# STANDARD CONJECTURES FOR THE ARITHMETIC GRASSMANNIAN $G(2,N)$ AND RACAH POLYNOMIALS

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Abstract. We prove the arithmetic Hodge index and hard Lefschetz conjectures for the Grassmannian $G = G(2,N)$ parametrizing lines in projective space, for the natural arithmetic Lefschetz operator defined via the Plücker embedding of $G$ in projective space. The analysis of the Hodge index inequality involves estimates on values of certain Racah polynomials.

## 0. Introduction

Let $X$ be an arithmetic variety, by which we mean a regular, projective and flat scheme over $\operatorname{Spec}\mathbb{Z}$, of absolute dimension $d+1$. Assume that $\overline{M} = (M, \|\cdot\|)$ is a hermitian line bundle on $X$ which is arithmetically ample, in the sense of [Z] and [So, §5.2]. For each $p \geqslant 0$ the line bundle $\overline{M}$ defines an arithmetic Lefschetz operator

$$\begin{array}{rccc} \widehat{L}: & \widehat{CH}^p(X)_{\mathbb{R}} & \longrightarrow & \widehat{CH}^{p+1}(X)_{\mathbb{R}} \\ & \alpha & \longmapsto & \alpha \cdot \widehat{c}_1(\overline{M}). \end{array}$$

Here $\widehat{CH}^*(X)_{\mathbb{R}}$ is the real arithmetic Chow ring of [GS] and $\widehat{c}_1(\overline{M})$ is the arithmetic first Chern class of $\overline{M}$.

In this setting, Gillet and Soulé [GS] proposed arithmetic analogues of Grothendieck's standard conjectures [Gr] on algebraic cycles. A more precise version of the conjectures was formulated in [So, §5.3]; assuming $2p \leqslant d+1$, the statement is

**Conjecture 1.** (a) *(Hard Lefschetz) The map*

$$\widehat{L}^{d+1-2p} : \widehat{CH}^p(X)_{\mathbb{R}} \longrightarrow \widehat{CH}^{d+1-p}(X)_{\mathbb{R}}$$

*is an isomorphism;*
(b) *(Hodge index) If the nonzero $x \in \widehat{CH}^p(X)_{\mathbb{R}}$ satisfies $\widehat{L}^{d+2-2p}(x) = 0$, then*

$$(-1)^p \, \widehat{\deg}(x \, \widehat{L}^{d+1-2p}(x)) > 0.$$

Notice that Conjecture 1 (for all $p$) implies that the intersection pairing

$$\widehat{CH}^p(X)_{\mathbb{R}} \otimes \widehat{CH}^{d+1-p}(X)_{\mathbb{R}} \longrightarrow \mathbb{R}$$

is nondegenerate. When $d = 1$, the Conjecture follows from the Hodge index theorem for arithmetic surfaces due to Faltings [Fa] and Hriljac [Hr].

We study these conjectures when $X = G(r,N)$ is the arithmetic Grassmannian, parametrizing $r$-dimensional subspaces of an $(r+N)$-dimensional vector space, over any field, and $\overline{M} = \overline{\mathcal{O}}(1)$ is the very ample line bundle giving the Plücker

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embedding, equipped with its natural hermitian metric. The latter is the metric induced from the standard metric on complex affine space, so that the first Chern form $c_1(\overline{M})$ on $X(\mathbb{C})$ is $U(r+N)$-invariant and dual to the hyperplane class.

Our main result is that, for $G(r,N)$ and $\overline{\mathcal{O}}(1)$, Conjecture 1 holds when $r=2$. For projective space ($r=1$) this was shown by Künnemann [Ku]. Moreover, it is proved in [KM] and [Ta] that Conjecture 1 holds for $G(r,N)$ after a suitable scaling of the metric on $\overline{\mathcal{O}}(1)$. To obtain the precise result for $G(2,N)$ we use the arithmetic Schubert calculus of [T] and linear algebra to reduce the problem to combinatorial estimates. In this case the inequality in part (b) asserts the positivity of a linear combination of harmonic numbers with coefficients certain Racah polynomials. The latter are a system of orthogonal polynomials in a discrete variable introduced by Wilson [Wi] [AW] which generalize the classical Racah coefficients or 6-$j$ symbols [Ra] of quantum physics.

The results of Künnemann [Ku] show that each statement in Conjecture 1 (for given $X$, $p$ and $\overline{M}$) is true if and only if it holds when $\widehat{CH}^p(X)_{\mathbb{R}}$ is replaced by the Arakelov subgroup $CH^p(\overline{X})_{\mathbb{R}}$ associated to the Kähler form $c_1(\overline{M})$. We therefore restrict attention to this subgroup throughout the paper. In Section 1 we study arithmetic Lefschetz theory for varieties which admit a cellular decomposition and derive a cohomological criterion (Corollary 1) which we use to check Conjecture 1. This criterion does not suffice to check the Hodge index inequality on more general Grassmannians. In Section 2 we apply classical and arithmetic Schubert calculus to reduce the conjecture for $G(2,N)$ to estimates for a class of Racah polynomials. The required bounds for these polynomials are established in Section 3.

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## 1. Arithmetic standard conjectures on cellular spaces

We study Conjecture 1 for arithmetic varieties $X$ which have a cellular decomposition over $\operatorname{Spec}\mathbb{Z}$, in the sense of [Fu, Ex. 1.9.1]; the Grassmannian $G(r,N)$ is a typical example. See [KM] for more information on these spaces and an approach to a weaker version of the conjecture. Recall that for each $p$ the class map

$$\mathrm{cl}:\, CH^p(X)_{\mathbb{R}} \longrightarrow H^{p,p}(X_{\mathbb{R}})$$

is an isomorphism of the real Chow ring $CH^p(X)_{\mathbb{R}} = CH^p(X)\otimes_{\mathbb{Z}}\mathbb{R}$ with the space $H^{p,p}(X_{\mathbb{R}})$ of real harmonic differential $(p,p)$-forms on $X(\mathbb{C})$. We denote by

$$\begin{array}{cccc} L: & CH^p(X)_{\mathbb{R}} & \longrightarrow & CH^{p+1}(X)_{\mathbb{R}} \\ & \alpha & \longmapsto & \alpha\cdot c_1(M) \end{array}$$

the classical Lefschetz operator associated to an ample line bundle $M$ over $X$.

Let us equip the holomorphic line bundle $M(\mathbb{C})$ with a smooth positive hermitian metric, invariant under complex conjugation, to obtain a hermitian line bundle $\overline{M}$. As we have indicated, to check Conjecture 1 for the operator $\widehat{L}(\alpha)=\alpha\cdot\widehat{c}_1(\overline{M})$ it suffices to work with the Arakelov Chow group $CH^p(\overline{X})_{\mathbb{R}}$ defined using the Kähler form $c_1(\overline{M})$. Since $X$ has a cellular decomposition, we have an exact sequence

$$0 \longrightarrow CH^{p-1}(X)_{\mathbb{R}} \xrightarrow{\widetilde{a}} \widehat{CH}^p(\overline{X})_{\mathbb{R}} \xrightarrow{\zeta} CH^p(X)_{\mathbb{R}} \longrightarrow 0 \tag{1}$$

(see [KM, Prop. 6]). Here $\widetilde{a} = a \circ \mathrm{cl}$ is the composite of the class map with the natural inclusion $a : H^{p-1,p-1}(X_{\mathbb{R}}) \hookrightarrow \widehat{CH}^p(\overline{X})_{\mathbb{R}}$ and $\zeta$ is the projection defined in [GS, §1]. We choose a splitting

$$s_p \colon CH^p(X)_{\mathbb{R}} \longrightarrow \widehat{CH}^p(\overline{X})_{\mathbb{R}}$$

for the sequence (1) and thus arrive at a direct sum decomposition

$$\widehat{CH}^p(\overline{X})_{\mathbb{R}} \cong CH^p(X)_{\mathbb{R}} \oplus CH^{p-1}(X)_{\mathbb{R}}. \tag{2}$$

for every $p$.

Summing (1) over all $p$ produces a sequence

$$0 \longrightarrow CH^{*-1}(X)_{\mathbb{R}} \xrightarrow{\widetilde{a}} \widehat{CH}^*(\overline{X})_{\mathbb{R}} \xrightarrow{\zeta} CH^*(X)_{\mathbb{R}} \longrightarrow 0 \tag{3}$$

which is compatible with the actions of $L$ and $\widehat{L}$. The splitting $s := \oplus_p s_p$ of (3) does not commute with $\widehat{L}$ in general. Rather, the image of $\widehat{L} \circ s - s \circ L$ is contained in $\mathrm{Ker}(\zeta)$, hence

$$\widehat{L} \circ s - s \circ L = \widetilde{a} \circ U \tag{4}$$

for a uniquely defined degree-preserving linear operator $U$ on $CH^*(X)_{\mathbb{R}}$.

We now give some conditions equivalent to the arithmetic hard Lefschetz theorem (Theorem 1). When checking these for $G(2,N)$, we obtain something stronger, which establishes the arithmetic Hodge index theorem as well; this is quantified in Theorem 2. Recall the classical Lefschetz decomposition on $CH^m(X)_{\mathbb{R}} \simeq H^{2m}(X(\mathbb{C}), \mathbb{R})$:

$$CH^m(X)_{\mathbb{R}} = \bigoplus_{p \geqslant 0} L^{m-p} CH^p_{prim}(X)_{\mathbb{R}},$$

where the group of primitive codimension $p$ classes is

$$CH^p_{prim}(X)_{\mathbb{R}} = \mathrm{Ker}(L^{d+1-2p} : CH^p(X)_{\mathbb{R}} \to CH^{d+1-p}(X)_{\mathbb{R}}).$$

For $m = d - p$ this decomposition induces a projection map

$$\pi_p : CH^{d-p}(X)_{\mathbb{R}} \longrightarrow CH^{d-p}_{coprim}(X)_{\mathbb{R}}$$

where $CH^{d-p}_{coprim}(X)_{\mathbb{R}} = L^{d-2p} CH^p_{prim}(X)_{\mathbb{R}}$.

**Theorem 1.** *Let $X$ be an arithmetic variety of dimension $d+1$ which admits a cellular decomposition. Let $\widehat{L}$ be the arithmetic Lefschetz operator associated to an ample hermitian line bundle on $X$. Then the following statements are equivalent:*

(i) $\widehat{L}^{d+1-2p} : \widehat{CH}^p(\overline{X})_{\mathbb{R}} \longrightarrow \widehat{CH}^{d+1-p}(\overline{X})_{\mathbb{R}}$ *is an isomorphism for all $p$.*

(ii) *There exists a linear map $\widehat{\Lambda} \colon \widehat{CH}^*(\overline{X})_{\mathbb{R}} \longrightarrow \widehat{CH}^{*-1}(\overline{X})_{\mathbb{R}}$ such that for every $p$ and $\alpha \in \widehat{CH}^p(\overline{X})_{\mathbb{R}}$ we have $[\widehat{\Lambda}, \widehat{L}]\alpha = (d+1-2p)\alpha$.*

(iii) *For some (equivalently, any) choice of splitting $s$ of (3), with $U$ as in (4), the map*

$$\delta_p := \pi_p \sum_{i=0}^{d-2p} L^{d-2p-i} U L^i : CH^p_{prim}(X)_{\mathbb{R}} \longrightarrow CH^{d-p}_{coprim}(X)_{\mathbb{R}}$$

*is an isomorphism for all $p$.*

*Proof.* The equivalence of (i) and (ii) follows from standard Lefschetz theory; this is described e.g. in [Kl, §4]. Now consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & CH^{p-1}(X)_{\mathbb{R}} & \xrightarrow{\widetilde{a}} & CH^p(\overline{X})_{\mathbb{R}} & \xrightarrow{\zeta} & CH^p(X)_{\mathbb{R}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle L^{d+1-2p}} & & \downarrow{\scriptstyle \widehat{L}^{d+1-2p}} & & \downarrow{\scriptstyle L^{d+1-2p}} & & \\ 0 & \longrightarrow & CH^{d-p}(X)_{\mathbb{R}} & \xrightarrow{\widetilde{a}} & CH^{d+1-p}(\overline{X})_{\mathbb{R}} & \xrightarrow{\zeta} & CH^{d+1-p}(X)_{\mathbb{R}} & \longrightarrow & 0 \end{array}$$

By classical Lefschetz theory the left vertical map is injective and the right vertical map is surjective. The snake lemma gives an exact sequence

$$0 \to \mathrm{Ker}(\widehat{L}^{d+1-2p}) \to CH^p_{prim}(X)_{\mathbb{R}} \xrightarrow{\delta} CH^{d-p}_{coprim}(X)_{\mathbb{R}} \to \mathrm{Coker}(\widehat{L}^{d+1-2p}) \to 0.$$

The connecting homomorphism $\delta$ is characterized by the property that $\widetilde{a} \circ \delta = \widehat{L}^{d+1-2p} \circ s$ modulo the subspace $\widetilde{a}(\mathrm{Ker}(\pi_p))$. Note also that (4) implies

$$\widehat{L}^k \circ s - s \circ L^k = \widetilde{a} \sum_{i=0}^{k-1} L^{k-1-i} U L^i$$

for all $k$. We deduce that $\delta$ coincides with the map $\delta_p$ in (iii), and hence that statements (i) and (iii) are equivalent. This also shows that $\delta_p$ does not depend on the splitting $s$ of (3) and the associated linear operator $U$ on $CH^*(X)_{\mathbb{R}}$. □

**Remark.** Assuming statement (iii), it is possible to give an explicit construction of the map $\widehat{\Lambda}$ in (ii), as follows. We first claim that there exists a splitting $s'$ of (3), with associated operator $U'$, such that for any $p$ and $\alpha \in CH^p_{prim}(X)_{\mathbb{R}}$ we have

$$(5) \qquad U' L^i \alpha = 0 \ \text{ for all } \ i < d-2p \quad \text{and} \quad U' L^{d-2p} \alpha \in CH^{d-p}_{coprim}(X)_{\mathbb{R}}.$$

Indeed, if we let $D$ be the linear transformation such that $s' - s = \widetilde{a} \circ D$, then

$$U' = U + [L, D],$$

and it is an exercise to check that the space of transformations $[L, D]$ is equal to the set of operators $V$ on $CH^*(X)_{\mathbb{R}}$ satisfying $\pi_p \sum_{i=0}^{d-2p} L^{d-2p-i} V L^i(\alpha) = 0$ for all $\alpha \in CH^p_{prim}(X)_{\mathbb{R}}$ and every $p$.

Now choose a primitive basis for $CH^*(X)_{\mathbb{R}}$ and apply $s'$ to get half of a basis for $CH^*(\overline{X})_{\mathbb{R}}$. By (iii), we may apply $(L^{d-2p})^{-1} \pi_p U' L^{d-2p}$ to the basis elements in $CH^p_{prim}(X)_{\mathbb{R}}$ for each $p$ to obtain another basis for $CH^*(X)_{\mathbb{R}}$, which we view (via $\widetilde{a}$) as the other half of our basis for $CH^*(\overline{X})_{\mathbb{R}}$. Let $v \in CH^p_{prim}(X)_{\mathbb{R}}$ be one of the basis elements, and let $r = d - 2p$. By our conditions on $s'$, a subset of our basis for $CH^*(\overline{X})_{\mathbb{R}}$ consists of $\widehat{v} := s'(v)$, the iterates $\widehat{L}^i(\widehat{v}) = s'(L^i v)$ of $\widehat{L}$ applied to $\widehat{v}$, the primitive element $w$ satisfying $L^r(w) = \pi_p(U' L^r(v))$, and the iterates of $\widehat{L}$ applied to $w$:

$$(6) \qquad \widehat{v},\ \widehat{L}\widehat{v},\ \ldots,\ \widehat{L}^r\widehat{v},\ w,\ Lw,\ \ldots,\ L^r w.$$

The action of $\widehat{L}$ is to send each element in (6) to the element on its right, except that $\widehat{L}^r\widehat{v}$ is sent to $L^r w$, and $L^r w$ to 0. We now define $\widehat{\Lambda}$ by

$$\widehat{\Lambda}(\widehat{L}^i\widehat{v}) = i(r+2-i)\widehat{L}^{i-1}\widehat{v},$$
$$\widehat{\Lambda}(L^i w) = (r+1)\widehat{L}^i\widehat{v} + i(r-i)L^{i-1}w.$$

Then $\widehat{\Lambda}$ (defined this way for every basis element $v$) satisfies the condition of (ii).

**Theorem 2.** *Suppose the arithmetic variety $X$ and $p$ are such that $CH^{p-1}_{prim}(X)_{\mathbb{R}} = 0$. If, for each nonzero $\alpha \in CH^p_{prim}(X)_{\mathbb{R}}$, we have*

$$(-1)^p \sum_{i=0}^{d-2p} \int_X L^{d-2p-i}\alpha \wedge UL^i\alpha > 0 \tag{7}$$

*then the statements in the arithmetic hard Lefschetz and Hodge index conjectures are true for that $X$, $p$ and $\overline{M}$.*

*Proof.* Let $(\alpha,\beta) \in CH^p(\overline{X})_{\mathbb{R}}$ be a nonzero element of the kernel of $\widehat{L}^{d+2-2p}$; the notation $(\alpha,\beta)$ refers to the direct sum decomposition (2), with respect to some splitting. We claim that $\alpha$ must be in $CH^p_{prim}(X)_{\mathbb{R}}$. Indeed, $\widehat{L}^{d+2-2p}(\alpha,\beta) = (L^{d+2-2p}\alpha,\gamma)$ for some $\gamma$ and $L^{d+2-2p}\alpha = 0$ implies $L^{d+1-2p}\alpha = 0$ since $CH^{p-1}_{prim}(X)_{\mathbb{R}}$ vanishes. Also, by the classical hard Lefschetz theorem, $\alpha \neq 0$. Now, if

$$\langle\ ,\ \rangle : CH^*(\overline{X})_{\mathbb{R}} \otimes CH^*(\overline{X})_{\mathbb{R}} \longrightarrow \mathbb{R}$$

denotes the arithmetic intersection pairing, then we have

$$\begin{aligned}\langle\, (\alpha,\beta)\,,\, \widehat{L}^{d+1-2p}(\alpha,\beta)\,\rangle &= \langle\, (\alpha,\beta)\,, (0, \sum_i L^{d-2p-i}UL^i\alpha + L^{d+1-2p}\beta)\,\rangle \\ &= \frac{1}{2}\sum_i \int_X L^{d-2p-i}\alpha \wedge UL^i\alpha.\end{aligned}$$

Hence, assuming $CH^{p-1}_{prim}(X)_{\mathbb{R}} = 0$, we have $\widehat{L}^{d+1-2p}(\alpha,\beta) \neq 0$ for every nonzero $(\alpha,\beta) \in CH^p(\overline{X})_{\mathbb{R}}$. Moreover, if $(\alpha,\beta)$ is primitive, then the pairing of $(\alpha,\beta)$ with $\widehat{L}^{d+1-2p}(\alpha,\beta)$ has the required sign. □

**Corollary 1.** *Suppose $X$ is such that, for every $p$,*

$$CH^p_{prim}(X)_{\mathbb{R}} \neq 0 \qquad \textit{implies} \qquad CH^{p-1}_{prim}(X)_{\mathbb{R}} = 0. \tag{8}$$

*If condition (7) holds for every $p$ and each nonzero $\alpha \in CH^p_{prim}(X)_{\mathbb{R}}$, then both the arithmetic hard Lefschetz and Hodge index conjectures are true for $X$, $\overline{M}$.*

*Proof.* If $CH^{p-1}_{prim}(X)_{\mathbb{R}} = 0$ then the conjectures hold for $X$, $\overline{M}$ and $p$ by Theorem 2. Assume now that $CH^{p-1}_{prim}(X)_{\mathbb{R}} \neq 0$. It suffices to prove the Hodge index inequality for a nonzero $x \in CH^p(\overline{X})_{\mathbb{R}}$ in the kernel of $\widehat{L}^{d+2-2p}$. Since, by hypothesis, we have $CH^p_{prim}(X)_{\mathbb{R}} = 0$, it follows that $x = \widehat{L}(y) + \widetilde{a}(\eta)$ for some $\eta \in CH^{p-1}(X)_{\mathbb{R}}$ and $y \in CH^{p-1}(\overline{X})_{\mathbb{R}}$, with $y \neq 0$ and $\widehat{L}^{d+4-2p}(y) = 0$. Moreover, the condition $\widehat{L}^{d+2-2p}(x) = 0$ implies $\widehat{L}^{d+2-2p}(\widetilde{a}(\eta)) = -\widehat{L}^{d+3-2p}(y)$. Now we find

$$\langle\, x,\, \widehat{L}^{d+1-2p}(x)\,\rangle = -\langle\, y,\, \widehat{L}^{d+3-2p}(y)\,\rangle,$$

and the required Hodge index inequality is a consequence of the degree $p-1$ case of Theorem 2. □

**Example.** We illustrate the previous results for projective space $\mathbb{P}^n$ and the canonical hermitian line bundle $\overline{\mathcal{O}}(1)$ (compare [Ku, §4]). In this case we choose the splitting

$$CH^*(\overline{\mathbb{P}^n})_{\mathbb{R}} = \bigoplus_{i=0}^{n} \mathbb{R}\cdot\widehat{\omega}^i \,\oplus\, \bigoplus_{i=0}^{n} \mathbb{R}\cdot\omega^i$$

where $\widehat{\omega}^i = \widehat{c}_1(\overline{\mathcal{O}}(1))^i$ and $\omega^i = \widetilde{a}(c_1(\mathcal{O}(1))^i)$. Then the sequence (6) is given by

$$\widehat{1},\, \widehat{\omega},\, \ldots\, ,\, \widehat{\omega}^n,\, \tau_n,\, \tau_n\omega,\, \ldots\, ,\, \tau_n\omega^n.$$

Here $\tau_n = \sum_{k=1}^n \mathcal{H}_k$, where each $\mathcal{H}_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$ is a harmonic number. The Remark after Theorem 1 exhibits an explicit adjoint map $\widehat{\Lambda}$ for the arithmetic Lefschetz operator $\widehat{L}(x) = \widehat{\omega} \cdot x$; in our example it is given by

$$\widehat{\Lambda}(\widehat{\omega}^i) = i(n+2-i)\,\widehat{\omega}^{i-1},$$
$$\widehat{\Lambda}(\omega^i) = \frac{n+1}{\tau_n}\,\widehat{\omega}^i + i(n-i)\,\omega^{i-1}.$$

Observe that the nonzero primitive elements of $CH^*(\mathbb{P}^n)_{\mathbb{R}}$ are multiples of $1 \in CH^0(\mathbb{P}^n)_{\mathbb{R}}$; hence $\mathbb{P}^n$ satisfies (8). The operator $U$ is given by $U(\omega^i) = \delta_{i,n}\tau_n\omega^n$, and condition (7) for $p = 0$, $\alpha = 1$ becomes

$$\sum_{i=0}^n \int_{\mathbb{P}^n} \omega^{n-i} \wedge U(\omega^i) = \tau_n \int_{\mathbb{P}^n} \omega^n = \sum_{k=1}^n \mathcal{H}_k > 0. \tag{9}$$

The arithmetic Hodge index conjecture for $\mathbb{P}^n$ and $\overline{\mathcal{O}}(1)$ follows by applying Corollary 1.

A hermitian line bundle $\overline{M} = (M, \|\cdot\|)$ on an arithmetic variety $X$ is *arithmetically ample* if $M$ is an ample invertible sheaf on $X$ such that the first Chern form $c_1(\overline{M(\mathbb{C})})$ is nonnegative on $X(\mathbb{C})$ and for all nonempty irreducible closed subsets $Y \subset X$ the height $h_{\overline{M}}(Y)$ is positive [So, §5.2]. We say that $\overline{M}$ is a *limit for arithmetic ampleness* if (i) $(M, t\,\|\cdot\|)$ is arithmetically ample for all positive scalars $t < 1$, and (ii) $h_{\overline{M}}(Y) = 0$ for some (nonempty) irreducible closed $Y \subset X$.

The line bundle $\overline{\mathcal{O}}(1)$ on $\mathbb{P}^n$ considered above is a limit for arithmetic ampleness. Indeed, if $Y \in Z_1(\mathbb{P}^n)$ is the cycle attached to the rational point $[1 : 0 : \cdots : 0]$, then $h_{\overline{\mathcal{O}}(1)}(Y) = 0$ (see for instance [BGS, (3.1.6)]). Furthermore, property (i) for $\overline{\mathcal{O}}(1)$ follows from the argument in [BGS, Prop. 3.2.4]. On the other hand, one sees that the arithmetic Hodge index inequality (9) does not fail when the natural metric on $\mathcal{O}(1)$ is scaled by a factor $t \in (1-\epsilon, 1+\epsilon)$ for small $\epsilon > 0$ (see also [BGS, Prop. 3.2.2]). Observe however that (9) becomes sharp at $t = 1$ if we insist that it should hold for *any* positive real constants $\mathcal{H}_k$ (following the point of view in [T, §6]). Further evidence for this statement on more general Grassmannians is given in Section 3.

## 2. The arithmetic Grassmannian $G(2,N)$

In this section we study Conjecture 1 for the Grassmannian of lines in projective space. For computational purposes we will work with the isomorphic Grassmannian $G = G(N,2)$ parametrizing $N$-planes in $(N+2)$-space throughout. Note that $d = \dim_{\mathbb{C}} G(\mathbb{C}) = 2N$. There is a universal exact sequence of vector bundles

$$0 \longrightarrow S \longrightarrow E \longrightarrow Q \longrightarrow 0$$

over $G$; the complex points of $E$ and $Q$ are metrized by giving the trivial bundle $E(\mathbb{C})$ the trivial hermitian metric and the quotient bundle $Q(\mathbb{C})$ the induced metric. The hermitian vector bundles that result are denoted $\overline{E}$ and $\overline{Q}$.

The real vector space $CH^*(G)_{\mathbb{R}} \cong H^{2*}(G(\mathbb{C}),\mathbb{R})$ decomposes as

$$CH^*(G)_{\mathbb{R}} = \bigoplus_{a,b} \mathbb{R} \cdot s_{a,b}(Q),$$

summed over all partitions $\lambda = (a,b)$ with $a \leqslant N$, i.e., whose Young diagrams are contained in the $2 \times N$ rectangle $(N,N)$. Moreover $s_\lambda(Q) = s_{a,b}(Q)$ is the characteristic class coming from the Schur polynomial $s_{a,b}$ in the Chern roots of $Q$; this is dual to the class of a codimension $|\lambda| = a+b$ Schubert variety in $G$. In the following $s_1$ denotes the Schur polynomial $s_{1,0}$ and is thus just the first elementary symmetric function.

The line bundle $M = \det(Q)$ giving the Plücker embedding has $c_1(M) = s_1(Q)$; let $L : CH^p(G)_{\mathbb{R}} \to CH^{p+1}(G)_{\mathbb{R}}$ be the associated classical Lefschetz operator. Further for all $p$ let $* : CH^p(G)_{\mathbb{R}} \to CH^{2N-p}(G)_{\mathbb{R}}$ denote the Hodge star operator induced by the Kähler form $s_1(\overline{Q})$. We then have

**Proposition 1.** *The space $CH^p_{prim}(G)_{\mathbb{R}}$ is nonzero if and only if $p = 2k \leqslant N$. In the latter case it is one dimensional and spanned by the class*

$$\alpha_k = \sum_{j=0}^{k} (-1)^j \binom{N+1-j}{N-2k} \binom{N-2k+j}{N-2k} s_{2k-j,j}(Q).$$

*Proof.* By computing the Betti numbers for $G$ one sees that

$$\dim CH^p(G)_{\mathbb{R}} - \dim CH^{p-1}(G)_{\mathbb{R}} > 0$$

if and only if $p = 2k \leqslant N$, and in this case the above difference equals 1. For such $p$ we have

(10)

$$\operatorname{Ker}(L : CH^{2N-2k}(G)_{\mathbb{R}} \to CH^{2N-2k+1}(G)_{\mathbb{R}}) = \operatorname{Span}\{\sum_{j=0}^{k} (-1)^j s_{N-j,N-2k+j}(Q)\}.$$

One checks (10) easily using the Pieri rule:

$$L(s_{a,b}(Q)) = s_{a+1,b}(Q) + s_{a,b+1}(Q)$$

where it is understood that $s_{c,c'}(Q) = 0$ if $c < c'$ or $c > N$.

From [KT] we know the action of the Hodge star operator on $CH^*(G)_{\mathbb{R}}$ is given by

$$* \, s_{a,b}(Q) = \frac{(a+1)!\, b!}{(N-a)!\,(N-b+1)!}\, s_{N-b,N-a}(Q). \tag{11}$$

Since

$$CH^{2k}_{prim}(G)_{\mathbb{R}} = * \operatorname{Ker}\left(L : CH^{2N-2k}(G)_{\mathbb{R}} \to CH^{2N-2k+1}(G)_{\mathbb{R}}\right),$$

the proof is completed by applying (11) to (10) and noting that the result is proportional to $\alpha_k$. □

We now pass to the arithmetic setting, where we use the arithmetic Schubert calculus of [T, §3,4]. The real Arakelov Chow group $CH^p(\overline{G})_{\mathbb{R}}$ decomposes as

$$CH^p(\overline{G})_{\mathbb{R}} = \bigoplus_{a+b=p} \mathbb{R} \cdot \widehat{s}_{a,b}(\overline{Q}) \oplus \bigoplus_{a'+b'=p-1} \mathbb{R} \cdot s_{a',b'}(\overline{Q}). \tag{12}$$

Here the indexing sets satisfy $N \geqslant a \geqslant b \geqslant 0$, $\widehat{s}_{a,b}(\overline{Q})$ is an arithmetic characteristic class and we identify the harmonic differential form $s_{a',b'}(\overline{Q})$ with its image in

$CH^p(\overline{G})_{\mathbb{R}}$. The decomposition (12) is induced by the splitting map $s_{a,b}(Q) \longmapsto \widehat{s}_{a,b}(\overline{Q})$ which agrees with the one used in [T].

The hermitian line bundle $\overline{M}$ has $\widehat{c}_1(\overline{M}) = \widehat{s}_1(\overline{Q})$ and is a limit for arithmetic ampleness; the latter property follows as in the remarks at the end of Section 1. We now apply the arithmetic Pieri rule of [T, §4] to compute the action of the arithmetic Lefschetz operator $\widehat{L}(x) = \widehat{s}_1(\overline{Q}) \cdot x$ on the above basis elements. The induced operator $U : CH^*(G)_{\mathbb{R}} \to CH^*(G)_{\mathbb{R}}$ of (4) satisfies $U(s_{a,b}) = 0$ for $a < N$ and

$$U(s_{N,b}) = \Big(\sum_{i=0}^{N+1} \mathcal{H}_i\Big) s_{N,b} - \sum_{i=0}^{\lfloor (N-b)/2 \rfloor} (\mathcal{H}_{N-b+1-i} - \mathcal{H}_i) s_{N-i,b+i}. \tag{13}$$

Here and in the rest of this section $s_{a,b}$ will denote the Schubert class $s_{a,b}(Q) \in CH^{a+b}(G)_{\mathbb{R}}$ and $\mathcal{H}_i$ is a harmonic number; by convention $\mathcal{H}_0 = 0$. Recall that the classical intersection pairing on $CH^*(G)_{\mathbb{R}}$ satisfies

$$\langle s_{a,b},\, s_{a',b'} \rangle = \int_G s_{a,b} \wedge s_{a',b'} = \delta_{(a,b),(N-b',N-a')}.$$

The sequence of Betti numbers for $G$ shows that $G$ satisfies condition (8) of Corollary 1. We proceed to check the inequality (7) for all even $p = 2k$; this will establish Conjecture 1 for $G(N,2)$. In our case (7) may be written as

$$\Sigma(N,k) := \int_G \sum_{b=0}^{N-2k} L^{N-2k-b} \alpha_k \wedge U L^{N-2k+b} \alpha_k > 0.$$

To compute iterates of the classical Lefschetz operator $L$ on the Schubert basis, note that

$$L^r s_\mu = \sum_{\substack{\lambda \supset \mu \\ |\lambda| = |\mu| + r}} f^{\lambda/\mu} s_\lambda. \tag{14}$$

When $\lambda = (\lambda_1, \lambda_2)$ and $\mu = (\mu_1, \mu_2)$ are partitions with at most two parts the skew $f$-number in (14) satisfies

$$f^{\lambda/\mu} = \binom{|\lambda| - |\mu|}{\lambda_1 - \mu_1} - \binom{|\lambda| - |\mu|}{\lambda_1 - \mu_2 + 1}. \tag{15}$$

This follows from the determinantal formula for $f^{\lambda/\mu}$, given for example in [St, Corollary 7.16.3].

We now apply Proposition 1 and (14), (15) to calculate

$$L^{c-2k} \alpha_k = \sum_{j=0}^{k} (-1)^j \binom{N+1-j}{N-2k} \binom{N-2k+j}{N-2k} L^{c-2k} s_{2k-j,j}$$

$$= \sum_{j=0}^{k} \sum_{i \geqslant j} (-1)^j \binom{N+1-j}{N-2k} \binom{N-2k+j}{N-2k} \left[ \binom{c-2k}{i-j} - \binom{c-2k}{i-2k-1+j} \right] s_{c-i,i}.$$

Therefore,

$$L^{c-2k} \alpha_k = \sum_{i,j} (-1)^j \binom{N+1-j}{N-2k} \binom{N-2k+j}{N-2k} \binom{c-2k}{i-j} s_{c-i,i}. \tag{16}$$

We use (16) with $c = N + b$ to identify the coefficient of $s_{N,b}$ in the expansion of $L^{N+b-2k}\alpha_k$ as

$$\begin{aligned}\langle L^{N+b-2k}\alpha_k,\, s_{N-b}\rangle &= \sum_j (-1)^j \binom{N+1-j}{N-2k}\binom{N-2k+j}{j}\binom{N+b-2k}{N-2k+j}\\ &= \binom{N-2k+b}{N-2k}\sum_j (-1)^j \binom{b}{j}\binom{N+1-j}{N-2k}\\ &= \binom{N+1-b}{2k+1}\binom{N-2k+b}{N-2k} =: C_b.\end{aligned}$$

It now follows from (13) that

$$UL^{N-2k+b}\alpha_k = C_b\Big(\sum_{i=0}^{N+1}\mathcal{H}_i\Big)s_{N,b} - \sum_{i=0}^{\lfloor (N-b)/2\rfloor} C_b(\mathcal{H}_{N-b+1-i}-\mathcal{H}_i)s_{N-i,b+i}. \tag{17}$$

Note also that we have the identity

$$\sum_{b=0}^{N-2k} C_b = \binom{2N-2k+2}{N+2}. \tag{18}$$

Now we substitute $c = N - b$ in (16), pair with (17) and use (18) to sum over $b$ and obtain

$$\Sigma(N,k) = A_{N,k}\sum_{i=1}^{N+1}\mathcal{H}_i + \sum_{b=0}^{N-2k}\sum_{i=0}^{\lfloor (N-b)/2\rfloor} C_{N,k}^{b,i}, \tag{19}$$

where

$$A_{N,k} = \binom{N+1}{N-2k}\binom{2N-2k+2}{N+2}$$

and

$$C_{N,k}^{b,i} = \sum_j (-1)^j \binom{N+1-j}{N-2k}\binom{N-2k+j}{N-2k}\binom{N-2k-b}{i-j}C_b(\mathcal{H}_i - \mathcal{H}_{N-b+1-i}).$$

By dividing the expression for $C_{N,k}^{b,i}$ into two sums and substituting in (19) one gets

$$\Sigma(N,k) = A_{N,k}\sum_{i=1}^{N+1}\mathcal{H}_i + \sum_{i=1}^{N+1} B_{N,k}^i\mathcal{H}_i, \tag{20}$$

where

$$B_{N,k}^i = \sum_{j,b}(-1)^j\binom{N+1-j}{N-2k}\binom{N-2k+j}{N-2k}\binom{N-2k-b}{i-j}C_b.$$

(Notice that when $N - b = 2r - 1$ is odd, there is a missing summand (for $i = r$)

$$\sum_j (-1)^j \binom{N+1-j}{N-2k}\binom{N-2k+j}{N-2k}\binom{2r-2k-1}{r-j}C_b\mathcal{H}_r$$

which vanishes, as can be seen by the change of variable $j \mapsto 2k+1-j$.)

At this point it is convenient to introduce the variable change

$$n = N - 2k \qquad \text{and} \qquad T = N + 2$$

and write equation (20) in the new coordinates as

$$\Sigma(n,T) = A_{n,T}\sum_{i=1}^{T-1}\mathcal{H}_i + \sum_{i=1}^{T-1} B^i_{n,T}\mathcal{H}_i. \tag{21}$$

Observe that

$$\begin{aligned} B^i_{n,T} &= \sum_j (-1)^j \binom{n+j}{n}\binom{T-1-j}{n}\sum_b \binom{n-b}{i-j}\binom{T-1-b}{n-b}\binom{n+b}{n} \\ &= \sum_j (-1)^j \binom{n+j}{n}\binom{T-1-j}{n}\binom{T-1-n+i-j}{i-j}\binom{T+n}{n-i+j}. \end{aligned}$$

We now substitute $r = i-j$ and write the resulting sum in hypergeometric notation [Ro] [VK, Chap. 3]:

$$\begin{aligned} (-1)^i B^i_{n,T} &= \sum_r (-1)^r \binom{n+i-r}{n}\binom{T-1+r-i}{n}\binom{T-1-n+r}{r}\binom{T+n}{n-r} \\ &= \binom{n+i}{n}\binom{T+n}{n}\binom{T-1-i}{n}{}_4F_3\left(\begin{matrix} -n,\, -i,\, T-n,\, T-i \\ -n-i,\, T+1,\, T-n-i \end{matrix}\,\middle|\, 1\right). \end{aligned}$$

The Whipple transformation [Wh, §10] applied to the above ${}_4F_3$ gives

$$(-1)^i \frac{B^i_{n,T}}{A_{n,T}} = {}_4F_3\left(\begin{matrix} -n,\, n+1,\, -i,\, i+1 \\ 1,\, 1+T,\, 1-T \end{matrix}\,\middle|\, 1\right). \tag{22}$$

The hypergeometric term (22) belongs to a class of orthogonal polynomials called Racah polynomials, which are studied in the next section.

## 3. Bounds for Racah polynomials

The Racah coefficients [Ra] or 6-$j$ symbols have long been used by physicists as the transformation coefficients between two different coupling schemes of three angular momenta; see [BL] for an exposition. In mathematical language they are the entries of a change of basis matrix for the tensor product of three irreducible representations of $SU(2)$; the two bases involved come from the associativity relation for this product (see [VK, §8.4]). It was recognized later by Wilson [Wi] that these coefficients are special cases of a class of orthogonal polynomials $R_n(x;\alpha,\beta,\gamma,\delta)$, called Racah polynomials [VK, §8.5]:

$$R_n(s(s+\gamma+\delta+1);\alpha,\beta,\gamma,\delta) = {}_4F_3\left(\begin{matrix} -n,\, n+\alpha+\beta+1,\, -s,\, s+\gamma+\delta+1 \\ \alpha+1,\, \beta+\delta+1,\, \gamma+1 \end{matrix}\,\middle|\, 1\right).$$

The Racah polynomials in (22) have $\alpha = \beta = \gamma+\delta = 0$, with $\gamma = T$, a positive integer. We let

$$R_n(s,T) = R_n(s(s+1);0,0,T,-T).$$

Observe that $R_n(s,T)$ is symmetric in $n$ and $s$. The orthogonality condition (loc. cit. or [AW]) reads:

$$\sum_{s=0}^{T-1}(2s+1)R_n(s,T)R_m(s,T) = \frac{T^2}{(2n+1)}\delta_{nm}. \tag{23}$$

The arithmetic Hodge index inequality $\Sigma(n,T)>0$ can be rephrased using (21) and (22) as

$$\sum_{s=1}^{T-1}(-1)^{s+1}R_n(s,T)\mathcal{H}_s<\sum_{s=1}^{T-1}\mathcal{H}_s. \tag{24}$$

We give a proof of (24) which does not depend on the precise values of the harmonic numbers. Let us say that a sequence $\{\mathcal{H}_k\}_{k\geqslant 1}$ of positive real numbers (with $\mathcal{H}_0=0$) is *concave increasing* if $\mathcal{H}_k=\sum_{i=1}^k h_i$ for some monotone decreasing sequence $\{h_i\}$ of positive reals.

**Theorem 3.** *Let $\{\mathcal{H}_k\}$ be any concave increasing sequence of real numbers and $n$, $T$ integers with $0\leqslant n\leqslant T-1$ and $T\geqslant 3$. Then inequality* (24) *holds.*

We believe that, in fact, (24) holds for an arbitrary sequence of positive real numbers $\mathcal{H}_k$, that is, the arithmetic standard conjectures for $G(2,N)$ do not depend on the relative sizes of the harmonic numbers involved:

**Conjecture 2.** *For any integers $n,s$ with $0\leqslant n,s\leqslant T-1$ we have $|R_n(s,T)|\leqslant 1$.*

In Proposition 2 we check this conjecture for some values of $n$ near the endpoints 0 and $T-1$. Computer calculations support the validity of Conjecture 2 for general $n$.

*Proof of Theorem 3.* We shall see that (23) implies (24) except when $T$ is exponentially large compared to $n$. For large $T$, the Racah polynomials are close approximations of classical orthogonal polynomials, in this case the Legendre polynomials, and we know how to bound these.

By Cauchy's inequality, (23) gives

$$\left(\sum_{s=0}^{T-1}|R_n(s,T)|\,\mathcal{H}_s\right)^2\leqslant\frac{T^2}{2n+1}\sum_{s=0}^{T-1}\frac{\mathcal{H}_s^2}{2s+1}.$$

So, (24) holds whenever

$$\sum_{s=0}^{T-1}\frac{\mathcal{H}_s^2}{2s+1}<(2n+1)\left(\frac{1}{T}\sum_{s=0}^{T-1}\mathcal{H}_s\right)^2. \tag{25}$$

Since $\{\mathcal{H}_k\}$ is concave increasing, the average value of $\mathcal{H}_0,\ \ldots,\ \mathcal{H}_{T-1}$ is at least $\mathcal{H}_{T-1}/2$. As $\sum_{s=1}^{T-1}2/(2s+1)\leqslant\log T$, the inequality (25) holds whenever

$$\log T<n+\frac{1}{2}. \tag{26}$$

To analyze the case where $T$ is exponentially large compared to $n$, it is convenient to introduce the change of variable

$$x=s(s+1)=-1/4+T^2(1+t)/2 \tag{27}$$

and the rescaling

$$p_n(t)=(-1)^n\prod_{i=1}^n\frac{T^2-i^2}{T^2}R_n(x;0,0,T,-T).$$

Let $P_n(t)=P_n^{(0,0)}(t)$ denote the $n^{\text{th}}$ Legendre polynomial. It is known [NSU, §3.8] that

$$p_n(t)=P_n(t)+O(1/T^2)$$

where the constant in the error term depends on both $n$ and $t$. For our purposes, we demonstrate

**Lemma 1.** a) *Let $n$ and $T$ be positive integers such that $1 + 2n + 2n^2 < T^2/10$. Then*

$$|p_n(t) - P_n(t)| \leqslant (3/2) \cdot 4^n/T^2 \tag{28}$$

*for all $t$ with $-1 \leqslant t \leqslant 1$.*

b) *We have $|p_n(t) - P_n(t)| \leqslant 1/10$ whenever $T \geqslant 90$ and $n < \log T$.*

*Proof.* We have the following recurrences (loc. cit.; for $P_n(t)$ this is classical)

$$tP_n(t) = \frac{n+1}{2n+1}P_{n+1}(t) + \frac{n}{2n+1}P_{n-1}(t) \tag{29}$$

$$tp_n(t) = \frac{n+1}{2n+1}p_{n+1}(t) - \frac{2n^2+2n+1}{2T^2}p_n(t) + \Big(1 - \frac{n^2}{T^2}\Big)^2 \frac{n}{2n+1}p_{n-1}(t) \tag{30}$$

with initial data

$$\begin{aligned} P_0(t) &= 1 \quad ; \quad P_1(t) = t \\ p_0(t) &= 1 \quad ; \quad p_1(t) = t + 1/(2T^2). \end{aligned} \tag{31}$$

Subtracting (29) from (30) leads to a recurrence in $p_n(t) - P_n(t)$. Then (28) follows by induction on $n$, using the known bound $|P_n(t)| \leqslant 1$ for all $n$ and all $t$ with $-1 \leqslant t \leqslant 1$. The statement (b) is a corollary of (a). ◻

**Proposition 2.** *We have $|R_n(s,T)| \leqslant 1$ when $n \leqslant 3$ or $n = T - 1$.*

*Proof.* For $n = 0$ we have $R_0(s,T) = 1$. When $n = 1$, we see from (31) that $p_1(t)$ is a increasing linear function in $t$, attaining minimum when $s = 0$, giving $R_1(0,T) = 1$, and maximum when $s = T-1$, giving $R_1(T-1,T) = (1-T)/(1+T)$, hence the inequality holds. For $n = T - 1$ the Pfaff-Saalschütz identity [Ro] [VK, 8.3.3] gives

$$\begin{aligned} R_{T-1}(s,T) &= \sum_j (-1)^j \frac{T}{T+j}\binom{s}{j}\binom{s+j}{j} \\ &= {}_3F_2\left( \begin{matrix} -s,\ s+1,\ T \\ 1,\ T+1 \end{matrix} \,\middle|\, 1 \right) \\ &= \frac{(1-T)(2-T)\cdots(s-T)}{(1+T)(2+T)\cdots(s+T)} \end{aligned}$$

so the inequality is clear.

When $n = 2$ or $n = 3$, $p_n(t)$ is a quadratic or cubic polynomial, and it is a calculus exercise to check that $|R_n(s,T)| \leqslant 1$ for every integer $s$ with $0 \leqslant s \leqslant T - 1$. In fact, the integrality condition on $s$ is required only when $n = 3$, $T = 4$. ◻

**Lemma 2.** a) *We have $|P_n(t)| \leqslant 3/4$ for $t \in \mathbb{R}$, $|t| \leqslant 0.9$ and $n \geqslant 2$.*

b) *For $T \geqslant 10$, we have $|t| \leqslant 0.9$ in (27) whenever $\sqrt{5}/10 \leqslant s/T \leqslant 4/5$.*

c) *Assume $T \geqslant 90$ and $n < \log T$. Then* $\displaystyle\frac{1}{T^{2n}}\prod_{i=1}^{n}(T^2 - i^2) > 40/41.$

*Proof.* The indicated bound on Legendre polynomials is evident for $n = 2$, and for larger $n$ it follows from the inequality

$$\sqrt{\sin\theta}\,|P_n(\cos\theta)| < \sqrt{\frac{2}{\pi n}}, \qquad 0 \leqslant \theta \leqslant \pi. \tag{32}$$

One obtains (32) by using the transformed differential equation for $\sqrt{\sin\theta}\,P_n(\cos\theta)$ [Sz, IV.4.24.2]; this is indicated in [Ho, Chap. 5, Exer. 15–16]. The proofs of (b) and (c) are routine; for the latter, one may use the inequality

$$-\log(T^{-2n}\prod_{i=1}^{n}(T^2 - i^2)) \leqslant \frac{2}{T^2}\sum_{i=1}^{n} i^2.$$

□

To complete the proof of Theorem 3, assume that (26) fails, so that $n < \log T - 1/2$. If $T \leqslant 90$ then $n \leqslant 3$ and (24) follows from Proposition 2 (note that the inequality in the proposition is strict unless $n = 0$ or $s = 0$). When $T \geqslant 90$ we combine Lemma 1(b) with Lemma 2 to deduce the inequality (24). Indeed, $(40/41)|R_n(s,T)|$ is bounded by $1 + 1/10$ for every $s$, and by $3/4 + 1/10$ over the middle half of the summation range. By pairing terms $\mathcal{H}_s$ with $\mathcal{H}_{T-1-s}$ and using the fact that $\mathcal{H}_s + \mathcal{H}_{T-1-s}$ is monotone increasing for $0 \leqslant s \leqslant (T-1)/2$, we obtain (24). □

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