lambda^1, \ldots, \lambda^{p+q-1})$ with λ^q a_1-strict and typed, where

$$g_{\underline{\lambda}}^w := \sum_{u_1 \cdots u_{p+q-1} = w} a_{\lambda^1}^{u_1} \cdots a_{\lambda^{q-1}}^{u_{q-1}} d_{\lambda^q}^{u_q} a_{\lambda^{q+1}}^{u_{q+1}} \cdots a_{\lambda^{p+q-1}}^{u_{p+q-1}}$$

summed over all reduced factorizations $u_1 \cdots u_{p+q-1} = w$ compatible with $\mathfrak{a}$, $\mathfrak{b}$.

Proof. This follows from Proposition 4.13 by using equations (1.9) and (4.33). $\Box$

5. Geometrization

In this section, we discuss the precise way in which the Schubert polynomials given in the previous sections represent degeneracy loci of vector bundles in the sense of [Fu2]. This has been addressed in earlier work (see [T5, §4] and [T6, §6, 7]), but our aim here is to provide some more detailed historical comments, which include the author's papers [T2, T3]. We restrict attention to the symplectic case, as the orthogonal types are analogous to type C, and the situation in type A has been understood since [Fu1]. Throughout the section $\mathrm{X} := (\mathrm{x}_1, \mathrm{x}_2, \ldots)$ and $\mathrm{Y} := (\mathrm{y}_1, \mathrm{y}_2, \ldots)$ will denote two sequences of commuting independent variables, and for every integer $n \geq 1$, we set $\mathrm{X}_n := (\mathrm{x}_1, \ldots, \mathrm{x}_n)$ and $\mathrm{Y}_n := (\mathrm{y}_1, \ldots, \mathrm{y}_n)$.

Consider a vector bundle $E \to M$ of rank $2n$ over a smooth algebraic variety M, equipped with an everywhere nondegenerate skew-symmetric form $E \otimes E \to \mathbb{C}$. Assume that we are given two complete flags of subbundles of E

$$E_\bullet \,:\, 0 \subset E_1 \subset \cdots \subset E_{2n} = E \quad \text{and} \quad F_\bullet \,:\, 0 \subset F_1 \subset \cdots \subset F_{2n} = E \tag{5.1}$$

with $\operatorname{rank} E_i = \operatorname{rank} F_i = i$ for each i, while $E_{n+i} = E_{n-i}^{\perp}$ and $F_{n+i} = F_{n-i}^{\perp}$ for $0 \leq i < n$. For any w in the Weyl group W_n, we have the degeneracy locus

$$\mathfrak{X}_w := \{x \in M \mid \dim(E_i(x) \cap F_j(x)) \geq d_w(i,j) \ \ \forall\, i \in [1,n], j \in [1,2n]\}, \tag{5.2}$$

where $d_w(i,j) \in \mathbb{Z}$ and the inequalities in (5.2) are exactly those which define the Schubert variety $X_w(F_\bullet)$ in the flag variety $\mathrm{IF}_n := \mathrm{Sp}_{2n}/B$ (the precise values $d_w(i,j)$ are given in [T6, §6.2]). We assume that $\mathfrak{X}_w$ has pure codimension $\ell(w)$ in M, and seek a formula which expresses the cohomology class $[\mathfrak{X}_w] \in \mathrm{H}^*(M)$ as a universal polynomial in the Chern classes of the vector bundles E_i and F_j.

When the vector bundles F_j are trivial, the answer to the above *degeneracy locus problem* coincides with the answer to the *Giambelli problem* for $\mathrm{H}^*(\mathrm{IF}_n)$, which amounts to a theory of (single) *symplectic Schubert polynomials*. In this setting, the E_i are the universal (or tautological) vector bundles over $M = \mathrm{IF}_n$. The cohomology ring of IF_n has a standard Borel presentation [Bo] as a quotient ring

$$\mathrm{H}^*(\mathrm{IF}_n, \mathbb{Z}) \cong \mathbb{Z}[\mathrm{x}_1, \ldots, \mathrm{x}_n]/\mathrm{J}_n, \tag{5.3}$$

where the variables x_i represent the characters of the Borel subgroup B, or the Chern roots of the dual of the Lagrangian subbundle E_n, and J_n denotes the ideal generated by the W_n-invariant polynomials of positive degree. The aim of a theory of Schubert polynomials is to provide a combinatorially explicit and natural set of polynomial representatives $\{\mathfrak{C}_w(\mathrm{X}_n)\}_{w \in W_n}$ for the Schubert classes $\{[X_w]\}_{w \in W_n}$ in the presentation (5.3) of $\mathrm{H}^*(\mathrm{IF}_n, \mathbb{Z})$.

Among the many desirable attributes of the Schubert polynomials $\mathfrak{C}_w(\mathrm{X}_n)$, the *stability property* is perhaps the most important. This states that we have

$$\mathfrak{C}_{j_n(w)}(\mathrm{x}_1, \ldots, \mathrm{x}_n, 0) = \mathfrak{C}_w(\mathrm{x}_1, \ldots, \mathrm{x}_n), \quad \forall\, w \in W_n,$$

where $j_n : W_n \hookrightarrow W_{n+1}$ is the natural inclusion map of Weyl groups. The significance of this property was already recognized in the work of Lascoux and Schützenberger [LS] on type A Schubert polynomials, where – together with the fact that they represent Schubert classes – it completely characterizes them.

The inclusions j_n induce surjections

$$\cdots \to \mathrm{H}^*(\mathrm{IF}_{n+1}, \mathbb{Z}) \to \mathrm{H}^*(\mathrm{IF}_n, \mathbb{Z}) \to \cdots \tag{5.4}$$

and the inverse limit of the system (5.4) in the category of graded rings is the *stable cohomology ring* $\mathbb{H}(\mathrm{IF})$. The stability property implies that the symplectic Schubert polynomials lift to give representatives $\mathfrak{C}_w(\mathrm{X})$ of the *stable Schubert classes* $\sigma_w := \varprojlim [X_w]$, one for every $w \in W_\infty$. Unlike the situation in type A, the $\mathfrak{C}_w(\mathrm{X})$ will no longer be polynomials in X, but formal power series (see Example 5.1 below). Moreover, a special role is played by the subring of $\mathbb{H}(\mathrm{IF})$ invariant under the action of the symmetric group S_∞, whose elements are represented by symmetric power series, and which is isomorphic to the stable cohomology ring $\mathbb{H}(\mathrm{LG})$ of the *Lagrangian Grassmannian* $\mathrm{LG}(n, 2n)$.

The Giambelli problem for the cohomology ring of $\mathrm{LG}(n, 2n)$ was solved by Pragacz [P] by using the theory of Schur Q-functions, and the resulting isomorphism between $\mathrm{H}^*(\mathrm{LG}(n, 2n), \mathbb{Z})$ and a certain quotient of the ring Γ of Schur Q-functions was further studied by Józefiak [Jo]. Let Λ denote the ring of symmetric functions in the variables X, so that $\Lambda = \mathbb{Z}[e_1(\mathrm{X}), e_2(\mathrm{X}), \ldots]$, and let I be the ideal of Λ generated by the homogeneous symmetric functions in $\mathrm{X}^2 := (\mathrm{x}_1^2, \mathrm{x}_2^2, \ldots)$ of positive degree. According to [Jo, Cor. 2.3], the surjective map $\eta : \Lambda \to \Gamma$ with $\eta(e_i(\mathrm{X})) := q_i(X)$ for all $i \geq 1$ induces an isomorphism $\Lambda/\mathrm{I} \cong \Gamma$.

Define a map $\phi_n : \mathbb{Z}[\mathrm{X}] \to \mathbb{Z}[\mathrm{X}_n]$ by $\mathrm{x}_i \mapsto \mathrm{x}_i$ for $i \leq n$, while $\mathrm{x}_i \mapsto 0$ for $i > n$. If $\Lambda_n := \mathbb{Z}[e_1(\mathrm{X}_n), \ldots, e_n(\mathrm{X}_n)] = \mathbb{Z}[\mathrm{X}_n]^{S_n}$ is the ring of symmetric polynomials in X_n, then ϕ_n induces an homonymous map $\Lambda \to \Lambda_n$. Setting $\mathrm{I}_n := \phi_n(\mathrm{I}) = \Lambda_n(e_1(\mathrm{X}_n^2), \ldots, e_n(\mathrm{X}_n^2))$, we then have a commutative diagram of rings

$$\begin{array}{ccccc} \Gamma & \xrightarrow{\pi} & \Lambda/\mathrm{I} & \longrightarrow & \mathbb{H}(\mathrm{LG}) \\ & & \downarrow{\scriptstyle \phi_n} & & \downarrow \\ & & \Lambda_n/\mathrm{I}_n & \xrightarrow{\psi_0} & \mathrm{H}^*(\mathrm{LG}(n, 2n)) \end{array}$$

where the horizontal arrows are isomorphisms. The map ψ_0 sends $e_i(\mathrm{X}_n)$ to the i-th Chern class $c_i(E/E_n)$ of the universal quotient bundle over $\mathrm{LG}(n, 2n)$. The resulting surjection $\Gamma \to \mathrm{H}^*(\mathrm{LG}(n, 2n))$ maps $Q_\lambda(X)$ to the Schubert class $[X_\lambda]$, for any strict partition λ with $\lambda_1 \leq n$, and to zero, otherwise.

Since the combinatorial theory of Schur Q-functions was well understood and analogous to the type A theory of Schur S-functions, the above picture provided a satisfactory way to do classical Schubert calculus on LG. The study of related problems in the theory of degeneracy loci [PR, LP, KT1], Arakelov theory [T1], and quantum cohomology [KT2], however, required representatives for the Schubert classes in the Borel presentation of $\mathrm{H}^*(\mathrm{LG}(n, 2n))$, and hence in the ring Λ_n. The answer was provided by Pragacz and Ratajski's theory [PR] of $\widetilde{Q}$-polynomials $\widetilde{Q}_\lambda(\mathrm{X}_n)$, which were extended to the $\widetilde{Q}$-functions $\widetilde{Q}_\lambda(\mathrm{X})$ in [T2, §1.1]. For each strict partition λ, $\widetilde{Q}_\lambda(\mathrm{X})$ and $\widetilde{Q}_\lambda(\mathrm{X}_n)$ are defined by the raising operator expressions

$$\widetilde{Q}_\lambda(\mathrm{X}) := R^\infty\, e_\lambda(\mathrm{X}) \quad \text{and} \quad \widetilde{Q}_\lambda(\mathrm{X}_n) := \phi_n(\widetilde{Q}_\lambda(\mathrm{X})) = R^\infty\, e_\lambda(\mathrm{X}_n),$$

where as usual $e_\lambda := \prod_i e_{\lambda_i}$. The *geometrization* of the Schur Q-functions $Q_\lambda(X)$ is then displayed in the diagram

$$\begin{array}{ccccc} Q_\lambda(X) & \overset{\pi}{\longmapsto} & \widetilde{Q}_\lambda(\mathrm{X}) & \longmapsto & \sigma_\lambda \\ & & \downarrow{\scriptstyle \phi_n} & & \downarrow \\ & & \widetilde{Q}_\lambda(\mathrm{X}_n) & \overset{\psi_0}{\longmapsto} & [X_\lambda]. \end{array}$$

In other words, by geometrization we mean the choice of substitution $Q_\lambda(X) \mapsto \widetilde{Q}_\lambda(\mathrm{X})$ shown above, which lifts the ring homomorphisms

$$\Gamma \xrightarrow{\pi} \Lambda/\mathrm{I} \xrightarrow{\phi_n} \Lambda_n/\mathrm{I}_n$$

to maps of abelian groups

$$\Gamma \xrightarrow{\pi} \Lambda \xrightarrow{\phi_n} \Lambda_n.$$

The next step in the story was to extend the above picture to the entire Weyl group W_n, and thus obtain a theory of symplectic Schubert polynomials $\mathfrak{C}_w(\mathrm{X}_n)$ for $\mathrm{H}^*(\mathrm{IF}_n, \mathbb{Z})$. In a fundamental paper which built on the work of Lascoux-Schützenberger [LS] and Pragacz [P], Billey and Haiman [BH] found the combinatorially explicit family of type C Schubert polynomials $\mathfrak{C}_w(X\,;Y)$ of §3.1. These objects are actually formal power series, realized as nonnegative integer linear combinations of products $Q_\lambda(X)\mathfrak{S}_\varpi(Y)$ of Schur Q-functions and type A Schubert polynomials. The $\mathfrak{C}_w(X\,;Y)$ for $w \in W_\infty$ form a $\mathbb{Z}$-basis of a ring $\Gamma[Y]$ isomorphic to $\mathbb{H}(\mathrm{IF})$, and map to the stable Schubert classes σ_w under this isomorphism.

The problem with the Billey-Haiman power series $\mathfrak{C}_w(X\,;Y)$ was that they were not related in [BH] to the Borel presentation (5.3) in a way that retained combinatorial control over their coefficients. The λ-ring substitution used in [BH, §2] involved the odd power sums, and led to Schubert polynomials in the root variables x_i which were quite complicated (see [FK, §7] for a discussion of this). In 2006, motivated in part by an application to arithmetic intersection theory, the author resolved this issue by constructing a family of symplectic Schubert polynomials $\mathfrak{C}_w(\mathrm{X}_n)$ which are a *geometrization* of the $\mathfrak{C}_w(X\,;Y)$, in the same sense that the polynomials $\widetilde{Q}_\lambda(\mathrm{X}_n)$ are a geometrization of the power series $Q_\lambda(Y)$.

Here are the details of this work, which eventually appeared in [T2]. Let J be the ideal of $\Lambda[\mathrm{X}]$ generated by the elementary symmetric functions $e_p(\mathrm{X}^2)$ for $p \geq 1$. The map $\phi_n : \Lambda[\mathrm{X}] \to \mathbb{Z}[\mathrm{X}_n]$ sends J to the ideal $\mathrm{J}_n = (e_1(\mathrm{X}_n^2), \ldots, e_n(\mathrm{X}_n^2))$. Define an isomorphism $\pi : \Gamma[Y] \to \Lambda[\mathrm{X}]/\mathrm{J}$ by setting $\pi(q_i(X)) := e_i(\mathrm{X})$ and $\pi(y_i) := -\mathrm{x}_i$ for all $i \geq 1$. We then have a diagram

$$\begin{array}{ccccc} \Gamma[Y] & \xrightarrow{\pi} & \Lambda[\mathrm{X}]/\mathrm{J} & \longrightarrow & \mathbb{H}(\mathrm{IF}) \\ & & \downarrow{\scriptstyle \phi_n} & & \downarrow \\ & & \mathbb{Z}[\mathrm{X}_n]/\mathrm{J}_n & \xrightarrow{\psi} & \mathrm{H}^*(\mathrm{IF}_n) \end{array}$$

where the horizontal arrows are again ring isomorphisms. The map ψ is determined by $\psi(\mathrm{x}_i) = -c_1(E_{n+1-i}/E_{n-i})$ for $1 \leq i \leq n$. Given any $w \in W_n$, apply equations

(3.3) and (3.6) to write

$$\mathfrak{C}_w(X\,;Y) = \sum_{v,\varpi,\lambda} e^v_\lambda\, Q_\lambda(X)\mathfrak{S}_\varpi(Y)$$

where the sum is over all reduced factorizations $v\varpi = w$ and strict partitions λ such that $\varpi \in S_n$ and $|\lambda| = \ell(v)$. Following [T2, §2.2], define the symplectic Schubert polynomials $\mathfrak{C}_w(\mathrm{X}_n)$ and power series $\mathfrak{C}_w(\mathrm{X})$ by the equations

$$\mathfrak{C}_w(\mathrm{X}) := \sum_{v,\varpi,\lambda} e^v_\lambda\, \widetilde{Q}_\lambda(\mathrm{X})\mathfrak{S}_\varpi(-\mathrm{X}_n) \ \text{ and } \ \mathfrak{C}_w(\mathrm{X}_n) := \phi_n(\mathfrak{C}_w(\mathrm{X})).$$

The *geometrization* of the Schubert polynomials $\mathfrak{C}_w(X\,;Y)$ is then displayed in the diagram

$$\begin{array}{ccccc} \mathfrak{C}_w(X\,;Y) & \overset{\pi}{\longmapsto} & \mathfrak{C}_w(\mathrm{X}) & \longmapsto & \sigma_w \\ & & \big\downarrow \phi_n & & \big\downarrow \\ & & \mathfrak{C}_w(\mathrm{X}_n) & \overset{\psi}{\longmapsto} & [X_w]. \end{array}$$

Example 5.1. The linear Schubert polynomials $\mathfrak{C}_{s_i}$ are indexed by the simple reflections s_i in W_∞. For each $i \in \mathbb{N}_0$, we have

$$\mathfrak{C}_{s_i}(X\,;Y) = q_1(X) + \mathfrak{S}_{s_i}(Y) = 2\left(\sum_{j=1}^{\infty} x_j\right) + (y_1 + \cdots + y_i)$$

for the Billey-Haiman polynomials, while

$$\mathfrak{C}_{s_i}(\mathrm{X}) = \sum_{j=i+1}^{\infty} \mathrm{x}_j \quad \text{and} \quad \mathfrak{C}_{s_i}(\mathrm{X}_n) = \begin{cases} \mathrm{x}_{i+1} + \cdots + \mathrm{x}_n & \text{if } i < n, \\ 0 & \text{otherwise.} \end{cases}$$

There remained one missing ingredient to fully solve the problem of Schubert polynomials in the classical Lie types: define double versions of the Billey-Haiman polynomials, and extend the above picture to that setting. Fortunately, there was progress in this direction, as in 2005 Ikeda [I] had shown how Ivanov's factorial Schur Q-functions [Iv3] may be used to represent the torus-equivariant Schubert classes in the equivariant cohomology of the Lagrangian Grassmannian. At the March 2007 workshop on Schubert calculus in Banff, the author suggested to Ikeda that he should work on creating a double version of the Billey-Haiman theory. Ikeda was also informed about the author's theory of symplectic Schubert polynomials $\mathfrak{C}_w(\mathrm{X}_n)$, and asked to include an extension of the substitution $\mathfrak{C}_w(X\,;Y) \mapsto \mathfrak{C}_w(\mathrm{X}_n)$ to the double case, as it was important for geometric applications. In 2008, the work [IMN1] was announced, which used localization techniques as in [I, IN] to construct the required theory of polynomials $\mathfrak{C}_w(X\,;Y,Z)$. Moreover, the information necessary to extend the author's geometrization of the $\mathfrak{C}_w(X\,;Y)$ to equivariant cohomology was provided in [IMN1, §10]. This was done explicitly in 2009, with the announcement of our paper [T5], and simultaneously generalized, to deal with the degeneracy loci coming from any isotropic partial flag variety.

Following [T5, §4], to describe how the $\mathfrak{C}_w(X\,;Y,Z)$ represent degeneracy loci of vector bundles, recall from [Gr] and [T6] that these loci and their cohomology classes pull back from the Borel mixing space $BM_n := BB \times_{B\,\mathrm{Sp}_{2n}} BB$, which is the universal case of (5.2) and the degeneracy locus problem. Let

$$\mathbb{H}(BM) := \varprojlim \mathrm{H}^*(BM_n, \mathbb{Z})$$

be the stable cohomology ring of BM_n. Define the supersymmetric functions $e_p(\mathrm{X}/\mathrm{Y})$ for $p \geq 0$ by

$$e_p(\mathrm{X}/\mathrm{Y}) := \sum_{j=0}^{p} e_j(\mathrm{X}) h_{p-j}(\mathrm{Y}),$$

set $\widetilde{\Lambda} := \mathbb{Z}[e_1(\mathrm{X}/\mathrm{Y}), e_2(\mathrm{X}/\mathrm{Y}), \ldots]$, and let $\widetilde{\mathrm{J}}$ be the ideal of $\widetilde{\Lambda}[\mathrm{X},\mathrm{Y}]$ generated by the fundamental relations

$$e_p^2(\mathrm{X}/\mathrm{Y}) + 2\sum_{j=1}^{p}(-1)^j e_{p+j}(\mathrm{X}/\mathrm{Y}) e_{p-j}(\mathrm{X}/\mathrm{Y})$$

for all $p \geq 1$. Define an isomorphism

$$\widetilde{\pi} : \Gamma[Y,Z] \to \widetilde{\Lambda}[\mathrm{X},\mathrm{Y}]/\widetilde{\mathrm{J}}$$

by setting $\widetilde{\pi}(q_i(X)) := e_i(\mathrm{X}/\mathrm{Y})$, $\widetilde{\pi}(y_i) := -\mathrm{x}_i$, and $\widetilde{\pi}(z_i) := \mathrm{y}_i$, for each $i \geq 1$. Moreover, let $\widetilde{\phi}_n : \widetilde{\Lambda}[\mathrm{X},\mathrm{Y}] \to \mathbb{Z}[\mathrm{X}_n, \mathrm{Y}_n]$ be the map determined by $\mathrm{x}_i \mapsto \mathrm{x}_i$ and $\mathrm{y}_i \mapsto \mathrm{y}_i$ for $i \leq n$, while $\mathrm{x}_i \mapsto 0$ and $\mathrm{y}_i \mapsto 0$ for $i > n$, and set $\widetilde{\mathrm{J}}_n := \phi_n(\widetilde{\mathrm{J}}) \subset \mathbb{Z}[\mathrm{X}_n, \mathrm{Y}_n]$. (It is not hard to show that $\widetilde{\mathrm{J}}_n$ is equal to the ideal of $\mathbb{Z}[\mathrm{X}_n, \mathrm{Y}_n]$ generated by the differences $e_p(\mathrm{X}_n^2) - e_p(\mathrm{Y}_n^2)$ for $1 \leq p \leq n$.) We then have a commutative diagram of rings

$$\begin{array}{ccccc}
\Gamma[Y,Z] & \xrightarrow{\widetilde{\pi}} & \widetilde{\Lambda}[\mathrm{X},\mathrm{Y}]/\widetilde{\mathrm{J}} & \longrightarrow & \mathbb{H}(BM) \\
 & & \Big\downarrow{\scriptstyle \widetilde{\phi}_n} & & \Big\downarrow \\
 & & \mathbb{Z}[\mathrm{X}_n, \mathrm{Y}_n]/\widetilde{\mathrm{J}}_n & \xrightarrow{\widetilde{\psi}} & \mathrm{H}^*(BM_n, \mathbb{Z})
\end{array}$$

extending the previous ones, where again the horizontal maps are isomorphisms. The map $\widetilde{\psi}$ satisfies $\widetilde{\psi}(\mathrm{x}_i) = -c_1(E_{n+1-i}/E_{n-i})$ and $\widetilde{\psi}(\mathrm{y}_i) = -c_1(F_{n+1-i}/F_{n-i})$ for each i with $1 \leq i \leq n$.

Given $w \in W_n$, apply equations (3.3) and (3.6) to write

$$\mathfrak{C}_w(X\,; Y, Z) = \sum_{u,v,\varpi,\lambda} e^v_\lambda\, \mathfrak{S}_{u^{-1}}(-Z) Q_\lambda(X) \mathfrak{S}_\varpi(Y)$$

where the sum is over all reduced factorizations $uv\varpi = w$ and strict partitions λ such that $u, \varpi \in S_n$ and $|\lambda| = \ell(v)$. Define the *supersymmetric $\widetilde{Q}$-function* $\widetilde{Q}_\lambda(\mathrm{X}/\mathrm{Y})$ by the equation

$$\widetilde{Q}_\lambda(\mathrm{X}/\mathrm{Y}) := R^\infty\, e_\lambda(\mathrm{X}/\mathrm{Y}).$$

and the *double symplectic Schubert polynomials* $\mathfrak{C}_w(\mathrm{X}_n, \mathrm{Y}_n)$ and power series $\mathfrak{C}_w(\mathrm{X}, \mathrm{Y})$ by the equations

$$\mathfrak{C}_w(\mathrm{X}, \mathrm{Y}) := \sum_{u,v,\varpi,\lambda} e^v_\lambda\, \mathfrak{S}_{u^{-1}}(-\mathrm{Y}_n) \widetilde{Q}_\lambda(\mathrm{X}/\mathrm{Y}) \mathfrak{S}_\varpi(-\mathrm{X}_n)$$

and $\mathfrak{C}_w(\mathrm{X}_n, \mathrm{Y}_n) := \widetilde{\phi}_n(\mathfrak{C}_w(\mathrm{X}/\mathrm{Y}))$. The *geometrization* of the double Schubert polynomials $\mathfrak{C}_w(X\,;Y,Z)$ is exhibited in the diagram

$$\begin{array}{ccccc} \mathfrak{C}_w(X\,;Y,Z) & \overset{\widetilde{\pi}}{\longmapsto} & \mathfrak{C}_w(\mathrm{X},\mathrm{Y}) & \longmapsto & \varprojlim[\mathfrak{X}_w] \\ & & \downarrow{\scriptstyle \widetilde{\phi}_n} & & \downarrow \\ & & \mathfrak{C}_w(\mathrm{X}_n,\mathrm{Y}_n) & \overset{\widetilde{\psi}}{\longmapsto} & [\mathfrak{X}_w]. \end{array}$$

As before, we observe that the substitutions $Q_\lambda(X) \mapsto \widetilde{Q}_\lambda(\mathrm{X}/\mathrm{Y})$, $y_i \mapsto -\mathrm{x}_i$, and $z_i \mapsto \mathrm{y}_i$ lift the ring homomorphisms

$$\Gamma[Y,Z] \xrightarrow{\widetilde{\pi}} \widetilde{\Lambda}[\mathrm{X},\mathrm{Y}]/\widetilde{\mathrm{J}} \xrightarrow{\widetilde{\phi}_n} \mathbb{Z}[\mathrm{X}_n,\mathrm{Y}_n]/\widetilde{\mathrm{J}}_n$$

to maps of abelian groups

$$\Gamma[Y,Z] \xrightarrow{\widetilde{\pi}} \widetilde{\Lambda}[\mathrm{X},\mathrm{Y}] \xrightarrow{\widetilde{\phi}_n} \mathbb{Z}[\mathrm{X}_n,\mathrm{Y}_n].$$

Since the variables $\{-\mathrm{x}_i\}$ and $\{-\mathrm{y}_i\}$ for $1 \leq i \leq n$ represent the Chern roots of the isotropic vector bundles in (5.1), it is now a simple matter to translate the formulas in this paper into Chern class formulas for degeneracy loci. For each $r \geq 0$, define $c_r(E - E_i - F_j)$ by the equation of total Chern classes

$$c(E - E_i - F_j) := c(E)c(E_i)^{-1}c(F_j)^{-1}.$$

Then the *geometrization map* $\omega_n := \widetilde{\psi}\widetilde{\phi}_n\widetilde{\pi}$ sends $q_r(X)$ to $c_r(E-E_n-F_n)$ for each $r \geq 0$, and $Q_\lambda(X)$ to $Q_\lambda(E - E_n - F_n)$ for every strict partition λ.

More generally, following [T5, Thm. 3], the substitution which maps the theta polynomial $\Theta_\lambda(X\,;Y_{(k)})$ to $\Theta_\lambda(E-E_{n-k}-F_n)$ for each k-strict partition λ is applied to treat the degeneracy loci which come from any symplectic partial flag variety (see also [T6, Remark 4]). Computing the image of formula (3.38), we thus obtain that the class of the degeneracy locus $\mathfrak{X}_w$ in $\mathrm{H}^*(M)$ is equal to

$$\sum_{\underline{\lambda}} f^w_{\underline{\lambda}}\, s_{\lambda^1}(F_{n+b_{q-1}} - F_{n+b_q}) \cdots \Theta_{\lambda^q}(E - E_{n-a_1} - F_n) \cdots s_{\lambda^{p+q-1}}(E_{n-a_{p-1}} - E_{n-a_p})$$

with the coefficients $f^w_{\underline{\lambda}}$ given by equation (3.39).

The polynomials ${}^k c^r_p$ defined in (3.15) are particularly useful to work with, as we have the Chern class equation

$$\omega_n({}^k c^r_p) = c_p(E - E_{n-k} - F_{n+r})$$

(compare with [TW, Eqn. (31)]). For the top Schubert polynomial $\mathfrak{C}_{w_0}(X\,;Y,Z)$, equation (3.16) maps to the Pfaffian formula

$$[\mathfrak{X}_{w_0}] = Q_{\delta_n+\delta_{n-1}}(E - E_{(1,2,\ldots,n)} - F_{(1,2,\ldots,n)}) \tag{5.5}$$

in $\mathrm{H}^*(M)$. Following [TW, Cor. 1], the Chern polynomial in (5.5) is defined as the image of the polynomial $R^\infty c_{\delta_n+\delta_{n-1}}$ under the $\mathbb{Z}$-linear map which sends the noncommutative monomial c_α to $\prod_j c_{\alpha_j}(E - E_j - F_j)$, for every integer sequence α. The geometric substitutions of this section can thus be used to relate some of the equations for Schubert polynomials found in the present paper to the Chern class formulas in [Ka, AF1, AF2] and elsewhere.

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