TOPOLOGICAL SYMMETRY GROUPS OF GRAPHS EMBEDDED IN THE 3-SPHERE

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1. Introduction

In this paper, we begin a systematic study of topological symmetry groups of graphs embedded in the 3-sphere. The notion of a topological symmetry group of a graph in S^3 was introduced by Simon [S], who was motivated by the Longuet-Higgins symmetry groups of non-rigid molecules [L]. The significance of such groups in chemistry stems from the fact that the chemical properties of a molecule depend on the symmetries of its graph model (where the vertices represent atoms and the edges represent bonds).

The study of graphs as geometric objects necessarily involves an investigation of their symmetries. The symmetries of an abstract graph γ are described by the group $\mathrm{Aut}(\gamma)$ of automorphisms of γ. The automorphism group of a graph has been the subject of much study, with roots in the nineteenth century (see [B3] and [B4] for surveys). In contrast, the group of those symmetries of an embedded graph in S^3 which are induced by diffeomorphisms of the ambient space has received little attention.

By a *graph* we shall mean a finite, connected graph, such that each edge has two distinct vertices and there is at most one edge with a given pair of vertices. An *embedded graph* Γ is a pair (V, E) of sets of vertices V and edges E such that V is a set of points in S^3, every edge is an embedded arc in S^3 between two vertices, and the interior of each edge contains no vertex and no point of any other edge. When we write $h : (S^3, \Gamma) \to (S^3, \Gamma)$ or $h(\Gamma) = \Gamma$, we shall mean that $h(V) = V$ and $h(E) = E$. The restriction of h to V induces an automorphism of the abstract graph γ underlying Γ. The *topological symmetry group* $\mathrm{TSG}(\Gamma)$ is the subgroup of $\mathrm{Aut}(\gamma)$ consisting of those automorphisms which are induced by some diffeomorphism of (S^3, Γ). Allowing only orientation preserving diffeomorphisms of S^3 defines the orientation preserving topological symmetry group $\mathrm{TSG}_+(\Gamma)$. For any embedded graph Γ, either $\mathrm{TSG}_+(\Gamma) = \mathrm{TSG}(\Gamma)$ or $\mathrm{TSG}_+(\Gamma)$ is a normal subgroup of $\mathrm{TSG}(\Gamma)$ with index 2. Starting with a particular embedded graph Γ, we can re-embed it by tying the same invertible chiral knot in every edge of Γ to get an embedded graph Γ' such that $\mathrm{TSG}(\Gamma') = \mathrm{TSG}_+(\Gamma') = \mathrm{TSG}_+(\Gamma)$. Thus every group which is the orientation preserving topological symmetry group of some embedded graph is also the topological symmetry group of some (possibly different) embedded graph.

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Frucht [Fr] showed that any finite group is the automorphism group of some connected graph; moreover, restricting to k-connected graphs for a fixed $k \geq 2$ does not affect the conclusion [Sa] (a graph is *k-connected* if at least k vertices together with their incident edges must be removed in order to disconnect the graph or reduce it to a single vertex). Since every graph admits an embedding in S^3, it is natural to ask whether every finite group can be realized as $\mathrm{TSG}(\Gamma)$ (or $\mathrm{TSG}_+(\Gamma)$) for some embedded graph Γ. Using the terminology of [B3], the question becomes whether the class of embedded graphs and their topological symmetry groups is *universal* for finite groups. We show that the answer is negative, and we characterize the class of all orientation preserving topological symmetry groups for 3-connected graphs.

In general, $\mathrm{TSG}_+(\Gamma)$ will depend on the particular embedding of the abstract graph underlying Γ in S^3. For example, consider θ_n consisting of two vertices of valence $n > 2$ which are joined together by n edges. Since θ_n is not a graph, we add a vertex of valence 2 to each edge to get a graph γ_n (see Figure 1). Starting with the image of a planar embedding of γ_n, we add identical non-invertible knots to each of the arcs to obtain an embedded graph Γ_n such that $\mathrm{TSG}_+(\Gamma_n)$ is the symmetric group S_n. On the other hand, if Γ'_n is an embedded graph obtained from the image of a planar embedding of γ_n by tying distinct non-invertible knots in each edge, then $\mathrm{TSG}_+(\Gamma'_n)$ is trivial.

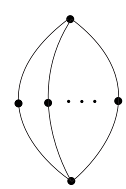

FIGURE 1. γ_n

Given any finite abelian group H, we can construct an embedded graph Γ such that $\mathrm{TSG}_+(\Gamma) = H$. For example, the embedded graph Γ which is illustrated in Figure 2 has $\mathrm{TSG}_+(\Gamma) = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$. If H contains more than one factor of a given $\mathbb{Z}_n$, we can add knots on the spokes of the "wheels", so that no diffeomorphism takes one "wheel" to another "wheel".

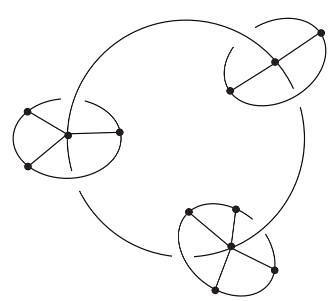

FIGURE 2. $\mathrm{TSG}_+(\Gamma) = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$

Another source of topological symmetry groups comes from planar embeddings. For a planar graph γ realized as an embedded graph Γ via the natural inclusion of S^2 in S^3, it can be shown using results of [Ma] and [D, Thm 4.3.1] that $\mathrm{TSG}(\Gamma) = \mathrm{TSG}_+(\Gamma) = \mathrm{Aut}(\gamma)$.

The automorphism groups of planar graphs have been characterized by Mani [M] and Babai [B1] [B2]. In particular, these groups do not exhaust all finite groups, and for 3-connected planar graphs the automorphism groups are precisely the finite subgroups of $O(3)$. In contrast to the case of planar embeddings, for an arbitrary embedded graph Γ it can happen that $\mathrm{TSG}_+(\Gamma) \neq \mathrm{Aut}(\gamma)$. In fact, it was shown in [F1] that for $n \geqslant 6$, no matter how the complete graph K_n is embedded in S^3, the cycle automorphism (1234) of K_n cannot be induced by a diffeomorphism of S^3. Thus for any embedded graph Γ which has underlying abstract graph K_n with $n \geqslant 6$, $\mathrm{TSG}(\Gamma)$ is a proper subgroup of $\mathrm{Aut}(K_n)$.

In general, it is not possible to induce each element of $\mathrm{TSG}_+(\Gamma)$ by a finite order diffeomorphism of S^3. For example, consider the graph Γ_n with $n \geqslant 4$ described above whose underlying abstract graph γ_n is illustrated in Figure 1. Then as seen above, $\mathrm{TSG}_+(\Gamma_n) = S_n$; however, many of the diffeomorphisms which induce the elements of $\mathrm{TSG}_+(\Gamma)$ cannot be of finite order. Indeed, it follows from the proof of the Smith Conjecture [MB] that no finite order diffeomorphism of (S^3, Γ_n) can interchange two vertices of valence two and fix the remaining vertices, since the fixed point set of such a diffeomorphism would include a non-trivial knot.

In fact, there exist 3-connected embedded graphs Γ such that some element of $\mathrm{TSG}(\Gamma)$ cannot be induced by any finite order diffeomorphism of S^3. An example of such a graph is illustrated in Figure 3. There is no order 3 diffeomorphism of S^3 which takes a figure eight knot to itself ([Ha] and [Mu]). Hence the automorphism (123)(456) is induced by a diffeomorphism of S^3 (by sliding the graph along itself), but cannot be induced by a finite order diffeomorphism of S^3.

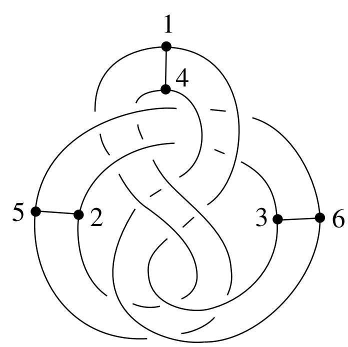

FIGURE 3. (123)(456) cannot be induced by a finite order diffeomorphism of S^3

The above examples indicate that a priori, the classification of all possible topological symmetry groups could be rather complicated. The three main theorems which follow help to clarify the situation.

Theorem 1. *Let* Γ *be an embedded graph.*
a) If $\mathrm{TSG}_+(\Gamma)$ *is a simple group, then it is either the alternating group* A_5 *or a cyclic group of prime order.*
b) In general, the sequence of quotient groups in any composition series for $\mathrm{TSG}_+(\Gamma)$ *contains only alternating groups* A_n *with* $n \geqslant 5$ *and cyclic groups of prime order.*

We note that the same conclusion holds for the automorphism groups of planar graphs; in fact Theorem 1 implies the corresponding results in [B2]. However, there exist embedded graphs Γ such that $\mathrm{TSG}_+(\Gamma)$ is not isomorphic to the automorphism group of any planar graph (see Section 2).

It follows from Theorem 1 that the class of orientation preserving topological symmetry groups of embedded graphs is not universal for finite groups. Furthermore, if $\mathrm{TSG}(\Gamma)$ is a simple group then $\mathrm{TSG}_+(\Gamma) = \mathrm{TSG}(\Gamma)$, and hence the class of topological symmetry groups is also not universal.

Theorem 2. *Let Γ be an embedded 3-connected graph. Then $\mathrm{TSG}_+(\Gamma)$ is isomorphic to a finite subgroup of the group $\mathrm{Diff}_+(S^3)$ of orientation preserving diffeomorphisms of S^3.*

Thus the topological symmetry groups for 3-connected graphs are very special (see below). This stands in contrast with the fact that every finite abelian group is the orientation preserving topological symmetry group of some embedded graph that is not 3-connected, as we saw in Figure 2.

In Section 5, we study when a graph γ may be embedded in S^3 in such a way that a given subgroup of $\mathrm{Aut}(\gamma)$ is induced by an isomorphic subgroup of $\mathrm{Diff}_+(S^3)$. We also prove the following converse to Theorem 2.

Theorem 3. *For every finite subgroup G of $\mathrm{Diff}_+(S^3)$, there is an embedded 3-connected graph Γ such that $G \cong \mathrm{TSG}_+(\Gamma)$. Moreover, Γ can be chosen to be a complete bipartite graph $K_{n,n}$ for some n.*

The finite subgroups of $\mathrm{Diff}_+(S^3)$ consist of the finite subgroups of $SO(4)$, possibly together with the Milnor groups $Q(8k, m, n)$ in the case where the subgroup acts freely on S^3 (see [DV] for the finite subgroups of $SO(4)$, and [Mi] and [Z] for groups that could act freely on S^3). We note that Thurston's geometrization program [Th] would imply that the groups $Q(8k, m, n)$ do not occur.

We deduce from Theorems 2 and 3 that the set of orientation preserving topological symmetry groups of 3-connected embedded graphs in S^3 is exactly the set of finite subgroups of $\mathrm{Diff}_+(S^3)$. The proofs of Theorems 1 and 2 follow from two propositions. We start with the proofs of these theorems in Section 2, and then we prove the propositions in Sections 3 and 4. The heart of the argument lies in the proof of Proposition 1 in Section 3, which uses the Characteristic Submanifold Theorem of Jaco-Shalen [JS] and Johannson [Jo] and Thurston's Hyperbolization Theorem [Th], in the context of pared manifolds. These results were applied in a similar fashion in [F2]. In the case of a 3-connected embedded graph Γ, the strategy is to re-embed Γ in a "nicer" way as Δ such that $\mathrm{TSG}_+(\Gamma)$ is a subgroup of $\mathrm{TSG}_+(\Delta)$ and $\mathrm{TSG}_+(\Delta)$ is induced by a finite subgroup of $\mathrm{Diff}_+(S^3)$. Finally, in Section 5, Theorem 3 is proved by a direct construction.

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2. Proofs of Theorems 1 and 2

Let Γ be a graph embedded in S^3. We shall use spheres and pinched spheres to decompose Γ into smaller pieces. We begin with several definitions.

Definition 1. Let Σ be a 2-sphere embedded in S^3. If Σ intersects Γ in a single vertex v of valence more than two and each component of $S^3 - \Sigma$ contains part of Γ, then we say that Σ is a *type I sphere* and v is a *type I vertex* of Γ. (See Figure 4.)

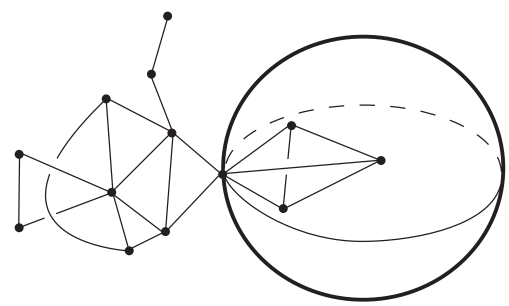

Figure 4. An embedded graph with a type I sphere

Observe that removing a type I vertex from Γ separates Γ, but not every vertex which separates Γ is a type I vertex.

Definition 2. Let Σ be a 2-sphere embedded in S^3. If Σ intersects Γ in two vertices and the closure of no component of $(S^3 - \Sigma) \cap \Gamma$ is a single arc or a single vertex then we say that Σ is a *type II sphere* of Γ. (See Figure 5.)

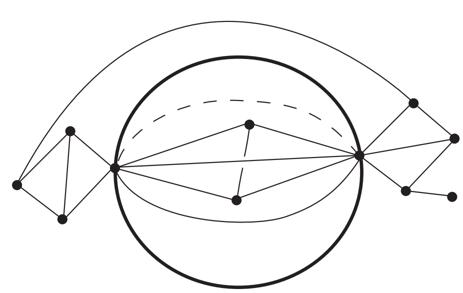

Figure 5. An embedded graph with a type II sphere

Definition 3. Let Σ be a 2-sphere with two points identified to a single point p. We say that Σ is a *pinched sphere* and p is the *pinch point*. Let Σ be a pinched sphere in S^3, with pinch point p. Suppose that p is a vertex of Γ, such that $\Sigma \cap \Gamma = \{p\}$ and each component of $S^3 - \Sigma$ contains part of Γ. Then we say that Σ is a *type III sphere* of Γ. (See Figure 6.)

We remark that our definition of a type I sphere is similar to that of Suzuki [Su], however, our definition of a type II sphere is different from his.

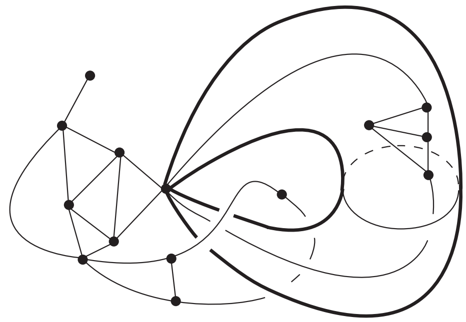

FIGURE 6. An embedded graph with a type III sphere

Let Γ be a graph embedded in S^3 and let G be a group of orientation preserving diffeomorphisms of (S^3, Γ). Let H be the image of G under the natural homomorphism from G to $\mathrm{TSG}_+(\Gamma)$. Then H is said to be the subgroup of $\mathrm{TSG}_+(\Gamma)$ *induced* by G.

A group H is said to be *realizable* if there is an embedded graph Γ with $\mathrm{TSG}_+(\Gamma) \cong H$. We also say that H is *realized* by Γ.

We will use the following two propositions to prove Theorems 1 and 2. The proofs of these propositions will be given in the next two sections.

Proposition 1. *Let Γ be an embedded graph with no type I spheres and let $H = \mathrm{TSG}_+(\Gamma)$. Then either H is isomorphic to a finite subgroup of $\mathrm{Diff}_+(S^3)$, $H \cong S_r$ for some r, or H has a non-trivial normal subgroup N such that both N and H/N are realizable. Furthermore, if Γ has no type II or type III spheres, then Γ can be re-embedded as Δ such that H is a finite subgroup of $\mathrm{TSG}_+(\Delta)$, and H is induced on Δ by an isomorphic finite subgroup of $\mathrm{Diff}_+(S^3)$.*

It follows from Proposition 1 that if Γ has no type I spheres and $\mathrm{TSG}_+(\Gamma)$ is a simple group then $\mathrm{TSG}_+(\Gamma)$ is isomorphic to a finite subgroup of $\mathrm{Diff}_+(S^3)$.

Proposition 2. *Let Γ be an embedded graph which has at least one type I sphere, and let $H = \mathrm{TSG}_+(\Gamma)$. Then either H is realizable by a graph with no type I spheres, $H \cong S_r$ for some r, or H has a non-trivial normal subgroup N such that both N and H/N are realizable.*

If $H \cong S_r$ and H is simple, then $H \cong \mathbb{Z}_2$. Hence H is realized by the graph consisting of a single edge. Thus it follows from Proposition 2 that any simple group which is realizable, is realizable by an embedded graph with no type I spheres.

We now prove Theorems 1 and 2.

Theorem 1. *Let Γ be an embedded graph.*
a) If $\mathrm{TSG}_+(\Gamma)$ is a simple group, then it is either the alternating group A_5 or a cyclic group of prime order.
b) In general, the sequence of quotient groups in any composition series for $\mathrm{TSG}_+(\Gamma)$ contains only alternating groups A_n with $n \geqslant 5$ and cyclic groups of prime order.

Proof. To prove part (a), we observe that by Proposition 2, there is an embedded graph Δ with no type I spheres such that $\mathrm{TSG}_+(\Gamma) \cong \mathrm{TSG}_+(\Delta)$. Then by Proposition 1, $\mathrm{TSG}_+(\Delta)$ is isomorphic to a finite subgroup of $\mathrm{Diff}_+(S^3)$. However, it is shown in [Z, Thm. 1] that the alternating group A_5 is the only non-abelian finite simple group which acts faithfully by diffeomorphisms on a homology 3-sphere. The result follows.

To prove part (b), we use induction on the number of elements in $H = \mathrm{TSG}_+(\Gamma)$. Let $|H| = k$ and assume the result is true for all realizable groups with fewer than k elements. If H is simple, then we are done by part (a). Otherwise Propositions 1 and 2 imply that H is a finite subgroup of $\mathrm{Diff}_+(S^3)$ or $H \cong S_r$, in which case the result follows from [Z, Thm. 2] , or H has a normal subgroup N such that both N and H/N are realizable. In the latter case, we know by induction that both N and H/N have composition series with all simple quotients either alternating or cyclic, and putting these two series together gives a composition series for H with the same property. The Jordan-Hölder theorem implies that this also holds for any other composition series for H. □

By a similar argument, we can prove that for any realizable group H, the number of quotients in a composition series for H which are isomorphic to $\mathbb{Z}_2$ is at least as large as the number of quotients which are isomorphic to any A_r with $r \geqslant 6$. To see this, observe that if H is a finite subgroup of $\mathrm{Diff}_+(S^3)$ or $H \cong S_r$, then the result follows; otherwise, the proof uses induction on the order of H, as above. A complete characterization of all realizable groups may be possible, working along the lines of [B2].

According to [B2, Cor. 9.15], the group $G = A_5 \times \mathbb{Z}_3$ is not the automorphism group of any planar graph. However, G is realizable. To see this, recall from [M] that A_5 is the automorphism group of a 3-connected planar graph which can be realized as the 1-skeleton X of a convex polytope P in $\mathbb{R}^3$, such that all abstract automorphisms of X are induced by isometries of P. Let Γ_1 be obtained from X by connecting each vertex to a fixed point v_1 in the interior of P. Now let Γ_2 be the 1-skeleton of a tetrahedron disjoint from P, and tie a non-invertible knot to each of the the three edges of Γ_2 which do not emanate from a particular vertex v_2, all oriented in the same way. Finally, form Γ by connecting v_1 to v_2 by a line segment which does not meet the rest of Γ_1 and Γ_2. Then $\mathrm{TSG}_+(\Gamma) = A_5 \times \mathbb{Z}_3$. Thus there exist realizable groups which are not the automorphism group of any planar graph.

Theorem 2. *Let* Γ *be an embedded* 3*-connected graph. Then* $\mathrm{TSG}_+(\Gamma)$ *is isomorphic to a finite subgroup of* $\mathrm{Diff}_+(S^3)$*.*

Proof. Let Γ be an embedded graph. If Γ has a type I or III sphere then Γ can be disconnected be removing a single vertex and the edges incident to it. If Γ has a type II sphere then Γ can be disconnected by removing two vertices and the edges incident to them. Thus if Γ is 3-connected, Γ has no type I, II, or III spheres. So the result follows by Proposition 1. □

We also use Proposition 1 to prove the following strengthening of Theorem 2.

Proposition 3. *Let $H = \mathrm{TSG}_+(\Gamma)$ for some embedded 3-connected graph Γ. Then Γ can be re-embedded as Δ such that H is a subgroup of $\mathrm{TSG}_+(\Delta)$ and $\mathrm{TSG}_+(\Delta)$ is induced by an isomorphic subgroup of $\mathrm{Diff}_+(S^3)$.*

Proof. Let $H_1 = \mathrm{TSG}_+(\Gamma)$. Then, as above, Γ has no type I, II or III spheres. Hence by Proposition 1 we can re-embed Γ as Γ_1 such that H_1 is a subgroup of $\mathrm{TSG}_+(\Gamma_1)$, and H_1 is induced by an isomorphic subgroup K_1 of $\mathrm{Diff}_+(S^3)$. If $\mathrm{TSG}_+(\Gamma_1) = H_1$, then we are done by letting $\Delta = \Gamma_1$. If H_1 is a proper subgroup of $\mathrm{TSG}_+(\Gamma_1)$, we let $H_2 = \mathrm{TSG}_+(\Gamma_1)$ and again apply Proposition 1 to the 3-connected embedded graph Γ_1. Continue this process. Let γ denote the underlying abstract graph of Γ. Then for each i, $\mathrm{TSG}_+(\Gamma_i)$ is a subgroup of the finite group $\mathrm{Aut}(\gamma)$. Hence this process cannot go on indefinitely. $\square$

3. Proof of Proposition 1

We begin with some notation that we will use throughout the rest of the paper. Let V denote the set of embedded vertices of an embedded graph Γ, and let E denote the set of embedded edges of Γ. We shall construct a closed regular neighborhood $N(\Gamma)$ as the union of two sets, $N(V)$ and $N(E)$, which have disjoint interiors. For each vertex $v \in V$, let $N(v)$ denote a small ball around v, and let $N(V)$ denote the union of all of these balls. For each embedded edge $\varepsilon \in E$, let $N(\varepsilon)$ denote a tube $D^2 \times I$ whose core is $\varepsilon - N(V)$, such that $N(\varepsilon)$ contains no other part of Γ, and $N(\varepsilon)$ meets $N(V)$ in a pair of disks. Observe that ε is not contained in $N(\varepsilon)$. Let $N(E)$ denote the union of all the tubes $N(\varepsilon)$. Let $N(\Gamma) = N(V) \cup N(E)$. Throughout the paper we shall use $\partial' N(\varepsilon)$ to denote the annulus $\partial N(\Gamma) \cap N(\varepsilon)$ in order to distinguish it from the sphere $\partial N(\varepsilon)$.

We will use *cl* to denote the closure of a set and *int* to denote the interior of a set. Finally, by a *chain of length* n we shall mean an arc in Γ containing n vertices of valence two and no vertices of higher valence in its interior such that neither endpoint of the arc has valence two.

We assume the reader is familiar with standard 3-manifold topology. However, we will need to to use some terminology and results about pared manifolds which we give below.

Definition 4. A *pared* 3*-manifold* (M, P) is an orientable 3-manifold M together with a family P of disjoint incompressible annuli and tori in ∂M.

A pared manifold is a special case of a manifold with boundary patterns in the sense of Johannson [Jo] or a 3-manifold pair in the sense of Jaco-Shalen [JS]. The following definitions agree with those of [Jo] and [JS].

Definition 5. A pared manifold (M, P) is said to be *simple* if it satisfies the following three conditions:
1) M is irreducible and $\partial M - P$ is incompressible
2) Every incompressible torus in M is parallel to a torus component of P
3) any annulus A in M with ∂A contained in $\partial M - P$ is either compressible or parallel to an annulus A' in ∂M with $\partial A' = \partial A$ and such that $A' \cap P$ consists of zero or one annular component of P.

Definition 6. A pared manifold (M, P) is said to be *Seifert fibered* if there is a Seifert fibration of M for which P is a union of fibers. A pared manifold (M, P) is said to be *I-fibered* if there is an I-bundle map of M over a surface B such that P is in the preimage of ∂B.

We will use the following results about pared manifolds.

Characteristic Submanifold Theorem for Pared Manifolds. ([JS] and [Jo]) *Let (X, P) be a pared manifold with X irreducible and $\partial X - P$ incompressible. Then, up to an isotopy of (X, P), there is a unique family Ω of disjoint incompressible tori and annuli with $\partial\Omega$ contained in $\partial X - P$ such that the following two conditions hold:*
1) If W is the closure of a component of $X - \Omega$, then the pared manifold $(W, W \cap (P \cup \Omega))$ is either simple, Seifert fibered, or I-fibered.
2) There is no family Ω' with fewer components than Ω which satisfies the above.

Thurston's Theorem for Pared Manifolds. [Th] *If (M, P) is simple, M is connected, and ∂M is non-empty, then either $M - P$ admits a finite volume complete hyperbolic metric with totally geodesic boundary, or (M, P) is Seifert fibered or I-fibered.*

Now we are ready to prove Proposition 1. Since the proof of Proposition 1 is long, we begin with an outline. In Step 1, we will follow the proof that we gave of [F2, Thm. 1], and use the Characteristic Submanifold Theorem ([JS] and [Jo]) to split the complement of $N(\Gamma)$ along a minimal family Θ of incompressible tori which is unique up to ambient isotopy and such that each component is either simple or Seifert fibered. We let X denote the component which contains $\partial N(\Gamma)$. Since Γ has no type I spheres, we can then use the Characteristic Submanifold Theorem for Pared Manifolds to split X along a minimal family Ω of incompressible annuli which is unique up to ambient isotopy and such that as a pared manifold each component is either simple, Seifert fibered, or I-fibered. We then define a group G of orientation preserving diffeomorphisms of (S^3, Γ) such that for every $g \in G$, $g(\Theta) = \Theta$ and $g(\Omega) = \Omega$, and every $a \in \mathrm{TSG}_+(\Gamma)$ is induced by some $g_a \in G$.

In Step 2, we cap off each annulus $A_i \in \Omega$ to obtain a sphere or pinched sphere Σ_i which meets Γ in two or one points respectively. Then we choose a particular component W of $X - \Omega$ which is setwise invariant under G, and has the property that G permutes some of the A_i's which meet ∂W or some of the components of $\partial N(V) \cap W$. For each A_i in ∂W, we let B_i denote the closure of the component of $S^3 - \Sigma_i$ which does not contain W and we let $\Gamma_i = \Gamma \cap B_i$.

In Step 3, we show that the proof can be reduced to analyzing the action that G induces on W. In Steps 4, 5, and 6 we obtain our result in the cases where W is Seifert fibered, I-fibered, and simple, respectively.

Proof of Proposition 1. Let Γ be as given in the hypothesis of Proposition 1 and let γ denote the underlying abstract graph of Γ. The result is clear if $\mathrm{TSG}_+(\Gamma)$ is trivial so we assume it is non-trivial. We begin by considering the special cases where Γ is an arc or a circle. In these cases, the full automorphism group $\mathrm{Aut}(\gamma)$ is either a finite cyclic group or a dihedral group, and hence $\mathrm{TSG}_+(\Gamma)$ is a finite subgroup of $\mathrm{Diff}_+(S^3)$. If γ is an arc with n vertices, then $\mathrm{Aut}(\gamma) \cong \mathbb{Z}_2$. In this case, we let Δ be a straight line segment with n uniformly spaced vertices, then $\mathrm{TSG}_+(\Gamma) \cong \mathbb{Z}_2$. If γ is a circle with n vertices, we let Δ be a planar circle with n uniformly spaced vertices. In either of these two cases, there is a subgroup G of $\mathrm{Diff}_+(S^3)$ such that $G \cong H$ and G induces H on Δ.

From now on, we shall assume that γ is not a simple closed curve or an arc. Hence Γ has some vertex with valence at least three. Also since Γ has no type I spheres, Γ cannot have any vertices of valence one. Let m be a number larger than the number of vertices in Γ. We will use m at several places in the proof.

Step 1. *We split the complement of* Γ *along characteristic tori and annuli.*

Let M denote the closure of the complement of $N(\Gamma)$ in S^3. Since Γ is a connected graph, M is irreducible. So we can apply the Characteristic Submanifold Theorem to M to get a minimal family of incompressible tori, Θ, in M such that the closure of every component of $M - \Theta$ is either simple or Seifert fibered, and Θ is unique up to an isotopy fixing ∂M pointwise. It follows from the uniqueness of Θ that, for every automorphism $a \in \mathrm{TSG}_+(\Gamma)$, there is an orientation preserving diffeomorphism $g : (S^3, \Gamma) \to (S^3, \Gamma)$ which induces a, such that $g(\Theta) = \Theta$, $g(N(V)) = N(V)$, and $g(N(E)) = N(E)$. Since ∂M is connected, there is a unique component of $M - \Theta$ which contains ∂M. Let X denote this component. Then for each diffeomorphism $g : (S^3, \Gamma) \to (S^3, \Gamma)$ such that $g(\Theta) = \Theta$, we have $g(X) = X$. If X is Seifert fibered, then ∂X is a collection of tori, and hence Γ is homeomorphic to a circle, which is contrary to our assumption above. Thus X must be simple. So every incompressible torus in X is boundary parallel. Also note that since each torus boundary component of X is incompressible in M and M is irreducible, X must be irreducible.

Now ∂X consists of $\partial N(\Gamma)$ together with a collection of tori in Θ. Let P denote the union of the annuli in $\partial' N(E)$ and the torus boundary components of X. Observe that $\partial X - P = \partial N(V) \cap \partial X$ consists of a collection of spheres with holes.

We show as follows that $\partial X - P$ is incompressible in X. Suppose that there is a non-trivial loop L in some component of $\partial X - P$ which compresses in X. Then L is contained in some $\partial N(v) \cap \partial X$ and L bounds a disk D_1 in X. Also L bounds a disk D_2 in $N(v)$ such that D_2 intersects Γ only in v. Now $\Sigma = D_1 \cup D_2$ is a sphere, whose intersection with Γ is the vertex v. Since L is non-trivial in $\partial N(v) \cap \partial X$, each component of $S^3 - \Sigma$ contains part of Γ. Since Γ has no vertices of valence one, if the valence of v is two, we can slide Σ along a path in Γ until Σ intersects Γ at a vertex with valence at least three. But this gives us a type I sphere for Γ, contrary to hypothesis. Hence $\partial X - P$ must be incompressible in X.

Since X is irreducible and $\partial X - P$ is incompressible in X, we can now apply the Characteristic Submanifold Theorem for Pared Manifolds to the pair (X, P). This gives us a minimal family Ω of incompressible tori and annuli in X with the boundary of each component of Ω contained in $\partial X - P$, such that if W is the closure of any component of $X - \Omega$, then the pared manifold $(W, W \cap (P \cup \Omega))$ is either simple, Seifert fibered, or I-fibered, and Ω is unique up to an isotopy of (X, P). Since every incompressible torus in X is boundary parallel, and the family Ω is minimal, Ω cannot contain any tori. Thus Ω is a (possibly empty) family of incompressible annuli in X. Furthermore, for any W, we can show that $\partial W - (W \cap (P \cup \Omega))$ is incompressible in W by using an argument analogous to our proof that $\partial X - P$ is incompressible in X.

Now let G denote the group of all orientation preserving diffeomorphisms $g : (S^3, \Gamma) \to (S^3, \Gamma)$, such that $g(N(V)) = N(V)$, $g(N(E)) = N(E)$, $g(\Theta) = \Theta$ and $g(\Omega) = \Omega$. Observe that P and X are each setwise invariant under G. Also by the uniqueness of each of the sets $N(V)$, $N(E)$, and Θ, up to isotopy fixing Γ and the uniqueness of Ω up to an isotopy of (X, P), we know that every $a \in \mathrm{TSG}_+(\Gamma)$ is induced by some $g_a \in G$.

For each A in Ω, the boundary of A is contained in $\partial X - P$. So each component of ∂A is contained in some $\partial N(v)$. Thus each component of ∂A bounds a disk D in some $N(v)$ such that $D \cap \Gamma = \{v\}$. Furthermore, we can choose the collection of these disks to be pairwise disjoint except possibly at the vertex v. For each boundary component of each A in Ω choose one such disk, and let Q be the collection of all of these disks. Thus for each A in Ω, there is a pair of disks D_1 and D_2 in Q such that $\Sigma = A \cup D_1 \cup D_2$ is either a sphere meeting Γ in two vertices or a pinched sphere with its pinch point at a vertex. Observe that if Σ is a sphere and the closure of neither component of $(S^3 - \Sigma) \cap \Gamma$ is a single arc or a single vertex, then Σ is a type II sphere; and if Σ is a pinched sphere and the closure of neither component of $(S^3 - \Sigma) \cap \Gamma$ is single vertex then Σ is a type III sphere. Let Λ denote the collection of these spheres and pinched spheres. Since the collection Q is unique up to an isotopy of $N(V)$ fixing both Γ and $\partial N(V)$, we can assume that we chose G such that for every $g \in G$, $g(\Lambda) = \Lambda$.

Step 2. *We choose a component W of $X - \Omega$ which is setwise invariant under G, such that G permutes some of the elements of Ω that are contained in ∂W or some of the components of $\partial N(V) \cap W$.*

If Ω is empty, let $W = X$. Then every component of $\partial N(V)$ meets W. So we suppose that Ω is non-empty.

It follows from the proof of F2, Thm. 1] (which uses the same notation that we use here) that if Γ is a 3-connected graph then there is a unique component of $X - \Omega$ whose closure W has the properties that: every element of Ω is contained in ∂W and for every $\Sigma_i \in \Lambda$ the closure of the component of $S^3 - \Sigma_i$ which is disjoint from W meets Γ in an arc if Σ_i is a sphere and in a single vertex if Σ_i is a pinched sphere. The proof that there is such a W is analogous if we replace the hypothesis that Γ is 3-connected by the hypothesis that Γ has no type I, II, or III spheres. Thus if Γ has no type I, II or III spheres, then we choose this W.

Since W is the only closed up component of $X - \Omega$ with these properties, W is setwise invariant under G. It follows from the above properties that for every non-trivial $a \in \mathrm{TSG}_+(\Gamma)$, g_a induces a non-trivial permutation of either the elements of Ω or the components of $\partial N(V) \cap W$. Thus we are done with Step 2 in the case when Γ has no type II or III spheres.

In order to choose W when Γ does have a type II or type III sphere, we first associate an abstract graph λ to the set Λ of spheres and pinched spheres as follows. For each component Y of $S^3 - \Lambda$, let y be a vertex in λ, and for every pair of components Y and Z of $S^3 - \Lambda$ let there be an edge in λ between the vertices y and z if and only if there is some $\Sigma \in \Lambda$ which is contained in the boundary of both Y and Z. Observe that because every element of Λ separates S^3, the graph λ is a tree. Since G takes Λ to itself, every $g \in G$

unambiguously defines an automorphism g' of λ. Let G' be the group of automorphisms of λ induced by G. Since λ is a tree, it follows from an elementary result in graph theory that there is a vertex or an edge of λ that is invariant under G'.

Claim 1: Suppose that there is no vertex of λ which is fixed by G'. Then either $H \cong \mathbb{Z}_2$ or H has a non-trivial normal subgroup N such that both N and H/N are realizable.

Proof of Claim 1: Since no vertex is fixed by G', there is an edge e of λ which is invariant under G' such that the vertices of e are interchanged by some element of G'. So there is a $\Sigma \in \Lambda$ which is invariant under G, and some element $g \in G$ which interchanges the two components of $S^3 - \Sigma$. The closures of the components of $S^3 - \Sigma$ intersect Γ in subgraphs α_1 and α_2 neither of which is a single arc.

Since Σ is invariant under G, we can define a homomorphism $\Phi : \mathrm{TSG}_+(\Gamma) \to \mathbb{Z}_2$ as follows. For each $a \in \mathrm{TSG}_+(\Gamma)$, let $\Phi(a) = 0$ if a takes each α_i to itself, and let $\Phi(a) = 1$ if a interchanges α_1 and α_2. Since some $a \in \mathrm{TSG}_+(\Gamma)$ interchanges α_1 and α_2, Φ must be onto.

If $\ker(\Phi)$ is trivial then $\mathrm{TSG}_+(\Gamma) \cong \mathbb{Z}_2$, so we may assume $\ker(\Phi)$ is non-trivial. We modify Γ by adding m vertices to every edge in α_1 without changing α_2. Let Π denote Γ with these additional vertices. Then every $a' \in \mathrm{TSG}_+(\Pi)$ induces some $a \in \mathrm{TSG}_+(\Gamma)$ which does not interchange α_1 and α_2, and every such a is induced by some $a' \in \mathrm{TSG}_+(\Gamma)$. It follows that $\mathrm{TSG}_+(\Pi) \cong \ker(\Phi)$. Also, let Π' denote the graph consisting of a single edge, then $\mathrm{TSG}_+(\Pi') \cong \mathbb{Z}_2 = \mathrm{im}(\Phi)$. Now $N = \ker(\Phi)$ and $H/N \cong \mathrm{im}(\Phi)$ are both realizable. This proves Claim 1.

By Claim 1, we can assume there is some vertex of λ which is invariant under G', since otherwise we would be done. Now suppose that G' does not act trivially on λ, then there is some vertex x of λ which is fixed by G' and which is adjacent to a vertex of λ which is not fixed by G'. In this case, we choose W to be the closure of the component of $X - \Omega$ which corresponds to the vertex x of λ. Then W will be setwise invariant under G, however some element of Ω which is contained in ∂W is not setwise invariant under G.

Finally, suppose that G' acts trivially on λ. Then every component of $X - \Omega$ is setwise invariant under G. Since $\mathrm{TSG}_+(\Gamma)$ is not trivial, there is some vertex v of Γ which is not fixed by G. Let W be the closure of some component of $X - \Omega$ which meets $\partial N(v)$. Thus we are done with Step 2.

Now we introduce some notation. Let $A_1, \dots, A_n$ denote those annuli in Ω which are contained in ∂W, and let $\Sigma_1, \dots, \Sigma_n$ denote the spheres or pinched spheres of Λ associated with $A_1, \dots, A_n$ respectively. If Σ_i is a sphere, let $\Sigma_i \cap \Gamma = \{v_i, w_i\}$, and if Σ_i is a pinched sphere, let $\Sigma_i \cap \Gamma = \{u_i\}$. For each i, let c_i and d_i denote the boundary components of A_i, such that in the former case c_i is on $\partial N(v_i)$ and d_i is on $\partial N(w_i)$, and in the latter case both c_i and d_i are on $\partial N(u_i)$. For each A_i in ∂W, we let B_i denote the closure of the component of $S^3 - \Sigma_i$ whose interior is disjoint from W and let $\Gamma_i = B_i \cap \Gamma$. Then the sets $\{\Gamma_1, \dots, \Gamma_n\}$ and $\{B_1, \dots, B_n\}$ are each setwise invariant under G, since W is setwise invariant under G. It follows that $\Gamma' = \mathrm{cl}(\Gamma - (\Gamma_1 \cup \cdots \cup \Gamma_n))$ is setwise invariant under G. (Note that Γ' is not necessarily connected.)

Suppose that Γ has no type II or type III spheres. The conditions on W imply that if Σ_i is a sphere then Γ_i is an arc, and if Σ_i is a pinched sphere then Γ_i is a single vertex.

Note it also follows from these conditions that for every vertex v of Γ with valence at least three, $\partial N(v) \cap W$ contains a disk with at least two holes (this will be used in Step 4).

Step 3. *We reduce the proof of the proposition to analyzing the action that G induces on W.*

More precisely, we prove the following claim.

Claim 2: Suppose that there is some $g \in G$ and some $A_i \subset \partial W$ such that $g(A_i) = A_i$ and g does not interchange the boundary components of A_i. If either Γ has no type II or III spheres or $\mathrm{TSG}_+(\Gamma)$ is a simple group, then g induces the trivial automorphism on Γ_i. Furthermore, if g induces a non-trivial automorphism on Γ_i, then H has a non-trivial normal subgroup N such that both N and H/N are realizable.

Proof of Claim 2: Let $g \in G$ and some $A_i \subset \partial W$ such that $g(A_i) = A_i$ and g does not interchange the boundary components of A_i. If Γ has no type II or type III spheres, then Γ_i is an arc or a single vertex, hence g must induce a trivial automorphism on Γ_i. So in this case we are done.

Now we suppose that Γ has a type II or III sphere. We use g to define a new orientation preserving diffeomorphism h of (S^3, Γ) as follows. Let $h|B_i = g|B_i$. Since g is orientation preserving and takes B_i to itself without interchanging the boundary components of A_i, it follows that $g|\Sigma_i$ is orientation preserving. Hence $h|\Sigma_i$ is orientation preserving, and thus is isotopic to the identity on Σ_i fixing both v_i and w_i or fixing u_i. Let C_i be a ball or a pinched ball containing B_i such that $C_i \cap \Gamma = \Gamma_i$ and ∂C_i is parallel to Σ_i in $C_i - B_i$. Extend h within $C_i - B_i$ such that h is the identity on ∂C_i. Then extend h to the rest of S^3 by the identity. Now let $a \in \mathrm{TSG}_+(\Gamma)$ be induced by h. It follows that $a|\mathrm{cl}(\Gamma - \Gamma_i)$ is the identity and g induces a on Γ_i.

Let N be the set of all $\alpha \in \mathrm{TSG}_+(\Gamma)$ such that $\alpha|\Gamma'$ is the identity, and for each $j \leqslant n$, $\alpha(\Gamma_j) = \Gamma_j$ and $g_\alpha(c_j) = c_j$ and $g_\alpha(d_j) = d_j$. Then N is a normal subgroup of H containing the element a. Observe that for each $\alpha \in N$, g_α does not permute any of the A_j's or any of the components of $\partial N(V) \cap W$. Thus by our choice of W, $N \neq \mathrm{TSG}_+(\Gamma)$. So if $\mathrm{TSG}_+(\Gamma)$ is simple, then a is trivial. Thus g induces the trivial automorphism on Γ_i, and we are done.

So we assume that a is non-trivial. Then Γ_i is not a single vertex and N is non-trivial. We prove as follows that N and H/N are both realizable by constructing embedded graphs Π and Π' with $\mathrm{TSG}_+(\Pi) \cong N$ and $\mathrm{TSG}_+(\Pi') \cong H/N$.

First we construct Π. For each k such that Γ_k is a single vertex let $\Gamma'_k = \Gamma_k$. If Γ_k is not a single vertex, let ε_k be an edge in Γ_k which has a vertex x_k in Γ', and let y_k be a point in $\mathrm{int}(\varepsilon_k)$. We add a chain ε'_k of length km inside $N(\varepsilon_k)$ with endpoints at x_k and y_k, such that ε'_k is isotopic rel $\{x_k, y_k\}$ to the arc in ε_k from x_k to y_k. For each edge ε in the orbit of ε_k under N, there is an $\alpha \in N$ such that $\varepsilon = \alpha(\varepsilon_k)$. For each such ε, we add the chain $\varepsilon' = g_\alpha(\varepsilon'_k)$. For each k, let Γ'_k denote Γ_k after adding this collection of chains of length km. Then for all $l \neq k$, Γ'_l contains no chain of length km.

We create Π by stringing the collection Γ'_1, ..., Γ'_n together with edges as follows. We add disjoint edges β_0, β_1, ..., β_n on the outside of B_1, ..., B_n, such that β_0 is attached to Γ'_1 only at the vertex w_1 or u_1; for each $k < n$, β_k has one vertex at v_k or u_k and the other vertex at w_{k+1} or u_{k+1}; and β_n is attached to Γ'_n only at v_n or u_n. If $n = 1$, we attach β_0

at v_1 and β_1 at w_1 if Σ_1 is a sphere, and attach β_0 at u_1 if Σ_i is a pinched sphere. Now each β_i is invariant under $\mathrm{TSG}_+(\Pi)$ and for each k, the collection of chains of length km is setwise invariant under $\mathrm{TSG}_+(\Pi)$. It follows that there is an isomorphism from N to $\mathrm{TSG}_+(\Pi)$. Thus N is realizable.

Now we will construct an embedded graph Π' such that $\mathrm{TSG}_+(\Pi') \cong H/N$ by replacing each Γ_j by an arc with two chains attached as follows. Without loss of generality, let $\{\Gamma_1, ..., \Gamma_s, ..., \Gamma_r\}$ consist of one representative from each orbit of $\{\Gamma_1, ..., \Gamma_n\}$ under $\mathrm{TSG}_+(\Gamma)$, such that there is some $g \in G$ which interchanges the boundary components of A_k if and only if $1 \leqslant k \leqslant s$. For each k, let ε_k be an arc properly embedded in B_k (or a simple closed curve if ∂B_k is a pinched sphere) such that $\varepsilon_k \cap \partial B_k = \Gamma' \cap \partial B_k$, and ε_k is isotopic rel boundary to an arc in ∂B_k. We add vertices x_k and y_k to $\mathrm{int}(\varepsilon_k)$, and let a_k and b_k be the closures of the non-adjacent segments of $\varepsilon_k - \{x_k, y_k\}$. We add chains ε'_k and ε''_k in B_k such that ε'_k is isotopic rel boundary to a_k and has length km, and ε''_k is isotopic rel boundary to b_k and has length km if $k < s$ and length $km + 1$ otherwise. Now for each j such that $r < j \leqslant n$, there is some $k \leqslant r$ and some $a \in \mathrm{TSG}_+(\Pi)$ such that $\Gamma_j = g_a(\Gamma_k)$. Then we add $\varepsilon_j = g(\varepsilon_k)$ and the chains $\varepsilon'_j = g(\varepsilon'_k)$ and $\varepsilon''_j = g(\varepsilon''_k)$.

Now for every $j \leqslant n$, we replace Γ_j by $\Gamma'_j = \varepsilon_j \cup \varepsilon'_j \cup \varepsilon''_j$. Observe that for each j,there is no non-trivial element of $\mathrm{TSG}_+(\Gamma'_j)$ which takes ε_j to itself and does not interchange ε'_j and ε''_j. Let $\Pi' = \Gamma' \cup \Gamma'_1 \cup ... \cup \Gamma'_n$. Then Γ'_j and Γ'_k are in the same orbit under $\mathrm{TSG}_+(\Pi')$ if and only if Γ_j and Γ_k are in the same orbit under $\mathrm{TSG}_+(\Gamma)$; and there is an $\alpha' \in \mathrm{TSG}_+(\Pi')$ which interchanges ε'_j and ε''_j if and only if there is a $g \in G$ which interchanges the boundary components of A_j.

For each $\alpha \in \mathrm{TSG}_+(\Gamma)$, there exists $g_\alpha \in G$ such that $g_\alpha(\Pi') = \Pi'$. If another element $g'_\alpha \in G$ also induces α on Γ and $g'_\alpha(\Pi') = \Pi'$, then g_α and g'_α induce the same automorphism of Π'. Define $\Phi : \mathrm{TSG}_+(\Gamma) \to \mathrm{TSG}_+(\Pi')$ by letting $\Phi(\alpha)$ denote the automorphism that g_α induces on Π'. Observe that $N = \ker(\Phi)$.

To see that Φ is onto, let $\alpha' \in \mathrm{TSG}_+(\Pi')$. By our construction, $\{\Gamma'_1, ..., \Gamma'_n\}$ is setwise invariant under $\mathrm{TSG}_+(\Pi')$. Since each Γ'_j can be isotoped rel $(\Gamma'_j \cap \partial B_j)$ into ∂B_j, there is an orientation preserving diffeomorphism $g' : (S^3, \Pi') \to (S^3, \Pi')$ inducing α' on Π', such that $g'(\{B_1, ..., B_n\}) = \{B_1, ..., B_n\}$. For every j, let C_j be a ball or pinched ball containing B_j such that $C_j \cap \Gamma = \Gamma_j$ and ∂C_j is parallel to ∂B_j in $C_j - B_j$. Now there is an orientation preserving diffeomorphism $g' : (S^3, \Pi') \to (S^3, \Pi')$ inducing α' on Π', such that $g'(\{C_1, ..., C_n\}) = \{C_1, ..., C_n\}$. Let $h|S^3 - \mathrm{int}(C_1 \cup ... \cup C_n) = g'|S^3 - \mathrm{int}(C_1 \cup ... \cup C_n)$.

For each j, we define h on C_j as follows. If $g'(C_j) = C_j$ and g' does not interchange ε'_j and ε''_j, then $g'|\partial C_j$ is isotopic to the identity fixing $\Pi' \cap \partial C_j$. So we can extend h homeomorphically within C_j such that $h|\Gamma_j$ is the identity. If $g'(C_j) = C_j$ and g' does interchange ε'_j and ε''_j, then there is some $g \in G$ such that $g(B_j) = B_j$ and g interchanges the boundary components of A_j. Also, $g'|\partial C_j$ and $g'|\partial B_j$ are each orientation preserving. Now define $h|B_j = g|B_j$, and extend h to a diffeomorphism of $C_j - B_j$. Finally, if $g'(C_j) = C_k$ for some $k \neq j$, then there is $g \in G$ such that $g(B_j) = B_k$ and $g(\Gamma'_j) = g'(\Gamma'_j)$. In this case, we define $h|B_j = g|B_j$, and again extend h to a diffeomorphism taking $C_j - B_j$ to $C_k - B_k$. Now $h : (S^3, \Gamma) \to (S^3, \Gamma)$ and h induces some automorphism α on Γ. Then $\Phi(\alpha) = \alpha'$. Hence $\mathrm{TSG}_+(\Pi') = \mathrm{im}(\Phi) \cong H/N$. Thus both N and H/N are realizable, and Claim 2 is proven.

Because of Claim 2, from now on we assume that if $g \in G$ such that $g(A_i) = A_i$ and g does not interchange the boundary components of A_i, then g induces a trivial automorphism on Γ_i since otherwise we would be done. Thus we have completed Step 3.

Recall from Step 1 that the pared manifold $(W, W \cap (P \cup \Omega))$ is either Seifert fibered, I-fibered, or simple. We shall consider each of these cases in a separate step, making use of the above assumption.

Step 4. *We prove the proposition when* $(W, W \cap (P \cup \Omega))$ *is Seifert fibered.*

Since there is only one component of ∂W that meets $\partial N(\Gamma)$, we know this component is a torus.

Recall from the end of Step 2 that if Γ has no type II or III spheres, then $\partial N(V) \cap W$ contains a disk with at least two holes. Thus in this case, the component of ∂W which meets $\partial N(\Gamma)$ could not be a torus, and thus $(W, W \cap (P \cup \Omega))$ could not be Seifert fibered. So we focus the case where Γ has some type II or III sphere.

Claim 3: Let T be the boundary component of W which meets $\partial N(\Gamma)$. If T is a torus, then $\mathrm{TSG}_+(\Gamma)$ *is a subgroup of a dihedral group.*

Proof of Claim 3: Let $\{x_1, \dots, x_r\}$ denote those vertices of Γ such that $\partial N(x_i)$ meets W. Let the components of $\partial N(V) \cap W$ be $J_1, \dots, J_q$. (Note that for a given vertex x_i, the set $\partial N(x_i) \cap W$ may have more than one component, so we may have $q > r$.) Now each J_i is a sphere with holes, and each boundary component of J_i is either a boundary component of $\partial' N(\varepsilon)$ for some edge ε, or a boundary component of some A_j.

We saw in Step 1, that $\partial W - (W \cap (P \cup \Omega))$ is incompressible in W. Thus for each i, J_i is incompressible in W, and hence each boundary component of J_i is essential in T. Thus, since T is a torus, every J_i has exactly two boundary components.

Recall from Step 2 that G permutes some of the A_j's or some components of $\partial N(V) \cap W$. Thus W must contain at least two A_j's or at least two J_i's. In either case, $q > 1$ and T is made up of alternating annuli $R_1, \dots, R_q$ (which are each either A_j's or components of $\partial' N(E)$), and spheres with two holes, $J_1, \dots, J_q$. Also, G leaves $\{J_1, \dots, J_q\}$ and $\{R_1, \dots, R_q\}$ setwise invariant. It follows that the group of automorphisms that G induces on the set $\{J_1, \dots, J_q, R_1, \dots, R_q\}$ is a subgroup of the dihedral group D_q.

Define $\Phi : \mathrm{TSG}_+(\Gamma) \to D_q$ by letting $\Phi(a)$ denote the automorphism that g_a induces on the set $\{J_1, \dots, J_q, R_1, \dots, R_q\}$. We see that Φ is well-defined as follows. Suppose that g_a and g'_a are both elements of G which induce a on Γ. Then g_a and g'_a induce the same permutation on the set of the components of $\partial N(V)$, on the set of annuli $\{A_1, \dots, A_n\}$, on the set of circles $\{c_1, d_1, \dots, c_n, d_n\}$, and on the set of components of $\partial' N(E)$. Thus g_a and g'_a both induce the same permutation on $\{J_1, \dots, J_q, R_1, \dots, R_q\}$. Therefore, Φ is a well-defined homomorphism.

We show as follows that Φ is one-to-one. Suppose that $\Phi(a)$ is the identity. Then for each $i = 1, \dots, q$, we have $g_a(J_i) = J_i$ and $g_a(R_i) = R_i$. Hence $a(x_i) = x_i$ for each vertex x_i such that $\partial N(x_i)$ meets W. Let v be a vertex of Γ such that $v \notin \{x_1, \dots, x_r\}$. Then for some j, the vertex v is in Γ_j, and A_j is one of the R_i. Also since T contains more than one J_i and g_a does not permute the J_i, the boundaries of A_j cannot be interchanged by g_a. In particular, $g_a(c_j) = c_j$ and $g_a(d_j) = d_j$. Thus by our assumption at the end of

Step 3, $a|\Gamma_j$ is the identity. In particular, $a(v) = v$, so Φ is one-to-one. Hence $\mathrm{TSG}_+(\Gamma)$ is a subgroup of D_q. This completes the proof of Claim 3.

Thus if the component of ∂W that meets $\partial N(\Gamma)$ is a torus, then $\mathrm{TSG}_+(\Gamma)$ is isomorphic to a finite subgroup of $SO(4)$ and hence of $\mathrm{Diff}_+(S^3)$, so we are done. In particular, if $(W,(W \cap (P \cup \Omega))$ is Seifert fibered then we are done.

From now on we shall assume that the boundary component of W which meets $\partial N(\Gamma)$ is not a torus. Since $\partial W - (W \cap (P \cup \Omega))$ is incompressible in W, the boundary component of W which meets $\partial N(\Gamma)$ is also not a sphere. Hence the boundary component of W which meets $\partial N(\Gamma)$ has genus at least three. It follows that there is some vertex v of Γ such that $\partial N(v) \cap W$ contains a sphere with at least three holes. We will need this fact in the proof of Claim 5 in Step 6.

Step 5. *We prove the proposition when $(W, W \cap (P \cup \Omega))$ is I-fibered.*

Since W is an I-fibered subspace of S^3, $W = Y \times I$ where Y is a surface with holes. Furthermore, since the pared manifold $(W, W \cap (P \cup \Omega))$ is I-fibered, by definition $W \cap (P \cup \Omega)$ is contained in $\partial Y \times I$. Thus $Y \times \{0\}$ must be contained in some component of $\partial N(V) \cap W$, and thus Y must be a sphere with holes. Since the boundary of W has only one component, $X = \mathrm{cl}(S^3 - N(\Gamma))$. Also, there are vertices v_0 and v_1 (not necessarily distinct) in Γ such that either $v_0 \neq v_1$, and $Y \times \{0\} = \partial N(v_0) \cap W$ and $Y \times \{1\} = \partial N(v_1) \cap W$, or $v_0 = v_1$ and $Y \times \{0\}$ and $Y \times \{1\}$ are the components of $\partial N(v_0) \cap W$.

Suppose that Y has only one boundary component. Then $\partial Y \times I$ is compressible in W, so ∂W contains no elements of Ω, and hence $\partial N(\Gamma) = \partial W$. This implies that Γ is a tree. But this is impossible since Γ has no vertices of valence one. We deduce that ∂Y has at least two components.

If ∂Y has exactly two components then ∂W would be a torus, contrary to our assumption at the end of Step 5. Hence ∂Y has at least three components. Let $b_1, \dots, b_r$ denote the boundary components of Y, let $Y_0 = Y \times \{0\}$ and $Y_1 = Y \times \{1\}$, and for each i, let C_i denote the annulus $b_i \times I$. Every $g \in G$ restricts to a map of the pair $(W, W \cap (P \cup \Omega))$. So for every $g \in G$, $g(\{Y_0, Y_1\}) = \{Y_0, Y_1\}$ and $g(\{C_1, \dots, C_r\}) = \{C_1, \dots, C_r\}$. Then for each i, let F_i denote the sphere or pinched sphere obtained from C_i by adding disks containing v_0 and v_1 within $N(v_0)$ and $N(v_1)$ respectively, such that the disks are disjoint from Γ and from each other except possibly at v_0 and v_1. Let E_i denote the closure of the component of $S^3 - F_i$ which is disjoint from W, and let $\gamma_i = \Gamma \cap E_i$. Then for each i, either $\gamma_i = \Gamma_j$ for some j, or γ_i is an arc of Γ, and $\Gamma = \gamma_1 \cup \dots \cup \gamma_r$.

Suppose $v_0 = v_1$ and some Γ_k is a single vertex. Let b denote one boundary component of A_k. Then b bounds disks D_0 and D_1 in $\partial N(v_0)$ such that $Y_0 \subset D_0$ and $Y_1 \subset D_1$, and every boundary component of Y_0 bounds a disk in D_0. By Step 1, Y_0 is incompressible in $X = \mathrm{cl}(S^3 - N(\Gamma))$, so each such disk must meet Γ. It follows that Γ_k is the only γ_i which is a single vertex. In particular, since $r > 2$, if $v_0 = v_1$, then some F_j is a type III sphere.

Suppose that Γ has no type II or III spheres. Then $v_0 \neq v_1$. Hence, by Step 2, every γ_i is an arc. For the sake of contradiction suppose that ∂Y has at least four components. Let c be a simple closed curve on $Y \times \{0\}$ which separates two of the boundary components of Y from the rest of the boundary components of Y. Let A denote the annulus $c \times I$ in $Y \times I$. Now $c \times \{0\}$ bounds a disk D_0 in $N(v_0)$ such that $D_0 \cap \Gamma = \{v_0\}$ and $c \times \{1\}$ bounds a disk D_1 in $N(v_1)$ such that $D_1 \cap \Gamma = \{v_1\}$. Let $\Sigma = A \cup D_0 \cup D_1$. Then Σ is a type II sphere, contrary to our hypothesis. Thus $r = 3$. It follows that Γ is a theta graph θ_3 with some additional vertices of valence 2 and $\mathrm{Aut}(\theta_3) \cong S_3 \times \mathbb{Z}_2$. Thus $\mathrm{TSG}_+(\Gamma)$ is a subgroup of the dihedral group $D_6 \cong S_3 \times \mathbb{Z}_2$ and hence is isomorphic to a finite subgroup of $SO(4)$ and hence of $\mathrm{Diff}_+(S^3)$. Furthermore, we can re-embed Γ as Δ in S^2 such that the vertices of Δ of valence three are at the poles of the sphere, and the components of $S^2 - \Delta$ are three identical wedges. Because of the symmetry of this embedding $\mathrm{TSG}_+(\Delta) = \mathrm{Aut}(\gamma)$. Now $\mathrm{TSG}_+(\Delta)$ is induced by an isomorphic finite subgroup K of $SO(4)$ and hence of $\mathrm{Diff}_+(S^3)$. Thus the subgroup $H = \mathrm{TSG}_+(\Gamma)$ is induced on Δ by an isomorphic subgroup of K. So if Γ has no type II or III spheres, then we are done.

Now suppose that there is some $g \in G$ which interchanges Y_0 and Y_1. Let $\Phi : \mathrm{TSG}_+(\Gamma) \to \mathbb{Z}_2$ be defined as follows. For each $a \in \mathrm{TSG}_+(\Gamma)$, let $\Phi(a)$ be the permutation that g_a induces on the set $\{Y_0, Y_1\}$. Then Φ is onto. If $\ker(\Phi)$ is trivial then it follows that $\mathrm{TSG}_+(\Gamma) \cong \mathbb{Z}_2$. Hence in this case we are done.

If $N = \ker(\Phi)$ is non-trivial, then we construct an embedded graph Π as follows. Let ε_1 be an edge of Γ containing v_0, such that its other vertex is not v_1. Let $\{\varepsilon_1, ..., \varepsilon_s\}$ denote the orbit of ε_1 under N. For each ε_i we add a vertex x_i in the interior of ε_i and add a new edge β_i attached only at x_i. Let Π denote the embedded graph obtained from Γ in this way. Now the set $\{\beta_1, ..., \beta_s\}$ is setwise invariant under $\mathrm{TSG}_+(\Pi)$. It follows that $\mathrm{TSG}_+(\Pi) \cong \ker(\Phi)$. Hence both N and $H/N \cong \mathrm{im}(\Phi) = \mathbb{Z}_2$ are realizable.

Thus from now on we assume that for every $g \in G$ we have $g(Y_0) = Y_0$ and $g(Y_1) = Y_1$.

Claim 4:
a) If $a \in \mathrm{TSG}_+(\Gamma)$ such that $a(\gamma_i) = \gamma_j$ with $i \neq j$, then there exists $a' \in \mathrm{TSG}_+(\Gamma)$ such that $a'(\gamma_i) = \gamma_j$ and $a'|\Gamma - (\gamma_i \cup \gamma_j)$ is the identity.

b) If $a \in \mathrm{TSG}_+(\Gamma)$ such that $a(\gamma_j) = \gamma_j$ for some j, then $a|\gamma_j$ is the identity.

Proof of Claim 4:
a) Let $a \in \mathrm{TSG}_+(\Gamma)$ such that $a(\gamma_i) = \gamma_j$ with $i \neq j$. We will define an element $g \in G$ as follows. Let $g|E_i = g_a|E_i$ and $g|E_j = g_a^{-1}|E_j$. Then g interchanges E_i and E_j. Let b denote a simple closed curve in Y which separates b_i and b_j from all the other boundary components of Y. Let F denote the disk with two holes in Y bounded by the three curves b_i, b_j, and b. Extend g homeomorphically to $F \times I$ such that $g|(b \times I)$ is the identity. Now extend g to $S^3 - ((F \times I) \cup N(v_0) \cup N(v_1)$ by the identity. Finally, extend g within $N(v_0)$ and $N(v_1)$ such that $g(\Gamma) = \Gamma$. Now $g : (S^3, \Gamma) \to (S^3, \Gamma)$ and $g|\Gamma - (\gamma_i \cup \gamma_j)$ is the identity. Let a' denote the automorphism of Γ induced by g. Then a' interchanges γ_i and γ_j, and $a'|\Gamma - (\gamma_i \cup \gamma_j)$ is the identity.

b) Now suppose that $a \in \mathrm{TSG}_+(\Gamma)$ such that $a(\gamma_j) = \gamma_j$ for some j. If $C_j = A_i$, for some A_i, then $\gamma_j = \Gamma_i$, $g_a(A_i) = A_i$, $g_a(c_i) = c_i$ and $g_a(d_i) = d_i$. Hence by the end of Step 3, $a|\gamma_j$ is the identity. If $C_j \notin \Omega$, then γ_j is an arc if $v_0 \neq v_1$ and a simple closed curve if $v_0 = v_1$. In either case, since $g_a(Y_0) = Y_0$ and $g_a(Y_1) = Y_1$, we again conclude that $a|\gamma_j$ is the identity. Thus we have proven Claim 4.

It follows from Claim 4b) that for every non-trivial $a \in \mathrm{TSG}_+(\Gamma)$, there is some γ_j such that $a(\gamma_j) \neq \gamma_j$. Because $\mathrm{TSG}_+(\Gamma)$ is non-trivial, without loss of generality we can assume that γ_1 is not setwise invariant under $\mathrm{TSG}_+(\Gamma)$. Let $\{\gamma_1, \ldots, \gamma_q\}$ be the orbit of γ_1 under $\mathrm{TSG}_+(\Gamma)$.

Now define $\Phi : \mathrm{TSG}_+(\Gamma) \to S_q$ by letting $\Phi(a)$ be the permutation that g_a induces on the set $\{C_1, \dots, C_q\}$. Then Φ is well-defined. We prove as follows that Φ is onto. Let (ij) be any transposition in the symmetric group S_q. Since $\{\gamma_1, \dots, \gamma_q\}$ is the orbit of γ_1, there is some $a \in \mathrm{TSG}_+(\Gamma)$, such that $a(\gamma_i) = \gamma_j$. Now by Claim 4a), there exists $a' \in \mathrm{TSG}_+(\Gamma)$ such that $a'(\gamma_i) = \gamma_j$ and $a'|\Gamma - (\gamma_i \cup \gamma_j)$ is the identity. Hence $\Phi(a') = (ij)$. Thus Φ is onto. If $\ker(\Phi)$ is trivial then $\mathrm{TSG}_+(\Gamma) \cong S_q$ and we are done with Step 5.

If $N = \ker(\Phi)$ is non-trivial, then there is some γ_k with $k > q$ such that $a(\gamma_k) \neq \gamma_k$. There is at most one edge of Γ containing both v_0 and v_1. Let ε_1 be an edge of γ_k containing v_0. Since $a(\gamma_k) \neq \gamma_k$, the other vertex of ε_1 is not v_1. Let $\{\varepsilon_1, ..., \varepsilon_s\}$ denote the orbit of ε_1 under N. We obtain Π from Γ as follows. First remove γ_1, ..., γ_q from Γ. Then, add a vertex x_i in the interior of each ε_i and add a new edge β_i attached only at x_i. Let Π be the embedded graph we obtain in this way. Now the set $\{\beta_1, ..., \beta_s\}$ is setwise invariant under $\mathrm{TSG}_+(\Pi)$. It follows that $\mathrm{TSG}_+(\Pi) \cong N$.

Let Π' denote the graph consisting of q edges joined together at a single vertex. Then $\mathrm{TSG}_+(\Pi') \cong S_q = \mathrm{im}(\Phi) = H/N$. Hence both N and H/N are realizable. Thus we are done in the case where $(W, W \cap (P \cup \Omega))$ is I-fibered.

Step 6. *We prove the proposition when* $(W, W \cap (P \cup \Omega))$ *is simple.*

Note the argument in the beginning of this step will be similar to the analogous part of the proof of Theorem 1 in [F2].

We begin by applying Thurston's Hyperbolization Theorem for Pared Manifolds [Th] to $(W, W \cap (P \cup \Omega))$ to conclude that $W - (W \cap (P \cup \Omega))$ admits a finite volume complete hyperbolic metric with totally geodesic boundary. Recall that every $g \in G$ restricts to a map of $(W, W \cap (P \cup \Omega))$. Let D denote the double of $W - (W \cap (P \cup \Omega))$ along its boundary. Then D is a finite volume hyperbolic manifold. For every $a \in \mathrm{TSG}_+(\Gamma)$, the diffeomorphism g_a induces a diffeomorphism g'_a of D which restricts to g_a on each side of D. Now we use Mostow's Rigidity Theorem [Mo], to find an isometry h'_a of D that restricts to an isometry h_a of $(W, W \cap (P \cup \Omega))$ such that h_a is homotopic to g_a on $(W, W \cap (P \cup \Omega))$. It follows from Waldhausen's Isotopy Theorem [Wa] that h_a is actually isotopic to g_a on $(W, W \cap (P \cup \Omega))$.

Before we extend h_a to a finite order diffeomorphism of S^3, we re-embed W in S^3 as follows. Let T_j be a torus in Θ which is contained in ∂W. The closure of one component of $S^3 - T_j$ is a solid torus Z_j. Because T_j is incompressible in M, Z_j cannot be contained in M. Thus $\partial N(\Gamma)$ is contained in Z_j, and $R_j = \mathrm{cl}(S^3 - Z_j)$ is a knot complement contained in M. Now, up to isotopy, there is a well-defined longitude ℓ_j of T_j, which bounds a Seifert surface in R_j. Observe that for every $g \in G$, there is some R_k in M such that $g(R_j) = R_k$, and $g(\ell_j)$ is a longitude of R_k. We re-embed W in S^3 by replacing each R_j by a solid torus U_j such that a meridian of U_j is glued to ℓ_j.

Recall that $\Gamma' = \mathrm{cl}(\Gamma - (\Gamma_1 \cup ... \cup \Gamma_n))$. Let W' be the union of W together with all of the solid tori U_j. Then W' is contained in S^3, and the boundary of W' has a single component contained in the union of $\partial N(\Gamma')$ and a collection of annuli in Ω. We glue $N(\Gamma')$ to W' as it was attached to W. This gives us a manifold W'' in S^3 which contains W such that each boundary component of W'' is either a sphere made up of an annulus A_i together with a pair of disks in $\partial N(V)$ or a torus made up of an A_i together with an annulus in $\partial N(V)$. For each such sphere we glue in a solid cylinder C_i to W'' such that

C_i has boundary A_i together with these two disks. Then we glue in solid tori to all the torus boundary components of W'' so that W'' together with the solid cylinders and solid tori is S^3. Let Δ' denote the result of re-embedding Γ' in this way.

Now for every $a \in \mathrm{TSG}_+(\Gamma)$, we shall extend the isometry h_a of $(W, W \cap (P \cup \Omega))$ to a finite order orientation preserving diffeomorphism of S^3 as follows. Because g_a is isotopic to h_a on $(W, W \cap (P \cup \Omega))$, h_a will take a meridian of each U_j to a meridian of some U_k, since g_a took a longitude of R_j to a longitude of R_k. Thus we can extend h_a radially within each solid torus U_j. Now extend h_a radially within each solid cylinder component of $N(E)$ and each ball component of $N(V)$. Finally extend h_a radially within each solid cylinder or solid torus C_i. Thus for each $a \in \mathrm{TSG}_+(\Gamma)$, we have defined a finite order orientation preserving diffeomorphism h_a of S^3 such that $h_a(\Delta') = \Delta'$ and h_a induces the automorphism a on Δ'.

Suppose that Γ has no type II or type III spheres. Then each Γ_i is either an arc or a single vertex. If Γ_i is an arc then C_i is a solid cylinder. In this case, we re-embed Γ_i as Γ_i' so that $\Gamma_i' \cap C_i$ is the core of C_i and we embed the segments in $\Gamma_i - C_i$ as radii of some $N(v)$. If Γ_i is a single vertex let $\Gamma_i' = \Gamma_i$. The union of Δ' together with Γ_1', … , Γ_n' gives us an embedded graph Δ in S^3 which has underlying abstract graph γ. Now for each $a \in \mathrm{TSG}_+(\Gamma)$, $h_a(\{\Gamma_1', \dots, \Gamma_n'\}) = \{\Gamma_1', \dots, \Gamma_n'\}$, since each Γ_i is the core of C_i and h was defined radially in C_i. Thus for each $a \in \mathrm{TSG}_+(\Gamma)$, $h_a(\Delta) = \Delta$ and h_a induces a on Δ. It follows that $H = \mathrm{TSG}_+(\Gamma)$ is a subgroup of $\mathrm{TSG}_+(\Delta)$.

Now we return to the general case. Let $K = \{h_a | a \in H\}$ where h_a is the diffeomorphism of S^3 defined above. Define $\Phi : H \to K$ by $\Phi(a) = h_a$.

Claim 5: K is a well-defined group and Φ is a well-defined isomorphism.

Proof of Claim 5: We begin by proving that Φ is well-defined. Let h_a and h_a' be diffeomorphisms of S^3 which both come from a given $a \in \mathrm{TSG}_+(\Gamma)$ by the above construction. Then both h_a and h_a' induce the same permutation of the set of components of $\partial N(V) \cap W$, the same permutation of the set of components of $\partial' N(E)$ in ∂W, and the same permutation of the A_i's. We show as follows that $h_a = h_a'$.

By the end of Step 4 we know there is some vertex v of Γ such that a component J of $\partial N(v) \cap W$ is a sphere with $r \geqslant 3$ holes. Let α_1, … , α_r denote the boundary components of J. Now $h_a(J) = h_a'(J)$ and $h_a(\alpha_i) = h_a'(\alpha_i)$ for each $i = 1$, … , r. Since J is a sphere with at least three holes and $h_a^{-1} \circ h_a' | J$ is an orientation preserving isometry which setwise fixes all of the boundary components of J, $h_a^{-1} \circ h_a' | J$ is the identity. It follows that $h_a | J = h_a' | J$. Now since $h_a | W$ and $h_a' | W$ are isometries which are identical on J, we must have $h_a | W = h_a' | W$. Since h_a and h_a' are both determined by their restrictions to W, we can conclude that $h_a = h_a'$. Hence K contains exactly one h_a for each $a \in H$ and Φ is well-defined.

Now let h_a, $h_b \in K$, and let $f = h_a \circ h_b$. Both f and $h_{a \circ b}$ are diffeomorphisms of S^3 which can be constructed by radially extending isometries of W. Also, f and $h_a \circ h_b$ both induce the same permutation of the set of components of $\partial N(V)$ which meet ∂W, the set of components of $N(E)$ in ∂W, and the set of components of Ω in ∂W. So by the same argument as in the above paragraph $f = h_{a \circ b}$. Hence K is a group.

Clearly, Φ is onto. To see that Φ is one-to-one, suppose that $a \in \mathrm{TSG}_+(\Gamma)$ such that $\Phi(a) = h_a$ is the identity on S^3. Thus $h_a | W$ is the identity. So for every vertex v of Γ such that $\partial N(v)$ meets W, $a(v) = v$. Let v be a vertex of Γ such that $\partial N(v)$ does not meet W. Then v is a vertex in some Γ_i. Since A_i is contained in ∂W, $h_a|A_i$ is the identity. So, $h_a(c_i) = c_i$, and $h_a(d_i) = d_i$. Also, h_a is isotopic to g_a on $(W, W \cap (P \cup \Omega))$. Hence $g_a(A_i) = A_i$, $g_a(c_i) = c_i$, and $g_a(d_i) = d_i$. Now by the end of Step 3, $a|\Gamma_i$ is the identity. So $a(v) = v$. It follows that Φ is an isomorphism. Hence we have proven Claim 5.

Now H is finite group that is a finite group that is isomorphic to the subgroup K of $\mathrm{Diff}_+(S^3)$. Furthermore, if Γ has no type II or type III spheres, then we saw before Claim 5 that every automorphism $a \in H$ is induced on Δ by the element h_a of K. Also, every element of K induces some element of H on Δ. Thus H is induced on Δ by an isomorphic finite subgroup of $\mathrm{Diff}_+(S^3)$.

This completes the proof of Proposition 1. $\square$

4. Proof of Proposition 2

A key part of the proof of Proposition 2 involves showing that if $\mathrm{TSG}_+(\Gamma)$ is a simple group, then we can find an embedded graph Δ with no type I vertices such that $\mathrm{TSG}_+(\Delta) \cong \mathrm{TSG}_+(\Gamma)$. Our strategy is to first find a subgraph Γ' which is setwise invariant under $\mathrm{TSG}_+(\Gamma)$ such that Γ' has fewer type I vertices than Γ. We then will create a new graph Λ with the same type I vertices as Γ' by adding vertices or chains to Γ' in order to make certain that $\mathrm{TSG}_+(\Lambda) \cong \mathrm{TSG}_+(\Gamma)$. Continuing this process until no type I vertices remain, we arrive at the desired embedded graph Δ.

We begin with some definitions and lemmas. The proofs in this section make use of combinatorial arguments and cut and paste techniques.

Definition 7. Let Γ be a graph embedded in S^3, and let Σ and Σ' be spheres which each meet Γ in a single vertex. Then Σ and Σ' are said to be *almost disjoint* if $\Sigma \cap \Sigma'$ is either empty or a single vertex of Γ. If $\Sigma \cap \Sigma'$ is the single vertex v and Σ bounds a ball B containing Σ' such that Σ' is isotopic to Σ in B fixing Γ, then Σ and Σ' are said to be *parallel*.

We can consider those type I spheres which have the property that there are no other non-parallel type I spheres on one side of the sphere.

Definition 8. Let Γ be a graph embedded in S^3, and let Σ be a type I sphere for Γ. Suppose that a ball B bounded by Σ has the property that any other type I sphere Σ' which is contained in B and almost disjoint from Σ, is parallel to Σ. Then we say that Σ is an *innermost type I sphere*, B is an *innermost ball*, and $\Gamma \cap B$ is an *innermost subgraph* with *attaching vertex* $\Gamma \cap \Sigma$.

We can also focus on a particular type I vertex, and consider the type I spheres at that vertex which have the property that there are no other non-parallel type I spheres at that vertex on one side of the sphere. In the following definitions, we use the term *rel* v to mean that we are only considering those type I spheres that meet Γ at the vertex v.

Definition 9. Let Γ be a graph embedded in S^3, and let Σ be a type I sphere for Γ with $\Sigma \cap \Gamma = \{v\}$. Suppose that the closure B of one component of $S^3 - \Sigma$ has the property that any other type I sphere Σ' which is contained in B and such that $\Sigma \cap \Sigma' = \{v\}$, is parallel to Σ. Then we say that Σ is an *innermost type I sphere rel* v, B is an *innermost ball rel* v, and $\Gamma \cap B$ is an *innermost subgraph rel* v.

Definition 10. Let Γ be a graph embedded in S^3, and let $F = \{\Sigma_1, \dots, \Sigma_n\}$ be a collection of innermost type I spheres (respectively rel v) for Γ. We say F is a *maximal collection of innermost type I spheres* (respectively *rel* v) if the following three conditions hold:
1) The spheres in F are pairwise almost disjoint.
2) No two spheres in F are parallel.
3) If Σ' is an innermost type I sphere (respectively rel v) for Γ which is almost disjoint from every Σ_i, then Σ' is parallel to some Σ_i.

Observe that if a maximal collection of innermost type I spheres contains only a single sphere, then there are two innermost subgraphs and two innermost balls.

In the next few lemmas we shall prove that every embedded graph, has a maximal collection of innermost type I spheres (possibly rel v), and this collection is unique up to isotopy fixing Γ.

Lemma 1. *(Existence) Let Γ be a graph embedded in S^3. Then Γ has a maximal collection of innermost type I spheres. Furthermore, if v is a type I vertex, then Γ has a maximal collection of innermost type I spheres rel v.*

Proof. We prove that Γ has a maximal collection of innermost type I spheres. The proof that Γ has a maximal collection of innermost type I spheres rel v is analogous.

If Γ has no type I spheres, then the empty set satisfies the conclusion of the lemma. Suppose that Σ is a type I sphere for Γ. Let B be the closure of one component of $S^3 - \Sigma$. Then $\Gamma' = B \cap \Gamma$ is a proper subgraph of Γ containing at least one edge. Suppose that Σ is not innermost. Then Γ has a type I sphere $\Sigma' \subset B$ which is almost disjoint from Σ, such that Σ' is not parallel to Σ. Let B' denote the closure of the component of $S^3 - \Sigma'$ which does not contain Σ. Then $\Gamma'' = B' \cap \Gamma'$ is a proper subgraph of Γ' containing at least one edge. Suppose that Σ' is not innermost. Then Γ has a type I sphere $\Sigma'' \subset B'$ which is almost disjoint from Σ' and not parallel to Σ'. Since the graph Γ has a finite number of edges, this process cannot continue indefinitely, so eventually we obtain an innermost type I sphere Σ_0.

Let $F_0 = \{\Sigma_0\}$, and let Σ_0 have innermost ball B_0. Suppose Γ has an innermost type I sphere Σ_1 which is almost disjoint from Σ_0 and not parallel to Σ_0. Let B_1 be the innermost ball for Σ_1 and let $A = \mathrm{cl}(S^3 - (B_0 \cup B_1))$. Then $\Lambda = A \cap \Gamma$ is a proper subgraph of Γ. Let $F_1 = \{\Sigma_0, \Sigma_1\}$. Suppose Γ has an innermost type I sphere Σ_2 which is almost disjoint from both spheres in F_1 and not parallel to either sphere in F_1. Let B_2 be the innermost ball for Σ_2, and let $A' = \mathrm{cl}(A - B_2)$. Then $\Lambda' = A' \cap \Lambda$ is a proper subgraph of Λ. Let $F_2 = \{\Sigma_0, \Sigma_1, \Sigma_2\}$. Again since Γ is finite, we eventually obtain a maximal collection F_k of pairwise non-parallel, almost disjoint, innermost type I spheres for Γ. $\square$

We will use the following lemma to prove the Uniqueness Lemma, as well as in subsequent proofs. In particular, this lemma will allow us to move one set of pairwise almost disjoint type I spheres so that it is pairwise almost disjoint from another such set.

Lemma 2. *Let Γ be a graph embedded in S^3, and let $S_1, \ldots, S_m$ be pairwise almost disjoint type I spheres for Γ. Let $\Sigma_1, \ldots, \Sigma_n$ be pairwise almost disjoint spheres such that for each i, $\Sigma_i \cap \Gamma = \{v_i\}$ and Σ_i bounds a ball B_i such that $B_i \cap \Gamma$ is either $\{v_i\}$ or an innermost subgraph of Γ (possibly rel v_i). Then there is an isotopy of S^3 fixing Γ which takes $\{S_1, \ldots, S_m\}$ to a collection $\{T_1, \ldots, T_m\}$ of pairwise almost disjoint type I spheres that are each almost disjoint from each Σ_i.*

Proof. By moving the S_j's slightly if necessary, we can assume that each S_j meets each Σ_i transversely in a finite collection of disjoint circles, possibly together with a vertex. Consider $J = \{\Sigma_i \cap S_j | \ 1 \leqslant i \leqslant n, \ 1 \leqslant j \leqslant m\}$. Observe that the intersection of any pair of circles in J is either empty or a single vertex. If J contains no circles, then we are done. We show as follows how to isotop the family $\{S_1, \ldots, S_m\}$, fixing Γ, to a new family $\{S_1', \ldots, S_m'\}$ such that $J' = \{\Sigma_i \cap S_j' | \ 1 \leqslant i \leqslant n, \ 1 \leqslant j \leqslant m\}$ contains fewer circles than J.

Case 1: J contains some circle which does not contain any vertex.

Pick q such that $\Sigma_q - \{v_q\}$ contains at least one circle of J. Pick c to be a circle of J which is innermost on $\Sigma_q - \{v_q\}$. Then c bounds a disk D_q on Σ_q disjoint from Γ and whose interior is disjoint from any S_j. Now for some r, c is contained in S_r. Choose D to be the disk on S_r which is bounded by c and does not contain a vertex.

Now the sphere $D \cup D_q$ is disjoint from Γ and all S_j with $j \neq r$. Thus $D \cup D_q$ bounds a ball B which is disjoint from Γ. It follows that B is disjoint from all S_j with $j \neq r$. Let $N(B)$ denote a neighborhood of B which is disjoint from Γ and from all S_j with $j \neq r$. Isotop S_r, fixing $S^3 - N(B)$ (and hence fixing every S_j with $j \neq r$), by moving D across B and past D_q to a disk which is parallel to D_q and disjoint from every Σ_i. Let S_r' denote S_r after this isotopy, and for each $j \neq r$ let $S_j' = S_j$. Then S_r' is a sphere which is almost disjoint from the other S_j'. Also, S_r' intersects Σ_q in fewer circles than S_r does, and for each i, every circle of $S_r' \cap \Sigma_i$ is a circle of $S_r \cap \Sigma_i$. Now the collection $\{S_1, \ldots, S_m\}$ is isotopic, fixing Γ, to the collection $\{S_1', \ldots, S_m'\}$ of pairwise almost disjoint type I spheres, and $J' = \{\Sigma_i \cap S_j' | \ 1 \leqslant i \leqslant n, \ 1 \leqslant j \leqslant m\}$ contains fewer circles than J.

Case 2: Every circle of J contains a vertex.

Pick q and r such that $\Sigma_q \cap S_r$ contains at least one circle of J. Then $\Sigma_q \cap S_r$ will consist of a single circle of J, since any such circle contains v_q. Thus $B_q \cap S_r$ is a disk D_r. Now D_r splits B_q into two balls B' and B''. Since Γ_q is either an innermost subgraph (possibly rel v_q) or a single vertex, the intersection of at least one of B' or B'' with Γ is a single vertex. Without loss of generality, $B' \cap \Gamma$ is a single vertex. Let D' denote the disk in Σ_q such that $\partial B' = D' \cup D_r$. Let c be a circle of J which is innermost on D' (possibly $c = \partial D_r'$). Then J contains v_q. For some p, $c = \Sigma_q \cap S_p$. Then $B' \cap S_p$ is a disk D_p bounded by c. Let D_q denote the disk in D' which is bounded by c.

Now the sphere $D_p \cup D_q$ meets Γ in the single vertex v_q. Since $B' \cap \Gamma = \{v_q\}$, $D_p \cup D_q$ bounds a ball B whose interior is disjoint from Γ. It follows that B is disjoint from all S_j with $j \neq p$. Isotop S_p, fixing Γ and every S_j with $j \neq p$, by moving D_p across B and past D_q to a disk which is parallel to D_q and whose intersection with every Σ_i and every S_j with $j \neq p$ is either empty or the single vertex v_q. Let S'_q denote S_q after this isotopy, and for each $j \neq q$ let $S'_j = S_j$. Then S'_p is a sphere which is almost disjoint from the other S_j and from Σ_q. Also, for each i, every circle of $S'_p \cap \Sigma_i$ is a circle of $S_p \cap \Sigma_i$. Now the collection $\{S_1, \dots, S_m\}$ is isotopic, fixing Γ, to the collection $\{S'_1, \dots, S'_m\}$ of pairwise almost disjoint type I spheres, and $J' = \{\Sigma_i \cap S'_j | \; 1 \leqslant i \leqslant n, \, 1 \leqslant j \leqslant m\}$ contains fewer circles than J. $\square$

Lemma 3. *(Uniqueness) Let Γ be a graph embedded in S^3 which has some type I vertex v. Let $F = \{\Sigma_1, \dots, \Sigma_n\}$ denote either a maximal collection of innermost type I spheres for Γ, or a maximal collection of innermost type I spheres for Γ rel v. Then F is unique up to an isotopy of S^3 fixing Γ.*

Proof. We prove this when $F = \{\Sigma_1, \dots, \Sigma_n\}$ is a maximal collection of innermost type I spheres; the argument rel v is analogous.

Let $v_1, \dots, v_n$ be the respective attaching vertices of $\Sigma_1, \dots, \Sigma_n$. Let $E = \{S_1, \dots, S_m\}$ denote another maximal collection of innermost type I spheres for Γ with attaching vertices $w_1, \dots, w_m$. By applying Lemma 2, E can be isotoped fixing Γ to a maximal collection $E' = \{T_1, \dots, T_m\}$ of innermost type I spheres that are each almost disjoint from every sphere in F. Now by definition of maximality, T_1 is either parallel or equal to some Σ_i. Without loss of generality, T_1 is either parallel or equal to Σ_1. In particular, $v_1 = w_1$. Now T_2 is either parallel or equal to some Σ_i. However, T_2 cannot be equal or parallel to Σ_1, since T_2 is not equal or parallel to T_1. Thus without loss of generality, T_2 is equal or parallel to Σ_2. Continue in this way, so that for each j, $v_j = w_j$ and T_j is equal or parallel to Σ_j. This implies that $m \leqslant n$. However, reversing the roles of the T_j's and the Σ_i's we conclude that $m = n$. Since every T_j is equal or parallel to Σ_j, there is an isotopy fixing Γ taking the collection E' to the collection F. Now by composing this isotopy with the isotopy taking E to E', we obtain an isotopy fixing Γ taking E to F. $\square$

Let $F = \{\Sigma_1, \dots, \Sigma_n\}$ be a maximal collection of innermost type I spheres (respectively rel v) for Γ with $n > 1$. Then F uniquely determines a set of innermost balls $\{B_1, \dots, B_n\}$ (respectively rel v) which in turn uniquely determines a family of innermost subgraphs $\{\Gamma_1, \dots, \Gamma_n\}$ (respectively rel v). Thus it follows from Lemma 3 that there is a well-defined set $\{\Gamma_1, \dots, \Gamma_n\}$ of all the innermost subgraphs (respectively rel v) of Γ independent of the choice of the family F, and for each $i \neq j$, $\Gamma_i \cap \Gamma_j$ is either empty or a single vertex.

Lemma 4. *Let Γ be an embedded graph, and suppose that $\{\Gamma_1, \dots, \Gamma_r\}$ is a set of innermost subgraphs of Γ, or innermost subgraphs rel v for a particular type I vertex v. Let $\Gamma' = \mathrm{cl}(\Gamma - (\Gamma_1 \cup \dots \cup \Gamma_r))$. Then every type I vertex of Γ' is a type I vertex of Γ.*

Proof. Let $\{B_1, \dots, B_r\}$ be a collection of innermost balls or innermost balls rel v, with $B_i \cap \Gamma = \Gamma_i$ for each i. Since Σ is a type I sphere for Γ', $\Sigma \cap \Gamma'$ is some vertex x. We can now apply Lemma 2 to the graph Γ' to conclude that there is an isotopy fixing Γ' which takes Σ to a type I sphere Σ' for Γ' that is disjoint from every B_i except possibly at x. Since Σ' is a type I sphere for Γ', both components of $S^3 - \Sigma'$ meet Γ'. If some B_i contained Σ', then $\mathrm{int}(B_i) \cap \Gamma'$ would be non-empty. But this contradicts the fact that Γ_i is innermost or innermost rel v and Σ' is a type I sphere for Γ'. Thus for each i, $\Sigma' \cap B_i$ is either empty or the vertex x. It follows that $\Sigma \cap \Gamma = \{x\}$. Also, since Γ contains Γ', we know that each component of $S^3 - \Sigma$ contains part of Γ. Thus Σ' is also a type I sphere for Γ. □

Lemma 5. *Let Γ be a graph embedded in S^3, and suppose that Γ has some type I sphere. Then Γ has an innermost type I vertex v with the property that at most one innermost subgraph rel v is not an innermost subgraph of Γ.*

Proof. Since Γ has a type I sphere, there is some type I vertex v_1 where an innermost subgraph is attached. Assume v_1 does not satisfy the conclusion of the lemma. Then there is some ball B_1 which is innermost rel v_1 but not innermost. Hence $\mathrm{int}(B_1)$ contains a type I vertex v_2 where another innermost subgraph is attached.

We shall continue this process inductively for $n \geqslant 2$ as follows. Let $v_1, \dots, v_n$ be distinct vertices and let B_{n-1} be an innermost ball rel v_{n-1} whose interior contains a type I vertex v_n where an innermost subgraph is attached such that $\mathrm{int}(B_{n-1})$ does not contain any v_i with $i < n$. Suppose that v_n does not satisfy the conclusion of the lemma.

By Lemma 1, there is a maximal collection F_n of innermost type I spheres rel v_n. Thus there is a collection E_n of innermost balls rel v_n whose boundaries are the spheres in F_n and whose interiors are pairwise disjoint. Now precisely one of the balls in the collection E_n contains the vertex v_{n-1} in its interior. Since v_n does not satisfy the conclusion of the lemma, there are at least two innermost subgraphs rel v_n which are not innermost. Hence at least two of the balls in E_n are not innermost. Thus we can choose a ball B_n in E_n which is not innermost and which does not contain v_{n-1}.

By applying Lemma 2 to Γ, we can assume that ∂B_n was chosen so that it was disjoint from ∂B_{n-1}. Since $v_n \in \mathrm{int}(B_{n-1})$ and ∂B_n contains v_n, it now follows that $B_n \subset \mathrm{int}(B_{n-1})$. Also since B_n is an innermost ball rel v_n which is not innermost, $\mathrm{int}(B_n)$ contains some type I vertex v_{n+1} where an innermost subgraph is attached. On the other hand, for all $i < n+1$, $v_i \notin \mathrm{int}(B_n)$. Thus $v_1, \dots, v_{n+1}$ are distinct vertices.

Since Γ has only finitely many vertices, this process must come to an end. Hence we eventually find some type I vertex which satisfies the conclusion of the lemma. □

Proposition 2. *Let Γ be an embedded graph which has at least one type I sphere, and let $H = \mathrm{TSG}_+(\Gamma)$. Then either H is realizable by a graph with no type I spheres, $H \cong S_r$ for some r, or H has a non-trivial normal subgroup N such that both N and H/N are realizable.*

Proof. If $\mathrm{TSG}_+(\Gamma) \cong \mathbb{Z}_2$, then H is realized by the graph consisting of a single edge. Thus we assume that $\mathrm{TSG}_+(\Gamma)$ is neither trivial nor $\mathbb{Z}_2$ and Γ has a type I sphere. As before, m

will denote a number larger than the number of vertices in Γ. We will use m throughout the proof. We will show that either $H \cong S_r$ for some r, H has a non-trivial normal subgroup N such that both N and H/N are realizable, or there exists an embedded graph Λ such that $\mathrm{TSG}_+(\Lambda) \cong \mathrm{TSG}_+(\Gamma)$ and Λ has fewer type I vertices than Γ.

Since Γ has a type I vertex, by Lemma 5 some innermost type I vertex v_1 has the property that at most one innermost subgraph rel v_1 is not an innermost subgraph of Γ. We will use this property in Step 3. If v_1 is fixed by $\mathrm{TSG}_+(\Gamma)$, let $F = \{\Sigma_1, \dots, \Sigma_n\}$ be a maximal family of innermost type I spheres rel v_1. Otherwise, let $F = \{\Sigma_1, \dots, \Sigma_n\}$ be a maximal family of innermost type I spheres for Γ with respective attaching vertices $\{v_1, \dots, v_n\}$ (not necessarily distinct). In either case, let the innermost balls and subgraphs associated with F be $\{B_1, \dots, B_n\}$ and $\{\Gamma_1, \dots, \Gamma_n\}$, respectively. If $n > 1$, then the B_i and Γ_i are uniquely determined by the Σ_i. If $n = 1$, then we arbitrarily choose B_1 to be the closure of one component of $S^3 - \Sigma_1$ and choose $\Gamma_1 = B_1 \cap \Gamma$.

Step 1. *We reduce the proof to the case where there is a group G of diffeomorphisms of (S^3, Γ) inducing* $\mathrm{TSG}_+(\Gamma)$ *leaving $\{B_1, \dots, B_n\}$ setwise invariant.*

For each $a \in \mathrm{TSG}_+(\Gamma)$, there is some orientation preserving diffeomorphism $h_a : (S^3, \Gamma) \to (S^3, \Gamma)$ which induces a. By Lemma 3, the set $\{h_a(\Sigma_1), \dots, h_a(\Sigma_n)\}$ is isotopic, fixing Γ, to the set $\{\Sigma_1, \dots, \Sigma_n\}$. So there is an orientation preserving diffeomorphism $f_a : S^3 \to S^3$ which pointwise fixes Γ such that $f_a(\{h_a(\Sigma_1), \dots, h_a(\Sigma_n)\}) = \{\Sigma_1, \dots, \Sigma_n\}$. Hence for each $a \in \mathrm{TSG}_+(\Gamma)$, there exists an orientation preserving diffeomorphism $f_a \circ h_a : (S^3, \Gamma) \to (S^3, \Gamma)$ which induces a and leaves $\{\Sigma_1, \dots, \Sigma_n\}$ setwise invariant. Let G denote the group of all orientation preserving diffeomorphisms $g : (S^3, \Gamma) \to (S^3, \Gamma)$ such that $g(\{\Sigma_1, \dots, \Sigma_n\}) = \{\Sigma_1, \dots, \Sigma_n\}$. Then G induces $\mathrm{TSG}_+(\Gamma)$, and for every $a \in \mathrm{TSG}_+(\Gamma)$, we can choose a $g_a \in G$ such that g_a induces a.

If $n > 1$, then the conclusion of Step 1 follows. Suppose that $n = 1$ and there is some $g \in G$ which interchanges the two components of $S^3 - \Sigma_1$. The closures of the components of $S^3 - \Sigma_1$ intersect Γ in subgraphs α_1 and α_2 and neither α_1 nor α_2 is an arc. Define $\Phi : \mathrm{TSG}_+(\Gamma) \to \mathbb{Z}_2$ by $\Phi(a) = 0$ if g_a does not interchange α_1 and α_2 and $\Phi(a) = 1$ otherwise. Then Φ is onto.

We know that Φ is not an isomorphism, so we may assume $\ker(\Phi)$ is not trivial. We create a new embedded graph Π from Γ by adding m vertices to every edge of α_1. Now every $a' \in \mathrm{TSG}_+(\Pi)$ induces an $a \in \mathrm{TSG}_+(\Gamma)$ which does not interchange α_1 and α_2. Conversely, every $a \in \mathrm{TSG}_+(\Gamma)$ which does not interchange α_1 and α_2 induces an $a' \in \mathrm{TSG}_+(\Pi)$. It follows that $\mathrm{TSG}_+(\Pi) \cong \ker(\Phi)$. Let Π' denote the graph consisting of a single edge, then $\mathrm{TSG}_+(\Pi') \cong \mathbb{Z}_2 \cong \mathrm{im}(\Phi)$. Now $N = \ker(\Phi)$ and $H/N \cong \mathrm{im}(\Phi)$ are both realizable, and we are done.

Thus we assume each component of $S^3 - \Sigma_1$ is setwise invariant under $\mathrm{TSG}_+(\Gamma)$. It follows that for each $g \in G$, we have $g(B_1) = B_1$.

Step 2. *We prove the proposition in the case where v_1 is fixed by* $\mathrm{TSG}_+(\Gamma)$ *and there is some innermost subgraph rel v_1 which is not setwise invariant under* $\mathrm{TSG}_+(\Gamma)$.

In this case, $\{\Gamma_1, \dots, \Gamma_n\}$ was chosen to be a maximal family of innermost subgraphs rel v_1. Without loss of generality, $\{\Gamma_1, \dots, \Gamma_r\}$ is the orbit of Γ_1, and $r > 1$. We define a homomorphism $\Phi : \mathrm{TSG}_+(\Gamma) \to S_r$ by letting $\Phi(a)$ be the permutation that a induces on the set $\{\Gamma_1, \dots, \Gamma_r\}$.

To see that Φ is onto, let (ij) be some transposition in S_r. Since Γ_i and Γ_j are both in the orbit of Γ_1, there is some $a \in \mathrm{TSG}_+(\Gamma)$ such that $a(\Gamma_i) = \Gamma_j$. We will use g_a to define an orientation preserving diffeomorphism $g : (S^3, \Gamma) \to (S^3, \Gamma)$ as follows. Let $g|B_i = g_a|B_i$ and $g|B_j = g_a^{-1}|B_j$. Let B denote a ball containing $B_i \cup B_j$ such that $\partial B \cap (B_i \cup B_j) = \{v_1\}$ and $B \cap \Gamma = \Gamma_i \cup \Gamma_j$. Extend g to a diffeomorphism of B such that $g|\partial B$ is the identity, and extend g to $S^3 - B$ by the identity. Now $g(\Gamma) = \Gamma$ and g takes vertices to vertices. Let a' denote the automorphism induced on Γ by g. Then a' interchanges Γ_i and Γ_j and $a'|\Gamma - (\Gamma_i \cup \Gamma_j)$ is the identity. So $\Phi(a')$ is the transposition (ij). Hence Φ is onto.

If $\ker(\Phi)$ is trivial, then $\mathrm{TSG}_+(\Gamma) \cong S_r$ and we are done, so suppose that $\ker(\Phi)$ is non-trivial. Starting with Γ, for each $i \leqslant r$ we add im vertices to every edge of Γ_i containing v_1. Let Π denote the embedded graph we obtain. Then every $a' \in \mathrm{TSG}_+(\Pi)$ induces an $a \in \mathrm{TSG}_+(\Gamma)$ that takes each Γ_i to itself. Conversely, each $a \in \mathrm{TSG}_+(\Gamma)$ which takes each Γ_i to itself induces an $a' \in \mathrm{TSG}_+(\Pi)$. It follows that $\mathrm{TSG}_+(\Pi) \cong \ker(\Phi)$. Now let Π' denote the embedded graph consisting of r edges joined together at a common vertex. Then $\mathrm{TSG}_+(\Pi') \cong S_r$. Thus if $N = \ker(\Phi)$, then both N and H/N are realizable. So we are done with Step 2.

From now on, we shall assume that one of the following two conditions holds.

(a) v_1 is not fixed by $\mathrm{TSG}_+(\Gamma)$

(b) v_1 is fixed by $\mathrm{TSG}_+(\Gamma)$ and every innermost subgraph rel v_1 is setwise invariant under $\mathrm{TSG}_+(\Gamma)$.

Step 3. *We choose a particular subgraph Γ' on which $\mathrm{TSG}_+(\Gamma)$ induces a non-trivial action.*

If condition (a) holds, then $F = \{\Sigma_1, \dots, \Sigma_n\}$ was chosen to be a maximal family of innermost type I spheres for Γ with respective innermost type I vertices $\{v_1, \dots, v_n\}$ and innermost subgraphs $\{\Gamma_1, \dots, \Gamma_n\}$, and $n > 1$. Recall from our choice of v_1 prior to Step 1 that at most one innermost subgraph rel v_1 is not one of the subgraphs $\Gamma_1, \dots, \Gamma_n$. Since v_1 is not fixed by $\mathrm{TSG}_+(\Gamma)$, there is exactly one innermost graph rel v_1 which is not one of these subgraphs. Let the orbit of the innermost graphs at v_1 be $O = \{\Gamma_1, \dots, \Gamma_r\}$, and let $\Gamma' = \mathrm{cl}(\Gamma - O)$.

Since Γ' contains the vertex v_1, which is not fixed by $\mathrm{TSG}_+(\Gamma)$, there is some $a \in \mathrm{TSG}_+(\Gamma)$ that induces a non-trivial automorphism on Γ'. Observe that Γ' is setwise invariant under G, and the homomorphism $\Psi : \mathrm{TSG}_+(\Gamma) \to \mathrm{TSG}_+(\Gamma')$ given by $\Psi(a) = a|\Gamma'$ is not trivial.

We see as follows that $\ker(\Psi)$ is realizable. If $\ker(\Psi)$ is trivial then it's realizable, so assume it is non-trivial. Thus there is more than one $\Gamma_j \in O$ which is attached at v_1. Let $\{x_1, \dots, x_s\}$ denote a set of distinct vertices representing the orbit of v_1 under $TSG_+(\Gamma)$. Then $s > 1$, hence $r \geqslant 4$. We create a new embedded graph Π from Γ as follows. For each $j \leqslant s$, we add jm vertices to every edge in O containing the vertex x_j. Then we collapse Γ' to a single vertex v. Since $r > 2$, v has valence at least three and is the only vertex in Π which is an endpoint of every chain of length at least m. Hence $\mathrm{TSG}_+(\Pi)$ fixes v. Now for every $a' \in \mathrm{TSG}_+(\Pi)$ we can uniquely define an automorphism a of Γ which fixes every vertex in Γ' and is induced by a' on O. Furthermore, since Γ' is contained in a ball whose interior is disjoint from O, it is not hard to see that the automorphism a is induced by an orientation preserving diffeomorphism of (S^3, Γ). Conversely, every $a \in \mathrm{TSG}_+(\Gamma)$ that fixes every vertex in Γ' induces a unique $a' \in \mathrm{TSG}_+(\Pi)$. It follows that $\mathrm{TSG}_+(\Pi) \cong \ker(\Psi)$.

If condition (b) holds, then $F = \{\Sigma_1, \dots, \Sigma_n\}$ was chosen to be a maximal family of innermost type I spheres rel v_1. Since $\mathrm{TSG}_+(\Gamma)$ is non-trivial, without loss of generality there is some $a \in \mathrm{TSG}_+(\Gamma)$ which induces a non-trivial automorphism on Γ_1. In this case, we let $\Gamma' = \Gamma_1$, and let $r = 1$.

As in the previous case, Γ' is setwise invariant under G, and Ψ (as defined above) is not trivial. We let Π denote $\mathrm{cl}(\Gamma - \Gamma')$ with two chains added at v_1 one of length m and the other of length $2m$. For every $a' \in \mathrm{TSG}_+(\Pi)$ we can define an automorphism $a \in \mathrm{TSG}_+(\Gamma)$ which fixes every vertex in Γ' and is equal to a' on Π. Conversely, every $a \in \mathrm{TSG}_+(\Gamma)$ that fixes every vertex in Γ' induces an $a' \in \mathrm{TSG}_+(\Pi)$. It follows that $\mathrm{TSG}_+(\Pi) \cong \ker(\Psi)$.

In both of the above cases, while the set $\{v_1, \dots, v_r\}$ of vertices is setwise invariant under $\mathrm{TSG}_+(\Gamma)$, it may not be setwise invariant under $\mathrm{TSG}_+(\Gamma')$. Thus, in general, Ψ may not be surjective. So in Step 4 we will create a new graph Λ by adding vertices and chains of vertices to Γ' such that $\mathrm{im}(\Psi) \cong \mathrm{TSG}_+(\Lambda)$. Then in Step 5 we will show that $\mathrm{TSG}_+(\Lambda) \cong \mathrm{im}(\Psi)$.

Step 4. *We construct* Λ.

Let P be a chain in Γ' containing v_1. Note that if no edge in Γ' containing v_1 has a vertex of valence two in Γ', then P will be a single edge. Suppose that $\Gamma' = P$, then $\mathrm{TSG}_+(\Gamma') \cong \mathbb{Z}_2$. Since there is some $a \in \mathrm{TSG}_+(\Gamma)$ which induces a non-trivial automorphism on Γ', $\mathrm{im}(\Psi) = \mathrm{TSG}_+(\Gamma')$. Thus if $N = \ker(\Psi)$, then both N and H/N are realizable. If N is trivial, then $\mathrm{TSG}_+(\Gamma) \cong \mathbb{Z}_2$. So in this case we are done. Thus we assume that Γ' is not a chain.

We will construct an embedded graph Λ that satisfies the following two conditions:

(1) For every $a \in \mathrm{TSG}_+(\Gamma)$, there is a $g_a \in G$ such that $g_a(\Lambda) = \Lambda$.

(2) For any diffeomorphism $g : (S^3, \Lambda) \to (S^3, \Lambda)$, $g(\{v_1, \dots, v_r\}) = \{v_1, \dots, v_r\}$, and $g(\Gamma') = \Gamma'$.

Case 1: v_1 has valence one in Γ'.

Let ε be the edge in Γ' containing v_1. We create Λ by adding m vertices to each edge in the orbit of ε under $\mathrm{TSG}_+(\Gamma)$. It is easy to see that condition (1) is satisfied for Λ.

Since P and its orbit under $\mathrm{TSG}_+(\Gamma)$ are the only chains of length at least m in Λ, the orbit of P under $\mathrm{TSG}_+(\Lambda)$ is the same as the orbit of P under $\mathrm{TSG}_+(\Gamma)$. Since $\Gamma' \neq P$, one endpoint x of P has valence at least three in Γ'. So there is no $a \in \mathrm{TSG}_+(\Lambda)$ which interchanges the endpoints of P or any arc in its orbit. Now for any diffeomorphism $g : (S^3, \Lambda) \to (S^3, \Lambda)$, we must have $g(\{v_1, \dots, v_r\}) = \{v_1, \dots, v_r\}$, and $g(\Gamma') = \Gamma'$.

Case 2: The valence of v_1 is at least two in Γ'.

In this case, we will add chains with m vertices in order to make sure that condition (2) is satisfied. For each $i \leqslant r$ and each ε_j in Γ' which contains v_i, let $f_j = N(v_i) \cap \varepsilon_j$ and let

$\{w_j\} = \varepsilon_j \cap \partial N(v_i)$. Now for each j choose an arc d_j in $N(v_i) - B_i$ with endpoints v_i and w_j such that: $\mathrm{int}(d_j)$ is disjoint from Γ, if $j \neq j'$ then $\mathrm{int}(d_j)$ is disjoint from $d_{j'}$, and there is an isotopy of $N(v_i)$ which is fixed on Γ' taking the d_j's to the corresponding f_j's. We create Λ from Γ' by adding each d_j together with the vertex w_j, and adding m vertices to d_j.

Since the d_j are isotopic to the f_j fixing Γ', for every $a \in \mathrm{TSG}_+(\Gamma)$, we can choose $g_a \in G$ such that $g_a(\Lambda) = \Lambda$. Also in Λ, each w_i has valence three. On the other hand, since we added at least two d_i's containing the vertex v_1, we know that $v_1, \dots, v_r$ each have valence at least four in Λ. Now $v_1, \dots, v_r$ are the only vertices of valence more than three in Λ which are endpoints of a chain of length at least m in Λ. So again for any diffeomorphism $g : (S^3, \Lambda) \to (S^3, \Lambda)$, we must have $g(\{v_1, \dots, v_r\}) = \{v_1, \dots, v_r\}$ and $g(\Gamma') = \Gamma'$.

Step 5. *We prove that* $\mathrm{im}(\Psi) \cong \mathrm{TSG}_+(\Lambda)$.

For both of our constructions of Λ, if g and g' are elements of G inducing the same automorphism on Γ' and taking Λ to itself sending vertices of Λ to vertices of Λ, then g and g' both induce the same automorphism of Λ. Since Λ satisfies condition (1), we can thus define a homomorphism $\Phi : \mathrm{im}(\Psi) \to \mathrm{TSG}_+(\Lambda)$ by letting $\Phi(a)$ denote the automorphism that g_a induces on Λ. Since Γ' is a subset of Λ, if g_a induces a trivial automorphism of Λ, then a must be trivial on Γ'. Thus $\ker(\Phi)$ is trivial.

To show that Ψ is onto, we let $a \in \mathrm{TSG}_+(\Lambda)$. Then there is an orientation preserving diffeomorphism $g : (S^3, \Lambda) \to (S^3, \Lambda)$ inducing a, and by condition (2), $g(\{v_1, \dots, v_r\}) = \{v_1, \dots, v_r\}$, and $g(\Gamma') = \Gamma'$. We will construct an orientation preserving diffeomorphism $h : (S^3, \Gamma) \to (S^3, \Gamma)$ that induces a on Λ as follows.

First suppose that Γ satisfies condition (a). Then, for each $i \leqslant r$, $B_i \cap \Gamma' = \{v_i\}$. By our construction of Λ, for each $i \leqslant r$, $B_i \cap \Lambda = \{v_i\}$. Now, for each $i \leqslant r$, we choose a slightly larger innermost ball D_i for Γ attached at v_i such that $B_i - \{v_i\} \subset \mathrm{int}(D_i)$ and $D_i \cap \Lambda = \{v_i\}$. Since $g(\{v_1, \dots, v_r\}) = \{v_1, \dots, v_r\}$ and $\{v_1, \dots, v_r\}$ is setwise invariant under G, for each $i \leqslant r$ there is a $j_i \leqslant r$ such that $g(D_i) \cap \Lambda = \{v_{j_i}\}$ and B_{j_i} is in the orbit of B_i under G. So there is a $g_i \in G$ such that $g_i(B_i) = B_{j_i}$. Also, there is an isotopy fixing Λ which takes each $g(D_i)$ to D_{j_i}. So there is some orientation preserving diffeomorphism $g' : (S^3, \Lambda) \to (S^3, \Lambda)$ which induces a on Λ such that for each $i \leqslant r$, $g'(D_i) = D_{j_i}$.

Define $h|\mathrm{cl}(S^3 - (D_1 \cup \cdots \cup D_r)) = g'|\mathrm{cl}(S^3 - (D_1 \cup \cdots \cup D_r))$, and define $h : (B_i, \Gamma_i) \to (B_{j_i}, \Gamma_{j_i})$ to be $g_i|B_i$. Finally, for each i, since $h|\partial D_i$ and $h|\partial B_i$ are both orientation preserving diffeomorphisms which take v_i to v_{j_i}, we can extend h to a diffeomorphism taking $D_i - B_i$ to $D_{j_i} - B_{j_i}$. Now $h : (S^3, \Lambda) \to (S^3, \Lambda)$ induces a on Λ and $h(\Gamma) = \Gamma$. Let b be the automorphism which h induces on Γ. Then $\Phi \circ \Psi(b) = a$, and hence Φ is onto.

Now suppose that Γ satisfies condition (b). Then $B_1 \cap \Gamma' = \Gamma'$, and by our construction of Λ, $B_1 \cap \Lambda = \Lambda$ and $\partial B_1 \cap \Lambda = \{v_1\}$. Since $g(\Lambda) = \Lambda$, we have $g(B_1) \cap \Lambda = \Lambda$. Now since $g(v_1) = v_1$, it follows that $g(\partial B_1)$ is isotopic to ∂B_1 by an isotopy fixing Λ. Hence there is some orientation preserving diffeomorphism $g' : (S^3, \Lambda) \to (S^3, \Lambda)$ inducing a on Λ such that $g'(B_1) = B_1$. Define $h|B_1 = g'|B_1$. Let D_1 be a slightly larger innermost ball for Γ attached at v_1 such that $B_1 - \{v_1\} \subset \mathrm{int}(D_1)$ and $D_1 \cap \Gamma = \Gamma_1$. Since $h|\partial B_1$ is orientation preserving we can extend h to a diffeomorphism of $D_1 - B_1$ such that $h|\partial D_1$ is the identity, and define h to be the identity on $S^3 - D_1$. Now $h : (S^3, \Lambda) \to (S^3, \Lambda)$ induces a on Λ, and

h is a diffeomorphism of (S^3, Γ). So as above, let b be the automorphism which h induces on Γ, then $\Phi \circ \Psi(b) = a$. Thus again Φ is onto.

Therefore in either case, $\mathrm{im}(\Psi) \cong \mathrm{TSG}_+(\Lambda)$. Now let $N = \ker(\Psi)$, then N and H/N are both realizable. If N is non-trivial then we are done. Otherwise, H is realized by Λ. Hence we will be done after Step 6.

Step 6. *We show that Λ has fewer type I vertices than Γ.*

First suppose Λ was created in Case 1. Then Λ is homeomorphic as a topological space to Γ'. So, Γ' and Λ have the same set of type I vertices, and by Lemma 4, every type I vertex of Γ' is a type I vertex of Γ. On the other hand, since the valence of v_1 is one in Λ, v_1 is not a type I vertex of Λ. So Λ has fewer type I vertices than Γ.

Assume we created Λ in Case 2, and x is a type I vertex of Λ with type I sphere Σ. Let α and β be the closures of the components of $\Lambda - \Sigma$. If α or β contains an edge in one of the d_i then it contains $d_i \cup f_i$. Thus each component of $S^3 - \Sigma$ contains part of Γ'.

Suppose $x = v_1$, then x is a type I vertex of Γ, so the valence of x in Γ is at least three. By applying Lemma 2 if necessary to Λ, we can choose Σ so that it is almost disjoint from each of the spheres $\Sigma_1, \dots, \Sigma_r$. It follows that $\Sigma \cap \Gamma = \{v_1\}$. Thus Σ is a type I sphere for Γ with attaching vertex v_1. Recall that if condition (a) is satisfied then γ is the unique innermost subgraph rel v_1 that is not innermost, and by definition, $\Gamma' \subset \gamma$. It follows that each component of $S^3 - \Sigma$ contains part of γ. If condition (b) is satisfied, then each component of $S^3 - \Sigma$ contains part of $\Gamma' = \Gamma_1$. Since both γ and Γ_1 are innermost subgraphs of Γ rel v_1 and Σ is a type I sphere for Γ with attaching vertex v_1, in either case we obtain a contradiction. Thus v_1 cannot be a type I vertex of Λ (or of Γ'). Now for each $i \leqslant r$, there is a diffeomorphism $g : (S^3, \Lambda) \to (S^3, \Lambda)$ such that $g(v_1) = v_i$. Thus, for any $i \leqslant r$, v_i is not a type I vertex of Λ (or of Γ').

Now suppose x is some w_j. Without loss of generality, $f_j \cup d_j$ is contained in α and $\varepsilon_j - f_j$ is contained in β. For some $i \leqslant r$, v_i is an endpoint of d_j. Now v_i and all of the edges of Λ incident to v_i are also in α. Let B denote the closure of the component of $S^3 - \Sigma$ which contains β. By our construction, $f_j \cup d_j$ bounds a disk whose interior is disjoint from Λ. So there is a ball B' consisting of B together with a ball containing $d_j \cup f_j$ such that $B' \cap \Lambda = \beta \cup f_j \cup d_j$, and $\partial B' \cap \Lambda = \{v_i\}$. Let $\Sigma' = \partial B'$. Thus Σ' is a type I sphere for Λ with attaching vertex v_i. But as we saw above this is impossible.

Thus x is neither v_i with $i \leqslant r$ nor some w_j. Also x has valence at least three in Λ. It follows that x is a vertex of Γ' with the same valence in Γ' that it has in Λ. Hence x is a type I vertex for Γ' with type I sphere Σ. Thus every type I vertex of Λ is a type I vertex of Γ', and by Lemma 4 every type I vertex of Γ' is a type I vertex of Γ. We saw above that $v_1, \dots, v_r$ are not type I vertices of Λ. Thus the set of type I vertices of Λ are a proper subset of those of Γ.

This completes the proof of Proposition 2. $\square$

5. Embedding graphs in S^3

In this section we prove the converse of Theorem 2. In particular, in Theorem 3 we will show that for every finite subgroup G of $\mathrm{Diff}_+(S^3)$, there is an embedded graph Γ with underlying abstract graph a complete bipartite graph such that $\mathrm{TSG}_+(\Gamma) \cong G$. A *complete bipartite graph* $K_{n,n}$ is the graph consisting of two sets V and W of n vertices each and edges joining every vertex in V to every vertex in W.

Our strategy to construct Γ will be as follows. Let n denote the order of G. Then the sets of vertices V and W will each be embedded as the orbit under G of a single point in S^3. We will embed the edges of $K_{n,n}$ by lifting paths from the orbit space obtained from S^3 under the action of G. Finally, by tying distinct knots in edges from different orbits we will ensure that $\mathrm{TSG}_+(\Gamma)$ is not larger than G.

We will use the following terminology. An edge e of a graph γ is said to be *invertible* if there exists some $a \in \mathrm{Aut}(\gamma)$ that interchanges the vertices of e. In this case we say that a *inverts* e. We have analogous definitions for embedded graphs: if ε is an edge in an embedded graph Γ, and there is some $a \in \mathrm{TSG}_+(\Gamma)$ such that a interchanges the vertices of ε, then we say ε is *invertible* and a *inverts* ε.

Graph Embedding Lemma. *Let γ be a graph. Let H be a subgroup of $\mathrm{Aut}(\gamma)$ that is isomorphic to a finite subgroup G of $\mathrm{Diff}_+(S^3)$. Suppose that no non-trivial element of H fixes any vertex or inverts any edge of γ. Then there is an embedded graph Γ with underlying abstract graph γ such that G induces H on Γ.*

Proof. Smith [Sm] has shown that the fixed point set of every finite order orientation preserving diffeomorphism of S^3 is either the empty set or a simple closed curve. Let Y denote the union of the fixed point sets of all of the non-trivial elements of G. Then Y is a union of finitely many simple closed curves whose pairwise intersection consists of finitely many points. So $M = S^3 - Y$ is path connected. Also, M is setwise invariant under G because Y is setwise invariant.

Let $\Psi : H \to G$ be an isomorphism and for each $a \in H$, define $g_a = \Psi(a)$. Let $\{w_1, \ldots, w_q\}$ be a set consisting of one representative from each orbit of the action of H on the vertices. Let $v_1, \ldots, v_q$ be distinct points in M which have disjoint orbits under G. For each $i \leqslant q$, we embed the vertex w_i as the point v_i.

Since no vertex is fixed by any non-trivial element of H, for every vertex w in γ, there are unique $a \in H$ and $i \leqslant q$ such that $w = a(w_i)$. Thus we can unambiguously embed every vertex $w = a(w_i)$ as the point $g_a(v_i)$. Since the embedded vertex representatives $v_1, \ldots, v_q$ are disjoint from Y and have disjoint orbits under G, all of the vertices of γ are embedded as distinct points in M. Let V denote the set of embedded vertices; then G leaves V setwise invariant. The map sending each $a \in H$ to the restriction $g_a|V$ is an isomorphism, since Ψ is an isomorphism and V is disjoint from Y. Thus G induces H on the set V.

Let $\{e_1, \ldots, e_n\}$ be a set consisting of one representative from each orbit of the action of H on the edges. For each i, let x_i and y_i be the embedded vertices of e_i. Since M is path connected, for each i there is a path α_i in M from x_i to y_i.

Let $\pi : M \to M/G$ denote the quotient map. Then π is a covering map, and the quotient space $Q = M/G$ is a 3-manifold. For each i, let $\alpha'_i = \pi \circ \alpha_i$. Then α'_i is a path or loop from $\pi(x_i)$ to $\pi(y_i)$. Using a general position argument in Q, we can homotop each α'_i rel its endpoints to a simple path or loop ρ'_i such that $\mathrm{int}(\rho'_1(I)), \ldots, \mathrm{int}(\rho'_n(I))$, and $\pi(V)$ are pairwise disjoint. For each i, let ρ_i be the lift of ρ'_i beginning at x_i. Since $\rho'_i = \pi \circ \rho_i$ is one-to-one except possibly on the set $\{0, 1\}$, we know that ρ_i must also be one-to-one except possibly on the set $\{0,1\}$. Since ρ_i' is homotopic rel its endpoints to α_i', we know that ρ_i is homotopic rel its endpoints to α_i. Thus ρ_i is a simple path in M from x_i to y_i.

For each i, we embed e_i as the image of the simple path ρ_i. Then for $i \neq j$, $\mathrm{int}(\rho_i(I))$ is disjoint from $\mathrm{int}(\rho_j(I))$.

We embed an arbitrary edge e as follows. We know that $e = a(e_i)$ for some $a \in H$ and some i. To show that a and i are uniquely determined by e, suppose that $b \in H$ such that $e = b(e_j)$. Then i must equal j, since when $i \neq j$, e_i and e_j have disjoint orbits under H. Also, since no non-trivial element of H fixes any vertex or inverts any edge, $a^{-1} \circ b$ must be trivial, and hence $b = a$. So we can unambiguously embed e as $\varepsilon = g_a(\rho_i(I))$. Then ε is a simple path from $g_a(x_i)$ to $g_a(y_i)$.

Let Γ consist of the vertices V together with the embedded edges described above. Then Γ is setwise invariant under G. We see that Γ is an embedded graph as follows. First, since each $\mathrm{int}(\rho_i'(I))$ is disjoint from $\pi(V)$, the orbit of $\mathrm{int}(\rho_i(I))$ must be disjoint from V. Similarly, since for $i \neq j$, $\rho_i'(I)$ and $\rho_j'(I)$ have disjoint interiors, for every g, $h \in G$ $g(\rho_i(I))$ and $h(\rho_j(I))$ have disjoint interiors. Furthermore, if $g \neq h$, then $g(\rho_i(I))$ and $h(\rho_i(I))$ have disjoint interiors.

Hence Γ is an embedded graph with underlying abstract graph γ such that G induces H on Γ. $\square$

In the proof of Proposition 4, we will use local knotting as a tool to modify our embedded graph. In particular, we would like to be able to add a local knot κ to a particular edge ε of Γ so that no element of $\mathrm{TSG}_+(\Gamma)$ can take ε to an edge which does not contain κ. Also we would like to be able to add a non-invertible local knot to an edge ε so that no element of $\mathrm{TSG}_+(\Gamma)$ can interchange the endpoints of ε.

We begin by formalizing our definition of local knotting. Let Γ be an embedded graph and let ε be some edge which is contained in a simple closed curve in Γ. We say that ε *contains the local knot* κ if there is a ball B such that $B \cap \Gamma$ is an arc α in the interior of ε, properly embedded in B, and the union of α and an arc in ∂B has knot type κ. We abbreviate this by saying ε contains the local knot κ *with ball* B. When we say an embedded graph Γ' is obtained from Γ by *adding the local knot* κ *to* ε we will mean that we replace an arc α in the interior of ε with an arc α' in a tubular neighborhood B of α such that α' is properly embedded in B, and the union of α' and an arc in ∂B has knot type κ.

Suppose we add a prime local knot κ to an edge ε of Γ and call the new embedded edge ε'. If $\kappa' \neq \kappa$ is a prime knot that is not a local knot of ε, then κ' is also not a local knot of ε'. This can be seen as follows. Suppose towards contradiction that ε' contains κ' with ball B'. Since the balls B and B' for κ and κ' are disjoint from $\Gamma - \varepsilon$, and ε is contained in a simple closed curve α in Γ, it is enough to prove the assertion in the case when $\Gamma = \alpha$. But in this case the assertion follows immediately from the prime decomposition theorem for knots.

Orienting an edge from one endpoint to the other naturally induces an orientation on any local knot it contains. If κ is non-invertible and ε does not contain κ, then, by an argument similar to the above paragraph, adding κ to ε to get ε' does not result in ε' containing the reverse of κ as a local knot. In particular, this means that if Γ does not contain the non-invertible local knot κ (or its reverse), and Γ' is the graph obtained from

Γ by adding κ to ε, then there is no $a \in \mathrm{TSG}_+(\Gamma')$ which inverts ε'.

We sometimes want to add a new local knot to one edge of an embedded graph without causing other edges to contain that local knot. Let ε_1 and ε_2 be edges of Γ. A *bridging sphere* for ε_1 and ε_2 is a sphere S such that S meets Γ transversely in $\{x_1, x_2\}$, where x_i is a point in the interior of ε_i.

Local Knotting Lemma. *Let Γ be an embedded graph with distinct edges ε_1 and ε_2, each contained in a simple closed curve in Γ. Let κ_1 and κ_2 be knot types, which are not necessarily distinct. Suppose ε_2 does not contain the local knot κ_2, and there is no bridging sphere for ε_1 and ε_2. Then adding κ_1 to ε_1 does not cause ε_2 to contain the local knot κ_2.*

Proof. Let Γ' be the graph obtained from Γ by adding the local knot κ_1 to ε_1. Let ε_1' denote the edge in Γ' obtained by adding the local knot κ_1 to ε_1 in Γ.

Suppose, in Γ', ε_1' contains the local knot κ_1 with ball B_1, and ε_2 contains the local knot κ_2 with ball B_2. By general position, we can assume that ∂B_1 and ∂B_2 intersect in a disjoint union of circles. We begin by eliminating as many circles of intersection as we can as follows. Suppose there is a circle of intersection that bounds a disk F on ∂B_1 disjoint from Γ'. Choose C to be an innermost circle of intersection in F, and let D_1 be the disk in F which is bounded by C. Suppose, for the sake of contradiction, that each component of $\partial B_2 - C$ contains one point of ε_2. Pick a disk D_2 bounded by C on ∂B_2. By the hypotheses of the lemma, ε_2 is contained in a simple closed curve in Γ'. Then the sphere $D_1 \cup D_2$ meets this simple closed curve transversely in a single point, which is impossible. Thus C bounds a disk D_2 on ∂B_2 disjoint from Γ'. Then the sphere $D_1 \cup D_2$ is disjoint from Γ', so it bounds a ball X which is also disjoint from Γ'. Therefore, while fixing Γ', we can isotop D_2 through X to a disk just past D_1, and thus eliminate the circle of intersection C. By repeating this process, we can isotop B_2, fixing Γ', to a new ball B_2' such that no circle of $\partial B_1 \cap \partial B_2'$ bounds a disk on ∂B_1 disjoint from Γ'. Now, as an edge of Γ', ε_2 contains the local knot κ_2 with ball B_2'.

Suppose, for the sake of contradiction, that $\partial B_1 \cap \partial B_2'$ is empty. Then B_1 and B_2' are disjoint. Hence we can re-embed ε_1' back as ε_1 in B_1 without creating any intersections between ε_1 and B_2'. Thus, in Γ, ε_2 contains the local knot κ_2 with ball B_2'. But this is contrary to the hypothesis of our lemma. Therefore $\partial B_1 \cap \partial B_2'$ must contain one or more circles of intersection, none of which bounds a disk on ∂B_1 disjoint from ε_1'. Let C be a circle of intersection that bounds an innermost disk D_1 on ∂B_1. Let D_2 be a disk bounded by C on $\partial B_2'$. It follows from our hypothesis that ε_1' is contained in a simple closed curve in Γ', which we know intersects the sphere $D_1 \cup D_2$ transversely in at least one point, in the interior of ε_1'. Therefore D_2 intersects ε_2 transversely in a single interior point. Thus $D_1 \cup D_2$ is a bridging sphere for ε_1' and ε_2 as edges of Γ'. Hence $D_1 \cup D_2$ is also a bridging sphere for ε_1 and ε_2 as edges of Γ. But this contradicts our hypothesis. $\square$

Observe that a 3-connected embedded graph Γ can have no bridging spheres. Thus, by the Local Knotting Lemma, adding a local knot κ_1 to any edge of Γ does not cause any other edge of Γ to contain a new local knot κ_2.

Proposition 4. *Let Δ be an embedded 3-connected graph. Let H be a subgroup of $\mathrm{TSG}_+(\Delta)$ which is induced by an isomorphic group G of diffeomorphisms of S^3, such that no non-trivial element of H fixes any vertex of Δ. Then Δ can be re-embedded as Γ such that $H = \mathrm{TSG}_+(\Gamma)$ and H is still induced by G.*

Proof. We will obtain Γ by adding local knots to the edges of Δ. For each $a \in H$, let g_a denote the unique element of G which induces a on Δ. We can choose $N(\Delta)$ so that $N(V)$ and $N(E)$ are each setwise invariant under G.

Let γ denote the underlying abstract graph of Δ, and let $\{e_1, \dots, e_n\}$ be a set of edges of γ consisting of one representative from each edge orbit under H. Without loss of generality, there is some $m \leqslant n$ such that e_i is invertible if and only if $i \leqslant m$. For each i, let δ_i denote the embedded edge corresponding to e_i. Since no non-trivial element of H fixes any vertex, for each $i \leqslant m$, there is a unique $a_i \in H$ which inverts e_i. So g_{a_i} is the unique element of G which inverts δ_i.

Let $\{\kappa_1, \dots, \kappa_n\}$ be a set of distinct knots none of which is contained in Δ, such that if $i \leqslant m$ then κ_i is strongly invertible and otherwise κ_i is non-invertible. For each i, we add the local knot κ_i to δ_i and call this new edge ε_i. For $i \leqslant m$, since κ_i is strongly invertible, we can add κ_i in such a way that $g_{a_i}(\varepsilon_i) = \varepsilon_i$. We embed each e_i as ε_i and each edge $e = a(e_i)$ as $g_a(\varepsilon_i)$. Thus, $g_a(\varepsilon_i)$ is $g_a(\delta_i)$ with the local knot κ_i added. Let Γ denote the embedded graph obtained in this way. Observe that in constructing Γ, we added the local knot κ_i to an edge δ if and only if δ is in the orbit of δ_i under G. Now it follows from the Local Knotting Lemma that an embedded edge ε in Γ contains the local knot κ_i if and only if ε is in the orbit of ε_i under G.

By our construction, for every $a \in H$, Γ is setwise invariant under g_a. Thus H is a subgroup of $\mathrm{TSG}_+(\Gamma)$ which is induced by G and $H \cong G$. We will show below that $H = \mathrm{TSG}_+(\Gamma)$.

Let φ be a non-trivial element of $\mathrm{TSG}_+(\Gamma)$. We will show that $\varphi \in H$. Since $\varphi \in \mathrm{TSG}_+(\Gamma)$, there is some diffeomorphism $h : (S^3, \Gamma) \to (S^3, \Gamma)$ such that h induces φ. Since φ is non-trivial, there is some edge ε which h does not leave setwise invariant. For some i, this ε is in the orbit of ε_i, and hence contains the local knot κ_i. Since h is a diffeomorphism, $h(\varepsilon)$ must contain the local knot κ_i, and therefore $h(\varepsilon)$ is in the orbit of ε_i under G. Hence there is some $g_1 \in G$ such that $g_1(\varepsilon) = h(\varepsilon)$. Thus $g_1^{-1} \circ h(\varepsilon) = \varepsilon$.

We define a new diffeomorphism $f : (S^3, \Gamma) \to (S^3, \Gamma)$ as follows. If $g_1^{-1} \circ h$ interchanges the vertices of ε, then the local knot κ_i must be invertible. In this case, there is some $g_2 \in G$ which inverts ε. So we let $f = g_2 \circ g_1^{-1} \circ h$. Otherwise we let $f = g_1^{-1} \circ h$. Thus in either case, $f(\varepsilon) = \varepsilon$, fixing both vertices.

We will show below that f actually fixes every vertex of Γ. Since we have shown that f fixes the vertices of ε, it suffices to show that if f fixes a vertex x, then f fixes every vertex adjacent to x. Suppose that there is some edge ε' containing the vertex x, such that $f(\varepsilon') \neq \varepsilon'$. By the same argument given two paragraphs up, since ε' and $f(\varepsilon')$ contain the same local knots, there is some $g_3 \in G$ such that $g_3(\varepsilon') = f(\varepsilon')$. By our hypothesis no non-trivial element of G fixes any embedded vertex. Let x' denote the vertex of $g_3(\varepsilon')$ other than x, then $g_3(x) = x'$. So $g_3^{-1} \circ f$ is a diffeomorphism that takes ε' to itself interchanging x and x'. It follows that the local knot κ_j which is contained in ε' must be invertible. Hence, as in the above paragraph, there is a $g_4 \in G$ which takes ε' to itself interchanging

x and x'. But this implies that $g_3 \circ g_4(\varepsilon') \neq \varepsilon'$ and $g_3 \circ g_4(x) = x$. So $g_3 \circ g_4$ is a non-trivial element of G that fixes an embedded vertex, which is impossible. Hence f takes every edge containing the vertex x to itself, and thus fixes every vertex adjacent to x.

Recall that either $f = g_2 \circ g_1^{-1} \circ h$ or $f = g_1^{-1} \circ h$. Since f fixes every vertex of Γ, h induces the same automorphism as either $g_1 \circ g_2^{-1}$ or g_1 does. Since the automorphisms induced by both $g_1 \circ g_2^{-1}$ and g_1 are elements of H, it follows that φ, the automorphism induced by h, is an element of H. Therefore $\mathrm{TSG}_+(\Gamma) = H$. $\square$

The following result follows immediately from Propositions 1 and 4, together with the fact that a 3-connected embedded graph has no type I, II, or III spheres.

Corollary. *Let Δ be an embedded 3-connected graph. Let H be a subgroup of $\mathrm{TSG}_+(\Delta)$ such that no non-trivial element of H fixes any vertex of Γ. Then Δ can be re-embedded as Γ such that $H = \mathrm{TSG}_+(\Gamma)$ and H is induced by an isomorphic finite subgroup of $\mathrm{Diff}_+(S^3)$.*

We shall use Proposition 4 to prove the following converse of Theorem 2. Note that the statement of Theorem 3 that we prove below is slightly stronger than that given in the introduction.

Theorem 3. *For every finite subgroup G of $\mathrm{Diff}_+(S^3)$, there is an embedded 3-connected graph Γ such that $G \cong \mathrm{TSG}_+(\Gamma)$ and $\mathrm{TSG}_+(\Gamma)$ is induced by G. Moreover, this Γ can be chosen to be a complete bipartite graph $K_{n,n}$ for some n.*

Proof. Suppose that G is the trivial group. The complete bipartite graph $K_{3,3}$ is 3-connected. By Proposition 4 there is an embedded graph Γ with underlying abstract graph $K_{3,3}$ such that $\mathrm{TSG}_+(\Gamma)$ is trivial. So from now on we assume that the group G is not trivial.

Now let n denote the order of G. First we suppose that $n > 2$. Let $\{v_1, v_2, \ldots, v_n\}$ and $\{w_1, w_2, \ldots, w_n\}$ denote the sets of vertices of an abstract complete bipartite graph $K_{n,n}$. Since $n > 2$ we know that $K_{n,n}$ is 3-connected. Pick x_1 to be a point in S^3 that is not fixed by any non-trivial element of G. Let $\{x_1, x_2, \ldots, x_n\}$ denote the orbit of x_1 under G. We will define a homomorphism $\Psi : G \to \mathrm{Aut}(K_{n,n})$ as follows. Let $g \in G$ and let $i \leqslant n$ be given. Then $g(x_i) = x_j$ for some j. We define $\Psi(g)(v_i) = v_j$ and $\Psi(g)(w_i) = w_j$. Let H denote the image of Ψ.

Since no non-trivial element of G fixes any x_i, Ψ is one-to-one, and hence $H \cong G$. Also, no non-trivial element of H takes any vertex to itself. Furthermore, since no element of H takes any v_i to any w_j, no edges of $K_{n,n}$ are inverted by any element of H. Now we can apply the Graph Embedding Lemma to obtain an embedded graph Γ with underlying abstract graph $K_{n,n}$ such that G induces H on Γ. Furthermore, by Proposition 4, Γ can be chosen so that $H = \mathrm{TSG}_+(\Gamma)$ and H is induced by G.

Finally, suppose that $n = 2$, so G is $\mathbb{Z}_2$. Let $\{v_1, v_2, v_3, v_4\}$ and $\{w_1, w_2, w_3, w_4\}$ denote the sets of vertices of an abstract complete bipartite graph $K_{4,4}$. Then $K_{4,4}$ is 3-connected. Let H be the subgroup of $\mathrm{Aut}(K_{4,4})$ generated by the 2-cycle $(v_1, v_2)(v_3, v_4)(w_1, w_2)(w_3, w_4)$. Then $H \cong \mathbb{Z}_2$, no vertex of $K_{4,4}$ is fixed by any non-trivial element of H, and there are no edges which are inverted by an element of H. Thus we can again apply the Graph Embedding Lemma and Proposition 4 to get an embedded graph Γ with underlying abstract graph $K_{4,4}$ such that $\mathrm{TSG}_+(\Gamma)$ is induced by G and $G \cong \mathrm{TSG}_+(\Gamma)$. $\square$

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