CMSC 351: Big Notation

Justin Wyss-Gallifent

May 25, 2023

1 Inspiration

Suppose two algorithms do exactly the same thing to lists of length n. We find out that the time they take in seconds is as follows. Note that these are just made up!

n	$A_1(n)$	$A_2(n)$
10	6	1
20	12	6
30	18	17
40	24	25
50	28	40
60	30	63
70	38	82
80	45	109
90	50	140
100	59	190

Observe that Algorithm 2 is better (faster) up until about $n = 40$, and then Algorithm 1 is better.

But can we formalize this more, both the comparison and the values themselves?

It turns out that the values satisfy:

$$0.4n \leq A_1(n) \leq 0.6n$$

and:

$$0.01n^2 \leq A_2(n) \leq 0.02n^2$$

Although we don't have an exact knowledge about other values we do certainly have a more rigorous way not only of comparing the two algorithms but of understandng each algorithm independently.

For example we can say that if $n = 150$ then Algorithm 2 will take at most $0.02(150)^2 = 450$ seconds. In this case we have an upper bound which is a multiple of n^2.

Our goal is to formalize these notions.

2 The Bigs

2.1 Big-O Notation

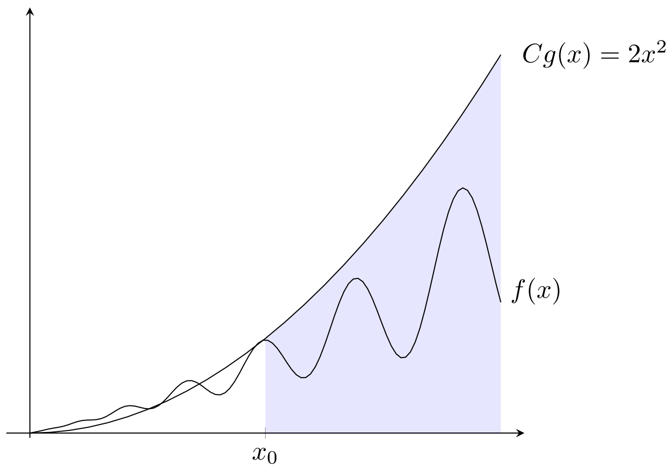
Recall the definition:

Definition 2.1.1. We say that:

$$f(x) = \mathcal{O}(g(x)) \text{ if } \exists x_0, C > 0 \text{ such that } \forall x \geq x_0\,,\ f(x) \leq Cg(x)$$

We think of this as stating that *eventually* $f(x)$ is smaller than some constant multiple of $g(x)$.

Example 2.1. For example, here $f(x) = \mathcal{O}(x^2)$ with $C = 2$ and x_0 as shown:



Note 2.1.1. There's frequently (but not always) a trade-off in that if C is large then x_0 might be smaller, or vice-versa. In light of this note that "eventually" could mean for a very large x_0.

Example 2.2. It's true that $42000x \lg x = \mathcal{O}(x^2)$ with $C = 10$ because eventually $42000x \lg x \leq 10x^2$. However "eventually" in this case means $x_0 \approx 67367$. In other words this is the smallest x_0 such that if $x \geq x_0$ then $42000x \lg x \leq 10x^2$.

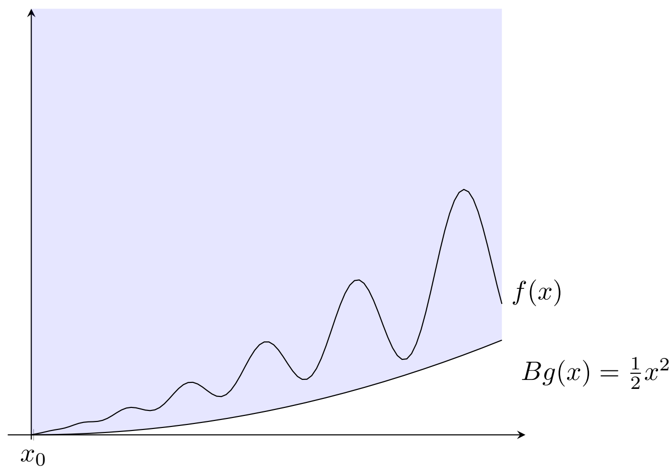
2.2 Big-Omega and Big-Theta Notations

We can extend upon this with:

Definition 2.2.1. We have:

$$f(x) = \Omega(g(x)) \text{ if } \exists x_0, B > 0 \text{ such that } \forall x \geq x_0\,,\ f(x) \geq Bg(x)$$

Example 2.3. For example, here $f(x) = \Omega(x^2)$ with $B = \frac{1}{2}$ and x_0 as shown:

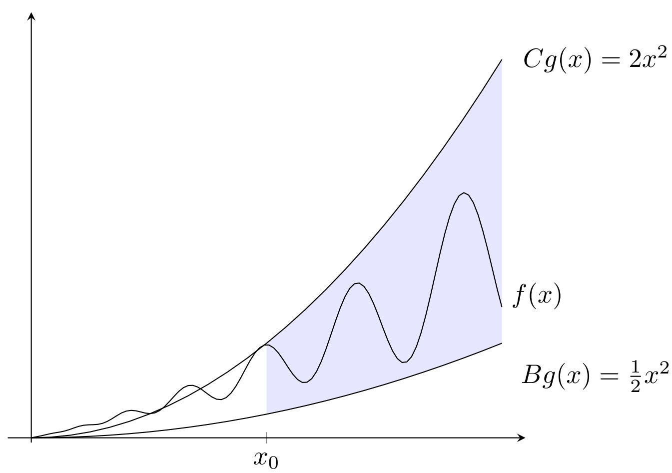


and with:

Definition 2.2.2. We have:

$$f(x) = \Theta(g(x)) \text{ if } \exists x_0, B > 0, C > 0 \text{ such that } \forall x \geq x_0\,,\ Bg(x) \leq f(x) \leq Cg(x)$$

Example 2.4. For example, here $f(x) = \Theta(x^2)$ with $B = \frac{1}{2}$ and $C = 2$ and x_0 as shown:



2.3 All Together

The basic idea is that $\mathcal{O}$ provides an upper bound for $f(x)$, Ω provides a lower bound and Θ provides a tight bound. Therefore $f(x) = \Theta(g(x))$ if and only if $f(x) = \mathcal{O}(g(x))$ and $f(x) = \Omega(g(x))$.

Moreover observe that $\Theta \Rightarrow \mathcal{O}$ and $\Theta \Rightarrow \Omega$ but the converses are false.

Example 2.5. We show: $3x \lg x + 17 = \mathcal{O}(x^2)$

Consider the expression:

$$3x \lg x + 17$$

Note two things:

- If $x > 0$ then $\lg x < x$.
- If $x \geq \sqrt{17} = 4.1231...$ then $x^2 \geq 17$.

Thus if $x \geq 5$ both of these are true and we have:

$$3x \lg x + 17 \leq 3x(x) + x^2 = 4x^2$$

Thus $x_0 = 5$ and $C = 4$ works.

Note: It's not necessary to pick an integer value of x_0 here. I just did it because it's pretty. Using $x_0 = \sqrt{17}$ would have been fine too.

Example 2.6. We show: $\frac{100}{x^2} + x^2 \lg x = \mathcal{O}(x^3)$

Consider the expression:

$$\frac{100}{x^2} + x^2 \lg x$$

Note two things:

- If $x > 0$ then $\lg x < x$.
- If $x \geq 10$ then $x^2 \geq 100$ and then $\frac{100}{x^2} \leq 1 < x < x^3$.

Thus if $x \geq 10$ both of these are true and we have:

$$\frac{100}{x^2} + x^2 \lg x = \mathcal{O}(x^3) \leq x^3 + x^3 = 2x^3$$

Thus $x_0 = 10$ and $C = 2$ works.

Example 2.7. We show: $0.001x \lg x + 0.0001x - 42 = \Omega(x)$

Consider the expression:

$$0.001x \lg x - 42$$

Note that if $x \geq 2$ then $\lg x \geq 1$ and then:

$$0.001x \lg x - 42 \geq 0.001x - 42$$

This is a line with slope 0.001 and any line with smaller slope will eventually be below it. For example the line $0.0001x$ is below it when:

$$\begin{aligned} 0.001x - 42 &\geq 0.0001x \\ 0.0009x &\geq 42 \\ x &\geq \frac{42}{0.0009} = 46666.66... \end{aligned}$$

Thus if we have $x \geq 46666.66...$ then:

$$0.001x \lg x - 42 \geq 0.001x - 42 \geq 0.0001x$$

Thus $x_0 = 46667$ and $B = 0.0001$ works.

Example 2.8. We show: $10x \lg x + x^2 = \Theta(x^2)$

Consider the expression:

$$10x \lg x + x^2$$

Observe that for all $x \geq 1$ we have $\lg x > 0$ and hence:

$$10x \lg x + x^2 \geq x^2$$

And we have:

$$10x \lg x + x^2 \leq 10x(x) + x^2 = 11x^2$$

thus $x_0 = 1$, $B = 1$ and $C = 11$ works.

For simple polynomials there's very little work to show Θ.

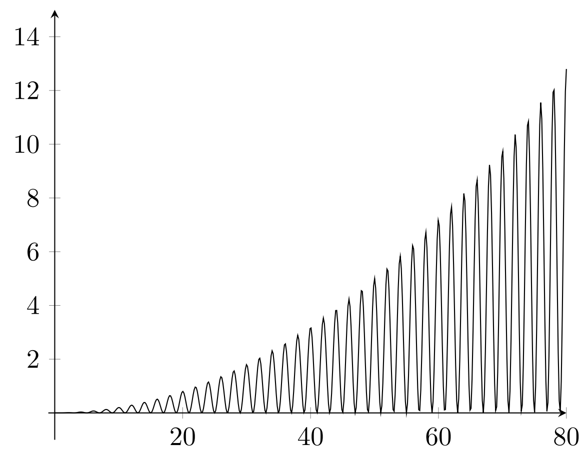
Example 2.9. Observe that $3x^2 = \Theta(x^2)$ because $x_0 = 0$ and $B = C = 3$ works.

Example 2.10. Consider $f(x) = 2x^2 - x$. Note that $2x^2 - x \leq 2x^2$ and

$2x^2 - x \geq 2x^2 - x^2 = 1x^2$ for $x \geq 1$ so that $x_0 = 1, B = 1, C = 2$ works for $2x^2 - x = \Theta(x^2)$.

Example 2.11. Consider $f(x) = 0.001x^2(1 + \cos(x\pi))$.

The graph of this function is:



The local maxima occur at $x = 0, 2, 4, 6, 8, ...$ and the local minima occur at $x = 1, 3, 5, 7, 9, ...$

Note that $0.001x^2(1 + \cos(x\pi)) \leq 0.001x^2(1 + 1) = 0.002x^2$ for $x \geq 0$ so that $f(x) = \mathcal{O}(x^2)$. However in addition note that when $x \in \mathbb{Z}$ is odd that $0.001x^2(1 + \cos(x\pi)) = 0.001x^2(1 - 1) = 0$ so that there is no $B > 0$ such that for large enough x we have $f(x) \geq Bx^2$. Consequently $f(x) \neq \Omega(x^2)$ and thus $f(x) \neq \Theta(x^2)$.

You might ask if there is any $g(x)$ such that $f(x) = \Theta(g(x))$ and the short answer is - yes, of course, because $f(x) = \Theta(f(x))$ but this is generally unsatisfactory. We are looking for *useful* $g(x)$ which help us understand $f(x)$. Saying essentially that $f(x)$ grows at the same rate as itself doesn't help much!

3 A Limit Theorem

There are a few alternative ways of proving $\mathcal{O}$, Ω and Θ. Here is one. Note that the following are unidirectional implications!

Theorem 3.0.1. Provided $\lim_{n\to\infty} f(n)$ and $\lim_{n\to\infty} g(n)$ exist (they may be ∞) then we have the following:

(a) If $\lim_{n\to\infty} \frac{f(n)}{g(n)} \neq 0, \infty$ then $f(n) = \Theta(g(n))$.
Note: Here we also have $f(n) = \mathcal{O}(g(n))$ and $f(n) = \Omega(g(n))$ as well.

(b) If $\lim_{n\to\infty} \frac{f(n)}{g(n)} \neq \infty$ then $f(n) = \mathcal{O}(g(n))$.

(c) If $\lim_{n\to\infty} \frac{f(n)}{g(n)} \neq 0$ then $f(n) = \Omega(g(n))$.

Proof. Here's a proof of (b). Suppose we have:

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = L \neq \infty$$

By the definition of the limit this means:

$$\forall \epsilon > 0, \exists n_0 \text{ st } n \geq n_0 \Longrightarrow L - \epsilon < \frac{f(n)}{g(n)} < L + \epsilon$$

Specifically, if $\epsilon = 1$ if we take only the right inequality this tell us that:

$$\exists n_0 \text{ st } n \geq n_0 \Longrightarrow \frac{f(n)}{g(n)} < L + 1$$

When $<$ is true, so is $\leq$ so this means that when $n \geq n_0$ we have:

$$f(n) < (L + 1)g(n)$$

This is exactly the definition of $\mathcal{O}$ using n_0 and $C = L + 1$. $\mathcal{QED}$

Example 3.1. Observe that:

$$\lim_{n\to\infty} \frac{n \ln n}{n^2} = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1/n}{1} = 0$$

Thus $n \ln n = \mathcal{O}(n^2)$.

The following example is far easier to prove using this theorem than from the definition of $\mathcal{O}$:

Example 3.2. We have $50n^{100} = \mathcal{O}(3^n)$.

Observe that 100 applications of L'hôpital's Rule yields:

$$\lim_{n\to\infty} \frac{50n^{100}}{3^n} = \lim_{n\to\infty} \frac{(100)(99)...(1)(50)}{(\ln 3)^{100} 3^n} = 0$$

The result follows.

4 Common Functions

In all of this you might wonder why we're always comparing functions to things like n^2 or $n \lg n$. We typically wouldn't say, for example, that $f(n) = \Theta(n^2 + 3n + 1)$.

The reason for this is that computer scientists have settled on a collection of "simple" functions, functions which are easy to understand and compare, and big-notation almost always uses these functions.

Here are a list of some of them, in order of increasing size:

$$1, \lg n, n, n \lg n, n^2, n^2 \lg n, n^3, ...$$

To say these are "increasing size" means, formally, that any of these is $\mathcal{O}$ of anything to the right, for example $n = \mathcal{O}(n \lg n)$ and $n^2 = \mathcal{O}(n^3)$ and so on.

There's a pattern there, that $n^k = \mathcal{O}(n^k \lg n)$ and $n^k \lg n = \mathcal{O}(n^{k+1})$, which is easy to prove.

In addition we have, for every positive integers k and $b \geq 2$:

$$n^k = \mathcal{O}(b^n)$$

These can be proved with the Limit Theorem.

Lastly, all of the above are $\mathcal{O}(n!)$, which is about the biggest one we ever encounter in this class.

5 Intuition

It's good to have some intuition here, and of course the following can be proved rigorously on a case-by-case basis, and you should try.

In essence the "largest term" always wins in a Θ sense. So for example if we have:

$$f(n) = n^2 - n \lg n + n + 1$$

The "largest term" is the n^2 so that wins and we can say:

$$n^2 - n \lg n + n + 1 = \Theta(n^2)$$

Likewise, for example:

$$n^2 \lg n + n \lg n - 100 = \Theta(n^2 \lg n)$$

6 Additional Facts

6.1 Use of n vs x

These statements about function of x are often phrased using the variable n instead. Typically this is done when n can only take on positive integers.

In this case it can still be helpful to draw the functions as if n could be any real number, otherwise we're left drawing a bunch of dots. In some cases though, like $f(n) = n!$, it's not entirely clear how we would sketch this for $n \notin \mathbb{Z}$.

Otherwise the calculations are basically identical, noting that the cutoff value n_0 must be a positive integer.

6.2 Cautious Comparisons

This notation brings a certain ordering to functions. Observe for example that $1000000 + n \lg n = \mathcal{O}(n^2)$ because *eventually* $1000000 + n \lg n \leq Cn^2$ for some $C > 0$. Thus we intuitively think of n^2 as "larger than" $1000000 + n \lg n$. However we have to make sure we understand that we really mean that a constant multiple of n^2 is eventually larger than $1000000 + n \lg n$.

> **Example 6.1.** For example *eventually* $1000000 + n \lg n \leq 17n^2$ but *eventually* here means for $n \geq n_0 = 243$.

We should also note that it's common to believe that if one function $g(x)$ has a larger derivative than another function $f(x)$ that eventually $f(x) \leq g(x)$. This is false.

7 Thoughts, Problems, Ideas

1. It's tempting to think that if $f(x)$ and $g(x)$ are both positive functions defined on $[0, \infty)$ with positive derivatives and if $f'(x) > g'(x)$ for all x then eventually $f(x)$ will be above $g(x)$. Show that this isn't true. Give explicit functions and sketches of those functions.
2. Find the value x_0 (approximately) which justifies $1234 + 5678x \lg x = \mathcal{O}(x^2)$ with $C = 42$. Use any technology you like but explain your process.
3. Find the value x_0 (approximately) which justifies $4758x + 789x^2 \lg x = \mathcal{O}(x^3)$ with $C = 17$. Use any technology you like but explain your process.
4. Find the value x_0 (approximately) which justifies $0.00357x^{2.01} \lg x = \Omega(x^2)$ with $C = 100$. Use any technology you like but explain your process.
5. Show from the definition that $5x^2 + 10x \lg x + \lg x = \mathcal{O}(x^2)$.
6. Show from the definition that:
$$\sum_{i=0}^{n-1} \left[2 + \sum_{j=i}^{n-1} 3 \right] = \mathcal{O}(n^2)$$
7. Show from the definition that $(475632)2^n = \mathcal{O}(5^n)$.
8. Show from the definition that $x + x \log x = \Theta(x \lg x)$.
9. Show from the definition that:
$$\sum_{i=0}^{n-1} \left[1 + i + \frac{1}{i+1} \right] = \Theta(n^2)$$
10. Show from the definition that $x^3 + 5x + \ln x + 100 = \Omega(x^2)$.
11. Show from the definition that:
$$\sum_{i=0}^{n} \left[i^2 + 3i \right] = \Omega(n^3)$$
12. Show from the definition that $5^n \neq \mathcal{O}(2^n)$.
13. Show that $5000 + 6000n^{1500} = \mathcal{O}(3^n)$.
14. Show that $5^n = \Omega(n^{1000})$
15. Show that $(0.001)5^n = \Omega(857n^{999})$
16. Show from the definition that $\log_2 n = \Theta(\log_5 n)$ and $\log_5 n = \Theta(\log_2 n)$.
17. Generalize the above problem. In other words prove that $\Theta(\log_b x) = \Theta(\log_c x)$ for any two bases $b, c > 1$.
18. In the previous question why do we need $b, c > 1$?

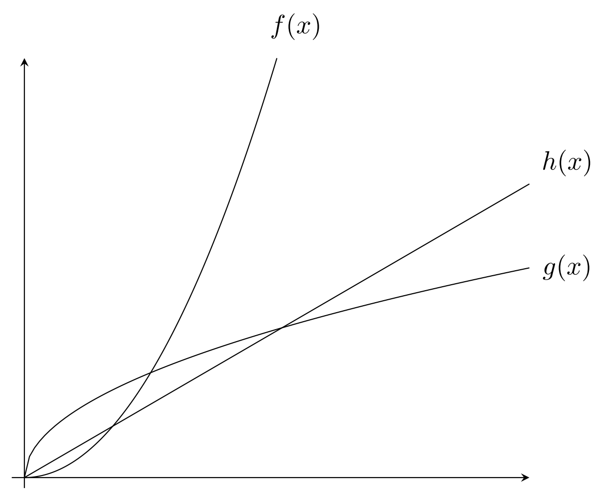
19. Give an example of two functions $f(x)$ and $g(x)$ which are not constant multiples of one another and which satisfy $f(x) = \mathcal{O}(g(x))$ and $g(x) = \mathcal{O}(f(x))$. Justify from the definitions.

20. Give an example of two functions $f(x)$ and $g(x)$ which are not constant multiples of one another and which satisfy $f(x) = \Omega(g(x))$ and $g(x) = \Omega(f(x))$. Justify from the definitions.

21. If $f(n) = \mathcal{O}(g(n))$ with C_0 and n_0 and $g(n) = \mathcal{O}(h(n))$ with C_1 and n_1 which constants would prove that $f(n) = \mathcal{O}(h(n))$?

22. The functions $f(x) = \log_b x$ for $b > 1$ and $g(x) = x^c$ for $0 < c < 1$ have similar shapes for increasing x. however $f(x) = \mathcal{O}(g(x))$ always. Prove this.
Note: This underlies the important fact that roots always grow faster than logarithms.

23. Consider the following three functions:



    (a) Write down as many possibilities as you can which satisfy $\square = \mathcal{O}(\lozenge)$ where $\square, \lozenge \in \{f(x), g(x), h(x)\}$.

    (b) Write down as many possibilities as you can which satisfy $\square = \Omega(\lozenge)$ where $\square, \lozenge \in \{f(x), g(x), h(x)\}$.

    (c) Write down as many possibilities as you can which satisfy $\square = \Theta(\lozenge)$ where $\square, \lozenge \in \{f(x), g(x), h(x)\}$.