TOPOLOGICAL SYMMETRY GROUPS OF COMPLETE GRAPHS IN THE 3-SPHERE

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Abstract. The orientation preserving topological symmetry group of a graph embedded in the 3-sphere is the subgroup of the automorphism group of the graph consisting of those automorphisms which can be induced by an orientation preserving homeomorphism of the ambient space. We characterize all possible orientation preserving topological symmetry groups of embeddings of complete graphs in the 3-sphere.

1. Introduction

The topological symmetry group of an embedded graph in the 3-sphere was first defined by Jon Simon [Si] as a way of describing the symmetries of flexible molecules. The paper [FNPT] began a systematic study of these groups for arbitrary finite and connected embedded graphs. Our aim here is to determine the set of all orientation preserving topological symmetry groups for embeddings of complete graphs in S^3.

An *embedded graph* Γ is a pair (V, E) of sets of vertices V and edges E such that V is a set of points in S^3, every edge is a smoothly embedded arc in S^3 between two vertices, and the interior of each edge contains no vertex and no point of any other edge. We shall abuse notation and say that an embedded graph Γ is an *embedding* of its underlying abstract graph γ.

We will be considering homeomorphisms g of S^3 which take an embedded graph Γ to itself. We will use the notation $g : (S^3, \Gamma) \to (S^3, \Gamma)$ to mean not only that $g(\Gamma) = \Gamma$, but that $g(V) = V$ and $g(E) = E$ as well. Any such g defines an automorphism of the underlying abstract graph γ; we shall say that this automorphism is *induced* by g. The *topological symmetry group* $\mathrm{TSG}(\Gamma)$ is defined to be the subgroup of $\mathrm{Aut}(\gamma)$ consisting of those automorphisms which are induced by some homeomorphism $g : (S^3, \Gamma) \to (S^3, \Gamma)$. If we allow only orientation preserving homeomorphisms of S^3, we obtain the orientation preserving topological symmetry group $\mathrm{TSG}_+(\Gamma)$.

Topological symmetry groups of embeddings of the complete graph K_n are interesting to consider because K_n has the largest possible automorphism group of any graph with n vertices. For every $n \leqslant 5$ it is not hard to find an embedding Γ of K_n with $\mathrm{TSG}(\Gamma) = S_n$ (the symmetric group). However,

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it was shown in [F] that for any embedding Γ of K_n with $n \geqslant 6$, the cycle automorphism (1234) of K_n cannot be induced by any homeomorphism of (S^3, Γ). Thus if Γ is an embedding of K_n with $n \geqslant 6$, then $\mathrm{TSG}(\Gamma)$ is a proper subgroup of S_n.

Other known examples of topological symmetry groups of complete graphs include the trivial group which can be obtained for any $n \geqslant 4$ by letting Γ be an embedding of K_n which has non-homeomorphic knots tied in every edge; and for every $n \geq 7$ the dihedral group D_n was shown by Noda [N] to be $\mathrm{TSG}(\Gamma)$ for a particular embedding Γ of K_n.

For $n \leqslant 3$, it is not hard to see that each of the non-trivial subgroups of $\mathrm{Aut}(K_n)$ can be $\mathrm{TSG}_+(\Gamma)$ for some embedding Γ of K_n. So we concentrate on the case where $n \geqslant 4$. In this case, Theorem 2 of [FNPT] shows that for any embedding Γ of K_n, $\mathrm{TSG}_+(\Gamma)$ is isomorphic to a finite subgroup of the group $\mathrm{Diff}_+(S^3)$ of orientation preserving diffeomorphisms of S^3. Our task is therefore to determine which of the finite subgroups of $\mathrm{Diff}_+(S^3)$ can actually occur for embeddings of complete graphs. This is the content of our main result.

Theorem 1. *A finite group H is $\mathrm{TSG}_+(\Gamma)$ for some embedding Γ of some complete graph if and only if H is isomorphic to a finite subgroup of either $SO(3)$ or $D_m \times D_m$ for some odd m.*

We begin by proving in Proposition 1 that we only need to consider those embeddings Γ of complete graphs such that $\mathrm{TSG}_+(\Gamma)$ is induced by an isomorphic group of orientation preserving isometries of S^3. Furthermore, in Lemma 2 we prove that for every such embedding Γ of a complete graph, the involutions in $\mathrm{TSG}_+(\Gamma)$ must satisfy a key condition. The proof that the groups listed in Theorem 1 are the only ones possible proceeds in Section 3 by considering all finite subgroups G of $SO(4)$ that satisfy this condition and examining two separate cases, according to whether or not G preserves a standard Hopf fibration of S^3. In Section 4, we prove that all of the groups listed in Theorem 1 can actually occur as $\mathrm{TSG}_+(\Gamma)$ for some embedding Γ of some complete graph K_n.

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2. Preliminary results

Let Γ be a graph embedded in S^3 and let H be a subgroup of $\mathrm{TSG}_+(\Gamma)$. Suppose G is a group of orientation preserving homeomorphisms of (S^3, Γ) such that for every $a \in H$ there is a $g_a \in G$ which induces a on Γ, and every $g \in G$ induces some element of H on Γ. Then we say that G *induces* H

on Γ. A finite group H is said to be *realizable* by an embedded graph Γ if $H \cong \mathrm{TSG}_+(\Gamma)$.

We begin by recalling a result proved in [FNPT].

Lemma 1 ([FNPT])**.** *Let Ω be an embedded 3-connected graph, and let $H = \mathrm{TSG}_+(\Omega)$. Then Ω can be re-embedded as Δ such that $H \leqslant \mathrm{TSG}_+(\Delta)$ and $\mathrm{TSG}_+(\Delta)$ is induced by an isomorphic subgroup of $\mathrm{Diff}_+(S^3)$.*

This lemma tells us that in order to classify all orientation preserving topological symmetry groups H of embeddings of complete graphs with at least four vertices, we only need to consider embeddings for which H is induced on the graph by an isomorphic finite subgroup G of $\mathrm{Diff}_+(S^3)$. In analyzing the possibilities for G, it is helpful to look at the fixed point sets of elements of G. Smith characterized the fixed point sets of finite order homeomorphisms of homology spheres [Sm]. In particular, he showed that the fixed point set, $\mathrm{fix}(g)$, of any non-trivial orientation preserving finite order homeomorphism g of S^3 is either empty or homeomorphic to S^1. This result will be a useful tool for us, which we refer to as *Smith theory.* Note that if g is an orientation preserving isometry of S^3, then this may be proved without making use of Smith's work. However, for consistency we will nonetheless refer to this result as Smith theory. Using Smith theory we are able to put strong conditions on the involutions in G.

Lemma 2. *Let Γ be an embedding of some complete graph K_n with $n \geq 4$, and let G be a finite subgroup of $\mathrm{Diff}_+(S^3)$ which leaves Γ setwise invariant. For every involution $g \in G$, $\mathrm{fix}(g) \cong S^1$, and no $h \in G$ distinct from g satisfies $\mathrm{fix}(h) = \mathrm{fix}(g)$.*

Proof. Let $g \in G$ be an involution. If g fixes any point of S^3, then by Smith Theory $\mathrm{fix}(g) \cong S^1$. Since g has finite order and $g(\Gamma) = \Gamma$, if g fixes every vertex of Γ then $\mathrm{fix}(g)$ would contain Γ. As $n \geq 4$, this would contradict Smith theory. Hence there is some vertex v which is not fixed by g. Let e be the edge connecting v and $g(v)$. Then $g(e) = e$, and hence g fixes precisely one point of e. Thus $\mathrm{fix}(g) \cong S^1$ and g fixes precisely one point of each edge whose vertices are interchanged by g. It follows that every vertex of Γ is either contained in $\mathrm{fix}(g)$ or is an endpoint of an edge meeting $\mathrm{fix}(g)$.

Now suppose that $h \in G$ such that $\mathrm{fix}(h) = \mathrm{fix}(g)$. Let v be a vertex of Γ. If $v \in \mathrm{fix}(g)$, then v is fixed by both g and h. Suppose that $v \notin \mathrm{fix}(g)$. Then v is a vertex of an edge e meeting $\mathrm{fix}(g) = \mathrm{fix}(h)$. Thus both g and h interchange the endpoints of e. Hence g and h must induce precisely the same automorphism on the vertices of Γ. Since G is a finite group, it follows that hg^{-1} fixes Γ pointwise. Now since $n > 3$, by Smith theory, $h = g$. □

We will use the above lemmas as follows to prove that we only need to consider finite subgroups of $SO(4)$ rather than finite subgroups of $\mathrm{Diff}_+(S^3)$.

Proposition 1. *Let Ω be an embedding of some K_n, and let $H = \mathrm{TSG}_+(\Omega)$. Then either H is a cyclic group of odd order or K_n can be re-embedded as Γ such that $H \leqslant \mathrm{TSG}_+(\Gamma)$ and $\mathrm{TSG}_+(\Gamma)$ is induced by an isomorphic subgroup of $SO(4)$.*

Proof. First suppose that $n \leqslant 3$. Let Γ be a symmetric planar embedding of K_n. Then $\mathrm{Aut}(\Gamma) = \mathrm{TSG}_+(\Gamma)$ is induced by an isomorphic subgroup of $SO(4)$.

Now assume that $n > 3$ and H is not a cyclic group of odd order. Since K_n is 3-connected, it follows from Lemma 1 that Ω can be re-embedded as Δ such that $H \leqslant \mathrm{TSG}_+(\Delta)$ and $\mathrm{TSG}_+(\Delta)$ is induced by an isomorphic finite subgroup F of $\mathrm{Diff}_+(S^3)$. Suppose F acts freely on S^3. By Lemma 2, $|F|$ must be odd. However, cyclic groups are the only odd order groups which can act freely on S^3 (see [O, 6.2.1 and 6.3 i-iii]); hence F cannot act freely on S^3. Thus by Smith theory, F contains some element whose fixed point set is homeomorphic to S^1. Now by Thurston's Orbifold Theorem [BP, BLP, CHK], F is conjugate to a subgroup G of $SO(4)$ by an orientation preserving diffeomorphism f. Let $\Gamma = f(\Delta)$, then G induces $\mathrm{TSG}_+(\Gamma)$ and $G \cong F \cong \mathrm{TSG}_+(\Delta) \cong \mathrm{TSG}_+(\Gamma)$. □

Let $H = \mathrm{TSG}_+(\Gamma)$ for some embedding Γ of some complete graph. By Proposition 1, we can assume that either H is a cyclic group of odd order or there is an isomorphic finite subgroup of $SO(4)$ which induces H on Γ. If H is a finite cyclic group, then it is one of the groups listed in Theorem 1. So we assume that H is induced on Γ by an isomorphic finite subgroup of $SO(4)$. This assumption, together with the conclusion of Lemma 2, is all that we will use about topological symmetry groups in order to prove that H is isomorphic to one of the groups listed in Theorem 1. Thus, rather than continuing to work with topological symmetry groups, we shall work with arbitrary finite subgroups of $SO(4)$ which satisfy the conclusion of Lemma 2. Since the condition on involutions specified in the conclusion of Lemma 2 will play an important role in our proofs, we shall henceforth refer to it as the *Involution Condition*. Specifically, we make the following definition.

Definition 1. *A finite subgroup G of $\mathrm{Diff}_+(S^3)$ is said to satisfy the* Involution Condition *if for every involution $g \in G$, $\mathrm{fix}(g) \cong S^1$, and no $h \in G$ distinct from g satisfies $\mathrm{fix}(h) = \mathrm{fix}(g)$.*

Observe that the Involution Condition implies that if $h \in G$ with even order greater than two, then h must be fixed point free.

3. Subgroups of $SO(4)$ that satisfy the Involution Condition

Let G be a finite subgroup of $SO(4)$ which satisfies the Involution Condition. In this section we show that G is isomorphic to a finite subgroup of either $SO(3)$ or $D_m \times D_m$ for some odd m.

We consider two cases according to whether or not G preserves a standard Hopf fibration of S^3. First we assume that G does preserve such a fibration of S^3. Note that the Hopf fibration has no exceptional fibers.

We will use the following terminology.

Notation. Let $\pi : G \to G_b$ denote the projection map from G to the group G_b acting on the base space of the fibration. Let $N = \ker(\pi)$, let G_b^+ denote the orientation preserving subgroup of G_b, and let $G^+ = \pi^{-1}(G_b^+)$. We thus have the following commutative diagram with exact rows of subgroups of $SO(4)$.

$$\begin{tikzcd}
1 \arrow[r] & N \arrow[r] \arrow[d, equal] & G^+ \arrow[r] \arrow[d, hook] & G_b^+ \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & N \arrow[r] & G \arrow[r, "\pi"] & G_b \arrow[r] & 1
\end{tikzcd}$$

Since G preserves a standard Hopf fibration, G_b is a group of isometries of a round S^2 with the usual Riemannian metric. Hence G_b^+ is a finite subgroup of $SO(3)$. Since any non-trivial element of $SO(3)$ fixes precisely two points of S^2, any element of $G^+ - N$ leaves precisely two fibers of S^3 setwise invariant. Also, since S^3 and S^2 are each orientable, every $h \in N$ preserves the orientation of every fiber. Thus since N is determined by its action on a single fiber, N is necessarily a cyclic group.

We begin by using Smith theory and the Involution Condition to prove three elementary facts.

Fact 1. *Let $g \in G$. Then the following are equivalent.*
1) *g reverses the orientation of some fiber.*
2) *g reverses the orientation of a circle of fibers and leaves no other fiber setwise invariant.*
3) *$\pi(g)$ is a reflection.*
4) *$g \notin G^+$ and* $\mathrm{order}(g) = 2$.

Proof. Clearly (2) implies (1). We prove that (1) implies (2) and (3), that (3) implies (1) and (4), and that (4) implies (3). Suppose that g reverses the orientation of some fiber l. Then g fixes precisely two points on l, and hence by Smith theory $\mathrm{fix}(g) \cong S^1$. Thus there must be a circle of fibers which are reversed by g, and hence $\pi(g)$ pointwise fixes a circle C which is the projection of these fibers. Since every element of N preserves the orientation of every fiber, $g \notin N$. Thus $\mathrm{fix}(\pi(g)) = C$, since $\pi(g)$ is an isometry of S^2. It follows that Conditions (2) and (3) hold.

Now suppose that $g' = \pi(g)$ is a reflection of S^2. Then $\mathrm{fix}(g')$ lifts to a fibered torus T in S^3 such that g takes each fiber on T to itself and interchanges the fibers in the two complementary solid tori. It follows that g reverses the orientation of the circle of fibers making up T. Hence Condition

(1) holds. Furthermore, g^2 pointwise fixes each fiber of T. Thus, by Smith theory, $\mathrm{order}(g) = 2$. Also since $\pi(g)$ is a reflection, $g \notin G^+$. So Condition (4) holds.

Finally, suppose $g \notin G^+$ and $\mathrm{order}(g) = 2$. By the Involution Condition, $\mathrm{fix}(g) \cong S^1$. Thus $\pi(g)$ has non-empty fixed point set. Now since $\pi(g)$ is an orientation reversing isometry of S^2, $\pi(g)$ must be a reflection. So Condition (4) holds. □

Fact 2. *Every element of G^+ commutes with every element of N.*

Proof. Let $g \in G^+$. Then $\pi(g)$ fixes two points, hence there are fibers k_1 and k_2 such that $g(k_i) = k_i$. By Fact 1, g preserves the orientation of each k_i. Let $h \in N$. Since h also preserves the orientation of each k_i, $ghg^{-1}h^{-1}|k_i$ is the identity for both $i = 1$ and $i = 2$. Hence it follows from Smith theory that $ghg^{-1}h^{-1}$ is the identity. □

Fact 3. *Every involution in G^+ pointwise fixes a fiber, and $|N|$ is odd.*

Proof. Let $g \in G^+$ be an involution. By the Involution Condition, $\mathrm{fix}(g) \cong S^1$. Since g is fiber preserving, if l is a fiber which intersects $\mathrm{fix}(g)$ then either g fixes two points of l or $l = \mathrm{fix}(g)$. If g fixes two points of l, then g reverses the orientation of l, and hence $g \notin G^+$ by Fact 1. Thus $\mathrm{fix}(g)$ must be a fiber.

Finally, if N contains an involution g, then $\mathrm{fix}(g)$ would be an exceptional fiber. Since the Hopf fibration has no exceptional fibers, $|N|$ is odd. □

Notation. Let T, O, and D denote the groups of orientation preserving symmetries of the regular tetrahedron, octahedron, and dodecahedron, respectively.

The finite subgroups of $SO(3)$ are cyclic, dihedral, or equal to T, O, or D.

Lemma 3. *There exist fibers A and B, such that $\{A, B\}$ is setwise invariant under G. Furthermore, let J denote the subgroup of G which takes each of A and B to itself, preserving the orientation of each. If G contains an involution which interchanges A and B, then $|J|$ is odd.*

Proof. If G_b is trivial, we can choose A and B to be any fibers. So we assume that G_b is non-trivial. Now suppose that G_b^+ is trivial. Then $G_b \cong \mathbb{Z}_2$ with some generator γ. Let A and B be any pair of fibers which project to a pair of points that are interchanged by γ. Since γ is the only non-trivial element of G_b, $J = N$. Since, $|N|$ is odd by Fact 3, in this case we are done.

We may therefore assume that G_b^+ is non-trivial. We begin by showing that there cannot be a pair of involutions $f_1 \in G_b^+$ and $f_2 \in G_b$ and distinct points $\{a_1, b_1, a_2, b_2\}$ in S^2 such that for $i \neq j$, f_i fixes a_i and b_i and interchanges a_j and b_j.

Suppose that there is such a pair of involutions f_1 and f_2. Let A_1, A_2, B_1 and B_2 be the fibers which project down to the points a_1, a_2, b_1 and b_2 respectively. For each i, let $g_i \in G$ be such that $\pi(g_i) = f_i$. It follows from our conditions on f_1 and f_2 that $g_i(A_i) = A_i$, $g_i(B_i) = B_i$, and for $j \neq i$, g_i interchanges A_j and B_j. Let $n = \mathrm{order}(g_1)$; then n is even since f_1 is an involution. So we let $h_1 = (g_1)^{n/2}$. Since h_1 is an involution and $|N|$ is odd, $h_1 \notin N$. Since $h_1 \in G^+ - N$, it follows that A_1 and B_1 are the only fibers which are left setwise invariant by h_1. In particular, h_1 interchanges A_2 and B_2. Also it follows from Fact 3 that $\mathrm{fix}(h_1)$ is a fiber. So without loss of generality, we can assume that $\mathrm{fix}(h_1) = A_1$, and $\mathrm{order}(h_1|B_1) = 2$.

Let $\alpha = (g_2h_1)^2$. Then $\alpha \in G^+$ and $\alpha(A_i) = A_i$ and $\alpha(B_i) = B_i$ for each i. It follows that $\alpha \in N$. Since $\mathrm{fix}(h_1) = A_1$ and g_2 interchanges A_1 and B_1, $\alpha|B_1 = h_1|B_1$. It follows that $\mathrm{order}(\alpha|B_1) = 2$, and hence α has even order. But this is a contradiction, since $|N|$ is odd.

Thus the involutions f_1 and f_2 cannot both exist. Each of the groups T, O, and D, as well as the dihedral group D_r where r is even, contains such a pair of involutions. Hence either G_b^+ is cyclic or $G_b^+ = D_r$ where r is odd. Thus we can define J_b to be the largest cyclic subgroup of G_b^+. Since we are assuming that G_b^+ is non-trivial, J_b is non-trivial. Hence J_b fixes precisely two points x and y of S^2, and for every $\gamma \in G_b^+$, $\gamma \in J_b$ if and only if γ fixes x and y.

Let $\gamma \in G_b$. We will see as follows that $\gamma(\{x, y\}) = \{x, y\}$. By hypothesis, there is some $\alpha \in J_b$ which is non-trivial. Now $\gamma^{-1}\alpha\gamma \in G_b^+$; and either $G_b^+ = J_b$ or $G_b^+ = D_r$ with r odd. If $\gamma^{-1}\alpha\gamma \notin J_b$, then $\gamma^{-1}\alpha^2\gamma$ is the identity. Hence α has order two. But this is not possible since in this case J_b has odd order r. Thus $\gamma^{-1}\alpha\gamma \in J_b$, and hence $\gamma^{-1}\alpha\gamma$ fixes both x and y. This implies that $\alpha(\gamma(x)) = \gamma(x)$ and $\alpha(\gamma(y)) = \gamma(y)$. However, since α is a non-trivial element of J_b, x and y are the only points of S^2 fixed by α; hence $\{\gamma(x), \gamma(y)\} = \{x, y\}$. Thus $\gamma(\{x, y\}) = \{x, y\}$. Now it follows that there is a simple closed curve C in $S^2 - \{x, y\}$ which is setwise invariant under G_b.

Let A and B be the fibers that project to x and y. Then for every $g \in G$, $g(\{A, B\}) = \{A, B\}$. By definition J preserves the orientations of A and B. Hence by Fact 1, no element of J projects to a reflection. For every $\beta \in J$, $\pi(\beta)$ fixes both x and y. Since $\pi(\beta)$ has fixed points but is not a reflection, $\pi(\beta) \in G_b^+$. Thus $\pi(\beta) \in J_b$. Conversely, let $\beta' \in J_b$ and let $\beta \in G$ such that $\pi(\beta) = \beta'$. Now $\beta(A) = A$ and $\beta(B) = B$. Furthermore, since β' is not a reflection, by Fact 1, β does not reverse the orientation of any fiber. Thus $\beta \in J$. It follows that $\pi(J) = J_b$.

Finally, suppose there is some involution $g \in G$ which interchanges A and B. Now $\pi(g)$ is an involution and $\pi(g)$ interchanges x and y. Since $\pi(g)$ leaves the simple closed curve C setwise invariant, there is a pair $\{a, b\}$ of antipodal points on C which are fixed by $\pi(g)$. If J_b contained an involution f_1, then f_1 would interchange a and b. But we showed above that the pair of involutions f_1 and $\pi(g)$ cannot both exist. Hence $|J_b|$ is odd. Finally, since $|N|$ is also odd, $|J|$ must be odd. □

We shall use the following notation which was introduced in the above proof.

Notation. Let A, B, and J be given by Lemma 3, let $J_b = \pi(J)$, and let x and y denote the projections of A and B. Let C denote a simple closed curve in $S^2 - \{x, y\}$ which is setwise invariant under G_b.

Next we analyze the group J.

Lemma 4. *J is cyclic or $J = \mathbb{Z}_q \times \mathbb{Z}_r$ with qr odd.*

Proof. It follows from Fact 1 that $J_b = \pi(J)$ is the subgroup of G_b^+ consisting of those elements which fix both x and y. Hence J_b is cyclic. Let α be a generator of the cyclic group N and let $\beta \in J$ such that $\pi(\beta)$ is a generator of J_b. Then α and β generate J. Since $J \leq G^+$, it follows from Fact 2 that α and β commute. Thus J is abelian, and hence $J = \mathbb{Z}_q \times \mathbb{Z}_r$, for some q and r.

Suppose $J = \mathbb{Z}_q \times \mathbb{Z}_r$ is not cyclic and qr is even. Then J contains an involution γ, and by Fact 3, fix(γ) is a fiber. Since $|N|$ is odd, $\gamma \notin N$. Hence A and B are the only fibers which are setwise invariant under γ. So without loss of generality, we assume that fix$(\gamma) = A$. Now by the Involution Condition, no other non-trivial element of G pointwise fixes A.

Suppose that $\delta \in J$ is also an involution and $\delta \neq \gamma$. Then fix$(\delta) \neq A$. Hence by the above argument, fix$(\delta) = B$ and no other non-trivial element of G fixes B pointwise. Now since J is abelian, $\gamma\delta$ is also an involution, and thus again by the above argument $\gamma\delta$ pointwise fixes either A or B. As this is impossible by the Involution Condition, we conclude that γ is the only involution in J.

Since $J = \mathbb{Z}_q \times \mathbb{Z}_r$ is not cyclic, $\gcd(q, r) \neq 1$. Now since qr is even, and J contains only one involution, without loss of generality, q is odd and r is even. Choose generators g_1 and g_2 for J such that order$(g_1) = q$, order$(g_2) = r$, and the subgroups generated by g_1 and g_2 have trivial intersection. By definition of J, A is setwise invariant under both g_1 and g_2. Now $\gamma = g_2^{\frac{r}{2}}$. Since γ is the only non-trivial element of G that pointwise fixes A, order$(g_1|A) = q$ and order$(g_2|A) = \frac{r}{2}$. Also since gcd(q, r) $\neq 1$, there exists an odd prime p such that $p|q$ and $p|\frac{r}{2}$. Let $i = \frac{q}{p}$, then order$(g_1^i|A) = p$. Let $j = \frac{r}{2p}$, then order$(g_2^j|A) = p$. It follows that for some $k < p$, $g_1^{ik}|A = g_2^j|A$. But $g_1^{ik} \neq g_2^j$, since the groups generated by g_1 and g_2 have trivial intersection. Thus order$(g_1^{ik}g_2^{-j}) = p$. Now since $g_1^{ik}g_2^{-j}$ is a non-trivial element of G which pointwise fixes A, $g_1^{ik}g_2^{-j}$ must be the involution γ. But this is impossible since p is an odd prime. Thus, if J is not cyclic, then qr is odd. □

Lemma 5. *If $G \neq J$, then we can choose an involution ϕ_1 which interchanges A and B and/or an involution ϕ_2 which reverses the orientation of A and B such that the following four conditions hold:*
1) *G is generated by J together with whichever of ϕ_1 and ϕ_2 exists.*
2) *If both ϕ_1 and ϕ_2 exist, then they commute.*
3) *If ϕ_2 exists, then for every $g \in J$, $\phi_2 g \phi_2 = g^{-1}$.*
4) *If ϕ_1 exists, then we can choose generators g_1 and g_2 for J such that $\phi_1 g_1 \phi_1 = g_1^{-1}$ and $\phi_1 g_2 \phi_1 = g_2$.*

Proof. Since $G \neq J$, either G contains an element ϕ which interchanges A and B, or G contains an element ψ which reverses the orientation of both A and B.

Suppose that G contains an element ϕ which interchanges A and B. Let ϕ_1 denote ϕ to an appropriate odd power so that order$(\phi_1) = 2^n$. Then ϕ_1 interchanges A and B. Suppose that $n > 1$. Let $\alpha = (\phi_1)^{2^{n-1}}$. Then $\alpha \in J$ and order$(\alpha) = 2$. By Fact 3, fix(α) is a fiber and $|N|$ is odd. Thus $\alpha \notin N$. It follows that A and B are the only fibers which are setwise invariant under α. So, without loss of generality we can assume fix$(\alpha) = A$. Since ϕ_1 interchanges A and B, $\phi_1 \alpha \phi_1^{-1} | B$ is the identity. But $\phi_1 \alpha \phi_1^{-1} = \alpha$, because $\alpha = (\phi_1)^{2^{n-1}}$. By Smith theory, α cannot pointwise fix both A and B. Thus $n = 1$, and ϕ_1 is an involution.

If G contains an element ψ which reverses the orientation of both A and B, then by Fact 1, $\psi \notin G^+$ and ψ is an involution. So let $\phi_2 = \psi$. Now G is generated by J together with whichever of ϕ_1 and ϕ_2 exists. Hence we have proved Property (1).

To prove Property (2), suppose that G contains an involution ϕ which interchanges A and B and an involution ϕ_2 which reverses the orientation of both A and B. By Fact 1, $\phi_2 \notin G^+$. In this case, we choose ϕ_1 as follows. If $\phi \notin G^+$, let $\phi_1 = \phi$. Otherwise, let $\psi_1 = \phi\phi_2$. Then $\psi_1 \notin G^+$ and ψ_1 interchanges A and B. Now let ϕ_1 denote ψ_1 to an appropriate odd power so that order$(\phi_1) = 2^n$. Then $\phi_1 \notin G^+$ and ϕ_1 interchanges A and B. Also as we saw in the proof of Property (1), ϕ_1 is an involution.

We see that the involutions ϕ_1 and ϕ_2 commute as follows. Since $\phi_1 \notin G^+$ and $\phi_2 \notin G^+$, by Fact 1, $\pi(\phi_1)$ and $\pi(\phi_2)$ are both reflections about great circles. These two great circles intersect in a pair of points p and q which are fixed by both $\pi(\phi_1)$ and $\pi(\phi_2)$. Let P and Q be the fibers projecting down to p and q respectively. Since ϕ_1 and ϕ_2 are involutions leaving each of P and Q setwise invariant, $\phi_1\phi_2\phi_1\phi_2$ pointwise fixes both P and Q. Hence by Smith theory ϕ_1 and ϕ_2 commute, and thus we have proved Property (2).

To prove Property (3), suppose the involution ϕ_2 exists. Now observe that since ϕ_2 reverses the orientation of A and B, for every $g \in J$, $\phi_2 g \phi_2 g$ fixes both A and B pointwise. Hence by Smith theory, for every $g \in J$, $\phi_2 g \phi_2 = g^{-1}$.

Finally, to prove Property (4) we suppose that the involution ϕ_1 exists. Thus by Lemma 3, $|J|$ is odd and hence $|J_b|$ is odd. In order to choose appropriate generators for J, we start with a generator α of N and some $\beta \in J$ such that $\pi(\beta)$ is a generator for J_b. Observe that $\pi(\beta)^2$ is also a generator of J_b, since $|J_b|$ is odd.

Case 1: $\phi_1 \in G^+$.

Since the simple closed curve C is setwise invariant under G_b and no non-trivial element of G_b^+ pointwise fixes C, G_b^+ induces an isomorphic group action on C. Since $\pi(\phi_1) \in G_b^+$ interchanges x and y, $\pi(\phi_1)$ reverses the orientation of C. Also since $\pi(\beta)$ generates J_b, $\pi(\beta)$ preserves the orientation of C. Thus $\pi(\phi_1\beta^{-1}\phi_1) = \pi(\beta)$. Let $g_1 = \beta\phi_1\beta^{-1}\phi_1$. Then $\pi(g_1) = \pi(\beta)^2$ since $\pi(\phi_1\beta^{-1}\phi_1) = \pi(\beta)$. Let $g_2 = \alpha$. Since $\pi(\beta)^2$ is a generator for J_b, g_1 and g_2 generate J. Now $\phi_1 g_1 \phi_1 = \phi_1(\beta\phi_1\beta^{-1}\phi_1)\phi_1 = \phi_1\beta\phi_1\beta^{-1} = g_1^{-1}$. Also by Fact 2, $\phi_1 g_2 \phi_1 = g_2$.

Case 2: $\phi_1 \notin G^+$.

Since ϕ_1 is an involution, $\pi(\phi_1)$ is a reflection by Fact 1. Now since $\pi(\phi_1)$ interchanges x and y, $\mathrm{fix}(\pi(\phi_1)) = C$. Also $\pi(\beta\phi_1\beta^{-1}\phi_1) \in G_b^+$ and fixes C pointwise. It follows that $\pi(\beta\phi_1\beta^{-1}\phi_1)$ is the identity and hence $\pi(\phi_1)$ and $\pi(\beta)$ commute. Let $g_2 = \beta\phi_1\beta\phi_1$. Then $\pi(g_2) = \pi(\beta)^2$ is a generator of J_b. Let $g_1 = \alpha$. Then g_1 and g_2 generate J. Now $\phi_1 g_2 \phi_1 = \phi_1(\beta\phi_1\beta\phi_1)\phi_1 = (\phi_1\beta\phi_1)\beta = \beta(\phi_1\beta\phi_1)$, since β and $\phi_1\beta\phi_1$ are both elements of J which is abelian by Lemma 4. Thus $\phi_1 g_2 \phi_1 = g_2$. Since $\pi(\phi_1)$ is a reflection, by Fact 1, there are fibers P and Q which are reversed by ϕ_1. Then $\phi_1 g_1 \phi_1 g_1 | P$ and $\phi_1 g_1 \phi_1 g_1 | Q$ are the identity, since $g_1 \in N$. By Smith theory, $\phi_1 g_1 \phi_1 = g_1^{-1}$, and we have proved Property (4). $\square$

We make the following observation which we will use throughout the proof of Proposition 2. Let G_1 be the group generated by J together with an involution ϕ_1 which interchanges A and B (if such an involution exists), and let G_2 be the group generated by J together with an involution ϕ_2 which reverses the orientation of both A and B (if such an involution exists). Let T_i be the group generated by ϕ_i. Then $T_i \cong \mathbb{Z}_2$ and the intersection of T_i and J is trivial. Also for every $j \in J$, $\phi_i j \phi_i \in J$, and hence J is normal in G_i. Now $\phi_i j = (\phi_i j \phi_i)\phi_i \in JT_i$, and hence $G_i = JT_i$. It follows that, for each i, $G_i = J \rtimes_{\Psi_i} \mathbb{Z}_2$, where Ψ_i acts on J by conjugation by ϕ_i.

Proposition 2. *Let G be a finite subgroup of $SO(4)$ which satisfies the Involution Condition and preserves a standard Hopf fibration of S^3. Then G is either cyclic, dihedral, or a subgroup of $D_m \times D_m$ for some odd number m.*

Proof. If $G = J$, then by Lemma 4, G is either cyclic or $\mathbb{Z}_q \times \mathbb{Z}_r$ where qr is odd, and the result is evident. So we assume $G \neq J$. Then G is generated by J together with the involution(s) ϕ_1 and/or ϕ_2 given by Lemma 5.

Case 1: ϕ_1 exists.

It follows from Lemma 3 that $|J|$ is odd. Let G_1 be the subgroup generated by J and ϕ_1. Then $G_1 = J \rtimes_{\Psi_1} \mathbb{Z}_2$, where Ψ_1 acts on J by conjugation by ϕ_1. Now let g_1 and g_2 be the generators for J given by Condition (4) of Lemma 5. Thus $\phi_1 g_1 \phi_1 = g_1^{-1}$ and $\phi_1 g_2 \phi_1 = g_2$. Let $q = \mathrm{order}(g_1)$ and $r = \mathrm{order}(g_2)$, so that $G_1 = D_q \times \mathbb{Z}_r$ and qr is odd. Furthermore, if ϕ_2 exists then by Lemma 5, G is generated by J together with ϕ_1 and ϕ_2, $\phi_2 g_i \phi_2 = g_i^{-1}$ and ϕ_1 and ϕ_2 commute. Hence $(\phi_1\phi_2)g_1(\phi_1\phi_2) = g_1$, $(\phi_1\phi_2)g_2(\phi_1\phi_2) = g_2^{-1}$, and $\phi_1\phi_2$ is an involution. This implies that $G = D_q \times D_r$, with generators g_1, g_2, ϕ_1, and $\phi_1\phi_2$. Thus, whether or not ϕ_2 exists, G is a subgroup of $D_m \times D_m$ where $m = qr$ is odd.

Case 2: ϕ_1 does not exist.

Since $G \neq J$, G is generated by J together with the involution ϕ_2. By Lemma 5, for every $g \in J$, $\phi_2 g \phi_2 = g^{-1}$. Also, $G = J \rtimes_{\Psi_2} \mathbb{Z}_2$, where Ψ_2 acts on J by conjugation by ϕ_2. If J is cyclic, it follows that G is dihedral. Otherwise, $J = \mathbb{Z}_q \times \mathbb{Z}_r$ and qr is odd by Lemma 4. It follows that G is a subgroup of $D_q \times D_r$. Hence as in Case 1, G is a subgroup of $D_m \times D_m$ where $m = qr$ is odd. □

Finally, we consider finite subgroups of $SO(4)$ which satisfy the Involution Condition and do not preserve a standard Hopf fibration of S^3.

Proposition 3. *Let G be a finite subgroup of $SO(4)$ which satifies the Involution Condition and does not preserve a standard Hopf fibration of S^3. Then G is isomorphic to either T, O, or D.*

Proof. Let $\rho : SO(4) \to SO(3) \times SO(3)$ be the canonical two-to-one homomorphism. Now $\rho|G$ is either one-to-one or two-to-one. If $\rho|G$ is two-to-one, then G contains the antipodal map, which is a fixed point free involution of S^3. As this would violate the Involution Condition, $\rho|G$ must be one-to-one, and hence $\rho(G) \cong G$.

Dunbar [D] shows that if G is a finite subgroup of $SO(4)$ which does not preserve a standard Hopf fibration of S^3, then $\rho(G)$ is conjugate to one of 16 explicit subgroups of $SO(3) \times SO(3)$ which he lists. Furthermore, he shows that if $\rho(G) \cong G$, then G is conjugate to one of only five of these subgroups, each of which is isomorphic to one of the groups T, O, or D. Hence the proposition now follows from Dunbar's result. □

4. Realizability of groups by embeddings of complete graphs

We would like to show that each of the groups listed in Proposition 2 and Proposition 3 is realizable by some embedding of a complete graph. We will use the following result from [FNPT] which gives conditions under which a subgroup of a realizable group is also realizable by a different embedding of the underlying graph.

Lemma 6 ([FNPT]). *Let Γ be an embedded 3-connected graph. Let H be a subgroup of $\mathrm{TSG}_+(\Gamma)$ which is induced by an isomorphic subgroup G of $\mathrm{Diff}_+(S^3)$, and no non-trivial element of H fixes any vertex of Γ. Then Γ can be re-embedded as Ω such that $H = \mathrm{TSG}_+(\Omega)$ and H is still induced by G.*

Let γ be a graph and let $J \leq \mathrm{Aut}(\gamma)$. If there is an element of J which interchanges the vertices of some edge e, then we will say that e is *invertible under J*. We define the *graph γ' associated with γ and J* to be the graph obtained from γ by adding a vertex in the interior of every edge which is invertible under J. Every automorphism of γ uniquely determines an automorphism of γ'. Thus we shall abuse notation slightly by treating J as a subgroup of both $\mathrm{Aut}(\gamma)$ and $\mathrm{Aut}(\gamma')$. Note that no edge of γ' is invertible under J.

Realizability Lemma. *Let γ be a 3-connected graph, and let $H \leqslant \mathrm{Aut}(\gamma)$ be such that no non-trivial element of H fixes any vertex of γ. Suppose that H is isomorphic to a subgroup G of $\mathrm{Diff}_+(S^3)$ which satisfies the Involution Condition. Then there is an embedding Γ of γ such that $H = \mathrm{TSG}_+(\Gamma)$ and H is induced by G.*

Proof. Let Y denote the union of the fixed point sets of all the non-trivial elements of G. Since G is a finite subgroup of $\mathrm{Diff}_+(S^3)$, Y is a union of finitely many simple closed curves whose pairwise intersection consists of finitely many points.

Let $\Psi : H \to G$ be an isomorphism and for each $a \in H$, define $g_a = \Psi(a)$. Let $\{w_1, \dots, w_q\}$ be a set consisting of one representative from each vertex orbit of γ under H. Let $v_1, \dots, v_q$ be distinct points in $S^3 - Y$ which have disjoint orbits under G. For each $i \leqslant q$, we embed the vertex w_i as the point v_i.

We embed an arbitrary vertex w of γ as follows. Since no vertex of γ is fixed by any non-trivial element of H, there is a unique $a \in H$ and $i \leqslant q$ such that $w = a(w_i)$. Thus every w determines a unique point $g_a(v_i)$ in $S^3 - Y$. So we embed w as $g_a(v_i)$. Let V denote the set of thus embedded vertices; then G leaves V setwise invariant. Since $v_1, \dots, v_q$ are in $S^3 - Y$ and have disjoint orbits under G, all of the vertices of γ are embedded as distinct points.

Let $\{a_1, \dots, a_r\}$ consist of one representative from each conjugacy class of the involutions in H. Now for each i, let $\{\epsilon_{i1}, \dots, \epsilon_{ir_i}\}$ be a (possibly empty) set consisting of one representative from each orbit of those edges of γ which are inverted by a_i. Since no vertex of γ is fixed by any non-trivial element of H, a given ϵ_{ij} is not setwise invariant under any non-trivial element of H other than a_i. Thus, ϵ_{ij} and ϵ_{kl} have disjoint orbits unless both $i = k$ and $j = l$. Now since G satisfies the Involution Condition, for each a_i, $\mathrm{fix}(g_{a_i})$ is a simple closed curve which is not fixed by any other non-trivial element of

G. Thus, if $i \neq j$, then $\text{fix}(g_{a_i})$ is not in the orbit of $\text{fix}(g_{a_j})$, since g_{a_i} and g_{a_j} are in distinct conjugacy classes of G.

For each a_i, we can choose a set of distinct points $\{z_{i1}, \ldots, z_{ir_i}\} \subset \text{fix}(g_{a_i})$ whose orbits are disjoint and none of which is fixed by any non-trivial element of G other than g_{a_i}. Thus, z_{ij} and z_{kl} have disjoint orbits unless both $i = k$ and $j = l$.

Let γ' denote the graph associated with γ and H. We will find an embedding Γ' of γ' such that $H = TSG_+(\Gamma')$ and H is induced by G. Let v be a vertex of γ' which is not a vertex of γ. Then v is on some invertible edge ϵ of γ, which is in the orbit of a unique ϵ_{ij}. Thus $\epsilon = a(\epsilon_{ij})$ for some $a \in H$. Suppose that for some $b \in H$, $\epsilon = b(\epsilon_{ij})$, then either $a = b$ or $a^{-1}b$ inverts ϵ_{ij}. In the latter case, $g_{a^{-1}b}$ fixes z_{ij}, and hence $g_a(z_{ij}) = g_b(z_{ij})$. Thus we can unambiguously embed v as the point $g_a(z_{ij})$. Let V' denote V together with the above embeddings of the vertices of $\gamma' - \gamma$. Then V' is setwise invariant under G.

Suppose that there are vertices of $\gamma' - \gamma$ on the edges ϵ and δ which are both embedded as the same point of S^3. Now $\epsilon = a(\epsilon_{ij})$ for some i and j, and $\delta = b(\epsilon_{kl})$ for some k and l, and $g_a(z_{ij}) = g_b(z_{kl})$. Thus $i = k$ and $j = l$, and hence $g_{a^{-1}b}$ fixes z_{ij}. It follows that $a^{-1}b = a_i$, and hence $a^{-1}b(\epsilon_{ij}) = \epsilon_{ij}$. Thus $\epsilon = a(\epsilon_{ij}) = b(\epsilon_{kl}) = \delta$. Hence we have shown that distinct vertices of $\gamma' - \gamma$ are embedded as distinct points of S^3. Now V' is an embedding of the vertices of γ', and G induces H on V'.

Next we will embed the edges of γ'. Let $\{e_1, \ldots, e_n\}$ be a set consisting of one representative from each orbit of the edges of γ' under H, and let x_i and y_i be the embedded vertices of e_i. For each i, let α_i be a path in S^3 from x_i to y_i whose interior is disjoint from $V' \cup Y$. Let $\pi : S^3 \to S^3/G$ denote the quotient map. Then $\pi|(S^3 - Y)$ is a covering map, and the quotient space $Q = (S^3 - Y)/G$ is a 3-manifold. For each i, let $\alpha_i' = \pi \circ \alpha_i$. Then α_i' is a path or loop from $\pi(x_i)$ to $\pi(y_i)$ whose interior is in Q. Using general position in Q, we can homotop each α_i' rel its endpoints to a simple path or loop ρ_i' such that the interiors of the $\rho_i'(I)$ are pairwise disjoint, and are each disjoint from $\pi(V' \cup Y)$. Now for each i, we can lift ρ_i' to a path ρ_i beginning at x_i such that $\text{int}(\rho_i)$ is disjoint from $V' \cup Y$. Since $\rho_i' = \pi \circ \rho_i$ is one-to-one except possibly on the set $\{0, 1\}$, ρ_i must also be one-to-one except possibly on the set $\{0, 1\}$. Also since ρ_i' is homotopic rel its endpoints to α_i', ρ_i is homotopic rel its endpoints to α_i. In particular, ρ_i is a simple path from x_i to y_i.

We embed each e_i as $\rho_i(I)$. We embed an arbitrary edge e of γ' as follows. Since no edge of γ' is setwise invariant under any non-trivial element of H there is a unique $a \in H$ and a unique $i \leq n$ such that $e = a(e_i)$. Hence e determines a unique arc $g_a(\rho_i(I))$ between $g_a(x_i)$ and $g_a(y_i)$. So we embed e as $\varepsilon = g_a(\rho_i(I))$. Let E' denote the set of thus embedded edges of γ'. Then G leaves E' setwise invariant.

Let Γ' consist of the vertices V' together with the embedded edges E'. Then Γ' is setwise invariant under G. We see that Γ' is an embedded graph as follows. First, since each $\mathrm{int}(\rho_i'(I))$ is disjoint from $\pi(V')$, each embedded edge must be disjoint from V'. Similarly, since for $i \neq j$, $\rho_i'(I)$ and $\rho_j'(I)$ have disjoint interiors, for every g, $h \in G$, $g(\rho_i(I))$ and $h(\rho_j(I))$ have disjoint interiors. Finally, since ρ_i' is a simple path or loop whose interior is disjoint from $\pi(Y)$, if $g \neq h$, then $g(\rho_i(I))$ and $h(\rho_i(I))$ have disjoint interiors.

Hence Γ' is an embedded graph with underlying abstract graph γ' such that G induces H on Γ'. Let Γ be the embedded graph obtained from Γ' by omitting those vertices of V' which are not in V. Then Γ has underlying abstract graph γ and G induces H on Γ. Now by Lemma 6, Γ can be re-embedded as Ω such that $H = \mathrm{TSG}_+(\Omega)$ and H is still induced by G. □

In the next two propositions, we use the Realizability Lemma to obtain our desired embeddings.

Proposition 4. *Let H be a group which is cyclic, dihedral, or a subgroup of $D_m \times D_m$ for some odd number m. Then there is an embedding Γ of some complete graph such that $H = \mathrm{TSG}_+(\Gamma)$ and H is induced by an isomorphic subgroup of $SO(4)$.*

Proof. **Case 1:** H is a subgroup of D_m, where m is even.

Let $F_1 \cup F_2$ be a Hopf link with components F_1 and F_2. Let g be a glide rotation of S^3 of order m which rotates about F_1 by $4\pi/m$ while rotating about F_2 by $2\pi/m$. Let f be a rotation by π around a circle which intersects each of F_1 and F_2 orthogonally in two points. Thus $f(F_i) = F_i$ for each i. Now let G be the subgroup of $SO(4)$ generated by g and f. It is not hard to check that $G \cong D_m$.

Observe that every involution in G other than $g^{m/2}$ has the form $g^i f$ for some non-negative $i < m$. Furthermore, $\mathrm{fix}(g^{m/2}) = F_2$, and for each $i < m$, $\mathrm{fix}(g^i f)$ is a circle which intersects F_1 orthogonally in a distinct pair of points. Also, for every non-trivial $g \in G$, $\mathrm{fix}(g)$ is non-empty if and only if g is an involution. It follows that G satisfies the Involution Condition.

We will find an embedding of K_{2m} in S^3 such that $\mathrm{TSG}_+(\Gamma) = H$. Observe that $\mathrm{Aut}(K_{2m})$ has a subgroup $J \cong D_m$ such that no non-trivial element of J fixes any vertex of K_{2m}. Since $H \leq J$, we can apply the Realizability Lemma to H to conclude that there is an embedding Γ of K_{2m} such that $H = \mathrm{TSG}_+(\Gamma)$ and H is induced by an isomorphic subgroup of G.

Case 2: H is a subgroup of $D_m \times D_m$ where m is odd.

Consider a torus T in S^3 which bounds two isometric solid tori. Let φ denote the glide rotation of S^3 which rotates each $(1,1)$ curve on T along itself by $2\pi/m$, and let ψ denote the glide rotation which rotates each $(-1,1)$ curve on T along itself by $2\pi/m$. Let f be a rotation by π about a particular $(1,1)$ curve X and let g be a rotation by π about a particular $(-1,1)$ curve Y. Let G be the subgroup of $SO(4)$ generated by ϕ, ψ, f and g.

The following two observations are not hard to show:

1. $\varphi\psi = \psi\varphi$, $fg = gf$, $f\varphi = \varphi f$, and $g\psi = \psi g$
2. $f\psi f = \psi^{-1}$ and $g\varphi g = \varphi^{-1}$

Using these observations we can show that $G = D_m \times D_m$. In fact, G is the image in $SO(4)$ of the subgroup of $S^3 \times S^3$ generated by $(e^{\frac{2\pi i}{m}}, 1)$, $(1, e^{\frac{2\pi j}{m}})$, $(i, -i)$, $(j, -j)$. Furthermore, since m is odd, every involution in G has the form $\psi^j f$, $\phi^i g$, or $\phi^i \psi^j fg$ for some i, $j < m$. Now for each i and j, $\mathrm{fix}(\psi^j f)$ is a distinct $(1,1)$ curve on T, $\mathrm{fix}(\phi^i g)$ is a distinct $(-1,1)$ curve on T, and $\mathrm{fix}(\phi^i\psi^j fg)$ is a distinct circle which meets T orthogonally in four points. Also, for every non-trivial $\alpha \in G$, $\mathrm{fix}(\alpha)$ is non-empty if and only if α is an involution. It follows that G satisfies the Involution Condition.

We will find an embedding Γ of K_{4m^2} in S^3 such that $\mathrm{TSG}_+(\Gamma) = H$. Observe that $\mathrm{Aut}(K_{4m^2})$ has a subgroup $J \cong D_m \times D_m$ such that no non-trivial element of J fixes any vertex of K_{4m^2}. Since $H \leq J$, we can apply the Realizability Lemma to H to conclude that there is an embedding Γ of K_{4m^2} such that $H = \mathrm{TSG}_+(\Gamma)$ and H is induced by an isomorphic subgroup of G. □

Proposition 5. *Let H be one of the groups T, O, or D. Then there is an embedding Γ of some complete graph such that $H = \mathrm{TSG}_+(\Gamma)$ and H is induced by an isomorphic subgroup of $SO(4)$.*

Proof. First let Λ be the 1-skeleton of a 4-simplex symmetrically embedded in $\mathbb{R}^4$ with the origin at its center; and let Γ be the radial projection of Λ to S^3. Then Γ is an embedding of K_5 in S^3. Let G denote the group of orientation preserving isometries of Λ in $\mathbb{R}^4$. It is not hard to show that G is isomorphic to the alternating group A_5. Since G induces an isomorphic action on (S^3, Γ), $\mathrm{TSG}_+(\Gamma) \cong A_5$. It is well known that A_5 is isomorphic to the dodecahedral group D.

Next we consider the octahedral group O. Below we define a subgroup G of $SO(4)$ which is isomorphic to O, but which is not the group of symmetries of a regular octahedron or cube. We begin by radially projecting a cube centered at the origin in S^3 to a 2-complex C which is contained in the unit 2-sphere. Let f be the rotation by $2\pi/3$ about an axis which runs through a pair of antipodal vertices of C. Let S be the square $a \cup d \cup b \cup e$ where a is a diagonal of one face of C, b is the parallel diagonal of the opposite face of C, and d and e are the edges of C which connect the endpoints of a and b. Let g be the rotation by π about S. Finally, let h be the glide rotation which rotates by $\pi/2$ about an axis X going through the center of two opposite faces of C, while rotating about a disjoint equator of C by π. We can choose f, g, and h so that ghf is the identity (see Figure 1). Let G be the subgroup of $SO(4)$ which is generated by f, g, and h. By inspection, we see that the group G has 24 elements and every non-trivial element of G either has order 2, 3, or 4 and is conjugate to g, f, or h respectively. It is not hard to check that $G \cong O$.

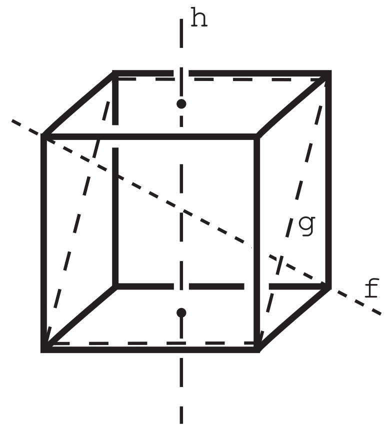

FIGURE 1. The generators of G

We consider the fixed point sets of the elements of G as follows. Let $\alpha \in G$. If α is an involution, then either $\mathrm{fix}(\alpha)$ is a circle which meets C in the centers of two opposite faces, or $\mathrm{fix}(\alpha)$ is a square made up of two parallel diagonals on opposite faces of C together with two edges connecting the endpoints of these diagonals. If α has order 3, then $\mathrm{fix}(\alpha)$ is an axis which runs through antipodal vertices. Finally, if α has order 4, then $\mathrm{fix}(\alpha)$ is empty. It follows from these observations that G satisfies the Involution Condition.

We will find an embedding Γ of K_{24} such that $\mathrm{TSG}_+(\Gamma) = O$. Observe that $\mathrm{Aut}(K_{24})$ has a subgroup $H \cong O$ such that no non-trivial element of H fixes any vertex of K_{24}. Thus we can apply the Realizability Lemma to get an embedding Γ of K_{24} such that $H = O = \mathrm{TSG}_+(\Gamma)$, and H is induced by G. Furthermore, since $T \leq O$, the Realizability Lemma also gives us an embedding Γ of K_{24} such that $T = \mathrm{TSG}_+(\Gamma)$, and T is induced by an isomorphic subgroup of G. □

5. CONCLUSION

We will now put together our five Propositions to complete our characterization of those groups which are $\mathrm{TSG}_+(\Gamma)$ for some embedding Γ of a complete graph.

Theorem 1. *A finite group H is $\mathrm{TSG}_+(\Gamma)$ for an embedding Γ of some complete graph if and only if H is isomorphic to a finite subgroup of either $SO(3)$ or $D_m \times D_m$ for some odd m.*

Proof. Recall that the finite subgroups of $SO(3)$ are precisely the finite cyclic groups, the dihedral groups, and T, O, and D.

Suppose that there is an embedding Γ of some K_n with $H = \mathrm{TSG}_+(\Gamma)$. By Proposition 1, either H is cyclic of odd order or there exists an embedding Γ' of K_n such that H is induced on Γ' by an isomorphic subgroup G of $SO(4)$. If H is cyclic we are done. So we assume that H is induced by an isomorphic subgroup G of $SO(4)$. Then, by Lemma 2, G satisfies the Involution Condition. If G preserves a standard Hopf fibration of S^3, then by Proposition 2, G is either cyclic, dihedral, or a subgroup of $D_m \times D_m$ for some odd number m. So in this case we are done. On the other hand, if G does not preserve a standard Hopf fibration of S^3, then by Proposition 3, G is isomorphic to T, O, or D. So again we are done.

In order to prove the converse, first suppose that H is cyclic, dihedral, or a subgroup of $D_m \times D_m$, for some odd number m. Then by Proposition 4, there is an embedding Γ of some K_n such that $H = \mathrm{TSG}_+(\Gamma)$. On the other hand, if H is T, O, or D, then by Proposition 5, there is an embedding Γ of some K_n such that $H = \mathrm{TSG}_+(\Gamma)$. □

Finally, observe that the following result follows immediately from Theorem 1 together with Propositions 4 and 5.

Corollary 1. *Let Ω be an embedding of some complete graph with $H = \mathrm{TSG}_+(\Omega)$. Then there exists an embedding Γ of some (possibly different) complete graph such that $H = \mathrm{TSG}_+(\Gamma)$ and H is induced by an isomorphic subgroup of $SO(4)$.*

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