CMSC 351: Depth-First Traverse

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1 Introduction:

Suppose we are given a graph G and a starting node s. Suppose we wish to simply traverse the graph in some way looking for a particular value associated with a node. We're not interested in minimizing distance or cost or any such thing, we're just interested in the traversal process.

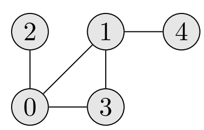
2 Intuition

One classic way to go about this is a depth-first traverse. The intuitive idea is that starting with a starting node s we follow one path as far as possible before backtracking. When we backtrack we only do so as little as possible until we can go deeper again.

Observe that this description does not lead to a unique traversal because there may be multiple paths that we can follow from a given vertex.

3 Visualization

Before writing down some explicit pseudocode let's look at an easy graph and look at how the above intuition might pan out. Consider this graph:



Suppose we start at the vertex $s = 0$. We have three edges we can follow, let's suppose we follow the edge to the vertex 3 first. From 3 we can only go to 1 (we can't go back to 0 since we've visited it already) and then to 4 (we can't go back to 0 or 3). At that point we have gone as deep as we can along that branch so we go back to the most recent branch for which there are other paths available. We have to go back to 0 and from there we go to 2. Thus our depth-first traverse follows the vertices in order $0, 3, 1, 4, 2$.

4 Algorithm Implementation

There are several classic approaches to constructing an algorithm for depth-first search to which we will add one more for reasons we shall make clear:

- Using recursion.
- Using a stack.
- Using a stack which is modified to be a doubly-linked-list.

5 Recursive Implementation

5.1 Pseudocode

The pseudocode for the recursive implementation is as follows:

```
// These are global.
VORDER = []
VISITED = list of length V full of FALSE
function dft(G,x):
    VORDER.append(x)
    VISITED[x] = TRUE
    for all adjacent nodes y (in some order)
        if VISITED[y] == FALSE
            dft(G,y)
        end if
    end for
end function
dft(G,s)
```

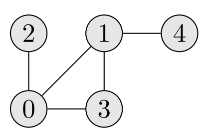
In this pseudocode the list `VORDER` will record the order in which we visit the vertices while the list `VISITED` will indicate whether or not a vertex has been visited.

The line:

`for all adjacent nodes y (in some order)`

does not suggest which order we should follow the adjacent nodes in. In what follows we'll follow them in decreasing order.

Example 5.1. Let's return to our example from earlier:



We'll start our traversal at the vertex $x = 0$. Before the function is called we have:

index	0	1	2	3	4
VORDER					
VISITED	F	F	F	F	F

Our first call is `dft(G,0)` and before the `for` loop we then have:

index	0	1	2	3	4
VORDER	0				
VISITED	T	F	F	F	F

The `for` loop cycles through the vertices 3,2,1 (call this depth 1) and the first call is `dft(G,3)` which yields, before its own `for` loop:

index	0	1	2	3	4
VORDER	0	3			
VISITED	T	F	F	T	F

From `dft(G,3)` the `for` loop cycles through the vertices 1,0 (call this depth 2) and the first call is `dft(G,1)` which yields, before its own `for` loop:

index	0	1	2	3	4
VORDER	0	3	1		
VISITED	T	T	F	T	F

From `dft(G,1)` the `for` loop cycles through the vertices 4,0 (call this depth 3) and the first call is `dft(G,4)` which yields, before its own `for` loop:

index	0	1	2	3	4
VORDER	0	3	1	4	
VISITED	T	T	F	T	T

From `dft(G,4)` the `for` loop cycles through the vertices 1 (call this depth 4) but since that vertex has been visited, `dft` is not called on it again and we are sent back to depth 3 and our loop is on vertex 0 but since that vertex has been visited, `dft` is not called on it again and we are sent back to depth 2 and our loop is on vertex 0 but since that vertex has been visited, `dft` is not called on it again and we are sent back to depth 1 and our loop is on vertex 2 so we call `dft(G,2)` which yields, before its own `for` loop:

index	0	1	2	3	4
VORDER	0	3	1	4	2
VISITED	T	T	T	T	T

From `dft(G,2)` the `for` loop cycles through the vertices 0 but since that vertex has been visited, `dft` is not called on it again and we are sent back to depth 1 and our loop is on vertex 1 but since that vertex has been visited, `dft` is not called on it again.

Then we are done. Observe that the order in which we visited the nodes is $0, 3, 1, 4, 2$.

5.2 Pseudocode Time Complexity

Suppose V is the number of nodes and E is the number of edges. What follows is exactly the same as breadth-first traverse so if that made sense you can possibly skip this.

- The initialization takes $\mathcal{O}(V)$. This could in fact take $\Theta(1)$ depending on the architecture but the choice has no effect on the result.
- Each node gets processed once so this is $V\Theta(1)$ each for a total of $\Theta(V)$.

- Since each edge is attached to two nodes the `for` loop will iterate a total of $2E$ times over the course of the entire algorithm. This gives a total of $\Theta(2E) = \Theta(E)$.

The time complexity is therefore $\mathcal{O}(V) + \Theta(V) + \Theta(E) = \mathcal{O}(V + E)$. If initialization is actually $\Theta(1)$ then this becomes $\Theta(V + E)$.

Note 5.2.1. Note that our pseudocode and analysis assumes we have direct access to a node's edges using something like an adjacency list. If we use an adjacency matrix then the inner loop becomes $\Theta(V)$ and the entire pseucode becomes $\Theta(V^2)$.

6 Stack Implementation

6.1 Pseudocode

The pseudocode for the stack implementation is as follows:

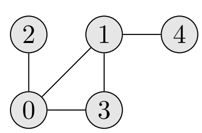
```
VORDER = []
VISITED = list of length V full of FALSE
STACK  = [s]
while STACK is not empty:
    x = STACK.pop()
    if VISITED[x] == FALSE:
        VISITED[x] = TRUE
        VORDER.append(x)
    end if
    for all nodes y adjacent to x:
        if VISITED[y] == FALSE:
            STACK.append(y)
        end if
    end for
end while
```

The line:

`for all nodes y adjacent to x`

does not suggest which order we should follow the adjacent nodes in. In what follows we'll follow them in increasing order.

Example 6.1. Let's return to our example from earlier:



We'll start our traversal at the vertex $x = 0$. We start with the following before the `while` loop:

S = [0]

index	0	1	2	3	4
VORDER					
VISITED	F	F	F	F	F

Iterate! We pop $x = 0$ off the stack. Since it's not visited we mark it as such and append it to the visiting order. We then iterate over the vertices $1, 2, 3$ and push them all onto the stack:

S = [1,2,3]

index	0	1	2	3	4
VORDER	0				
VISITED	T	F	F	F	F

Iterate! We pop $x = 3$ off the stack. Since it's not visited we mark it as such and append it to the visiting order. We then iterate over the vertices $0, 1$ and push 1 (but not 0) onto the stack:

S = [1,2,1]

index	0	1	2	3	4
VORDER	0	3			
VISITED	T	F	F	T	F

Iterate! We pop $x = 1$ off the stack. Since it's not visited we mark it as such and append it to the visiting order. We then iterate over the vertices $0, 3, 4$ and push 4 (but not $0, 3$) onto the stack:

S = [1,2,4]

index	0	1	2	3	4
VORDER	0	3	1		
VISITED	T	T	F	T	F

Iterate! We pop $x = 4$ off the stack. Since it's not visited we mark it as such and append it to the visiting order. We then iterate over the vertex 1 but nothing is pushed onto the stack:

S = [1,2]

index	0	1	2	3	4
VORDER	0	3	1	4	
VISITED	T	T	F	T	T

Iterate! We pop $x = 2$ off the stack. Since it's not visited we mark it as such and append it to the visting order. We then iterate over the vertex 0 but nothing is pushed onto the stack:

S = [1]

index	0	1	2	3	4
VORDER	0	3	1	4	2
VISITED	T	T	T	T	T

Iterate! We pop $x = 1$ off the stack. Since it's visited we do nothing with it. We then iterate over the vertices $0, 3, 4$ but nothing is pushed onto the stack:

S = []

index	0	1	2	3	4
VORDER	0	3	1	4	2
VISITED	T	T	T	T	T

Then we are done. Observe that the order in which we visited the nodes is $0, 3, 1, 4, 2$.

6.2 Pseudocode Time Complexity

It's not uncommon for various online sources to give essentially this pseudocode and state that it has time complexity $\Theta(V + E)$ but this is false as can be easily demonstrated.

Consider a graph with V vertices which is *complete*, meaning each vertex is connected to every other vertex. We start by pushing the starting vertex onto the stack.

When we pop this vertex there is 1 visited vertex and the `for` loop will push all of the remaining $V - 1$ (unvisited) vertices onto the stack.

When we pop the next vertex there are 2 visited vertices and the `for` loop will push $V - 2$ (unvisited) vertices onto the stack, all of which will be repeats.

This will repeat through the entire process and all together the number of vertices which get pushed onto the stack will be:

$$1 + (V - 1) + (V - 2) + ... + 2 + 1 + 0 = 1 + \frac{(V - 1)(V)}{2} = \Theta(V^2)$$

Since the `while` loop iterates once for each vertex on the stack it will iterate $\Theta(V^2)$ times.

The `for` loop will (as with the argument for the recursive implementation) iterate $2E$ times for a final time complexity of $\Theta(V^2 + E)$.

7 Stack/Doubly-Linked-List Implementation

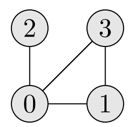
7.1 Introduction

It is possible to modify the stack version to bring it down to $\Theta(V + E)$. The trick is to find a way to ensure that an item is only ever popped off the stack once, but we can't be sloppy about it. In theory there are two ways to go about this. When we are about to push something onto the stack, other than checking if it has been visited:

(a) We check whether it has been or still is on the stack and if so, we don't push it.

(b) We check whether it has been on the stack, we don't push it, and if it is on the stack, we remove the earlier occurrence.

While (a) seems easier to do it does not work, as is easily demonstrated by this graph:



Examine approach (a). Let's start at 0, so $S = [0]$. We then pop $x = 0$ (marking it as visited) and suppose we push 3, 2, 1 onto the stack in that order, so $S = [3, 2, 1]$. We then pop $x = 1$ (marking it as visited) and don't push anything, so $S = [3, 2]$. We then pop $x = 2$ (marking it as visited) and don't push anything, so $S = [3]$. We then pop $x = 3$ (marking it as visited), and don't push anything, so $S = []$, and then we are done. However we have now visited the vertices in the order $0, 1, 2, 3$ which is not a BFT since after visiting 1 we should go to 3.

On the other hand examine approach (b). Let's start at 0, so $S = [0]$. We then pop $x = 0$ (marking it as visited) and suppose we push 3, 2, 1 onto the stack in that order, so $S = [3, 2, 1]$. We then pop $x = 1$ (marking it as visited) and we push 3, replacing the earlier occurrence, so $S = [2, 3]$. We then pop $x = 3$ (marking it as visited) and we push 2, replacing the earlier occurrence (a bit silly), so $S = [2]$. We then pop $x = 2$ (marking it as visited) and don't push anything, so $S = []$, and then we are done. Now we have visited the vertices in the order $0, 1, 3, 2$.

One way to accomplish (b) is to turn the stack into a doubly-linked-list and to keep an array of pointers, one for each vertex, which point to the vertex's location on the stack and are NULL by default. When a vertex is pushed onto the stack we check if its pointer is NULL and if not then we remove the previous vertex from within the stack (this is $\Theta(1)$ for a doubly-linked-list) and then push it on the end of the stack and update the pointer.

7.2 Pseudocode

The pseudocode for this new implementation is as follows:

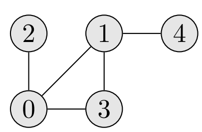
```
VORDER = []
STACK = [s] (functions as doubly-linked-list)
VISITED = list of length V full of FALSE
SP = list of length V full of NULL pointers
SP[x] = points to x in STACK
while STACK is not empty:
    x = STACK.pop()
    if VISITED[x] == FALSE:
        VISITED[x] = TRUE
        VORDER.append(x)
    end if
    for all nodes y adjacent to x:
        if VISITED[y] == FALSE:
            if SP[y] != NULL:
                delete y from STACK
            end if
            STACK.append(y)
            SP[y] = point to y in STACK
        end if
    end for
end while
```

The line:

`for all adjacent nodes y (in some order)`

does not suggest which order we should follow the adjacent nodes in. In what follows we'll follow them in increasing order.

Example 7.1. Let's return to our example from earlier:



Illustrating the pointers is a bit of a pain so we will avoid doing so. Instead the critical difference between this implementation and the stack implementation is that whenever a vertex is pushed onto the stack we check if it is already on the stack and if so we delete its earlier occurence. In what follows this is the only significant difference.

We'll start our traversal at the vertex $x = 0$. We start with the following before the `while` loop:

S = [0]

index	0	1	2	3	4
VORDER					
VISITED	F	F	F	F	F

Iterate! We pop $x = 0$ off the stack. Since it's not visited we mark it as such and append it to the visiting order. We then iterate over the vertices $1, 2, 3$ and push them all onto the stack:

S = [1,2,3]

index	0	1	2	3	4
VORDER	0				
VISITED	T	F	F	F	F

Iterate! We pop $x = 3$ off the stack. Since it's not visited we mark it as such and append it to the visiting order. We then iterate over the vertices $0, 1$ and push 1 (but not 0) onto the stack which deletes the earlier 1:

S = [2,1]

index	0	1	2	3	4
VORDER	0	3			
VISITED	T	F	F	T	F

Iterate! We pop $x = 1$ off the stack. Since it's not visited we mark it as such and append it to the visiting order. We then iterate over the vertices $0, 3, 4$ and push 4 (but not $0, 3$) onto the stack:

S = [2,4]

index	0	1	2	3	4
VORDER	0	3	1		
VISITED	T	T	F	T	F

Iterate! We pop $x = 4$ off the stack. Since it's not visited we mark it as such and append it to the visiting order. We then iterate over the vertex 1 but nothing is pushed onto the stack:

S = [2]

index	0	1	2	3	4
VORDER	0	3	1	4	
VISITED	T	T	F	T	T

Iterate! We pop $x = 2$ off the stack. Since it's not visited we mark it as such and append it to the visting order. We then iterate over the vertex 0 but nothing is pushed onto the stack:

S = []

index	0	1	2	3	4
VORDER	0	3	1	4	2
VISITED	T	T	T	T	T

Then we are done. Observe that the order in which we visited the nodes is $0, 3, 1, 4, 2$.

7.3 Pseudocode Time Complexity

With this modification each vertex is popped exactly once and so the `while` loop iterates V times. The `for` loop iterates $2E$ times and we then we have a total time complexity of $\Theta(V + E)$ once more.