# CMSC 351: InsertionSort

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# 1 What it Does

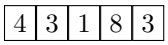
Sorts a list of integers or real numbers.

# 2 How it Works:

We pass through the list from left to right. If we encounter an entry which is smaller than some of its predecessors then we need to move it as far left as is appropriate. We shift as many elements to the right as needed to make room for it and then insert it (hence the name) in the proper position.

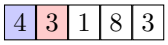
# 3 Visualization

Consider the following list:

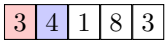



We don't bother to look at `A[0]==4` because nothing to the left could be larger, because there's nothing to the left of it.

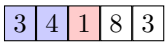
Thus we first look at `A[1]==3` and note that there is one larger element to the left:




We shift it to the right and insert the 3 where it belongs:




We then look at `A[2]==1` and note that there are two larger elements to the left:




We shift those to the right and insert the 1 where it belongs:


We next look at `A[3]==8` and note that there are no larger elements to the left:


We do nothing. Again officially the pseudocode will actually shift nothing and then assign `A[3]=8`, which is a bit wasteful but there we go.

We then look at `A[4]==3` and note that there are two larger elements to the left:


We shift those to the right and insert the 3 where it belongs:


Then we are done.

# 4 Outline of Pseudocode

To help see how the pseudocode work, here is the general outline. Pseudo-pseudocode, if you wish!

```
for i = 1 to n-1
    key = A[i], the value to potentially be moved left.
    j = i - 1, the first index to the left.
        shift A[j] to the right while key < A[j] and j >= 0
        j = j - 1
        this while loop is shifting things right
        to make room for key in the right place
    once this ends either key >= A[j] and should be left alone,
    meaning key should go into index j+1,
    or j = -1, meaning key should go into index 0 = j+1
    A[j+1] = key, so key is now in the right place
end
once this ends, everything is right
```

# 5 Pseudocode

Here is the pseudocode with time assignments. Here I've gone old-school and tagged the `while` loop condition check as it clarifies a bit.

```
\\PRE: A is a list of length n.
for i = 1 to n-1                    n - 1 iterations
    key = A[i]                     }
    j = i-1                        } c_1
    while j >= 0 and key < A[j]    c_2 each time there's a check
        A[j+1] = A[j]              }
        j = j - 1                  } c_3
    end
    A[j+1] = key                   c_4
end
\\POST: A is sorted.
```

Here `key` holds the value at index `i` needs to be stored as we check to the left of it and potentially shift any larger left element to the right. Once we have moved those to the right we insert `key` into its appropriate position.

Notice that the `while` loop checks not just whether the element to the left is larger but also whether that searching process falls off the front of the list. It stops in either case, of course. In the case where it falls off the front of the list we have `j==-1` and we need to insert `key` at index `0==j+1`. In the case where it encounters an element at index `j` which is smaller or equal than `key` and which does not need to be moved, it must insert `key` to the right of it at index `j+1`.

This is why the line after the `while` loop is `A[j+1] = key`.

# 6 Time Complexity Analysis

The time complexity analysis for InsertionSort is very different than BubbleSort and SelectionSort. This is due to the `while` loop which results in an unknown number of iterations. For a particular input of length $n$ this loop could iterate never or perhaps a number of times linearly dependent on $n$. We thus have to separate cases.

1. **Best-Case:**

   In a best-case scenario the `while` loop fails every time, which will happen iff the listed is sorted. In such a case we have:

   $$T(n) = (n-1)(c_1 + c_2 + c_4) = \Theta(n)$$

2. **Worst-Case:**

   In a worst-case scenario the `while` loop passes as many times as possible, which will happen iff the list is in reverse order.

   More specifically the `while` loops will pass for $j = i-1, i-2, ..., 0$ (a total of $i$ times) and fail at $j = -1$.

   In such a case we will have:

   $$T(n) = \sum_{i=1}^{n} [c_1 + (i)(c_2 + c_3) + c_2 + c_4] = ... = \Theta(n^2)$$

3. **Average Case:**

First we'll observe that the running time can be written in terms of the number of *inversions* which exist in the original list. An *inversion* is a pair of indices $(i, j)$ such that $i < j$ and $A[i] > A[j]$.

> **Example 6.1.** In our original list:
>
> | 4 | 3 | 1 | 8 | 3 |
> |---|---|---|---|---|
>
> In this list there are 5 inversions: $(0, 1)$ (because $A[0] > A[1]$), $(0, 2)$ (because $A[0] > A[2]$), $(0, 5)$ (because $A[0] > A[5]$), $(1, 2)$ (because $A[1] > A[2]$), and $(3, 4)$ (because $A[3] > A[4]$).

If we consider the pseudocode carefully we see that each time an element is shifted (to the right) in preparation for an insertion that we are essentially correcting/removing an inversion.

> **Example 6.2.** In our original list of $[4, 3, 1, 8, 3]$ we have five inversions and if we check back we see we shifted five elements.

Moreover since each shift corresponds to the `while` loop passing exactly once this means that the number of times the `while` loop passes is equal to the number of inversions in the list.

In light of this consider the time requirements again:

- For each of `i=1,...,n-1` we have $c_1 + c_4$ before and after the `while` loop.
- For each inversion the `while` loop criteria will pass exactly once, requiring time $c_2 + c_3$.
- For each of `i=1,...,n-1` the `while` loop criteria will fail exactly once, either after shifts have occurred or at the start if no shifts were necessary, requiring time $c_2$.

Thus if there are $I$ inversions then the total time requirement will be as follows, where C+P means "`while` loop checks and passes" and C+F means "`while` loop checks and fails".

$$T(n, I) = \underbrace{(n-1)(c_1 + c_4)}_{\text{Pre+Post-While}} + \underbrace{I(c_2 + c_3)}_{\text{C+P}} + \underbrace{(n-1)(c_2)}_{\text{C+F}}$$

For the sake of simplicity let's assume no duplicates in what follows.

In an arbitrary list if we look at all possible there are $C(n, 2)$ pairs of elements and therefore $C(n, 2)$ possible inversions. For example in a list of length $n = 5$ there are $C(5, 2) = 10$ possible inversions. If the list is sorted there are 0 and if the list is in reverse order there are 10.

If we look at all possible lists, it turns out that on average we will have

$I = I(n) = C(n,2)/2$ inversions and so the average the time requirement will be:

$$T\left(n, I(n)\right) = T\left(n, \frac{C(n,2)}{2}\right) = ... = \Theta(n^2)$$

## 7 Auxiliary Space

The auxiliary space is $\mathcal{O}(1)$.

## 8 Stability

InsertionSort is stable. This is because InsertionSort shifts elements to the left while they are smaller than those elements. Consequently an element will not be shifted to the left of an equal element and so the order of equal elements will be preserved.

## 9 In-Place

InsertionSort is in-place.

## 10 Notes

After $k$ iterations the first $k$ elements are sorted but not necessarily correctly placed. If InsertionSort were running on a very long list and it was forced to stop early this means that some amount of the beginning of the list would be sorted but in the wrong position and the end would not be sorted at all. This is different from both BubbleSort and SeledtionSort.

**Example 10.1.** Consider the list:

`[7,8,9,4,5,6,1,2,3]`

The first six iterations operate on the elements 7, 8, 9, 4, 5, and 6. The first three don't move but they still count as iterations as far as the psedocode is concerned.

The result is then:

`[4,5,6,7,8,9,1,2,3]`

Notice that the first six elements are sorted relative to one another but they are certainly not in their correct overall locations. It will take three more iterations to move the `1,2,3` for this to be the case.

# 11 Thoughts, Problems, Ideas

1. Show the step-by-step operation of InsertionSort on the list $\{5, 10, 3, 0, -6, 9\}$.
2. Assuming all constant times equal 1, fill in the details of the worst-case time calculation:
$$T(n) = \sum_{i=0}^{n-1} c_1 + i(c_2 + c_3) + c_2 + c_4 = ... = \Theta(n^2)$$
3. Assuming all constant times equal 1, fill in the details of the average-case time calculation:
$$T\left(n, I(n)\right) = T\left(n, \frac{C(n,2)}{2}\right) = ... = \Theta(n^2)$$
4. Modify InsertionSort so the list is sorted in decreasing order.
5. The basic premise of InsertionSort is that for each index `i` we effectively shift everything to the left of this index which smaller than the value at this index to the right by 1 and then insert (hence the name) the value in the empty left location. Suppose you had a function `rshift(A,a,b)` which shifted the subarray `A[a,...,b]` to the right by 1. Use this to develop a pseudocode version of InsertionSort. For each `i` this version should scan left until it finds where to put `A[i]`, then shift everything necessary to the right, and then insert. If `rshift` operates with constant time complexity what is the worst-case $\mathcal{O}$ time complexity of your new pseudocode?
6. Another way to think of InsertionSort is that for each index `i` we want to know where to the left to insert it. Note that for each `i` the sublist to the left is always sorted as the algorithm progresses. Call this target index `k`. The series of swaps effectively shifts `A[k,...,i-1]` to the right and inserts the original `A[i]` to index `k`. Suppose you had a function `locateindex(A,i)` which would find the correct index in `A[0,...,i-1]` to insert `A[i]`. Use this to develop a pseudocode version of InsertionSort. You'll still need to do the swapping manually.
7. Suppose you had both `rshift` and `locateindex` from the previous two problems. Now write the new pseudocode. If the time complexity of `locateindex` is $L(n)$ and the time complexity of `rshift` is $R(n)$ what would need to be true of these in order for the resulting search to be $\mathcal{O}$-better than InsertionSort?
8. Arguably the pseudocode is a bit confusing because the search index `j` is actually always one less than the target position. Tweak the pseudocode to change this.
9. We saw that in the average case the pseudocode has time requirement:
$$T(n, I) = n(c_1 + c_4) + I(c_2 + c_3) + n(c_2)$$

where $I$ is the number of inversions in the input list. The reason that the average case turns out to be $\mathcal{O}(n^2)$ is that on average the number of inversions satisfies $I(n) = \mathcal{O}(n^2)$. We'll say a list of length $n$ is *logarithmically sorted* (I made that term up) if the number of inversions is at most $\lceil (\lg n/n)C(n,2) \rceil$. For example a logarithmically sorted list of length 10 will have at most $\lceil (\lg 10/10)C(10,2) \rceil = 12$ inversions. Calculate the $\mathcal{O}$ average time complexity of the pseudocode if it only encounters such lists.

## 12 Python Test

Code:

```
import random
A = []
for i in range(0,10):
    A.append(random.randint(0,100))
n = len(A)
print(A)
for i in range(0,n):
    key = A[i]
    print('Checking ' + str(key) + ' at index ' + str(i))
    j = i-1
    howmanytoleft = 0
    while j>=0 and key<A[j]:
        howmanytoleft = howmanytoleft + 1
        A[j+1] = A[j]
        j = j - 1
    if howmanytoleft == 0:
        print('There are 0 larger elements to the left. Leave alone.')
    else:
        A[j+1] = key
        print('There are ' + str(howmanytoleft) + ' larger elements to the l
        print('Shift and insert: ' + str(A))
print(A)
```

Output:

```
[31, 24, 77, 79, 53, 20, 20, 75, 46, 0]
Checking 31 at index 0
There are 0 larger elements to the left. Leave alone.
Checking 24 at index 1
There are 1 larger elements to the left.
Shift and insert: [24, 31, 77, 79, 53, 20, 20, 75, 46, 0]
Checking 77 at index 2
There are 0 larger elements to the left. Leave alone.
Checking 79 at index 3
There are 0 larger elements to the left. Leave alone.
Checking 53 at index 4
There are 2 larger elements to the left.
Shift and insert: [24, 31, 53, 77, 79, 20, 20, 75, 46, 0]
Checking 20 at index 5
There are 5 larger elements to the left.
Shift and insert: [20, 24, 31, 53, 77, 79, 20, 75, 46, 0]
Checking 20 at index 6
There are 5 larger elements to the left.
Shift and insert: [20, 20, 24, 31, 53, 77, 79, 75, 46, 0]
Checking 75 at index 7
There are 2 larger elements to the left.
Shift and insert: [20, 20, 24, 31, 53, 75, 77, 79, 46, 0]
Checking 46 at index 8
There are 4 larger elements to the left.
Shift and insert: [20, 20, 24, 31, 46, 53, 75, 77, 79, 0]
Checking 0 at index 9
There are 9 larger elements to the left.
Shift and insert: [0, 20, 20, 24, 31, 46, 53, 75, 77, 79]
[0, 20, 20, 24, 31, 46, 53, 75, 77, 79]
```