THETA AND ETA POLYNOMIALS IN GEOMETRY, LIE THEORY, AND COMBINATORICS

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Abstract. The classical Schur polynomials form a natural basis for the ring of symmetric polynomials and have geometric significance since they represent the Schubert classes in the cohomology ring of Grassmannians. Moreover, these polynomials enjoy rich combinatorial properties. In the last decade, an exact analogue of this picture has emerged in the symplectic and orthogonal Lie types, with the Schur polynomials replaced by the theta and eta polynomials of Buch, Kresch, and the author. This expository paper gives an overview of what is known to date about this correspondence, with examples.

1. Introduction

The structure of the cohomology ring of the Grassmannian $\mathrm{G}(m,n)$ of m dimensional linear subspaces of $\mathbb{C}^n$ was first explored by Schubert [Sc], Pieri [Pi], and Giambelli [G]. The ring $\mathrm{H}^*(\mathrm{G}(m,n),\mathbb{Z})$ has an additive basis of Schubert classes, coming from the cell decomposition of $\mathrm{G}(m,n)$, and there is a natural choice of multiplicative generators for this ring, namely, the special Schubert classes. Giambelli showed that a general Schubert class, which may be indexed by a partition, can be expressed as a Jacobi-Trudi determinant [J, Tr] whose entries are special classes. It soon became apparent (see for example [Le]) that the resulting algebra is closely connected to the theory of Schur polynomials. The latter polynomials were originally defined by Cauchy [C], and studied by many others since then, motivated to a large extent by their applications to the representation theory of the symmetric and general linear groups [S1], and related combinatorics.

The theory of theta and eta polynomials, by contrast, has its origins in geometry, and specifically in the desire to extend the aforementioned work of Giambelli to the cohomology of symplectic and orthogonal Grassmannians. The first steps in this direction were taken in the 1980s by Hiller and Boe [HB] and Pragacz [P]. They proved Pieri and Giambelli formulas for the Grassmannians of maximal isotropic subspaces, with the Schubert classes indexed by strict partitions and the Jacobi-Trudi determinants replaced by Schur Pfaffians [S2].

In 2008, Buch, Kresch, and the author announced a series of works [BKT1, BKT2, BKT3, T3] which went beyond these hermitian symmetric (or cominuscule) examples. Two crucial insights from op. cit. were the identification of the correct set of special Schubert class generators to employ, and the realization of the essential role that Young's *raising operators* [Y] play in the theory. The latter becomes clear

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only when one attempts to understand the cohomology of *nonmaximal* isotropic Grassmannians. Our papers introduced *k-strict partitions* to index the Schubert classes and *theta polynomials* to represent them, in both the classical and quantum cohomology rings of general symplectic and odd orthogonal Grassmannians. Three years later, the companion papers [BKT4, T6] dealt with the even orthogonal case, using *typed k-strict partitions* and *eta polynomials*.

Theta and eta polynomials can be viewed as symmetric polynomials for the action of the corresponding Weyl group, but this is not obvious from their definition, and was pointed out only recently [T9]. In fact, a substantial part of the theory of Schur polynomials can be extended to the world of theta and eta polynomials, but this requires a change in perspective, as well as the introduction of new techniques of proof. It turns out that these objects can be applied to solve the (equivariant) Giambelli problem for the classical Lie groups, that is, to obtain intrinsic polynomial representatives for the (equivariant) Schubert classes on any classical G/P-space – so, any (isotropic) partial flag manifold [BKTY, T5, T7]. The resulting formulas (12), (38), and (60) are stated using solely the language of Lie theory.

The goal of this expository paper is to illustrate the correspondence between Schur polynomials and theta/eta polynomials in the case of single polynomials, where the story is most complete. There is ample room for further interesting connections to be found, and the reader is likely to discover more by just asking for the theta/eta analogue of their favorite statement about Schur polynomials. We have stopped short of discussing extensions of some of these results to the theory of degeneracy loci and equivariant cohomology, quantum cohomology, and K-theory.

This article is organized as follows. The Schur, theta, and eta polynomials are featured in Sections 2, 3, and 4, respectively. Each of these is split into parallel subsections on initial definitions and Pieri rules, the cohomology of Grassmannians, symmetric polynomials, algebraic combinatorics, and the cohomology of partial flag manifolds. Finally, Section 5 contains historical notes and references.

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2. Schur polynomials

2.1. **Definition using raising operators.** We will define the classical Schur polynomials by using the Jacobi-Trudi formula, but rewritten in the language of Young's raising operators. Let $u_1, u_2, \ldots$ be a sequence of commuting independent variables, and set $u_0 := 1$ and $u_i := 0$ for $i < 0$. Throughout the paper, $\alpha := (\alpha_1, \alpha_2, \ldots)$ will denote an integer sequence with only finitely many nonzero terms α_i. For any such α, we let $u_\alpha := u_{\alpha_1} u_{\alpha_2} \cdots$, which is a monomial in the variables u_i.

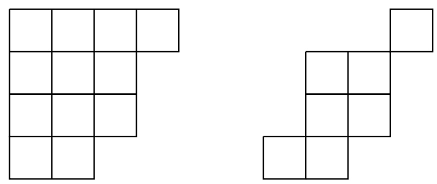
Given any integer sequence α and $i < j$, we define

$$R_{ij}(\alpha) := (\alpha_1, \ldots, \alpha_i + 1, \ldots, \alpha_j - 1, \ldots).$$

A *raising operator* R is any monomial in the R_{ij}'s. For any such R, we let $R\,u_\alpha := u_{R(\alpha)}$, where, as is customary, we regard the operator R as acting on the index α, and not on the monomial u_α.

An integer sequence α is a *partition* if $\alpha_i \geq \alpha_{i+1} \geq 0$ for each $i \geq 1$. If λ is a partition, we let $|\lambda|$ denote the sum of all its parts λ_i, and the length $\ell(\lambda)$ be the number of i such that $\lambda_i \neq 0$. A partition λ may be represented using its

Young diagram of boxes, placed in left justified rows, with λ_i boxes in the i-th row, for each $i \geq 1$. The diagram of the conjugate partition $\widetilde{\lambda}$ is obtained by taking the transpose of the diagram of λ. The containment relation $\lambda \subset \mu$ between two partitions is defined using their respective diagrams; in this case the set-theoretic difference $\mu \smallsetminus \lambda$ is the skew diagram μ/λ. For example, we illustrate below the diagram of the partition $(4,3,3,2)$ and the skew diagram $(4,3,3,2)/(3,1,1)$.



The *Schur polynomial* $s_\lambda(u)$ is defined by the raising operator formula

$$s_\lambda(u) := \prod_{i<j}(1-R_{ij})\, u_\lambda. \tag{1}$$

We regard the products in formulas such as (1) as being over all pairs (i,j) with $j \leq N$ for some N greater than or equal to the length $\ell(\lambda)$. The result will always be independent of any such N, so we may assume that $N = \ell(\lambda)$.

Example 1. For any partition $\lambda = (a,b)$ with two parts a and b, we have

$$s_{a,b}(u) = (1-R_{12})\, u_{a,b} = u_{a,b} - u_{a+1,b-1} = u_a u_b - u_{a+1}u_{b-1} = \begin{vmatrix} u_a & u_{a+1} \\ u_{b-1} & u_b \end{vmatrix}.$$

The formula of Example 1 generalizes: for any partition λ of length ℓ, we have

$$s_\lambda(u) = \det(u_{\lambda_i+j-i})_{1\leq i,j\leq \ell}. \tag{2}$$

The equivalence of (1) with (2) is a formal consequence of the Vandermonde identity

$$\prod_{1\leq i<j\leq \ell}(\mathrm{x}_i - \mathrm{x}_j) = \det(\mathrm{x}_i^{\ell-j})_{1\leq i,j\leq \ell}$$

The monomials u_λ and the polynomials $s_\lambda(u)$ as λ runs over all partitions form two $\mathbb{Z}$-bases of the graded polynomial ring $\mathbb{A} := \mathbb{Z}[u_1, u_2, \ldots]$. In addition to the Giambelli formula (1), these two bases of $\mathbb{A}$ interact via the *Pieri rule.* To state the latter, recall that a skew diagram is a *horizontal strip* (resp. a *vertical strip*) if it does not contain two boxes in the same column (resp. row). For any partition λ and $p \geq 0$, we then have the Pieri rule

$$u_p \cdot s_\lambda(u) = \sum_\mu s_\mu(u)$$

with the sum over all partitions $\mu \supset \lambda$ such that μ/λ is a horizontal strip with p boxes.

Example 2. We have

$$u_3 \cdot s_{2,2,1}(u) = s_{5,2,1}(u) + s_{4,2,2}(u) + s_{4,2,1,1}(u) + s_{3,2,2,1}(u).$$

2.2. **Cohomology of Grassmannians.** Let $\mathrm{G}(m,n)$ be the Grassmannian of all m-dimensional complex linear subspaces of $\mathbb{C}^n$. The general linear group $\mathrm{GL}_n(\mathbb{C})$ acts transitively on $\mathrm{G}(m,n)$, and the stabilizer of the point $\langle e_1,\ldots,e_m\rangle$ under this action is the parabolic subgroup P_m of matrices in $\mathrm{GL}_n(\mathbb{C})$ in the block form

$$\left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right)$$

where the lower left block is an $(n-m)\times m$ zero matrix. It follows that

$$\mathrm{G}(m,n) = \mathrm{GL}_n / P_m$$

and hence that $\mathrm{G}(m,n)$ is a complex manifold of dimension $m(n-m)$. Furthermore, $\mathrm{G}(m,n)$ has the structure of an algebraic variety, and the same is true of all the geometric objects which appear in this paper.

The Grassmannian $\mathrm{G}(m,n)$ has a natural decomposition into Schubert cells $\mathfrak{X}^\circ_\lambda$, one for each partition λ whose diagram is contained in an $m\times(n-m)$ rectangle. The Schubert variety $\mathfrak{X}_\lambda$ is the closure of the cell $\mathfrak{X}^\circ_\lambda$, and is an algebraic subvariety of $\mathrm{G}(m,n)$ of complex codimension equal to $|\lambda|$. Concretely, if $e_1,\ldots,e_n$ is the canonical basis of $\mathbb{C}^n$, and F_i is the $\mathbb{C}$-linear span of $e_1,\ldots,e_i$, then

$$\mathfrak{X}_\lambda := \{V\in \mathrm{G}(m,n) \mid \dim(V\cap F_{n-m+i-\lambda_i}) \geq i,\ 1\leq i\leq m\}.$$

If $[\mathfrak{X}_\lambda]$ denotes the cohomology class of $\mathfrak{X}_\lambda$, then the cell decomposition of $\mathrm{G}(m,n)$ implies that there is an isomorphism of abelian groups

$$\mathrm{H}^*(\mathrm{G}(m,n),\mathbb{Z}) \cong \bigoplus_\lambda \mathbb{Z}[\mathfrak{X}_\lambda]. \tag{3}$$

For every integer p with $1\leq p\leq n-m$, the variety $\mathfrak{X}_p$ is the locus of subspaces V in $\mathrm{G}(m,n)$ which meet the subspace $F_{n-m+1-p}$ nontrivially. The varieties $\mathfrak{X}_p$ are the *special Schubert varieties*, and their cohomology classes $[\mathfrak{X}_p]$ are the *special Schubert classes*. Let $Q\to \mathrm{G}(m,n)$ denote the universal quotient vector bundle over $\mathrm{G}(m,n)$. Then for each integer $p\geq 0$, the p-th Chern class $c_p(Q)$ of Q is equal to $[\mathfrak{X}_p]$ in $\mathrm{H}^{2p}(\mathrm{G}(m,n),\mathbb{Z})$. We can now state the *Giambelli formula*

$$[\mathfrak{X}_\lambda] = s_{\widetilde{\lambda}}(c(Q)) \tag{4}$$

where $\widetilde{\lambda}$ is the conjugate partition of λ, and the Chern class polynomial $s_{\widetilde{\lambda}}(c(Q))$ is obtained from $s_{\widetilde{\lambda}}(u)$ by performing the substitutions $u_p\mapsto c_p(Q)$ for each integer p. The discussion in §2.1 and equation (4) imply that the *Pieri rule*

$$[\mathfrak{X}_p]\cdot[\mathfrak{X}_\lambda] = \sum_\mu [\mathfrak{X}_\mu] \tag{5}$$

also holds in $\mathrm{H}^*(\mathrm{G}(m,n),\mathbb{Z})$, where the sum is over all indexing partitions μ containing λ such that μ/λ is a horizontal strip with p boxes.

2.3. **Symmetric polynomials.** Fix an integer $n\geq 1$ and let $X_n := (x_1,\ldots,x_n)$, where the x_i are independent variables. The Weyl group S_n of GL_n acts on the polynomial ring $\mathbb{Z}[X_n]$ by permuting the variables, and the invariant subring is the ring $\Lambda_n := \mathbb{Z}[X_n]^{S_n}$ of symmetric polynomials. Two important families of elements of Λ_n are the elementary symmetric polynomials $e_p(X_n)$ and the complete symmetric polynomials $h_p(X_n)$. These are defined the by the generating function equations

$$\sum_{p=0}^{\infty} e_p(X_n)t^p = \prod_{i=1}^{n}(1+x_it) \quad \text{and} \quad \sum_{p=0}^{\infty} h_p(X_n)t^p = \prod_{i=1}^{n}(1-x_it)^{-1},$$

respectively, where t is a formal variable. The fundamental theorem of symmetric polynomials states that

$$\Lambda_n = \mathbb{Z}[e_1(X_n),\ldots,e_n(X_n)].$$

For each partition λ, the Schur polynomial $s_\lambda(X_n)$ is obtained from $s_\lambda(u)$ by making the substitution $u_p \mapsto h_p(X_n)$ for every integer p. In this way, we obtain the *Jacobi-Trudi formula*

$$s_\lambda(X_n) = \prod_{i<j}(1-R_{ij})\, h_\lambda(X_n) = \det(h_{\lambda_i+j-i}(X_n))_{1\leq i,j\leq \ell(\lambda)},$$

where we have set $h_\alpha := h_{\alpha_1}h_{\alpha_2}\cdots$ for every integer sequence α. If we similarly let $e_\alpha := e_{\alpha_1}e_{\alpha_2}\cdots$, then we have the dual Jacobi-Trudi formula

$$s_\lambda(X_n) = \prod_{i<j}(1-R_{ij})\, e_{\widetilde{\lambda}}(X_n) = \det(e_{\widetilde{\lambda}_i+j-i}(X_n))_{1\leq i,j\leq \ell(\widetilde{\lambda})}. \tag{6}$$

We deduce that

$$\Lambda_n = \bigoplus_{\lambda} \mathbb{Z}\, s_\lambda(X_n)$$

where the sum is over all partitions $\lambda = (\lambda_1,\ldots,\lambda_n)$ of length at most n.

The classical definition of Schur polynomials is as a quotient of alternant determinants. For each integer $k\geq 1$, let $\delta_k := (k,\ldots,1,0)$, and, for any integer vector $\alpha=(\alpha_1,\ldots,\alpha_n)$, let $x^\alpha := x_1^{\alpha_1}\cdots x_n^{\alpha_n}$. Define the alternating operator $\mathcal{A}$ on $\mathbb{Z}[X_n]$ by

$$\mathcal{A}(f) := \sum_{\varpi\in S_n}(-1)^{\ell(\varpi)}\varpi(f),$$

where $\ell(\varpi)$ denotes the length of the permutation ϖ. Then we have

$$s_\lambda(X_n) = \det(x_i^{\lambda_i+n-j})_{i,j}\Big/\det(x_i^{n-j})_{i,j} = \mathcal{A}(x^{\lambda+\delta_{n-1}})\big/\mathcal{A}(x^{\delta_{n-1}}). \tag{7}$$

We restate equation (7) in a form closer to its analogue for theta polynomials. For each $r\geq 1$, embed S_r into S_{r+1} by adjoining the fixed point $r+1$. Let $S_\infty = \cup_r S_r$ denote the corresponding infinite symmetric group of bijections $\varpi : \mathbb{Z}_{>0}\to\mathbb{Z}_{>0}$ such that $\varpi_i = i$ for all but finitely many i. Here, and in the sequel, ϖ_i denotes the value $\varpi(i)$, for each $i\geq 1$. The *code* of ϖ is the the sequence $\gamma=\gamma(\varpi)$ with $\gamma_i := \#\{j>i \mid \varpi_j<\varpi_i\}$. The *shape* of ϖ is the partition $\lambda=\lambda(\varpi)$ whose parts are the nonzero entries γ_i arranged in weakly decreasing order.

Example 3. An n-Grassmannian permutation ϖ is an element of S_∞ such that $\varpi_i<\varpi_{i+1}$ for each $i\neq n$. The shape of any such ϖ is the partition $\lambda := (\varpi_n - n,\ldots,\varpi_1-1)$. Conversely, any partition $\lambda=(\lambda_1,\ldots,\lambda_n)$ of length at most n corresponds to a unique n-Grassmannian permutation ϖ with $\lambda(\varpi)=\lambda$.

Let ϖ be an n-Grassmannian element of S_∞ with corresponding partition $\lambda(\varpi)$. If $\varpi_0 := (n, n-1, \ldots, 1)$ denotes the longest permutation in S_n, then observe that $\lambda(\varpi_0) = \delta_{n-1}$ and $\lambda(\varpi\varpi_0) = \lambda(\varpi) + \delta_{n-1}$. Therefore, we have

$$s_{\lambda(\varpi)}(X_n) = \mathcal{A}(x^{\lambda(\varpi\varpi_0)}) \Big/ \mathcal{A}(x^{\lambda(\varpi_0)}).$$

2.4. **Algebraic combinatorics.** Although not strictly necessary, in this subsection we will extend our family of variables to be an infinite sequence $X := (x_1, x_2, \ldots)$, and work with formal power series instead of polynomials. Define $h_p(X)$ using the generating function expansion

$$\sum_{p=0}^{\infty} h_p(X) t^p = \prod_{i=1}^{\infty} (1 - x_i t)^{-1},$$

and, for any partition λ, the Schur function

$$s_\lambda(X) := \prod_{i<j} (1 - R_{ij})\, h_\lambda(X).$$

A *tableau* T on the shape λ is a filling of the boxes of λ with positive integers, so that the entries are weakly increasing along each row from left to right and strictly increasing down each column. The content vector $c(T) = (n_1, n_2, \ldots)$ of T has n_j equal to the number of entries j in T. We then have the *tableau formula*

$$s_\lambda(X) = \sum_T x^{c(T)} \tag{8}$$

with the sum over all tableaux T of shape λ. Equation (8) shows that $s_\lambda(X)$ is a formal power series with nonnegative integer coefficients, which have a combinatorial interpretation.

Example 4. A Young tableaux of shape $\lambda = (2, 1)$ is of the form

$$\begin{array}{|c|c|}\hline a & b \\ \hline c & \\ \cline{1-1}\end{array}$$

where the positive integers a, b, c satisfy $a \leq b$ and $a < c$. We therefore have

$$s_{2,1}(X) = \sum_{\substack{a\leq b \\ a<c}} x_a x_b x_c = \sum_{a\neq b} x_a^2 x_b + 2 \sum_{a<b<c} x_a x_b x_c.$$

In order to represent the Schubert classes not only on Grassmannians, but on any partial flag variety, we require a generalization of (8) which involves certain symmetric functions defined by Stanley. The group S_∞ is generated by the simple transpositions $s_i = (i, i+1)$ for $i \geq 1$. A *reduced word* of a permutation ϖ in S_∞ is a sequence $a_1 \cdots a_\ell$ of positive integers such that $\varpi = s_{a_1} \cdots s_{a_\ell}$ and ℓ is minimal, so (by definition) equal to the length $\ell(\varpi)$ of ϖ.

The *nilCoxeter algebra* $\mathcal{N}_n$ of the symmetric group S_n is the free associative algebra with unit generated by the elements $\xi_1, \ldots, \xi_{n-1}$, modulo the relations

$$\begin{array}{rcll} \xi_i^2 & = & 0 & i \geq 1\ ; \\ \xi_i\xi_j & = & \xi_j\xi_i & |i-j| \geq 2\ ; \\ \xi_i\xi_{i+1}\xi_i & = & \xi_{i+1}\xi_i\xi_{i+1} & i \geq 1. \end{array}$$

For any $\varpi \in S_n$, choose a reduced word $a_1 \cdots a_\ell$ for ϖ and define $\xi_\varpi := \xi_{a_1} \ldots \xi_{a_\ell}$. Then the ξ_ϖ for $\varpi \in S_n$ are well defined, independent of the choice of reduced word

$a_1 \cdots a_\ell$, and form a free $\mathbb{Z}$-basis of $\mathcal{N}_n$. We denote the coefficient of $\xi_\varpi \in \mathcal{N}_n$ in the expansion of the element $\zeta \in \mathcal{N}_n$ by $\langle \zeta, \varpi \rangle$. We therefore have

$$\zeta = \sum_{\varpi \in S_n} \langle \zeta, \varpi \rangle \, \xi_\varpi,$$

for all $\zeta \in \mathcal{N}_n$.

Recall that t denotes an indeterminate and define

$$A(t) := (1 + t\xi_{n-1})(1 + t\xi_{n-2}) \cdots (1 + t\xi_1) \ ;$$
$$A(X) := A(x_1)A(x_2) \cdots$$

and a function $G_\varpi(X)$ for $\varpi \in S_n$ by

$$G_\varpi(X) := \langle A(X), \varpi \rangle.$$

It turns out that $G_\varpi(X)$ is symmetric in the x_i variables; it is called a *Stanley symmetric function*. Clearly, G_ϖ has nonnegative integer coefficients.

When ϖ is the n-Grassmannian permutation associated to a partition λ of length at most n, then $G_\varpi(X) = s_\lambda(X)$. More generally, for any permutation ϖ, when G_ϖ is expanded in the basis of Schur functions, we have

$$G_\varpi(X) = \sum_{\lambda \,:\, |\lambda| = \ell(\varpi)} c_\lambda^\varpi \, s_\lambda(X) \tag{9}$$

for some nonnegative integers c_λ^ϖ. That is, the symmetric function G_ϖ is *Schur positive*. There exist several different combinatorial interpretations of the coefficients c_λ^ϖ, but the most important one for our purposes uses the *transition trees* of Lascoux and Schützenberger, as subsequently defined.

We say that a permutation $\varpi = (\varpi_1, \varpi_2, \ldots)$ has a *descent* at position $i \geq 1$ if $\varpi_i > \varpi_{i+1}$. For $i < j$, let t_{ij} denote the transposition which interchanges i and j. For any permutation $\varpi \in S_\infty$, the *transition tree* $T(\varpi)$ of ϖ is constructed as follows. The tree $T(\varpi)$ is a rooted tree with nodes given by permutations of the same length $\ell(\varpi)$, and root ϖ. If $\varpi = 1$ or ϖ is Grassmannian, then set $T(\varpi) := \{\varpi\}$. Otherwise, let r be the largest descent of ϖ, and set

$$s := \max(j > r \mid \varpi_j < \varpi_r).$$

Define

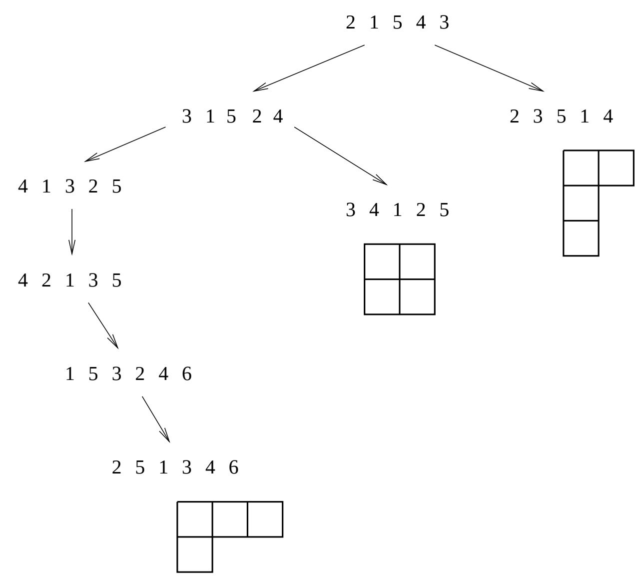
$$I(\varpi) := \{i \mid 1 \leq i < r \ \text{ and } \ \ell(\varpi t_{rs} t_{ir}) = \ell(\varpi)\}$$

and let

$$\Psi(\varpi) := \begin{cases} \{\varpi t_{rs} t_{ir} \mid i \in I(\varpi)\} & \text{if } I(\varpi) \neq \emptyset, \\ \Psi(1 \times \varpi) & \text{otherwise.} \end{cases}$$

We define $T(\varpi)$ recursively, by joining ϖ by an edge to each $v \in \Psi(\varpi)$, and attaching to each $v \in \Psi(\varpi)$ its tree $T(v)$. One can show that $T(\varpi)$ is a finite tree whose leaves are all Grassmannian permutations. Moreover, the Stanley coefficient c_λ^ϖ in equation (9) is equal to the number of leaves of shape λ in the transition tree $T(\varpi)$ associated to ϖ.

Example 5. The transition tree for the permutation $\varpi = (2, 1, 5, 4, 3)$ is shown below.



It follows that

$$G_{21543}(X) = s_{3,1}(X) + s_{2.2}(X) + s_{2,1,1}(X).$$

2.5. **Cohomology of flag manifolds.** Let $\{e_1, \ldots, e_n\}$ denote the standard basis of $E := \mathbb{C}^n$ and let $F_i = \langle e_1, \ldots, e_i \rangle$ be the subspace spanned by the first i vectors of this basis. The group $G = \mathrm{GL}_n(\mathbb{C})$ acts transitively on the space of all complete flags in E, and the stabilizer of the flag $F_\bullet$ is the Borel subgroup B of invertible upper triangular matrices. If $T \subset B$ denotes the maximal torus of diagonal matrices, then the Weyl group $W = S_n$ can be identified with $N_G(T)/T$.

The parabolic subgroups P of GL_n with $P \supset B$ correspond to sequences $a_1 < \cdots < a_p$ of positive integers with $a_p < n$. For any such P, the manifold GL_n /P parametrizes partial flags of subspaces

$$0 = E_0 \subset E_1 \subset \cdots \subset E_p \subset E = \mathbb{C}^n$$

with $\dim(E_r) = a_r$ for each $r \in [1, p]$. We agree that E_r and E will also denote the corresponding tautological vector bundles over GL_n /P. The associated parabolic subgroup W_P of S_n is generated by the simple transpositions s_i for $i \notin \{a_1, \ldots, a_p\}$.

There is a canonical presentation of the cohomology ring of GL_n /B, which gives geometric significance to the variables which appear in Section 2.3. Let $\mathrm{I}\Lambda_n$ denote the ideal of $\mathbb{Z}[X_n]$ generated by the homogeneous elements of positive degree in Λ_n, so that $\mathrm{I}\Lambda_n = \langle e_1(X_n), \ldots, e_n(X_n) \rangle$. We then have ring isomorphism

$$\mathrm{H}^*(\mathrm{GL}_n /B) \cong \mathbb{Z}[X_n]/\mathrm{I}\Lambda_n \tag{10}$$

which maps each variable x_i to $-c_1(E_i/E_{i-1})$. Moreover, for any parabolic subgroup P of GL_n, the projection map $\mathrm{GL}_n /B \to \mathrm{GL}_n /P$ induces an injection $\mathrm{H}^*(\mathrm{GL}_n /P) \hookrightarrow \mathrm{H}^*(\mathrm{GL}_n /B)$ of cohomology rings, and we have

$$\mathrm{H}^*(\mathrm{GL}_n /P) \cong \mathbb{Z}[X_n]^{W_P}/\mathrm{I}\Lambda_n^P, \tag{11}$$

where $\mathbb{Z}[X_n]^{W_P}$ denotes the W_P-invariant subring of $\mathbb{Z}[X_n]$, and $\mathrm{I}\Lambda_n^P$ is the ideal of $\mathbb{Z}[X_n]^{W_P}$ generated by $e_1(X_n), \ldots, e_n(X_n)$.

Consider the set

$$W^P := \{\varpi \in S_n \mid \ell(\varpi s_i) = \ell(\varpi)+1,\ \forall\, i \notin \{a_1,\ldots,a_p\},\ i<n\}$$

of minimal length W_P-coset representatives in S_n. We have a decomposition

$$\mathrm{GL}_n = \bigcup_{\varpi \in W^P} B\varpi P$$

and for each $\varpi \in W^P$, the B-orbit of ϖP in GL_n/P is the *Schubert cell* $\mathfrak{Y}^\circ_\varpi := B\varpi P/P$. Let $\mathfrak{Y}_\varpi$ be the closure of $\mathfrak{Y}^\circ_\varpi$ in GL_n/P, and set $\mathfrak{X}_\varpi := \mathfrak{Y}_{\varpi_0\varpi}$. The *Schubert class* $[\mathfrak{X}_\varpi]$ is the cohomology class of $\mathfrak{X}_\varpi$ in $\mathrm{H}^{2\ell(\varpi)}(\mathrm{GL}_n/P,\mathbb{Z})$. We thus obtain an isomorphism of abelian groups

$$\mathrm{H}^*(\mathrm{GL}_n/P,\mathbb{Z}) \cong \bigoplus_{\varpi\in W^P} \mathbb{Z}[\mathfrak{X}_\varpi]$$

which generalizes (3).

If V_1 and V_2 are complex vector bundles over a manifold M, define cohomology classes ϕ_p by the generating function equation

$$\sum_{p=0}^{\infty} \phi_p t^p = c_t(V_2^*)/c_t(V_1^*),$$

where $c_t(V_i^*) = 1 - c_1(V_i)t + c_2(V_i)t^2 - \cdots$ is the Chern polynomial of V_i^*, for $i=1,2$. Given any partition λ, the polynomial $s_\lambda(V_1-V_2)$ is obtained from $s_\lambda(u)$ via the substitution $u_p \mapsto \phi_p$ for every $p\in\mathbb{Z}$, so that

$$s_\lambda(V_1-V_2) := \prod_{i<j}(1-R_{ij})\,\phi_\lambda.$$

Recall that E_r for $r\in[1,p]$ and E denote the tautological and trivial rank n vector bundles over GL_n/P, respectively. For any $\varpi\in W^P$, we then have

$$[\mathfrak{X}_\varpi] = \sum_{\underline{\lambda}} c^\varpi_{\underline{\lambda}}\, s_{\widetilde{\lambda}^1}(E-E_1)s_{\widetilde{\lambda}^2}(E_1-E_2)\cdots s_{\widetilde{\lambda}^p}(E_{p-1}-E_p) \tag{12}$$

in $\mathrm{H}^*(\mathrm{GL}_n/P,\mathbb{Z})$, where the sum is over all sequences of partitions $\underline{\lambda} = (\lambda^1,\ldots,\lambda^p)$ and the coefficients $c^\varpi_{\underline{\lambda}}$ are given by

$$c^\varpi_{\underline{\lambda}} := \sum_{u_1\cdots u_p=\varpi} c^{u_1}_{\lambda^1}\cdots c^{u_p}_{\lambda^p} \tag{13}$$

summed over all factorizations $u_1\cdots u_p = \varpi$ such that $\ell(u_1)+\cdots+\ell(u_p) = \ell(\varpi)$ and $u_j(i) = i$ for all $j>1$ and $i\le a_{j-1}$. The nonnegative integers $c^{u_i}_{\lambda^i}$ which appear in the summands in (13) agree with the Stanley coefficients from equation (9). When $p=1$, the partial flag manifold GL_n/P is the Grassmannian $\mathrm{G}(a_1,n)$, and formula (12) specializes to equation (4).

Example 6. Let $P=B$ be the Borel subgroup, so that the flag manifold GL_n/B parametrizes complete flags of subspaces $0=E_0\subset E_1\subset\cdots\subset E_n = \mathbb{C}^n$. For each $i\in[1,n]$, let $x_i := -c_1(E_i/E_{i-1})$. Then for any partition λ, we have

$$s_{\widetilde{\lambda}}(E_{i-1}-E_i) = \begin{cases} x_i^r & \text{if } \lambda = r\ge 0,\\ 0 & \text{otherwise.}\end{cases}$$

Let $A_i(t) := (1+t\xi_{n-1})(1+t\xi_{n-2})\cdots(1+t\xi_i)$ in $\mathcal{N}_n[t]$, employing the notation of Section 2.4. Define the *Schubert polynomial* $\mathfrak{S}_\varpi(X_n)$ by

$$\mathfrak{S}_\varpi(X_n) := \langle A_1(x_1)\cdots A_{n-1}(x_{n-1}), \varpi\rangle. \tag{14}$$

It is then straightforward to show that formula (12) is equivalent to the statement that, for any permutation $\varpi \in S_n$, we have $[\mathfrak{X}_\varpi] = \mathfrak{S}_\varpi(X_n)$ in $\mathrm{H}^*(\mathrm{GL}_n/B, \mathbb{Z})$.

3. Theta polynomials

3.1. **Definition and Pieri rule.** Fix a nonnegative integer k. A partition λ is called *k-strict* if no part greater than k is repeated, that is, $\lambda_i > k \Rightarrow \lambda_i > \lambda_{i+1}$. A *strict* partition is the same as a 0-strict partition. For a general k-strict partition λ, we define the operator

$$R^\lambda := \prod_{i<j}(1-R_{ij}) \prod_{\lambda_i+\lambda_j>2k+j-i}(1+R_{ij})^{-1}$$

where the first product is over all pairs $i<j$ and second product is over pairs $i<j$ such that $\lambda_i+\lambda_j > 2k+j-i$. The *theta polynomial* $\Theta^{(k)}_\lambda(u)$ of level k is defined by

$$\Theta^{(k)}_\lambda(u) := R^\lambda u_\lambda. \tag{15}$$

We will write $\Theta_\lambda(u)$ for $\Theta^{(k)}_\lambda(u)$ when the level k is understood.

Example 7. (a) Suppose that $\lambda = (a,b)$ has two parts a and b with $a+b > 2k+1$. Then we have

$$\begin{aligned}\Theta_{a,b}(u) &= \frac{1-R_{12}}{1+R_{12}}\, u_{a,b} = (1-2R_{12}+2R_{12}^2-2R_{12}^3+\cdots)\, u_{a,b} \\ &= u_a u_b - 2u_{a+1}u_{b-1} + 2u_{a+2}u_{b-2} - 2u_{a+3}u_{b-3} + \cdots.\end{aligned}$$

(b) If $\lambda_i \leq k$ for each i, then

$$\Theta_\lambda(u) = \prod_{i<j}(1-R_{ij})\, u_\lambda = \det(u_{\lambda_i+j-i})_{1\leq i,j\leq \ell(\lambda)}. \tag{16}$$

(c) If $\lambda_i > k$ for all nonzero parts λ_i, then

$$\Theta_\lambda(u) = \prod_{i<j}\frac{1-R_{ij}}{1+R_{ij}}\, u_\lambda = \mathrm{Pfaffian}\left(\Theta_{\lambda_i,\lambda_j}(u)\right)_{1\leq i<j\leq \ell'} \tag{17}$$

where ℓ' is the least positive even integer such that $\ell' \geq \ell(\lambda)$.

Example 7 shows that as λ varies, the theta polynomial Θ_λ interpolates between the determinant (16) and the Pfaffian (17). The equality of Example 7(c) is a formal consequence of Schur's Pfaffian identity

$$\prod_{1\leq i<j\leq \ell'} \frac{\mathrm{x}_i-\mathrm{x}_j}{\mathrm{x}_i+\mathrm{x}_j} = \mathrm{Pfaffian}\left(\frac{\mathrm{x}_i-\mathrm{x}_j}{\mathrm{x}_i+\mathrm{x}_j}\right)_{1\leq i,j\leq \ell'}.$$

Example 8. Let $k=2$ and $\lambda=(5,2,1)$. Then we have

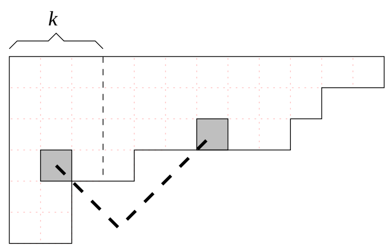
$$\begin{aligned}\Theta^{(2)}_{5,2,1}(u) &= \frac{1-R_{12}}{1+R_{12}}(1-R_{13})(1-R_{23})\, u_{5,2,1} \\ &= (1-2R_{12}+2R_{12}^2-2R_{12}^3)(1-R_{13}-R_{23})\, u_{5,2,1} \\ &= u_5u_2u_1 - u_5u_3 - 2u_6u_1^2 + u_6u_2 + 2u_7u_1.\end{aligned}$$

Recall that $\mathbb{A} = \mathbb{Z}[u_1, u_2, \ldots]$, and let $\mathbb{A}^{(k)}$ be the quotient of $\mathbb{A}$ by the ideal of relations

$$\frac{1-R_{12}}{1+R_{12}}\, u_{p,p} = u_p^2 + 2\sum_{i=1}^{p}(-1)^i u_{p+i}u_{p-i} = 0 \quad \text{for } p > k. \tag{18}$$

Then the monomials u_λ and the polynomials $\Theta_\lambda(u)$ as λ runs over all k-strict partitions form two $\mathbb{Z}$-bases of the graded ring $\mathbb{A}^{(k)}$. To state the Pieri rule for the $\Theta_\lambda(u)$, which holds modulo the relations (18), we need some further definitions.

We say that the box in row r and column c of a k-strict partition λ is *k-related* to the box in row r' and column c' if $|c-k-1|+r = |c'-k-1|+r'$. For example, the two grey boxes in the following partition are k-related.



For any two k-strict partitions λ and μ, we have a relation $\lambda \xrightarrow{p} \mu$ if $|\mu| = |\lambda| + p$ and μ is obtained by removing a vertical strip from the first k columns of λ and adding a horizontal strip to the resulting diagram, so that

(1) if one of the first k columns of μ has the same number of boxes as the same column of λ, then the bottom box of this column is k-related to at most one box of $\mu \smallsetminus \lambda$; and

(2) if a column of μ has fewer boxes than the same column of λ, then the removed boxes and the bottom box of μ in this column must each be k-related to exactly one box of $\mu \smallsetminus \lambda$, and these boxes of $\mu \smallsetminus \lambda$ must all lie in the same row.

If $\lambda \xrightarrow{p} \mu$, we let $\mathbb{D}$ be the set of boxes of $\mu \smallsetminus \lambda$ in columns $k+1$ and higher which are *not* mentioned in (1) or (2). Define $N(\lambda,\mu)$ to be the number of connected components of $\mathbb{D}$ which do not have a box in column $k+1$. Here we consider that two boxes are connected if they have at least one vertex in common.

For any k-strict partition λ and $p \geq 0$, we then have the *Pieri rule*

$$u_p \cdot \Theta_\lambda(u) = \sum_\mu 2^{N(\lambda,\mu)}\Theta_\mu(u) \tag{19}$$

in $\mathbb{A}^{(k)}$, where the sum over all k-strict partitions μ such that $\lambda \xrightarrow{p} \mu$.

Example 9. (a) When $k = 1$, we have the following equality in $\mathbb{A}^{(1)}$:

$$u_3 \cdot \Theta_{2,1}(u) = 2\,\Theta_6(u) + 4\,\Theta_{5,1}(u) + \Theta_{4,2}(u) + 2\,\Theta_{4,1,1}(u) + \Theta_{3,2,1}(u).$$

(b) If $|\lambda| + p \leq k$, then

$$u_p \cdot \Theta_\lambda(u) = \sum_\mu \Theta_\mu(u) \tag{20}$$

holds in $\mathbb{A}^{(k)}$, where the sum is over all partitions $\mu \supset \lambda$ such that $|\mu| = |\lambda| + p$ and μ/λ is a horizontal strip.

(c) If $\lambda_i > 2k$ for all nonzero parts λ_i and $p \geq 0$ is arbitrary, then

$$u_p \cdot \Theta_\lambda(u) = \sum_\mu 2^{N(\lambda,\mu)} \Theta_\mu(u) \tag{21}$$

holds in $\mathbb{A}^{(k)}$, where the sum is over all strict partitions $\mu \supset \lambda$ such that $|\mu| = |\lambda| + p$ and μ/λ is a horizontal strip, and $N(\lambda, \mu)$ equals the number of connected components of μ/λ which do not meet the first column.

In harmony with Example 7, Example 9 illustrates that as p and λ vary, the rule for the product $u_p \cdot \Theta_\lambda$ interpolates between the Pieri rule (20) for Schur polynomials and (21), which is the Pieri rule for the Schur Q-functions (see Example 11).

3.2. **Cohomology of Grassmannians.** Equip the vector space $\mathbb{C}^{2n}$ with the nondegenerate skew-symmetric bilinear form $(\ ,\ )$ defined by the conditions $(e_i, e_j) = 0$ for $i + j \neq 2n + 1$ and $(e_i, e_{2n+1-i}) = 1$ for $1 \leq i \leq n$. The symplectic group $\mathrm{Sp}_{2n}(\mathbb{C})$ is the subgroup of $\mathrm{GL}_{2n}(\mathbb{C})$ consisting of those elements g such that $(gv_1, gv_2) = (v_1, v_2)$, for every $v_1, v_2 \in \mathbb{C}^{2n}$. We say that a linear subspace V of $\mathbb{C}^{2n}$ is *isotropic* if the restriction of $(\ ,\ )$ to V vanishes identically. Since the form is nondegenerate, we have $\dim(V) \leq n$ for any isotropic subspace V. If V is isotropic and $\dim(V) = n$ then we call V a *Lagrangian* subspace.

Fix an integer k with $0 \leq k \leq n-1$. The isotropic Grassmannian $\mathrm{IG} = \mathrm{IG}(n-k, 2n)$ parametrizes all isotropic linear subspaces of dimension $n-k$ in $\mathbb{C}^{2n}$. The group Sp_{2n} acts transitively on $\mathrm{IG}(n-k, 2n)$, and the stabilizer of a fixed isotropic $(n-k)$-plane under this action is a maximal parabolic subgroup P_k of Sp_{2n}, so that $\mathrm{IG}(n-k, 2n) = \mathrm{Sp}_{2n}/P_k$.

The Schubert cells $\mathfrak{X}_\lambda^\circ$ in $\mathrm{IG}(n-k, 2n)$ are indexed by the k-strict partitions whose diagrams are contained in an $(n-k) \times (n+k)$ rectangle. The Schubert variety $\mathfrak{X}_\lambda$ is the closure of the $\mathfrak{X}_\lambda^\circ$, and has codimension $|\lambda|$ in IG. If F_i denotes the $\mathbb{C}$-linear span of $e_1, \ldots, e_i$ for each $i \in [1, 2n]$, then

$$\mathfrak{X}_\lambda := \{V \in \mathrm{IG} \mid \dim(V \cap F_{p_j(\lambda)}) \geq j \quad \forall\, 1 \leq j \leq n-k\},$$

where the strictly increasing index function $\{p_j(\lambda)\}_{1 \leq j \leq n-k}$ is defined by

$$p_j(\lambda) := n + k + j - \lambda_j - \#\{i < j \mid \lambda_i + \lambda_j > 2k + j - i\}.$$

If $[\mathfrak{X}_\lambda]$ denotes the cohomology class of $\mathfrak{X}_\lambda$ in $\mathrm{H}^{2|\lambda|}(\mathrm{IG}, \mathbb{Z})$, then we have a group isomorphism

$$\mathrm{H}^*(\mathrm{IG}(n-k, 2n), \mathbb{Z}) \cong \bigoplus_\lambda \mathbb{Z}[\mathfrak{X}_\lambda]. \tag{22}$$

The varieties $\mathfrak{X}_p$ for $1 \leq p \leq n+k$ are the *special Schubert varieties.* Let $Q \to \mathrm{IG}(n-k, 2n)$ denote the universal quotient vector bundle over IG, which has rank $n+k$. For every integer $p \geq 0$, the p-th Chern class $c_p(Q)$ is equal to the *special Schubert class* $[\mathfrak{X}_p]$ in $\mathrm{H}^{2p}(\mathrm{IG}(n-k, 2n), \mathbb{Z})$. We now have the *Giambelli formula*

$$[\mathfrak{X}_\lambda] = \Theta_\lambda(c(Q)) \tag{23}$$

where the Chern class polynomial $\Theta_\lambda(c(Q))$ is obtained from $\Theta_\lambda(u)$ by performing the substitutions $u_p \mapsto c_p(Q)$ for each integer p. Moreover, the *Pieri rule*

$$[\mathfrak{X}_p] \cdot [\mathfrak{X}_\lambda] = \sum_{\lambda \xrightarrow{p} \mu} 2^{N(\lambda,\mu)} [\mathfrak{X}_\mu] \tag{24}$$

holds in $\mathrm{H}^*(\mathrm{IG}(n-k,2n),\mathbb{Z})$, where the sum is over all partitions μ such that $\lambda \xrightarrow{p} \mu$ and the diagram of μ fits in an $(n-k)\times(n+k)$ rectangle.

The ring $\mathbb{A}^{(k)}$ is naturally isomorphic to the stable cohomology ring

$$\mathbb{H}(\mathrm{IG}_k) = \varprojlim \mathrm{H}^*(\mathrm{IG}(n-k,2n),\mathbb{Z})$$

of the isotropic Grassmannian IG. This is the inverse limit in the category of *graded* rings of the directed system

$$\cdots \leftarrow \mathrm{H}^*(\mathrm{IG}(n-k,2n),\mathbb{Z}) \leftarrow \mathrm{H}^*(\mathrm{IG}(n+1-k,2n+2),\mathbb{Z}) \leftarrow \cdots$$

Under this isomorphism, the variables u_p map to the Chern classes $c_p(Q)$ of the universal quotient bundle Q over IG. If S denotes the tautological subbundle of the trivial rank $2n$ vector bundle over $\mathrm{IG}(n-k,2n)$, then the symplectic form $(\ ,\ )$ gives a pairing $S\otimes Q\to\mathcal{O}$, which in turn produces an injection $S\hookrightarrow Q^*$. Using the Whitney sum formula $c(S)c(Q)=1$, we therefore obtain

$$c(Q^*)c(Q) = c(Q^*)c(S)^{-1} = c(Q^*/S).$$

The relations (18) arise from this, using the fact that $c_r(Q^*/S)=0$ for $r>2k$.

3.3. **Symmetric polynomials.** Let $c := (c_1, c_2, \ldots)$ be a sequence of commuting independent variables, set $c_0 := 1$ and $c_p = 0$ for $p<0$, and for every integer sequence α, let $c_\alpha := c_{\alpha_1}c_{\alpha_2}\cdots$. For any raising operator R, we let $R\,c_\alpha := c_{R\alpha}$. Consider the graded ring Γ which is the quotient of the polynomial ring $\mathbb{Z}[c]$ modulo the ideal generated by the relations

$$c_p^2 + 2\sum_{i=1}^{p}(-1)^i c_{p+i}c_{p-i} = 0, \quad \text{for all } p\geq 1.$$

The ring Γ is isomorphic to $\mathbb{A}^{(0)}$ and to the stable cohomology ring

$$\mathbb{H}(\mathrm{LG}) = \varprojlim \mathrm{H}^*(\mathrm{LG}(n,2n),\mathbb{Z})$$

of the Lagrangian Grassmannian LG, with the variables c_p mapping to the Chern classes $c_p(Q)$ of the universal quotient bundle $Q\to\mathrm{LG}$.

The Weyl group of the symplectic group Sp_{2n} is the *hyperoctahedral group* of signed permutations on the set $\{1,\ldots,n\}$, which is the semidirect product $S_n \ltimes \mathbb{Z}_2^n$ of S_n with $\mathbb{Z}_2^n = \{\pm 1\}^n$. We use a bar over an entry to denote a negative sign; thus $w = (\overline{2},\overline{3},1)$ maps $(1,2,3)$ to $(-2,-3,1)$. The group W_n is generated by the simple transpositions $s_i = (i,i+1)$ for $1\leq i\leq n-1$ and the sign change s_0 which satisfies $s_0(1)=\overline{1}$ and $s_0(j)=j$ for all $j\geq 2$. A *reduced word* of an element $w\in W_n$ is a sequence $a_1\cdots a_\ell$ of nonnegative integers of minimal length ℓ such that $w = s_{a_1}\cdots s_{a_\ell}$. The number ℓ is called the *length* of w, and denoted by $\ell(w)$.

Fix $n\geq 1$ and let $X_n := (x_1,\ldots,x_n)$, as in Section 2.3. There is a natural action of W_n on $\Gamma[X_n]$ which extends the action of S_n on $\mathbb{Z}[X_n]$, defined as follows. The simple transpositions s_i for $i\in[1,n-1]$ act by interchanging x_i and x_{i+1} while leaving all the remaining variables fixed. The reflection s_0 satisfies $s_0(x_1) = -x_1$ and $s_0(x_j) = x_j$ for all $j\geq 2$, while

$$s_0(c_p) := c_p + 2\sum_{j=1}^{p} x_1^j c_{p-j} \quad \text{for all } p\geq 1. \tag{25}$$

If t denotes a formal variable which is fixed by s_0, then we express equation (25) using generating functions as

$$s_0\left(\sum_{p=0}^{\infty} c_p t^p\right) = \frac{1+x_1t}{1-x_1t}\left(\sum_{p=0}^{\infty} c_p t^p\right).$$

For every integer p, define an element ${}^nc_p$ of $\Gamma[X_n]$ by

$${}^nc_p := \sum_{j=0}^{p} c_{p-j}e_j(X_n), \quad \text{for } p \geq 1$$

and let $\Gamma^{(n)} := \mathbb{Z}[{}^nc_1, {}^nc_2, \ldots]$. Let $\Gamma[X_n]^{W_n}$ denote the subring of W_n-invariants in $\Gamma[X_n]$. We claim that the generators ${}^nc_p$ of $\Gamma^{(n)}$ lie in $\Gamma[X_n]^{W_n}$. Indeed, we clearly have $s_j({}^nc_p) = {}^nc_p$ for each $j \geq 1$, while

$$\begin{aligned} s_0\left(\sum_{p=0}^{\infty} {}^nc_p t^p\right) &= s_0\left(\sum_{p=0}^{\infty} c_p t^p \cdot \prod_{j=1}^{n}(1+x_jt)\right) \\ &= \frac{1+x_1t}{1-x_1t}\left(\sum_{p=0}^{\infty} c_p t^p\right)\cdot(1-x_1t)\prod_{j=2}^{n}(1+x_jt) = \sum_{p=0}^{\infty} {}^nc_p t^p. \end{aligned}$$

In fact, there is an equality

$$\Gamma[X_n]^{W_n} = \Gamma^{(n)} = \mathbb{Z}[{}^nc_1, {}^nc_2, \ldots]. \tag{26}$$

The map which sends u_p to ${}^nc_p$ for every integer p induces a ring isomorphism $\mathbb{A}^{(n)} \cong \Gamma^{(n)}$. We therefore have

$$\Gamma^{(n)} = \bigoplus_{\lambda} \mathbb{Z}\,\Theta_\lambda(X_n) \tag{27}$$

where the sum is over all n-strict partitions λ, and the polynomial $\Theta_\lambda(X_n)$ is obtained from the theta polynomial $\Theta_\lambda^{(n)}(u)$ by making the substitution $u_p \mapsto {}^nc_p$ for all p. In other words, we have

$$\Theta_\lambda(X_n) = \Theta_\lambda^{(n)}(X_n) := R^\lambda\,({}^nc)_\lambda,$$

where we set $({}^nc)_\alpha := {}^nc_{\alpha_1}{}^nc_{\alpha_2}\cdots$ for any integer sequence α, and the raising operators R_{ij} in R^λ act on the subscripts α as usual.

For each $r \geq 1$, we embed W_r in W_{r+1} by adding the element $r+1$ which is fixed by W_r, and set $W_\infty := \cup_r W_r$. Let $w \in W_\infty$ be a signed permutation. To simpify the notation, let w_i denote the value $w(i)$, for each $i \geq 1$. Define a strict partition $\mu(w)$ whose parts are the absolute values of the negative entries of w, arranged in decreasing order. Let the *A-code* of w be the sequence γ with $\gamma_i := \#\{j > i \mid w_j < w_i\}$, and define a partition $\delta(w)$ whose parts are the nonzero entries γ_i arranged in weakly decreasing order. The *shape* of w is the partition $\lambda(w) := \mu(w) + \nu(w)$, where $\nu(w) := \widetilde{\delta(w)}$ is the conjugate of $\delta(w)$. One can show that the length $\ell(w)$ of w is equal to $|\lambda(w)|$.

Example 10. (a) An n-Grassmannian signed permutation w is an element of W_∞ such that $w_1 > 0$ and $w_i < w_{i+1}$ for each $i \neq n$. The shape of any such w is the

n-strict partition λ satisfying

$$\lambda_i = \begin{cases} n + |w_{n+i}| & \text{if } w_{n+i} < 0, \\ \#\{r \leq n \,:\, w_r > w_{n+i}\} & \text{if } w_{n+i} > 0. \end{cases}$$

Conversely, any n-strict partition λ corresponds to a unique n-Grassmannian permutation w with $\lambda(w) = \lambda$.

(b) Let $w_0 := (\overline{1}, \ldots, \overline{n})$ be the longest element of W_n. Then $\mu(w_0) = \delta_n$, $\nu(w_0) = \delta_{n-1}$, and $\lambda(w_0) = \delta_n + \delta_{n-1} = (2n-1, 2n-3, \ldots, 1)$.

Let w be an n-Grassmannian element of W_∞ with corresponding partition $\lambda(w)$, and let w_0 be the longest element of W_n. For any integer sequence α and composition β, let ${}^{\beta}c_\alpha := {}^{\beta_1}c_{\alpha_1}{}^{\beta_2}c_{\alpha_2}\cdots$, and set $R_{ij}{}^{\beta}c_\alpha := {}^{\beta}c_{R_{ij}\alpha}$ for each $i < j$. Consider the *multi-Schur Pfaffian*

$$(28) \qquad {}^{\nu(ww_0)}Q_{\lambda(ww_0)} := \prod_{i<j} \frac{1 - R_{ij}}{1 + R_{ij}}\, {}^{\nu(ww_0)}c_{\lambda(ww_0)}.$$

Define the alternating operator $\mathcal{A}'$ on $\Gamma[X_n]$ by

$$\mathcal{A}'(f) := \sum_{w \in W_n} (-1)^{\ell(w)} w(f),$$

where $\ell(w)$ is the length of the signed permutation w. We then have

$$(29) \qquad \Theta_{\lambda(w)}(X_n) = (-1)^{n(n+1)/2}\, \mathcal{A}'\left({}^{\nu(ww_0)}Q_{\lambda(ww_0)}\right) \Big/ \mathcal{A}'\left(x^{\lambda(w_0)}\right)$$

in $\Gamma[X_n]$.

3.4. **Algebraic combinatorics.** The combinatorial formulas discussed in this section require another incarnation of the ring Γ, using the formal power series known as Schur Q-functions. Fix a nonnegative integer k, set $X_k := (x_1, \ldots, x_k)$ and let $Z := (z_1, z_2, \ldots)$ be a sequence of variables. For any integer p, define $\vartheta_p = \vartheta_p(Z\,;X_k)$ by the generating function equation

$$(30) \qquad \sum_{p=0}^{\infty} \vartheta_p t^p = \prod_{i=1}^{\infty} \frac{1 + z_i t}{1 - z_i t} \prod_{j=1}^{k} (1 + x_j t).$$

By definition, for every k-strict partition λ, the theta polynomial $\Theta_\lambda(Z\,;X_k)$ is obtained from $\Theta^{(k)}_\lambda(u)$ by making the substitution $u_p \mapsto \vartheta_p$ for every integer p. In other words, we have

$$(31) \qquad \Theta_\lambda(Z\,;X_k) := R^\lambda\, \vartheta_\lambda$$

where, for any integer sequence α, $\vartheta_\alpha = \vartheta_{\alpha_1}\vartheta_{\alpha_2}\cdots$ and the raising operators R_{ij} in R^λ are applied to ϑ_λ as usual. Note that $\Theta_\lambda(Z\,;X_k)$ is a formal power series in the Z variables and a polynomial in the variables $x_1, \ldots, x_k$.

Example 11. Suppose that $k = 0$ and λ is a strict partition. Then the formal power series $Q_\lambda(Z) := \Theta_\lambda(Z)$ is a *Schur Q-function*. The map which sends c_p to $Q_p := Q_p(Z)$ for every integer p gives an isomorphism between the ring Γ defined in Section 3.3 and the ring $\mathbb{Z}[Q_1, Q_2, \ldots]$ of Schur Q-functions.

We wish to describe a tableau formula for $\Theta_\lambda(Z\,;X_k)$ which is analogous to the expression (8) for the Schur functions. The following key observation is used to define the relevant tableaux. In the Pieri rule (19) for the product $u_p\cdot\Theta_\lambda(u)$, all partitions μ which appear on the right hand side may be written as $\mu=(p+r,\nu)$ for some integer $r\geq 0$ and k-strict partition ν with $\nu\subset\lambda$. Moreover, if p is sufficiently large (for example $p>|\lambda|+2k$) and we write

$$u_p\cdot\Theta_\lambda(u)=\sum_{r,\nu}2^{n(\lambda/\nu)}\,\Theta_{(p+r,\nu)}(u), \tag{32}$$

with the sum over integers $r\geq 0$ and k-strict partitions $\nu\subset\lambda$ with $|\nu|=|\lambda|-r$, then the ν which appear in (32) and the exponents $n(\lambda/\nu)$ are independent of p. If this is the case, so that $\lambda\xrightarrow{p}(p+r,\nu)$ for any $p>|\lambda|+2k$, then we say that λ/ν is a *k-horizontal strip*.

Let λ and μ be any two k-strict partitions with $\mu\subset\lambda$. A *k-tableau* T of shape λ/μ is a sequence of k-strict partitions

$$\mu=\lambda^0\subset\lambda^1\subset\cdots\subset\lambda^r=\lambda$$

such that λ^i/λ^{i-1} is a k-horizontal strip for $1\leq i\leq r$. We represent T by a filling of the boxes in λ/μ with positive integers which is weakly increasing along each row and down each column, such that for each i, the boxes in T with entry i form the skew diagram λ^i/λ^{i-1}. For any k-tableau T we define $n(T):=\sum_i n(\lambda^i/\lambda^{i-1})$ and let $c(T)$ denote the content vector of T.

Let $\mathbf{P}$ denote the ordered alphabet $\{1'<2'<\cdots<k'<1<2<\cdots\}$. We say that the symbols $1',\ldots,k'$ are *marked*, while the rest are *unmarked*. A *k-bitableau* U of shape λ is a filling of the boxes in the diagram of λ with elements of $\mathbf{P}$ which is weakly increasing along each row and down each column, such that the marked entries are strictly increasing along each row, and the unmarked entries form a k-tableau T. We define

$$n(U):=n(T)\quad\text{and}\quad(zx)^{c(U)}:=z^{c(T)}\prod_{j=1}^{k}x_j^{m_j}$$

where m_j denotes the number of times that j' appears in U. For any k-strict partition λ, we then have the *tableau formula*

$$\Theta_\lambda(Z\,;X_k)=\sum_U 2^{n(U)}(zx)^{c(U)} \tag{33}$$

where the sum is over all k-bitableaux U of shape λ. Using the tableau formula (8) for Schur polynomials, we can rewrite equation (33) as

$$\Theta_\lambda(Z\,;X_k)=\sum_{\mu\subset\lambda}\sum_T 2^{n(T)}z^{c(T)}s_{\widetilde{\mu}}(X_k)$$

with the sums over all partitions $\mu\subset\lambda$ and k-tableaux T of shape λ/μ, respectively.

Example 12. If $k=0$ and λ is a strict partition, then (33) becomes

$$Q_\lambda(Z)=\sum_T 2^{n(T)}z^{c(T)} \tag{34}$$

summed over all 0-tableaux T of shape λ. Equation (34) is a tableau formula for the Schur Q-functions.

Example 13. (a) For any integer $p \geq 0$, we have

$$\Theta_p(Z\,;X_k) = \sum_{j=0}^{p} \sum_{|\alpha|=p-j} 2^{\#\alpha} z^\alpha s_{1^j}(X_k) = \sum_{j=0}^{p} Q_{p-j}(Z) e_j(X_k)$$

where the second sum is over all compositions α with $|\alpha| = p - j$, and $\#\alpha$ denotes the number of indices i such that $\alpha_i \neq 0$.

(b) Assume that $k \geq 1$. Then we have

$$\Theta_{1^p}(Z\,;X_k) = \sum_{j=0}^{p} \sum_{|\alpha|=p-j} 2^{\#\alpha} z^\alpha s_j(X_k) = \sum_{j=0}^{p} Q_{p-j}(Z) h_j(X_k).$$

The group W_∞ is generated by the reflections s_i for $i \geq 0$, and these generators are used to define reduced words and the length of signed permutations as in Section 2.4. The *nilCoxeter algebra* $\mathcal{W}_n$ of the hyperoctahedral group W_n is the free associative algebra with unit generated by the elements $\xi_0, \ldots, \xi_{n-1}$, modulo the relations

$$\begin{array}{rcll} \xi_i^2 & = & 0 & i \geq 0\,; \\ \xi_i\xi_j & = & \xi_j\xi_i & |i-j| \geq 2\,; \\ \xi_i\xi_{i+1}\xi_i & = & \xi_{i+1}\xi_i\xi_{i+1} & i \geq 1\,; \\ \xi_0\xi_1\xi_0\xi_1 & = & \xi_1\xi_0\xi_1\xi_0. & \end{array}$$

For any $w \in W_n$, choose a reduced word $a_1 \cdots a_\ell$ for w and set $\xi_w := \xi_{a_1} \ldots \xi_{a_\ell}$. Then the ξ_w are well defined and form a free $\mathbb{Z}$-basis of $\mathcal{W}_n$. We denote the coefficient of $\xi_w \in \mathcal{W}_n$ in the expansion of the element $\zeta \in \mathcal{W}_n$ by $\langle \zeta, w \rangle$.

Let t be an independent variable, define

$$C(t) := (1 + t\xi_{n-1}) \cdots (1 + t\xi_1)(1 + t\xi_0)(1 + t\xi_0)(1 + t\xi_1) \cdots (1 + t\xi_{n-1})$$

and let $C(Z) := C(z_1)C(z_2)\cdots$. Choose an integer k with $0 \leq k < n$ and set $A(X_k) := A(x_1)\cdots A(x_k)$. For any $w \in W_n$, the (restricted) type C *mixed Stanley function* $J_w(Z\,;X_k)$ is defined by

$$J_w(Z\,;X_k) := \langle C(Z)A(X_k), w \rangle.$$

Clearly, J_w is a power series in the Z variables and a polynomial in the X_k variables, and has nonnegative integer coefficients. One can show that J_w is symmetric in the Z and X_k variables separately.

When w is the k-Grassmannian permutation associated to a k-strict partition λ, then

$$J_w(Z\,;X_k) = \Theta_\lambda(Z\,;X_k).$$

This equality may be generalized as follows. We say that a signed permutation $w = (w_1, \ldots, w_n)$ has a *descent* at position $i \geq 1$ if $w_i > w_{i+1}$, and a descent at $i = 0$ if and only if $w_1 < 0$. A signed permutation $w \in W_n$ is *increasing up to* k if it has no descents less than k. This condition is automatically satisfied if $k = 0$, and for positive k it means that $0 < w_1 < w_2 < \cdots < w_k$. For any element $w \in W_n$ which is increasing up to k, we have

$$J_w(Z\,;X_k) = \sum_\lambda e_\lambda^w\, \Theta_\lambda(Z\,;X_k) \tag{35}$$

summed over k-strict partitions λ with $|\lambda| = \ell(w)$. The integers e_λ^w are nonnegative, that is, the function $J_w(Z\,;X_k)$ is *theta positive* when w is increasing up to k. These coefficients have only one known combinatorial interpretation, which is given below.

For positive integers $i < j$, define the reflections t_{ij} and $\bar{t}_{ij}, \bar{t}_{ii}$ in W_∞ by their right actions

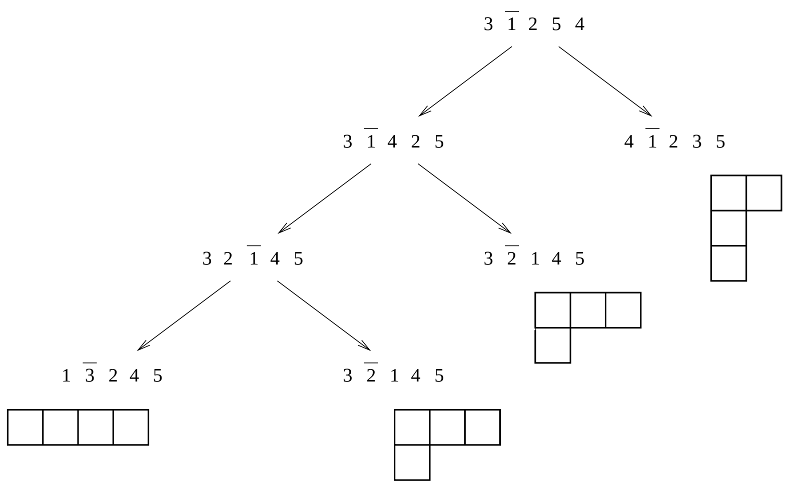
$$\begin{aligned}(w_1, \ldots, w_i, \ldots, w_j, \ldots)\, t_{ij} &= (w_1, \ldots, w_j, \ldots, w_i, \ldots),\\ (w_1, \ldots, w_i, \ldots, w_j, \ldots)\, \bar{t}_{ij} &= (w_1, \ldots, \overline{w}_j, \ldots, \overline{w}_i, \ldots), \quad \text{and}\\ (w_1, \ldots, w_i, \ldots)\, \bar{t}_{ii} &= (w_1, \ldots, \overline{w}_i, \ldots),\end{aligned}$$

and let $\bar{t}_{ji} := \bar{t}_{ij}$. For any $w \in W_\infty$ which is increasing up to k, we construct a rooted tree $T^k(w)$ with root w and whose nodes are elements of W_∞ as follows. Let r be the largest descent of w. If $w = 1$ or $r = k$, then set $T^k(w) := \{w\}$. Otherwise, let $s := \max(i > r \mid w_i < w_r)$ and $\Phi(w) := \Phi_1(w) \cup \Phi_2(w)$, where

$$\begin{aligned}\Phi_1(w) &:= \{wt_{rs}t_{ir} \mid 1 \leq i < r \ \text{ and } \ \ell(wt_{rs}t_{ir}) = \ell(w)\},\\ \Phi_2(w) &:= \{wt_{rs}\bar{t}_{ir} \mid i \geq 1 \ \text{ and } \ \ell(wt_{rs}\bar{t}_{ir}) = \ell(w)\}.\end{aligned}$$

We define $T^k(w)$ recursively, by joining w by an edge to each $v \in \Phi(w)$, and attaching to each $v \in \Phi(w)$ its tree $T^k(v)$. The finite tree $T^k(w)$ is the *k-transition tree* of w, and its leaves are all k-Grassmannian elements of W_∞. The coefficient e^w_λ in (35) is equal to the number of leaves of $T^k(w)$ which have shape λ.

Example 14. Suppose that $k = 1$. The 1-transition tree for the signed permutation $w = (3, \bar{1}, 2, 5, 4)$ in W_5 is shown below.



It follows that

$$J_{3\bar{1}254}(Z\,; X_1) = \Theta_4(Z\,; X_1) + 2\,\Theta_{3,1}(Z\,; X_1) + \Theta_{2,1,1}(Z\,; X_1).$$

3.5. **Cohomology of flag manifolds.** Let $\{e_1, \ldots, e_{2n}\}$ denote the standard symplectic basis of $E := \mathbb{C}^{2n}$ and let $F_i = \langle e_1, \ldots, e_i\rangle$ be the subspace spanned by the first i vectors of this basis, as in Section 3.2. The group $G = \mathrm{Sp}_{2n}$ acts transitively on the space of all complete isotropic flags in E, and the stabilizer of the flag $F_\bullet$ is a Borel subgroup B of G. Let $T \subset B$ denote the maximal torus of diagonal matrices in G, and the Weyl group $W = N_G(T)/T \cong W_n$.

The parabolic subgroups P of Sp_{2n} with $P \supset B$ correspond to sequences $a_1 < \cdots < a_p$ of nonnegative integers with $a_p < n$. For any such P, the manifold $\mathfrak{X} := \mathrm{Sp}_{2n}/P$ parametrizes partial flags of subspaces

$$E_\bullet \ : \ 0 \subset E_p \subset \cdots \subset E_1 \subset E = \mathbb{C}^{2n}$$

with E_1 isotropic and $\dim(E_r) = n - a_r$ for each $r \in [1,p]$. As usual, E_r and E will also denote the corresponding tautological vector bundles over $\mathfrak{X}$. The associated parabolic subgroup W_P of W_n is generated by the simple reflections s_i for $i \notin \{a_1, \ldots, a_p\}$.

There is a canonical presentation of the cohomology ring of Sp_{2n}/B, which gives geometric significance to the variables which appear in Section 3.3. Let $\mathrm{I}\Gamma^{(n)}$ denote the ideal of $\Gamma[X_n]$ generated by the homogeneous elements of positive degree in $\Gamma^{(n)}$, so that $\mathrm{I}\Gamma^{(n)} = \langle {}^n c_1, {}^n c_2, \ldots \rangle$. We then have ring isomorphism

$$\mathrm{H}^*(\mathrm{Sp}_{2n}/B, \mathbb{Z}) \cong \Gamma[X_n]/\mathrm{I}\Gamma^{(n)} \tag{36}$$

which maps each variable c_p to $c_p(E/E_n)$ and x_i to $c_1(E_{n+1-i}/E_{n-i})$ for $1 \leq i \leq n$. Furthermore, for any parabolic subgroup P of Sp_{2n}, the projection map $\mathrm{Sp}_{2n}/B \to \mathrm{Sp}_{2n}/P$ induces an injection $\mathrm{H}^*(\mathrm{Sp}_{2n}/P, \mathbb{Z}) \hookrightarrow \mathrm{H}^*(\mathrm{Sp}_{2n}/B, \mathbb{Z})$ on cohomology rings, and we have

$$\mathrm{H}^*(\mathfrak{X}, \mathbb{Z}) \cong \Gamma[X_n]^{W_P}/\mathrm{I}\Gamma_P^{(n)}, \tag{37}$$

where $\Gamma[X_n]^{W_P}$ denotes the W_P-invariant subring of $\Gamma[X_n]$, and $\mathrm{I}\Gamma_P^{(n)}$ is the ideal of $\Gamma[X_n]^{W_P}$ generated by ${}^n c_1, {}^n c_2, \ldots$

We have a decomposition

$$\mathrm{Sp}_{2n} = \bigcup_{w \in W^P} BwP$$

where

$$W^P := \{w \in W_n \mid \ell(ws_i) = \ell(w) + 1, \ \forall\, i \notin \{a_1, \ldots, a_p\}, \ i < n\}$$

is the set of minimal length W_P-coset representatives in W_n. For each $w \in W^P$, the B-orbit of wP in $\mathfrak{X}$ is the *Schubert cell* $\mathfrak{Y}_w^\circ := BwP/P$. The *Schubert variety* $\mathfrak{Y}_w$ is the closure of $\mathfrak{Y}_w^\circ$ in Sp_{2n}/P. Then $\mathfrak{X}_w := \mathfrak{Y}_{w_0 w}$ has codimension $\ell(w)$ in $\mathfrak{X}$, and its cohomology class $[\mathfrak{X}_w]$ is a *Schubert class*. The cell decomposition of $\mathfrak{X}$ implies that there is an isomorphism of abelian groups

$$\mathrm{H}^*(\mathfrak{X}, \mathbb{Z}) \cong \bigoplus_{w \in W^P} \mathbb{Z}[\mathfrak{X}_w]$$

which generalizes (22).

Recall that E_r for $r \in [1,p]$ and E denote the tautological and trivial vector bundles over $\mathfrak{X}$, of rank $n - a_r$ and $2n$, respectively. For any $w \in W^P$, we then have

$$[\mathfrak{X}_w] = \sum_{\underline{\lambda}} f_{\underline{\lambda}}^w \, \Theta_{\lambda^1}^{(a_1)}(E - E_1) s_{\lambda^2}(E_1 - E_2) \cdots s_{\lambda^p}(E_{p-1} - E_p) \tag{38}$$

in $\mathrm{H}^*(\mathfrak{X}, \mathbb{Z})$, where the sum is over all sequences of partitions $\underline{\lambda} = (\lambda^1, \ldots, \lambda^p)$ with λ^1 being a_1-strict, and the coefficients $f_{\underline{\lambda}}^w$ are given by

$$f_{\underline{\lambda}}^w := \sum_{u_1 \cdots u_p = w} e_{\lambda^1}^{u_1} c_{\lambda^2}^{u_2} \cdots c_{\lambda^p}^{u_p} \tag{39}$$

summed over all factorizations $u_1 \cdots u_p = w$ such that $\ell(u_1) + \cdots + \ell(u_p) = \ell(w)$, $u_j \in S_\infty$ for $j \geq 2$, and $u_j(i) = i$ for all $j > 1$ and $i \leq a_{j-1}$. The nonnegative integers $e^{u_1}_{\lambda^1}$ and $c^{u_i}_{\lambda^i}$ which appear in the summands in (39) are the same as the ones in equations (35) and (9), respectively. When $p = 1$, the partial flag manifold $\mathfrak{X}$ is the isotropic Grassmannian $\mathrm{IG}(n-a_1, 2n)$, and formula (38) specializes to equation (23). In general, the polynomial $\Theta^{(a_1)}_{\lambda^1}(E-E_1)$ in (38) is defined by pulling back the polynomial $\Theta^{(a_1)}_{\lambda^1}(c(E/E_1))$ under the natural projection map $\mathfrak{X} \to \mathrm{IG}(n-a_1, 2n)$ which sends a partial flag $E_\bullet$ to E_1.

Example 15. Let $P = B$ be the Borel subgroup, so that the flag manifold Sp_{2n}/B parametrizes flags of subspaces $0 \subset E_n \subset \cdots \subset E_1 \subset E = \mathbb{C}^{2n}$ with E_1 Lagrangian and $\dim(E_i) = n + 1 - i$ for each $i \in [1, n]$. For each i, let $x_i := -c_1(E_i/E_{i+1})$, and observe that since $E/E_1 \cong E_1^*$, for any integer p, we have $c_p(E/E_1) = e_p(x_1, \ldots, x_n) = e_p(X_n)$, using the definition of Chern classes. For every strict partition λ, define the *$\widetilde{Q}$-polynomial* $\widetilde{Q}_\lambda(X_n)$ by the formula

$$\widetilde{Q}_\lambda(X_n) := \prod_{i<j} \frac{1-R_{ij}}{1+R_{ij}}\, e_\lambda(X_n).$$

Since $a_1 = 0$, for any strict partition λ, we have

$$\Theta^{(a_1)}_\lambda(E - E_1) = \prod_{i<j} \frac{1-R_{ij}}{1+R_{ij}}\, c_\lambda(E/E_1) = \widetilde{Q}_\lambda(X_n).$$

For any $w \in W_n$, we define the *symplectic Schubert polynomial* $\mathfrak{C}_w(X_n)$ by

$$\mathfrak{C}_w(X_n) := \sum_{v,\varpi,\lambda} e^v_\lambda\, \widetilde{Q}_\lambda(X_n)\mathfrak{S}_\varpi(-X_n) \tag{40}$$

where the sum is over all factorizations $v\varpi = w$ and strict partitions λ with $\ell(v) + \ell(\varpi) = \ell(w)$, $\varpi \in S_n$, and $|\lambda| = \ell(v)$. Employing computations similar to those in Example 6, we then see that formula (38) is equivalent to the statement that for any element $w \in W_n$, we have $[\mathfrak{X}_w] = \mathfrak{C}_w(X_n)$ in $\mathrm{H}^*(\mathrm{Sp}_{2n}/B, \mathbb{Z})$.

4. Eta polynomials

4.1. **Definition and Pieri rule.** In the theory of eta polynomials, we must distinguish between the case of level zero and that of positive level. Given any strict partition λ, the *eta polynomial* $H^{(0)}_\lambda(u)$ of level 0 is defined by

$$H^{(0)}_\lambda(u) := 2^{-\ell(\lambda)} \prod_{i<j} \frac{1-R_{ij}}{1+R_{ij}}\, u_\lambda.$$

As a polynomial in the variables u_p, $H^{(0)}_\lambda(u)$ may have nonintegral coefficients. However, if we introduce new variables ω_p such that $u_p = 2\omega_p$ for each $p \geq 1$, then $H^{(0)}_\lambda$ is a polynomial in the ω_p with integer coefficients.

Let $\mathbb{B}^{(0)}$ be the quotient of the polynomial ring $\mathbb{Z}[\omega_1, \omega_2, \ldots]$ modulo the ideal of relations

$$\omega_p^2 + 2\sum_{i=1}^{p-1}(-1)^i \omega_{p+i}\omega_{p-i} + (-1)^p \omega_{2p} = 0 \quad \text{for } p \geq 1.$$

Then, for any $p \geq 0$, the Pieri rule

$$\omega_p \cdot H_\lambda^{(0)}(u) = \sum_\mu 2^{N'(\lambda,\mu)} H_\mu^{(0)}(u) \tag{41}$$

holds in $\mathbb{B}^{(0)}$, where the sum is over all strict partitions $\mu \supset \lambda$ such that $|\mu| = |\lambda| + p$ and μ/λ is a horizontal strip, and $N'(\lambda,\mu)$ is one less than the number of connected components of μ/λ.

Assume next that $k \geq 1$, and let $\omega_1, \ldots, \omega_{k-1}, \omega_k, \omega'_k, \omega_{k+1}, \ldots$ be independent variables related to the variables $u_1, u_2, \ldots$ by the equations

$$u_p = \begin{cases} \omega_p & \text{if } p < k, \\ \omega_k + \omega'_k & \text{if } p = k, \\ 2\omega_p & \text{if } p > k. \end{cases} \tag{42}$$

The eta polynomials $H_\lambda^{(k)}(u)$ lie in the ring $\mathbb{Z}[\omega_1, \ldots, \omega_{k-1}, \omega_k, \omega'_k, \omega_{k+1}, \ldots]$, and are indexed by *typed* k-strict partitions λ.

A *typed k-strict partition* is a pair consisting of a k-strict partition λ together with an integer in $\{0,1,2\}$ called the *type* of λ, and denoted $\mathrm{type}(\lambda)$, such that $\mathrm{type}(\lambda) = 0$ if and only if $\lambda_i \neq k$ for all $i \geq 1$. The type is usually omitted from the notation for the pair $(\lambda, \mathrm{type}(\lambda))$.

For a general typed k-strict partition λ, we define the operator

$$R^\lambda := \prod_{i<j} (1 - R_{ij}) \prod_{\lambda_i + \lambda_j \geq 2k+j-i} (1 + R_{ij})^{-1} \tag{43}$$

where the first product is over all pairs $i < j$ and second product is over pairs $i < j$ such that $\lambda_i + \lambda_j \geq 2k + j - i$. Let R be any finite monomial in the operators R_{ij} which appears in the expansion of the formal power series R^λ in (43). If $\mathrm{type}(\lambda) = 0$, then set $R \star u_\lambda := u_{R\lambda}$. Suppose that $\mathrm{type}(\lambda) \neq 0$, let r be the least index such that $\lambda_r = k$, and set

$$\widehat{\alpha} := (\alpha_1, \ldots, \alpha_{r-1}, \alpha_{r+1}, \ldots, \alpha_\ell)$$

for any integer sequence α of length ℓ. If R involves any factors R_{ij} with $i = r$ or $j = r$, then let $R \star u_\lambda := \frac{1}{2} u_{R\lambda}$. If R has no such factors, then let

$$R \star u_\lambda := \begin{cases} \omega_k \, u_{\widehat{R\lambda}} & \text{if } \mathrm{type}(\lambda) = 1, \\ \omega'_k \, u_{\widehat{R\lambda}} & \text{if } \mathrm{type}(\lambda) = 2. \end{cases}$$

We define the *eta polynomial* $H_\lambda^{(k)}(u)$ of level k by

$$H_\lambda^{(k)}(u) := 2^{-\ell_k(\lambda)} R^\lambda \star u_\lambda.$$

Here the *k-length* $\ell_k(\lambda)$ of a (typed) k-strict partition λ is the number of parts λ_i which are strictly greater than k. It is easy see that $H_\lambda^{(k)}(u)$ is a polynomial in the variables ω_p and ω'_k with integer coefficients. We will write $H_\lambda(u)$ for $H_\lambda^{(k)}(u)$ when the level k is understood.

Example 16. (a) Consider the typed 2-strict partition $\lambda = (3,2,2)$ with $\mathrm{type}(\lambda) = 2$. Then we have

$$\begin{aligned} H^{(2)}_\lambda(u) &= \frac{1}{2}\frac{1-R_{12}}{1+R_{12}}(1-R_{13})(1-R_{23}) \star u_{3,2,2} \\ &= \frac{1}{2}(1-2R_{12}+2R_{12}^2-2R_{12}^3)(1-R_{13}-R_{23}+R_{13}R_{23}) \star u_{3,2,2} \\ &= \omega_3\omega'_2(\omega_2+\omega'_2) - \omega_3^2\omega_1 + \omega_4\omega_3 - \omega_4\omega'_2\omega_1 + \omega_6\omega_1 - \omega_7. \end{aligned}$$

(b) Let λ be a k-strict partition with $\lambda_i = k \geq 1$ for some i, and let $H_\lambda(u)$ and $H'_\lambda(u)$ denote the eta polynomials of level k indexed by λ of type 1 and 2, respectively. Then we have

$$H_\lambda(u) + H'_\lambda(u) = 2^{-\ell_k(\lambda)} R^\lambda \, u_\lambda$$

where R^λ denotes the operator (43).

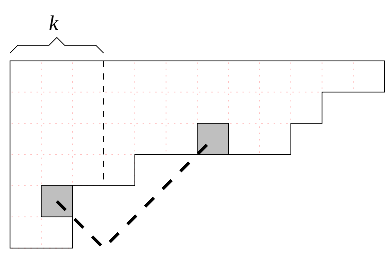
Let $\mathbb{B}^{(k)}$ be the quotient of polynomial ring $\mathbb{Z}[\omega_1, \ldots, \omega_{k-1}, \omega_k, \omega'_k, \omega_{k+1}, \ldots]$ modulo the ideal of relations

$$\omega_p^2 + \sum_{i=1}^{p}(-1)^i \omega_{p+i} u_{p-i} = 0 \quad \text{for } p > k, \tag{44}$$

$$\omega_k\omega'_k + \sum_{i=1}^{k}(-1)^i \omega_{k+i}\omega_{k-i} = 0, \tag{45}$$

where the u_i obey the equations (42). For every typed k-strict partition λ, we define a monomial ω_λ as follows. If $\mathrm{type}(\lambda) \neq 2$, then set $\omega_\lambda := \omega_{\lambda_1}\omega_{\lambda_2}\cdots$. If $\mathrm{type}(\lambda) = 2$ then define ω_λ by the same product formula, but replacing each occurrence of ω_k with ω'_k. The monomials ω_λ and the polynomials $H_\lambda(u)$ as λ runs over all typed k-strict partitions form two $\mathbb{Z}$-bases of the graded ring $\mathbb{B}^{(k)}$. The Pieri rule for the products $\omega_p \cdot H_\lambda(u)$ holds only modulo the relations (44) and (45); again we need some further definitions to state it.

We say that the box in row r and column c of a k-strict partition λ is k'-*related* to the box in row r' and column c' if $|c-k-1/2| + r = |c'-k-1/2| + r'$. For example, the two grey boxes in the following partition are k'-related.



For any two k-strict partitions λ and μ, the relation $\lambda \xrightarrow{p} \mu$ is defined as in Section 3.1, but replacing 'k-related' by 'k'-related' throughout. The set $\mathbb{D}$ of boxes of $\mu \smallsetminus \lambda$ is defined in the same way, and the integer $N'(\lambda,\mu)$ is equal to the number (respectively, one less than the number) of connected components of $\mathbb{D}$, if $p \leq k$ (respectively, if $p > k$).

If λ and μ are typed k-strict partitions, then we write $\lambda \xrightarrow{p} \mu$ if the underlying k-strict partitions satisfy $\lambda \xrightarrow{p} \mu$, with the added condition that $\mathrm{type}(\lambda)+\mathrm{type}(\mu) \neq 3$.

Let $c(\lambda,\mu)$ be the number of columns of μ among the first k which do not have more boxes than the corresponding column of λ, and

$$d(\lambda,\mu) := c(\lambda,\mu) + \max(\mathrm{type}(\lambda),\mathrm{type}(\mu)).$$

If $p \neq k$, then set $\delta_{\lambda\mu} = 1$. If $p = k$ and $N'(\lambda,\mu) > 0$, then set

$$\delta_{\lambda\mu} = \delta'_{\lambda\mu} := 1/2,$$

while if $N'(\lambda,\mu) = 0$, define

$$\delta_{\lambda\mu} := \begin{cases} 1 & \text{if } d(\lambda,\mu) \text{ is odd,} \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \delta'_{\lambda\mu} = \begin{cases} 1 & \text{if } d(\lambda,\mu) \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

For any typed k-strict partition λ and $p \geq 0$, we then have the *Pieri rule*

$$\omega_p \cdot H_\lambda(u) = \sum_\mu \delta_{\lambda\mu}\, 2^{N'(\lambda,\mu)}\, H_\mu(u), \tag{46}$$

in $\mathbb{B}^{(k)}$, where the sum over all typed k-strict partitions μ such that $\lambda \xrightarrow{p} \mu$. Furthermore, the product $\omega'_k \cdot H_\lambda(u)$ is obtained by replacing $\delta_{\lambda\mu}$ with $\delta'_{\lambda\mu}$ throughout.

4.2. **Cohomology of Grassmannians.** Equip the vector space $\mathbb{C}^{2n}$ with the nondegenerate symmetric bilinear form $(\ ,\ )$ defined by the conditions $(e_i,e_j) = 0$ for $i+j \neq 2n+1$ and $(e_i, e_{2n+1-i}) = 1$ for $1 \leq i \leq n$. The special orthogonal group $\mathrm{SO}_{2n}(\mathbb{C})$ is the subgroup of $\mathrm{SL}_{2n}(\mathbb{C})$ consisting of those elements g such that $(gv_1, gv_2) = (v_1,v_2)$, for every $v_1, v_2 \in \mathbb{C}^{2n}$. We say that a subspace V is *isotropic* if the restriction of $(\ ,\ )$ to V vanishes identically. Since the form is nondegenerate, we have $\dim(V) \leq n$ for any isotropic subspace V. For each $i \in [1,2n]$, let F_i denote the $\mathbb{C}$-linear span of $e_1,\ldots,e_i$.

Fix an integer k with $0 \leq k \leq n-1$. If $k \geq 1$, then the orthogonal Grassmannian $\mathrm{OG} = \mathrm{OG}(n-k,2n)$ parametrizes isotropic linear subspaces of dimension $n-k$ in $\mathbb{C}^{2n}$. The group SO_{2n} acts transitively on $\mathrm{OG}(n-k,2n)$, and we have $\mathrm{OG}(n-k,2n) = \mathrm{SO}_{2n}/P_k$, where P_k is a maximal parabolic subgroup of SO_{2n}. When $k=0$, the locus of maximal isotropic subspaces has two isomorphic connected components, called the two *families*, each of which is a single SO_{2n}-orbit. The orthogonal Grassmannian $\mathrm{OG}(n,2n) = \mathrm{SO}_{2n}/P_0$ parametrizes one of these components, which we take to be the family containing F_n.

We agree that when $k=0$, a typed 0-strict partition is the same as a strict partition, and that all such partitions have type 1. The Schubert cells $\mathfrak{X}^\circ_\lambda$ in $\mathrm{OG}(n-k,2n)$ are indexed by the typed k-strict partitions λ whose diagrams are contained in an $(n-k)\times(n+k-1)$ rectangle. We have

$$X^\circ_\lambda := \{V \in \mathrm{OG} \mid \dim(V \cap F_r) = \#\{j \mid p_j(\lambda) \leq r\} \quad \forall\, r\},$$

where the strictly increasing index function $\{p_j(\lambda)\}_{1\leq j\leq n-k}$ is defined by

$$p_j(\lambda) := n+k+j-\lambda_j - \#\{\, i<j \mid \lambda_i+\lambda_j \geq 2k+j-i\,\} \\ - \begin{cases} 1 & \text{if } \lambda_j > k, \text{ or } \lambda_j = k < \lambda_{j-1} \text{ and } n+j+\mathrm{type}(\lambda) \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

The Schubert variety $\mathfrak{X}_\lambda$ is the closure of the $\mathfrak{X}^\circ_\lambda$, and has codimension $|\lambda|$ in OG. If $[\mathfrak{X}_\lambda]$ denotes the cohomology class of $\mathfrak{X}_\lambda$ in $\mathrm{H}^{2|\lambda|}(\mathrm{OG},\mathbb{Z})$, then we have a group isomorphism

$$\mathrm{H}^*(\mathrm{OG}(n-k,2n),\mathbb{Z}) \cong \bigoplus_{\lambda} \mathbb{Z}[\mathfrak{X}_\lambda]. \tag{47}$$

The varieties $\mathfrak{X}_p$ for $1 \leq p \leq n+k-1$, together with $\mathfrak{X}'_k$ when $k \geq 1$, are the *special Schubert varieties*, and their classes in cohomology are the *special Schubert classes*. The convention here is that $\mathfrak{X}_k$ (respectively, $\mathfrak{X}'_k$) is indexed by the partition k of type 1 (respectively, type 2). As in the Lie types A and C, the special Schubert varieties $\mathfrak{X}_p$ and $\mathfrak{X}'_k$ can be viewed as the locus of all isotropic linear subspaces V which meet a given isotropic or coisotropic linear subspace nontrivially.

Let $Q \to \mathrm{OG}(n-k,2n)$ denote the universal quotient vector bundle over OG, of rank $n+k$. For $k=0$, we have $c_p(Q) = 2[\mathfrak{X}_p]$ for all $p \geq 1$, while for $k \geq 1$, we have

$$c_p(Q) = \begin{cases} [\mathfrak{X}_p] & \text{if } p < k, \\ [\mathfrak{X}_k] + [\mathfrak{X}'_k] & \text{if } p = k, \\ 2[\mathfrak{X}_p] & \text{if } p > k \end{cases}$$

in $\mathrm{H}^{2p}(\mathrm{OG}(n-k,2n),\mathbb{Z})$, in agreement with (42). We can now state the *Giambelli formula*

$$[\mathfrak{X}_\lambda] = H_\lambda(c(Q)) \tag{48}$$

where the polynomial $H_\lambda(c(Q))$ is obtained from $H_\lambda^{(k)}(u)$ by performing the substitutions $\omega_p \mapsto [\mathfrak{X}_p]$ and $\omega'_k \mapsto [\mathfrak{X}'_k]$ for every integer p.

Furthermore, the *Pieri rules* (41) and (46) hold in $\mathrm{H}^*(\mathrm{OG}(n-k,2n),\mathbb{Z})$. For instance, the latter rule is valid for $k \geq 1$ and states that

$$[\mathfrak{X}_p] \cdot [\mathfrak{X}_\lambda] = \sum_{\mu} \delta_{\lambda\mu}\, 2^{N'(\lambda,\mu)} [\mathfrak{X}_\mu] \tag{49}$$

summed over all typed k-strict partitions μ such that $\lambda \xrightarrow{p} \mu$ and the diagram of μ fits in an $(n-k)\times(n+k-1)$ rectangle. Moreover, the product $[\mathfrak{X}'_k]\cdot[\mathfrak{X}_\lambda]$ is obtained by replacing $\delta_{\lambda\mu}$ with $\delta'_{\lambda\mu}$ in (49).

Example 17. For the Grassmannian $\mathrm{OG}(5,14)$ we have $n=7$ and $k=2$. Let λ denote the partition $(8,7,2,1,1)$ of type 1. We then have the Pieri formulas

$$\begin{aligned} [\mathfrak{X}_2]\cdot[\mathfrak{X}_\lambda] &= [\mathfrak{X}_{8,7,4,1,1}] + [\mathfrak{X}_{8,7,3,2,1}] + [\mathfrak{X}_{8,7,6}] \\ [\mathfrak{X}'_2]\cdot[\mathfrak{X}_\lambda] &= [\mathfrak{X}_{8,7,4,1,1}] + [\mathfrak{X}_{8,7,3,2,1}] \end{aligned}$$

where the indexing partitions on the right hand side are all of type 0 or 1.

The ring $\mathbb{B}^{(k)}$ is naturally isomorphic to the stable cohomology ring

$$\mathbb{H}(\mathrm{OG}_k) = \varprojlim \mathrm{H}^*(\mathrm{OG}(n-k,2n),\mathbb{Z})$$

of the orthogonal Grassmannian OG, where the inverse limit is defined as in Section 3.2. Under this isomorphism, the variables ω_p and ω'_k map to the special Schubert classes $[\mathfrak{X}_p]$ and $[\mathfrak{X}'_k]$ in the cohomology ring of OG.

4.3. **Symmetric polynomials.** Let $b := (b_1, b_2, \ldots)$ be a sequence of commuting variables, and set $b_0 := 1$ and $b_p = 0$ for $p < 0$. Consider the graded ring Γ' which is the quotient of the ring $\mathbb{Z}[b]$ modulo the ideal generated by the relations

$$b_p^2 + 2\sum_{i=1}^{p-1}(-1)^i b_{p+i}b_{p-i} + (-1)^p b_{2p} = 0, \quad \text{for all } p \geq 1.$$

The ring Γ' is isomorphic to $\mathbb{B}^{(0)}$ and to the stable cohomology ring

$$\varprojlim \mathrm{H}^*(\mathrm{OG}(n, 2n), \mathbb{Z})$$

of the maximal orthogonal Grassmannian $\mathrm{OG}(n, 2n)$, with the variables b_p mapping to the special Schubert classes $[\mathfrak{X}_p]$. We regard Γ as a subring of Γ' using the injection which sends c_p to $2b_p$ for all $p \geq 1$.

The Weyl group $\widetilde{W}_n$ for the root system D_n is the subgroup of W_n consisting of all signed permutations with an even number of sign changes. The group $\widetilde{W}_n$ is an extension of S_n by the element $s_\Box = s_0 s_1 s_0$, which acts on the right by

$$(w_1, w_2, \ldots, w_n)s_\Box = (\overline{w}_2, \overline{w}_1, w_3, \ldots, w_n).$$

Fix $n \geq 2$ and let $X_n := (x_1, \ldots, x_n)$. There is a natural action of $\widetilde{W}_n$ on $\Gamma'[X_n]$ which extends the action of S_n on $\mathbb{Z}[X_n]$, defined as follows. The simple reflections s_i for $i > 0$ act by interchanging x_i and x_{i+1} and leaving all the remaining variables fixed. The reflection $s_\Box$ maps (x_1, x_2) to $(-x_2, -x_1)$, fixes the x_j for $j \geq 3$, and satisfies, for any $p \geq 1$,

$$s_\Box(b_p) := b_p + (x_1 + x_2)\sum_{j=0}^{p-1}\left(\sum_{a+b=j} x_1^a x_2^b\right) c_{p-1-j}. \tag{50}$$

If t is a formal variable which is fixed by $s_\Box$, then we express equation (50) using generating functions as

$$s_\Box\left(\sum_{p=0}^{\infty} c_p t^p\right) = \frac{1 + x_1 t}{1 - x_1 t} \cdot \frac{1 + x_2 t}{1 - x_2 t} \cdot \left(\sum_{p=0}^{\infty} c_p t^p\right).$$

For every integer p, define an element ${}^n b_p$ of $\Gamma'[X_n]$ by

$${}^n b_p := \begin{cases} e_p(X_n) + 2\sum_{i=0}^{p-1} e_i(X_n) b_{p-i} & \text{if } p < n, \\ \sum_{i=0}^{p} e_i(X_n) b_{p-i} & \text{if } p \geq n, \end{cases}$$

let

$${}^n b'_n := \sum_{i=0}^{n-1} e_i(X_n) b_{n-i}$$

and set $B^{(n)} := \mathbb{Z}[{}^n b_1, \ldots, {}^n b_{n-1}, {}^n b_n, {}^n b'_n, {}^n b_{n+1}, \ldots]$. Observe that we have

$${}^n c_p = \begin{cases} {}^n b_p & \text{if } p < n, \\ {}^n b_n + {}^n b'_n & \text{if } p = n, \\ 2 \cdot {}^n b_p & \text{if } p > n \end{cases}$$

and thus $\Gamma^{(n)}$ is a subring of $B^{(n)}$.

Let $\Gamma'[X_n]^{\widetilde{W}_n}$ denote the subring of $\widetilde{W}_n$-invariants in $\Gamma'[X_n]$. Then there is an equality

$$\Gamma'[X_n]^{\widetilde{W}_n} = B^{(n)} = \mathbb{Z}[{}^nb_1,\ldots,{}^nb_{n-1},{}^nb_n,{}^nb'_n,{}^nb_{n+1},\ldots]. \tag{51}$$

The map which sends ω_p to ${}^nb_p$ for every integer p and ω'_n to ${}^nb'_n$ induces a ring isomorphism $\mathbb{B}^{(n)} \cong B^{(n)}$. We therefore have

$$B^{(n)} = \bigoplus_\lambda \mathbb{Z}\, H_\lambda(X_n) \tag{52}$$

where the sum is over all typed n-strict partitions λ, and the polynomial $H_\lambda(X_n)$ is obtained from the eta polynomial $H_\lambda^{(n)}(u)$ by making the substitutions $\omega_p \mapsto {}^nb_p$ for all p and $\omega'_n \mapsto {}^nb'_n$. In other words, we have

$$H_\lambda(X_n) = H_\lambda^{(n)}(X_n) := 2^{-\ell_n(\lambda)} R^\lambda \star ({}^nc)_\lambda.$$

For each $r \geq 2$, we embed $\widetilde{W}_r$ in $\widetilde{W}_{r+1}$ by adding the element $r+1$ which is fixed by $\widetilde{W}_r$, and set $\widetilde{W}_\infty := \cup_r \widetilde{W}_r$. Let w be a signed permutation in $\widetilde{W}_\infty$. Define a strict partition $\mu(w)$ whose parts are the absolute values of the negative entries of w minus one, arranged in decreasing order. Let the *A-code* of w be the sequence γ with $\gamma_i := \#\{j > i \mid w_j < w_i\}$, define a partition $\delta(w)$ whose parts are the nonzero entries γ_i arranged in weakly decreasing order, and let $\nu(w)$ be the conjugate of $\delta(w)$. The *shape* of w is defined to be the partition $\lambda(w) := \mu(w) + \nu(w)$.

Example 18. (a) An element w of $\widetilde{W}_\infty$ is n-Grassmannian if $\ell(ws_i) > \ell(w)$ for all $i \neq n$. The type of an n-Grassmannian element w is 0 if $|w_1| = 1$, and 1 (respectively, 2) if $w_1 > 1$ (respectively, if $w_1 < -1$). There is a type preserving bijection between the n-Grassmannian elements of $\widetilde{W}_\infty$ and typed n-strict partitions, given as follows. If the element w corresponds to the typed n-strict partition λ, then for each $j \geq 1$, we have

$$\lambda_i = \begin{cases} n-1+|w_{n+i}| & \text{if } w_{n+i} < 0, \\ \#\{r \leq n \,:\, |w_r| > w_{n+i}\} & \text{if } w_{n+i} > 0. \end{cases}$$

The shape $\lambda(w)$ of w agrees with the typed n-strict partition associated to w, if w has type 0 or 1. However, this may fail if $w_1 < -1$, for instance the 2-Grassmannian element $v := \overline{3}5\overline{1}24$ in $\widetilde{W}_5$ is associated to the typed partition of shape $(2,2,1)$, while $\lambda(v) = (3,1,1)$.

(b) The longest element of $\widetilde{W}_n$ is given by

$$\widetilde{w}_0 = \begin{cases} (\overline{1},\ldots,\overline{n}) & \text{if } n \text{ is even}, \\ (1,\overline{2},\ldots,\overline{n}) & \text{if } n \text{ is odd}. \end{cases}$$

Then $\mu(\widetilde{w}_0) = \delta_{n-1}$, $\nu(\widetilde{w}_0) = \delta_{n-1}$, and $\lambda(\widetilde{w}_0) = 2\delta_{n-1} = (2n-2, 2n-4, \ldots, 2)$.

Let w be an n-Grassmannian element of $\widetilde{W}_\infty$ with corresponding partition $\lambda(w)$, and let $\widetilde{w}_0$ be the longest element of $\widetilde{W}_n$. Define

$${}^{\nu(w\widetilde{w}_0)}P_{\lambda(w\widetilde{w}_0)} := 2^{-r} \prod_{i<j} \frac{1-R_{ij}}{1+R_{ij}}\, {}^{\nu(w\widetilde{w}_0)}c_{\lambda(w\widetilde{w}_0)},$$

where r is the length of the partition $\lambda(w\widetilde{w}_0)$. The alternating operator $\mathcal{A}''$ on $\Gamma'[X_n]$ is given by

$$\mathcal{A}''(f) := \sum_{w\in\widetilde{W}_n} (-1)^{\ell(w)} w(f).$$

We then have

$$H_{\lambda(w)}(X_n) = (-1)^{n(n-1)/2}\cdot 2^{n-1}\,\mathcal{A}''\left({}^{\nu(w\widetilde{w}_0)}P_{\lambda(w\widetilde{w}_0)}\right)\Big/\,\mathcal{A}''\left(x^{\lambda(\widetilde{w}_0)}\right) \tag{53}$$

in $\Gamma'[X_n]$.

4.4. **Algebraic combinatorics.** Let k be a nonnegative integer. In this section, we will define the formal power series $H^{(k)}_\lambda(Z\,;X_k)$ which are the analogues in Lie type D of the power series $\Theta^{(k)}_\lambda(Z\,;X_k)$ discussed in Section 3.4. Their definition is easy when $k=0$: in this case, the index λ is a strict partition and we have

$$H^{(0)}_\lambda(Z) := 2^{-\ell(\lambda)}\Theta^{(0)}_\lambda(Z) = P_\lambda(Z),$$

where $P_\lambda(Z)$ is a Schur P-function. The map which sends ω_j to $P_j := P_j(Z) = Q_j(Z)/2$ for every positive integer j gives an isomorphism between the ring $\mathbb{B}^{(0)}\cong\Gamma'$ defined in Section 4.3 and the ring $\mathbb{Z}[P_1,P_2,\ldots]$ of Schur P-functions. Since $H^{(0)}_\lambda(Z)$ is a scalar multiple of $\Theta^{(0)}_\lambda(Z)=Q_\lambda(Z)$, we immediately obtain a tableau formula for $H^{(0)}_\lambda(Z)$ from the tableau formula (34) for the Schur Q-functions.

Assume next that $k\geq 1$. Set

$$\eta_r = \eta_r(Z\,;X_k) := \begin{cases} e_r(X_k) + 2\sum_{i=0}^{r-1} P_{r-i}(Z)e_i(X_k) & \text{if } r<k,\\ \sum_{i=0}^{r} P_{r-i}(Z)e_i(X_k) & \text{if } r\geq k\end{cases}$$

and

$$\eta'_k = \eta'_k(Z\,;X_k) = \sum_{i=0}^{k-1} P_{k-i}(Z)e_i(X_k).$$

For any $r\geq 0$, if ϑ_r is defined by equation (30), then we have

$$\vartheta_r = \begin{cases}\eta_r & \text{if } r<k,\\ \eta_k+\eta'_k & \text{if } r=k,\\ 2\eta_r & \text{if } r>k.\end{cases}$$

By definition, for every typed k-strict partition λ, the eta polynomial $H_\lambda(Z\,;X_k)$ is obtained from $H^{(k)}_\lambda(u)$ by making the substitutions $\omega_r\mapsto\eta_r$ for every integer r and $\omega'_k\mapsto\eta'_k$. In other words, we have

$$H_\lambda(Z\,;X_k) := 2^{-\ell_k(\lambda)}\,R^\lambda\star\vartheta_\lambda.$$

We proceed to give a tableau formula for $H_\lambda(Z\,;X_k)$ which is analogous to the formulas (8) and (33).

Let λ and μ be k-strict partitions with $\mu\subset\lambda$, and choose any $p>|\lambda|+2k-1$. If $|\lambda|=|\mu|+r$ and $\lambda\xrightarrow{p}(p+r,\mu)$, then we say that λ/μ is a *k'-horizontal strip.* We call a box in row r and column c of a Young diagram a *left box* if $c\leq k$ and a *right box* if $c>k$. If $\mu\subset\lambda$ are two k-strict partitions such that λ/μ is a k'-horizontal strip, we define $\lambda_0=\mu_0=+\infty$ and agree that the diagrams of λ and μ include all boxes $[0,c]$ in row zero. We let $\mathbb{E}$ denote the set of right boxes of μ (including boxes in row zero) which are bottom boxes of λ in their column and are not $(k-1)$-related to a left box of λ/μ.

If λ and μ are typed k-strict partitions with $\mu \subset \lambda$, we say that λ/μ is a *typed k'-horizontal strip* if the underlying k-strict partitions are such that λ/μ is a k'-horizontal strip and in addition $\mathrm{type}(\lambda) + \mathrm{type}(\mu) \neq 3$. In this case we let $n(\lambda/\mu)$ denote the number of connected components of $\mathbb{E}$ minus one.

Suppose that λ is any typed k-strict partition. Let $\mathbf{P}'$ denote the ordered alphabet $\{\widehat{1} < \widehat{2} < \cdots < \widehat{k} < 1, 1^\circ < 2, 2^\circ < \cdots\}$. We say that the symbols $\widehat{1}, \ldots, \widehat{k}$ are *marked*, while the rest are *unmarked*. A *typed k'-tableau* T of shape λ/μ is a sequence of typed k-strict partitions

$$\mu = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^r = \lambda$$

such that λ^i/λ^{i-1} is a typed k'-horizontal strip for $1 \leq i \leq r$. We represent T by a filling of the boxes in λ/μ with unmarked elements of $\mathbf{P}'$ which is weakly increasing along each row and down each column, such that for each i, the boxes in T with entry i or i° form the skew diagram λ^i/λ^{i-1}, and we use i (resp. i°) if and only if $\mathrm{type}(\lambda^i) \neq 2$ (resp. $\mathrm{type}(\lambda^i) = 2$), for every $i \geq 1$. For any typed k'-tableau T we define $n(T) = \sum_i n(\lambda^i/\lambda^{i-1})$ and let $c(T) = (r_1, r_2, \ldots)$ be the content vector of T, so that r_i denotes the number of times that i or i° appears in T, for each $i \geq 1$.

A *typed k'-bitableau* U of shape λ is a filling of the boxes in the diagram of λ with elements of $\mathbf{P}'$ which is weakly increasing along each row and down each column, such that the unmarked entries form a typed k'-tableau T of shape λ/μ with $\mathrm{type}(\mu) \neq 2$, and the marked entries are a filling of μ which is strictly increasing along each row. We define

$$n(U) = n(T) \quad \text{and} \quad (zx)^{c(U)} = z^{c(T)} \prod_{j=1}^{k} x_j^{m_j}$$

where m_j denotes the number of times that $\widehat{j}$ appears in U. For any typed k-strict partition λ, we then have the *tableau formula*

$$H_\lambda(Z\,; X_k) = \sum_U 2^{n(U)} (zx)^{c(U)} \tag{54}$$

where the sum is over all typed k'-bitableaux U of shape λ. Using the tableau formula (8), we can rewrite equation (54) as

$$H_\lambda(Z\,; X_k) = \sum_{\mu \subset \lambda} \sum_T 2^{n(T)} z^{c(T)} s_{\widetilde{\mu}}(X_k) \tag{55}$$

with the sums over all partitions $\mu \subset \lambda$ and typed k'-tableaux T of shape λ/μ, respectively.

Example 19. Suppose that $k = 1$, and for any $r \geq 1$, let $H_{1^r}(Z\,; X_1)$ and $H'_{1^r}(Z\,; X_1)$ denote the eta polynomials indexed by 1^r of type 1 and 2, respectively. Then for any integer $r \geq 0$, we deduce from equation (55) that

$$\begin{aligned} H_r(Z\,; X_1) &= P_r(Z) + P_{r-1}(Z)x_1, \\ H_{1^r}(Z\,; X_1) &= P_r(Z) + 2P_{r-1}(Z)x_1 + \cdots + 2P_1(Z)x_1^{r-1} + x_1^r, \text{ and} \\ H'_{1^r}(Z\,; X_1) &= P_r(Z). \end{aligned}$$

Consider the set $\mathbb{N}_\Box := \{\Box, 1, \ldots\}$ whose members index the simple reflections in $\widetilde{W}_\infty$. These elements generate the group $\widetilde{W}_\infty$ and are used to define reduced words and the length of signed permutations as in Section 3.4. The *nilCoxeter algebra* $\widetilde{\mathcal{W}}_n$

of the group $\widetilde{W}_n$ is the free associative algebra with unit generated by the elements $\xi_\square, \xi_1, \ldots, \xi_{n-1}$, modulo the relations

$$\begin{array}{rcll} \xi_i^2 & = & 0, & i \in \mathbb{N}_\square \ ; \\ \xi_\square \xi_1 & = & \xi_1 \xi_\square, & \\ \xi_\square \xi_2 \xi_\square & = & \xi_2 \xi_\square \xi_2, & \\ \xi_i \xi_{i+1} \xi_i & = & \xi_{i+1} \xi_i \xi_{i+1}, & i > 0 \ ; \\ \xi_i \xi_j & = & \xi_j \xi_i, & j > i+1, \text{ and } (i,j) \neq (\square, 2). \end{array}$$

As in the previous sections, for any $w \in \widetilde{W}_n$, choose a reduced word $a_1 \cdots a_\ell$ for w and define $\xi_w := \xi_{a_1} \ldots \xi_{a_\ell}$. The ξ_w form a free $\mathbb{Z}$-basis of $\widetilde{\mathcal{W}}_n$, and we denote by $\langle \zeta, w \rangle$ the coefficient of ξ_w in the expansion of the element $\zeta \in \widetilde{\mathcal{W}}_n$.

Let t be an independent variable, define

$$D(t) := (1 + t\xi_{n-1}) \cdots (1 + t\xi_2)(1 + t\xi_1)(1 + t\xi_\square)(1 + t\xi_2) \cdots (1 + t\xi_{n-1}),$$

and let $D(Z) = D(z_1)D(z_2)\cdots$. Choose an integer k with $0 \leq k < n$. For any $w \in W_n$, the (restricted) type D *mixed Stanley function* $I_w(Z\,;X_k)$ is defined by

$$I_w(Z\,;X_k) := \langle D(Z)A(X_k), w \rangle.$$

The power series $I_w(Z\,;X_k)$ is symmetric in the Z and X_k variables, separately, and has nonnegative integer coefficients.

When w is the k-Grassmannian element associated to a typed k-strict partition λ, then we have

$$I_w(Z\,;X_k) = H_\lambda(Z\,;X_k). \tag{56}$$

We say that an element $w = (w_1, \ldots, w_n)$ has a *descent* at position $i \in \mathbb{N}_\square$ if $\ell(ws_i) < \ell(w)$. If $k \geq 2$, we say that w is *increasing up to* k if it has no descents less than k; this means that $|w_1| < w_2 < \cdots < w_k$. By convention we agree that every element of $\widetilde{W}_\infty$ is increasing up to $\square$ and also increasing up to 1. We can now state the following generalization of equality (56): for any element $w \in \widetilde{W}_n$ which is increasing up to k, we have

$$I_w(Z\,;X_k) = \sum_\lambda d_\lambda^w \, H_\lambda(Z\,;X_k) \tag{57}$$

summed over typed k-strict partitions λ with $|\lambda| = \ell(w)$. The integers d_λ^w are nonnegative, in other words, the function $I_w(Z\,;X_k)$ is *eta positive* when w is increasing up to k. A combinatorial interpretation for these coefficients is provided below.

For any $w \in \widetilde{W}_\infty$ which is increasing up to k, we construct the k-transition tree $\widetilde{T}^k(w)$ with nodes given by elements of $\widetilde{W}_\infty$ and root w in a manner parallel to Section 3.4. Let r be the largest descent of w. If $w = 1$, or $k \neq 1$ and $r = k$, or $k = 1$ and $r \in \{\square, 1\}$, then set $\widetilde{T}^k(w) := \{w\}$. Otherwise, let $s := \max(i > r \mid w_i < w_r)$ and define $\widetilde{\Phi}(w) := \widetilde{\Phi}_1(w) \cup \widetilde{\Phi}_2(w)$, where

$$\widetilde{\Phi}_1(w) := \{wt_{rs}t_{ir} \mid 1 \leq i < r \ \text{ and } \ \ell(wt_{rs}t_{ir}) = \ell(w)\},$$
$$\widetilde{\Phi}_2(w) := \{wt_{rs}\overline{t}_{ir} \mid i \neq r \ \text{ and } \ \ell(wt_{rs}\overline{t}_{ir}) = \ell(w)\}.$$

To define $\widetilde{T}^k(w)$, we join w by an edge to each $v \in \widetilde{\Phi}(w)$, and attach to each $v \in \widetilde{\Phi}(w)$ its tree $\widetilde{T}^k(v)$. Then $\widetilde{T}^k(w)$ is a finite tree called the *k-transition tree* of w, and its leaves are all k-Grassmannian elements of $\widetilde{W}_\infty$. The coefficient d_λ^w in (57) is equal to the number of leaves of $\widetilde{T}^k(w)$ of shape λ.

4.5. **Cohomology of flag manifolds.** Let $\{e_1, \ldots, e_{2n}\}$ denote the standard orthogonal basis of $E := \mathbb{C}^{2n}$ and let $F_i = \langle e_1, \ldots, e_i \rangle$ be the subspace spanned by the first i vectors of this basis, as in Section 4.2. The group $G = \mathrm{SO}_{2n}$ acts on the space of all complete isotropic flags in E with two orbits, determined by the family of the maximal isotropic subspace in a given flag. The stabilizer of the flag $F_\bullet$ is a Borel subgroup B of G. Let $T \subset B$ denote the maximal torus of diagonal matrices in G, and the Weyl group $W = N_G(T)/T \cong \widetilde{W}_n$.

The parabolic subgroups P of SO_{2n} with $P \supset B$ correspond to sequences $a_1 < \cdots < a_p$ of elements of $\mathbb{N}_\Box$ with $a_p < n$. For any such P, the manifold $\mathfrak{X} := \mathrm{SO}_{2n}/P$ parametrizes partial flags of subspaces

$$E_\bullet \ : \ 0 \subset E_p \subset \cdots \subset E_1 \subset E = \mathbb{C}^{2n}$$

with E_1 isotropic, $\dim(E_r) = n - a_r$ for each $r \in [1, p]$, and E_1 in a given family if $a_1 = \Box$. The associated parabolic subgroup W_P of $\widetilde{W}_n$ is generated by the simple reflections s_i for $i \notin \{a_1, \ldots, a_p\}$. As usual, E_r for $r \in [1, p]$ and E will also denote the corresponding tautological vector bundles over $\mathfrak{X}$.

There is a canonical presentation of the cohomology ring of SO_{2n}/B, which gives geometric significance to the variables which appear in Section 4.3. Let $\mathrm{I}B^{(n)}$ denote the ideal of $\Gamma'[X_n]_{\mathbb{Q}} := \Gamma'[X_n] \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by the homogeneous elements of positive degree in $B^{(n)}$, so that $\mathrm{I}B^{(n)} = \langle {}^n b'_n, {}^n b_1, {}^n b_2, \ldots \rangle$. We then have ring isomorphism

$$\mathrm{H}^*(\mathrm{SO}_{2n}/B, \mathbb{Q}) \cong \Gamma'[X_n]_{\mathbb{Q}}/\mathrm{I}B^{(n)} \tag{58}$$

which maps each variable b_p to $c_p(E/E_n)/2$ and x_i to $c_1(E_{n+1-i}/E_{n-i})$ for $1 \leq i \leq n$. Furthermore, for any parabolic subgroup P of SO_{2n}, the projection map $\mathrm{SO}_{2n}/B \to \mathrm{SO}_{2n}/P$ induces an injection $\mathrm{H}^*(\mathrm{SO}_{2n}/P) \hookrightarrow \mathrm{H}^*(\mathrm{SO}_{2n}/B)$ on cohomology rings, and we have

$$\mathrm{H}^*(\mathfrak{X}, \mathbb{Q}) \cong \Gamma'[X_n]_{\mathbb{Q}}^{W_P}/\mathrm{I}B_P^{(n)}, \tag{59}$$

where $\Gamma'[X_n]_{\mathbb{Q}}^{W_P}$ denotes the W_P-invariant subring of $\Gamma'[X_n]_{\mathbb{Q}}$, and $\mathrm{I}B_P^{(n)}$ is the ideal of $\Gamma'[X_n]_{\mathbb{Q}}^{W_P}$ generated by ${}^n b'_n, {}^n b_1, {}^n b_2, \ldots$

We have a decomposition

$$\mathrm{SO}_{2n} = \bigcup_{w \in W^P} BwP$$

where

$$W^P := \{w \in \widetilde{W}_n \mid \ell(ws_i) = \ell(w) + 1, \ \forall\, i \notin \{a_1, \ldots, a_p\}, \ i < n\}$$

is the set of minimal length W_P-coset representatives in $\widetilde{W}_n$. For each $w \in W^P$, the B-orbit of wP in SO_{2n}/P is the *Schubert cell* $\mathfrak{Y}_w^\circ := BwP/P$. The *Schubert variety* $\mathfrak{Y}_w$ is the closure of $\mathfrak{Y}_w^\circ$ in $\mathfrak{X}$. Then $\mathfrak{X}_w := \mathfrak{Y}_{w_0 w}$ has codimension $\ell(w)$ in SO_{2n}/P, and its cohomology class $[\mathfrak{X}_w]$ is a *Schubert class*. We have an isomorphism of abelian groups

$$\mathrm{H}^*(\mathfrak{X}, \mathbb{Z}) \cong \bigoplus_{w \in W^P} \mathbb{Z}[\mathfrak{X}_w]$$

which generalizes (47).

Recall that E_r for $r \in [1,p]$ and E denote the tautological and trivial vector bundles over $\mathfrak{X}$, of rank $n - a_r$ and $2n$, respectively. For any $w \in W^P$, we then have

$$[\mathfrak{X}_w] = \sum_{\underline{\lambda}} g^w_{\underline{\lambda}} \, H^{(a_1)}_{\lambda^1}(E - E_1) s_{\lambda^2}(E_1 - E_2) \cdots s_{\lambda^p}(E_{p-1} - E_p) \tag{60}$$

in $\mathrm{H}^*(\mathfrak{X}, \mathbb{Z})$, where the sum is over all sequences of partitions $\underline{\lambda} = (\lambda^1, \ldots, \lambda^p)$ with λ^1 being typed a_1-strict, and the coefficients $g^w_{\underline{\lambda}}$ are given by

$$g^w_{\underline{\lambda}} := \sum_{u_1 \cdots u_p = w} d^{u_1}_{\lambda^1} c^{u_2}_{\lambda^2} \cdots c^{u_p}_{\lambda^p} \tag{61}$$

summed over all factorizations $u_1 \cdots u_p = w$ such that $\ell(u_1) + \cdots + \ell(u_p) = \ell(w)$, $u_j \in S_\infty$ for $j \geq 2$, and $u_j(i) = i$ for all $j > 1$ and $i \leq a_{j-1}$. The nonnegative integers $d^{u_1}_{\lambda^1}$ and $c^{u_i}_{\lambda^i}$ which appear in the summands in (61) are the same as the ones in equations (57) and (9), respectively. When $p = 1$, the partial flag manifold $\mathfrak{X}$ is the orthogonal Grassmannian $\mathrm{OG}(n - a_1, 2n)$, and formula (60) specializes to equation (48). In general, the polynomial $H^{(a_1)}_{\lambda^1}(E - E_1)$ in (60) is defined by pulling back the polynomial $H^{(a_1)}_{\lambda^1}(c(E/E_1))$ under the natural projection map $E_\bullet \mapsto E_1$ from $\mathfrak{X}$ to $\mathrm{OG}(n - a_1, 2n)$.

Example 20. Let $P = B$ be the Borel subgroup, so that the flag manifold SO_{2n}/B parametrizes flags of subspaces $0 \subset E_n \subset \cdots \subset E_1 \subset E = \mathbb{C}^{2n}$ with E_1 isotropic in a given family, and $\dim(E_i) = n + 1 - i$ for each $i \in [1, n]$. For any strict partition λ, define the *$\widetilde{P}$-polynomial* $\widetilde{P}_\lambda(X_n) := 2^{-\ell(\lambda)} \widetilde{Q}_\lambda(X_n)$. Since $a_1 = 0$, we have $H^{(a_1)}_\lambda(E - E_1) = \widetilde{P}_\lambda(X_n)$, where we set $x_i := -c_1(E_i/E_{i+1})$ for each i. For any element $w \in \widetilde{W}_n$, define the *orthogonal Schubert polynomial* $\mathfrak{D}_w(X_n)$ by

$$\mathfrak{D}_w(X_n) := \sum_{v, \varpi, \lambda} d^v_\lambda \, \widetilde{P}_\lambda(X_n) \mathfrak{S}_\varpi(-X_n)$$

where the sum is over all factorizations $v\varpi = w$ and strict partitions λ such that $\ell(v) + \ell(\varpi) = \ell(w)$, $\varpi \in S_n$, and $|\lambda| = \ell(v)$. Working as in Example 15, one can show that formula (60) is equivalent to the statement that for any element $w \in \widetilde{W}_n$, we have $[\mathfrak{X}_w] = \mathfrak{D}_w(X_n)$ in $\mathrm{H}^*(\mathrm{SO}_{2n}/B, \mathbb{Z})$.

5. Historical notes and references

5.1. **Schur polynomials.** The Schur polynomials $s_\lambda(X_n)$ were first defined in the early 19th century by Cauchy [C] using formula (7), as a quotient of two alternant determinants. These polynomials were studied further by Jacobi [J] and his student Trudi [Tr], who established the determinantal formula (2). The dual identity (6) was proved by Nägelsbach [N]. The reformulation of the Jacobi-Trudi identity using raising operators (1) originates in Young's work on the representation theory of the symmetric group; see [Y, Eqn. (I)] and compare with [R, Eqn. (2.26)].

The Pieri and Giambelli formulas (5) and (4) for Grassmannians were proved in [Pi] and [G], respectively. In his thesis [S1], Schur showed that the polynomials $s_\lambda(X_n)$ can be viewed as the characters of the irreducible polynomial representations of the general linear group GL_n. From the perspective of representation theory, equation (7) is a special case of the Weyl character formula. Finally, the tableau formula (8) for Schur polynomials was established by Littlewood [Li1], more than 100 years after Cauchy's definition appeared. For an approach to the theory of Schur polynomials starting from the raising operator definition (1), which includes the above topics and more, see [T4].

The code $\gamma(\varpi)$ of a permutation ϖ is the Lehmer code used in computer science, which was known in the 19th century (see for example [La]). The shape $\lambda(\varpi)$ of a permutation was defined in [LS1]. The Stanley symmetric functions $G_\varpi(X)$ were introduced in [St]; in the notation of op. cit., the function $G_{\varpi^{-1}}(X)$ is assigned to ϖ. Stanley's conjecture that the coefficients c_λ^ϖ in (9) are nonnegative was proved independently by Edelman and Greene [EG] and Lascoux and Schützenberger [LS2]; the latter authors introduced the transition trees of Section 2.4. The definition of $G_\varpi(X)$ using the nilCoxeter algebra is due to Fomin and Stanley [FS].

The presentations (10) and (11) of the cohomology of type A flag manifolds are due to Borel [Bo]. Formula (12) was proved in [BKTY], and grew out of a study of the Schubert polynomials $\mathfrak{S}_\varpi(X_n)$ of Lascoux and Schützenberger [LS1, M1] and their relation to the quiver polynomials of Buch and Fulton [BF]. The definition of $\mathfrak{S}_\varpi(X_n)$ given in formula (14) is found in [FS].

5.2. **Theta polynomials.** In order to clarify the relevant history, the arXiv announcement years are listed for the main papers below, since their publication dates in journals have little to do with when the work was completed. The story begins with the companion papers [BKT1] (arXiv:2008) and [BKT2] (arXiv:2008) which studied Schubert calculus on non maximal isotropic Grassmannians. The first paper [BKT1] proved the Pieri rule (24), while [BKT2] dealt with the Giambelli formula (23) and theta polynomials.

The parametrization of the Schubert classes on $\mathrm{IG}(n-k, 2n)$ by k-strict partitions was introduced in [BKT1]. As explained in op. cit., although the k-strict partitions are not really needed there, they are a key ingredient of its companion paper [BKT2], and in related works such as [T3] (arXiv:2008).

The paper [BKT2] was the first to realize that Young's raising operators play an essential role in geometry, in the Giambelli type formulas for isotropic Grassmannians, and to employ them in their proofs. Before [BKT2], these operators made occasional appearances, notably in the theory of Hall-Littlewood polynomials (see e.g. [Li2, Mo, M2]), but they were rarely used, even in representation theory and combinatorics. The solution to the Giambelli problem for the usual (type A), Lagrangian, and maximal orthogonal Grassmannians found in [G] and [P], respectively, employed the older language of Jacobi-Trudi determinants (in Lie type A) and Schur Pfaffians (in types B, C, and D), which goes back to [J] and [S2].

In the initial version of [BKT2], the theta polynomials were expressed as the formal power series in (31), although the intention in op. cit. (which justifies the term 'polynomial') was to regard the Θ_λ as Giambelli polynomials in the ϑ_p. The definition (15) of the theta polynomials $\Theta_\lambda(u)$ in independent variables u_i was first given explicitly in [T5, Eqn. (3)] (arXiv:2009).

The ring Γ of Section 3.3 is isomorphic to the ring of Schur Q-functions (see Example 11, [S2], and [M2, III.8]), whose elements are symmetric formal power series in $Z := (z_1, z_2, \ldots)$. Using the latter notation, the action of W_∞ on the ring $\Gamma[X]$ was studied by Billey and Haiman in [BH, Lemma 4.4]. The same authors obtained a natural $\mathbb{Z}$-basis of $\Gamma[X]$ consisting of type C Schubert polynomials (actually power series) $C_w(X)$, for $w \in W_\infty$. When w is an n-Grassmannian element of W_∞ of shape given by the n-strict partition λ, then $C_w(X)$ is equal to $\Theta_\lambda(X_n)$ in $\Gamma[X]$, so modulo the defining relations in Γ (see [BKT2, Prop. 6.2]).

It was shown in [T9] that the theta polynomials $\Theta_\lambda(X_n)$ of level n are symmetric for the W_n-action on $\Gamma[X_n]$, and form a $\mathbb{Z}$-basis for the W_n-invariants there, which is the content of (26) and (27). The paper [T9] goes on to define the shape $\lambda(w)$ of a signed permutation w, proves equation (29), and also obtains the presentations (36) and (37). We note that multi-Schur Pfaffians such as (28) first appeared in the work of Kazarian [K], resurfaced in [IMN] (arXiv:2008), and were subsequently used in the degeneracy locus formulas of [AF].

The tableau formula (33) for $\Theta_\lambda(Z\,;X_k)$ was established in [T3]. The fact that the *left weak Bruhat order* on the k-Grassmannian elements of W_∞ respects the inclusion relation $\lambda \subset \mu$ of k-strict Young diagrams is what makes such a formula possible. This was pointed out in [T3, Prop. 4]. Example 12 is well known in the combinatorial theory of Schur Q-functions; see for instance [M2, III.(8.16)].

The type C mixed Stanley functions and k-transition trees were defined and equation (35) was proved in [T5]. These constructions build on earlier results of Billey and Haiman [BH, B] and Fomin and Kirillov [FK], which had studied the $k = 0$ case. The paper [T5] also proves formula (38), which gives an intrinsic solution to the Giambelli problem of representing the Schubert classes in the cohomology of Sp_{2n}/P, for any parabolic subgroup P of Sp_{2n}.

The $\widetilde{Q}$-polynomials $\widetilde{Q}_\lambda(X_n)$ in Example 15 are due to Pragacz and Ratasjki [PR]. The symplectic Schubert polynomials $\mathfrak{C}_w(X_n)$ in (40) were defined in [T1] (arXiv:2008), and are a *geometrization* of the Billey-Haiman type C Schubert polynomials. For more on this history, we refer the reader to [T8, Section 5].

5.3. **Eta polynomials.** The paper [BKT1] introduced typed k-strict partitions and proved the Pieri rule (49) for non maximal even orthogonal Grassmannians. Young's raising operators were used in [BKT4] (arXiv:2011) to define eta polynomials and to prove equation (48), which solves the Giambelli problem for the same spaces. The work [T5] extended this to address the analogous question for all partial orthogonal flag manifolds, which gives equation (60). See [T7] for a detailed exposition, which includes the version of this result which holds in the more general setting of degeneracy loci of vector bundles for the classical Lie groups.

The ring Γ' of Section 4.3 is isomorphic to the ring $\mathbb{Z}[P_1, P_2, \ldots]$ of Schur P-functions. The action of $\widetilde{W}_\infty$ on the ring $\Gamma'[X]$ and the induced divided difference operators there were studied by Billey and Haiman [BH], who defined a $\mathbb{Z}$-basis of $\Gamma'[X]$ consisting of type D Schubert polynomials $D_w(X)$ for $w \in \widetilde{W}_\infty$, compatible with these operators. According to [BKT4, Prop. 6.3], when w is an n-Grassmannian element of $\widetilde{W}_\infty$ associated to the typed n-strict partition λ, then $D_w(X)$ is equal to $H_\lambda(X_n)$ in $\Gamma'[X]$. Again it is important to note that this equality takes place in a ring with relations, which come from the subring Γ'.

The fact that the eta polynomials of level n are symmetric for the $\widetilde{W}_n$-action on $\Gamma'[X_n]$ and provide a $\mathbb{Z}$-basis for the Weyl group invariants there, which is the content of (51) and (52), was explained in [T9]. The paper [T9] also defines the shape $\lambda(w)$ of an element w of $\widetilde{W}_\infty$, proves equation (53), and moreover obtains the presentations (58) and (59).

The tableau formula (54) for the eta polynomials $H_\lambda(Z\,;X_k)$ was established in [T6] (arXiv:2011). The type D mixed Stanley functions and k-transition trees were defined and equation (57) was proved in [T5]. This used the even orthogonal (type D) version of the nilCoxeter algebra approach of [FS, FK] to type A and B Stanley symmetric functions, which is found in [Lam]. The $\widetilde{P}$-polynomials $\widetilde{P}_\lambda(X_n)$ and orthogonal Schubert polynomials $\mathfrak{D}_w(X_n)$ of Example 20 were defined in [PR] and [T2] (arXiv:2009), respectively.

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