Abstract

We propose two different approaches generalizing the Karhunen-Loeve series expansion to model and simulate multi-correlated non-stationary stochastic processes. The muKL (multi-uncorrelated KL) method produces a series in terms an identical set of uncorrelated random variables, and mcKL (multi-correlated KL) relies on expansions in terms of correlated sets of random variables, both reflecting the cross-covariance structure of the processes. In particular, we study the accuracy and convergence rates of our series expansions and compare the results to other statistical techniques.

Methods

The effectiveness of KL expansion, for instance, optimal convergence in mean square error, is due to its bi-orthogonality, meaning that the eigenfunctions are orthogonal in L^2 and the random variables are uncorrelated. However, it restricts its application to single random process.

For multi-correlated random processes with following statistical structure, \( f(t;\omega) = \sum_{i=1}^{n} \lambda_i \phi_i(t) \psi_i(\omega) \), the KL expansion can be written as

\[
T(t;\omega) = f(t) + \sum_{k=1}^{n} \sqrt{\lambda_k} \phi_k(t) \tilde{\psi}_k(\omega)
\]

where \( \lambda_k, \phi_k \) are eigenvalue and orthogonal eigenvector of the covariance kernel, and \( \tilde{\psi}_k(\omega) \) are independent uncorrelated random variables.

Results

1. Sample path : Both muKL and mcKL methods properly capture the correlated structure of multiple processes. See Figure 3.

2. Convergence : muKL method has better convergence in terms of mean square error. Table 1. shows the number of random variable to achieve less than 3% error in the eigenvalues.

3. Computational cost : mcKL is more efficient than muKL due to its smaller size of eigenproblem and availability of some explicit results. See Figure 4.

Application

Tumor concentration with treatment modeled by random processes. \( f(t;\omega) = C(t) + g(x)f_1(t;\omega) + f_2(t;\omega) \), where \( C(t) \) is the tumor cell at time \( t \), \( f_1(t;\omega) \), \( f_2(t;\omega) \) are strength of the treatment (e.g., chemotherapy, radiotherapy) factors that determine the tumor cell (e.g., drugs, radiotherapy) where random processes are mutually correlated with the following structure, \( f_1(t;\omega) = \theta_1 e^{-t}, f_2(t;\omega) = \theta_2 e^{-t} \), and \( \theta_1, \theta_2 \) are parameters.

Conclusions

Two difference methods, muKL and mcKL, have been proposed to represent multi-correlated non-stationary random processes.

- muKL method usually provides better accuracy and convergence rate, but it is computationally more expensive than mcKL.
- mcKL method yields scalable algorithms and explicit result can be obtained. Also, it can be applied when each process are expanded in terms of random variables with different distribution.

These methods can be readily employed in stochastic simulation and we have demonstrated the importance of modeling the cross-covariance structure of the noise in a stochastic tumor model.

References