CMSC 351: Review

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1 Proofs

1.1 Weak Induction

To prove $\forall n \geq n_0\, P(n)$ we first prove $P(n_0)$ (this is the bae caes) and then we prove $\forall k \geq n_0\, P(k) \to P(k+1)$ (this is the inductive step). The assumption of $P(k)$ in the inductive step is the inductive hypothesis.

1.2 Strong Induction

The inductive hypothesis becomes $\forall k \geq n_0\,,\, P(n_0) \wedge P(n_0+1) \wedge ... \wedge P(k) \to P(k+1)$. We often need more than one base case. The quantify we need can be determined by examining how far back we go in the inductive step. If the inductive step refers back to $P(j)$ for $j < k$ then we must have $j \geq n_0$.

1.3 Constructive Induction

Useful when we have an idea (a guess) about a formula but we need to figure out some constants. We verify our guess while simultaneously finding the constants.

Example 1.1. Suppose we suspect that:

$$\sum_{k=1}^{n} k = an^2 + bn$$

To use constructive induction we first note that if this is going to be true for the base case then we need:

$$1 = \sum_{k=1}^{1} k = a(1)^2 + b(1)$$

Thus we know $a + b = 1$ and so $b = 1 - a$. Thus we can now suspect that:

$$\sum_{k=1}^{n} k = an^2 + (1-a)n$$

Then we note that if this is going to be true for the inductive step then we need:

$$\text{If } \sum_{k=1}^{n} k = an^2 + (1-a)n \text{ then } \sum_{k=1}^{n+1} k = a(n+1)^2 + (1-a)(n+1).$$

Well observe that:

$$\sum_{k=1}^{n+1} k = \left[\sum_{k=1}^{n} k\right] + (n+1) = an^2 + (1-a)n + (n+1)$$

Thus we need:

$$
\begin{aligned}
an^2 + (1-a)n + (n+1) &= a(n+1)^2 + (1-a)(n+1) \\
an^2 + n - an + n + 1 &= an^2 + 2an + a + n - an + 1 - a \\
an^2 + (2-a)n + 1 &= an^2 + (a+1)n + 1 \\
(2-a)n &= (a+1)n \\
2 - a &= a + 1 \\
2a &= 1 \\
a &= \frac{1}{2}
\end{aligned}
$$

Thus we have:

$$\sum_{k=1}^{n} k = \frac{1}{2}n^2 + \frac{1}{2}n$$

This is our familiar:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

There are two issues that often get questioned and while they're not issues we'll really worry about it's certainly worth addressing them.

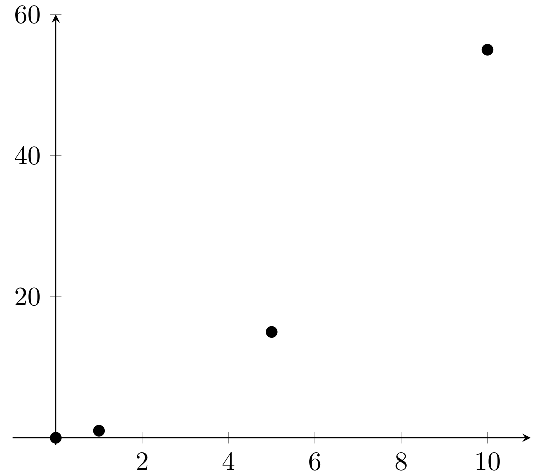
First, where does our guess come from? This is part art and part science and depends highly on the situation. Let's consider our example sum but let's start at 0:

$$\sum_{k=0}^{n} k$$

Suppose we do some tests:

$$
\begin{aligned}
\sum_{k=0}^{0} k &= 0 \\
\sum_{k=0}^{1} k &= 1 \\
\sum_{k=0}^{5} k &= 15 \\
\sum_{k=0}^{10} k &= 55
\end{aligned}
$$

If we plot these we perhaps guess that the result is quadratic with a y-intercept of 0:



Because of this we might guess that:

$$\sum_{k=0}^{n} k = an^2 + bn$$

Second, what happens if we make a wrong guess? Let's see. Suppose we suspect it's cubic instead:

$$\sum_{k=0}^{n} k = an^3 + bn$$

The base case $n = 0$ tells us $0 = 0 + 0$ or nothing.
The case $n = 1$ tells us that $1 = a + b$ so $b = 1 - a$ as before. So far so good.
Thus we can now suspect that:

$$\sum_{k=1}^{n} k = an^3 + (1 - a)n$$

Then we note that if this is going to be true for the inductive step then we need:

$$\text{If } \sum_{k=1}^{n} k = 3n^2 + (1 - a)n \text{ then } \sum_{k=1}^{n+1} k = a(n + 1)^3 + (1 - a)(n + 1).$$

Well observe that:

$$\sum_{k=1}^{n+1} k = \left[\sum_{k=1}^{n} k\right] + (n + 1) = an^3 + (1 - a)n + (n + 1)$$

Thus we need:

$$\begin{aligned} an^3 + (1-a)n + (n+1) &= a(n+1)^3 + (1-a)(n+1) \\ an^3 + (2-a)n + 1 &= a(n^3 + 3n^2 + 3n + 1) + (n - an + 1 - a) \end{aligned}$$

This needs to be true for all $n \geq 0$ which basically means they need to be equal as polynomials with n being the variable. But we have a problem because there is an n^2 on the right but not on the left. Thus we have chosen...poorly.

1.4 Structural Induction

Used when we are trying to prove some property about all items in a set where the set is defined recursively. We prove that the property holds for the base items and then we prove that the recursive addition of new items preserves the property.

Example 1.2. We define a binary tree as following:

- A single node is a binary tree.
- If B_1 and B_2 are binary trees then a single node as parent to B_1 and B_2 is also a binary tree.

Let's show that the number of nodes in a binary tree $N(T)$ satisfies $N(T) \leq 2^{H(T)+1} - 1$ where $H(t)$ is the height.
Base Case: A binary tree T consisting of a single node has $N(t) = 1$ and $H(T) = 0$ and hence satisfies $1 \leq 2^{0+1} - 1$.
Inductive step: Suppose T is constructed by taking a single node as parent to B_1 and B_2 and suppose we have:

$$N(B_1) \leq 2^{H(B_1)+1} - 1 \text{ and } N(B_2) \leq 2^{H(B_2)+1} - 1$$

We claim that:

$$N(T) \leq 2^{H(T)+1} - 1$$

To see this, note that $N(T) = 1 + N(B_1) + N(B_2)$ and $H(T) = 1 + \max\{H(B_1), H(B_2)\}$. From here note that:

$$\begin{aligned} N(T) &= 1 + N(B_1) + N(B_2) \\ &\leq 1 + 2^{H(B_1)+1} - 1 + 2^{H(B_2)+1} - 1 \\ &\leq 2^{\max\{H(B_1),H(B_2)\}+1} + 2^{\max\{H(B_1),H(B_2)\}+1} - 1 \\ &\leq 2\left(2^{\max\{H(B_1),H(B_2)\}+1}\right) - 1 \\ &\leq 2^{\max\{H(B_1),H(B_2)\}+1+1} - 1 \\ &\leq 2^{H(T)+1} - 1 \end{aligned}$$

This is as desired.

2 Combinatorics

2.1 Permutations and Combinations

Basic formulas:

$$
\begin{aligned}
n \text{ objects, permute } k &= \frac{n!}{(n-k)!} \\
n \text{ objects, choose } k &= \frac{n!}{k!(n-k)!} \\
n \text{ categories, permute } k &= n^k
\end{aligned}
$$

2.2 Probability and Expected Value

Suppose X is a random variable which takes on numerical outcomes x_1, ..., x_n with respective probabilities p_1, ..., p_n then the expected value of X is:

$$E(X) = p_1x_1 + ... + p_nx_n$$

Example 2.1. Suppose an algorithm sorts the values in a list and returns the alternating sum/difference of the result. For example if you give it $[5, 8, 4, 1]$ it first sorts to get $[1, 4, 5, 8]$ and then returns $1 - 4 + 5 - 8 = -6$.
If the possible inputs to the algorithm are $[5, 8, 4, 1]$, $[10, 20, 0]$, $[2, 1]$ and $[0, 5, 2, -3]$ all equally likely, what is the expected outcome?
Well there are four outcomes:

$$
\begin{aligned}
[5, 8, 4, 1] &\Rightarrow [1, 4, 5, 8] \Rightarrow 1 - 4 + 5 - 8 = -6 \\
[10, 20, 0] &\Rightarrow [0, 10, 20] \Rightarrow 0 - 10 + 20 = 10 \\
[2, 1] &\Rightarrow [1, 2] \Rightarrow 1 - 2 = -1 \\
[0, 5, 2, -3] &\Rightarrow [-3, 0, 2, 5] \Rightarrow -3 - 0 + 2 - 5 = -6
\end{aligned}
$$

Since all are equally likely they have probabilities 0.25 each and so the expected value is:

$$0.25(-6) + 0.25(10) + 0.25(-1) + 0.25(-6) = ...$$

3 Calculus Thread

3.1 Sequences and Sums

Some basic sums:

$$\sum_{i=1}^{n} 1 = n$$
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$
$$\sum_{i=0}^{n} r^i = \frac{r^{n+1}-1}{r-1}$$
$$\sum_{i=0}^{n} 2^i = 2^{n+1}-1$$
$$\sum_{i=1}^{n} i2^i = (n-1)2^{n+1}+2$$

Note: Most of these should be familiar. the only one that might not be is the final one.

Proof. We have:

$$\begin{aligned}
\sum_{i=1}^{n} i2^i &= 2\left[\sum_{i=1}^{n} i2^i\right] - \left[\sum_{i=1}^{n} i2^i\right] \\
&= \left[\sum_{i=1}^{n} i2^{i+1}\right] - \left[\sum_{i=1}^{n} i2^i\right] \\
&= \quad \left[1 \cdot 2^2 + 2 \cdot 2^3 + ... + (n-1)2^n + n2^{n+1}\right] \\
&\quad - \left[1 \cdot 2^1 + 2 \cdot 2^2 + ... + (n-1)2^{n-1} + n2^n\right] \\
&= n2^{n+1} - 2^n - 2^{n-1} - ... - 2^2 - 2^1 \\
&= n2^{n+1} - (2^n + 2^{n-1} + ... + 2^1) \\
&= n2^{n+1} - (2^{n+1} - 2) \\
&= (n-1)2^{n+1} + 2
\end{aligned}$$

$\mathcal{QED}$

These can be used in various ways:

Example 3.1. Consider the sum:

$$\sum_{i=2}^{n} \left(2i + 2^{-i} + i^2\right)$$

We split it up:

$$\sum_{i=2}^{n} \left(2i + 2^{-i} + i^2\right) = \sum_{i=2}^{n} 2i + \sum_{i=2}^{n} 2^{-i} + \sum_{i=2}^{n} i^2$$

Separately these are:

$$\sum_{i=2}^{n} 2i = 2\sum_{i=2}^{n} i = 2\left[\left[\sum_{i=1}^{n} i\right] - 1\right] = 2\left[\frac{n(n+1)}{2} - 1\right]$$

$$\sum_{i=2}^{n} 2^{-i} = \sum_{i=2}^{n} \left(\frac{1}{2}\right)^{i} = \left[\sum_{i=0}^{n} \left(\frac{1}{2}\right)^{i}\right] - 1 - \frac{1}{2} = \left[\frac{\left(\frac{1}{2}\right)^{n+1} - 1}{\frac{1}{2} - 1}\right] - \frac{1}{2}$$

$$\sum_{i=2}^{n} i^2 = \left[\sum_{i=1}^{n} i^2\right] - 1 = \left[\frac{n(n+1)(2n+1)}{6}\right] - 1$$

The result is then the sum of these:

$$2\left[\frac{n(n+1)}{2} - 1\right] + \left[\frac{\left(\frac{1}{2}\right)^{n+1} - 1}{\frac{1}{2} - 1}\right] - \frac{1}{2} + \left[\frac{n(n+1)(2n+1)}{6}\right] - 1$$

3.2 L'hôpital's Rule

It will be useful to remember two versions of L'hôpital's Rule:

Theorem 3.2.1. Suppose we are attempting to evaluate:

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}$$

- If $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$ then:

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{f'(x)}{g'(x)}$$

- If $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$ then:

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{f'(x)}{g'(x)}$$

Note 3.2.1. The theorem is valid for sequences as well as functions, we just treat f and g as continuous functions of n.

Example 3.2. We have:

$$\lim_{n\to\infty}\frac{n}{5n+1} = \lim_{n\to\infty}\frac{1}{5} = \frac{1}{5}$$

Oftentimes we'll need to use it repeatedly.

Example 3.3. We use it five times in a row here:

$$\lim_{n\to\infty}\frac{2^n}{n^5} = \lim_{n\to\infty}\frac{(\ln 2)^5 2^n}{(5)(4)(3)(2)(1)} = \infty$$

3.3 Manipulation of Logarithms

$$\begin{aligned}
\log_b a &= \frac{\log_c a}{\log_c b} \text{ (Change of Base)} \\
\log_b(xy) &= \log_b x + \log_b y \\
\log_b \left(\frac{x}{y}\right) &= \log_b x - \log_b y \\
\log_b \left(x^p\right) &= p \log_b x
\end{aligned}$$

Note 3.3.1. We use the Change of Base all the time when we play fast and loose with big notation and logarithms. For example:

$$\log n = \tfrac{1}{\lg 10} \lg n = \Theta(\lg n) \text{ and } \lg n = \tfrac{1}{\log 10} \log n = \Theta(\log n)$$

Which gives an example of why logarithms don't matter for big notation.

3.4 Differentiation

Some basic rules:

$$
\begin{aligned}
\frac{d}{dx}x^r &= rx^{r-1} \text{ for } r \neq 0 \\
\frac{d}{dx}\ln x &= \frac{1}{x} \\
\frac{d}{dx}a^x &= (\ln a)a^x \\
\frac{d}{dx}f(x)g(x) &= f'(x)g(x) + f(x)g'(x)
\end{aligned}
$$

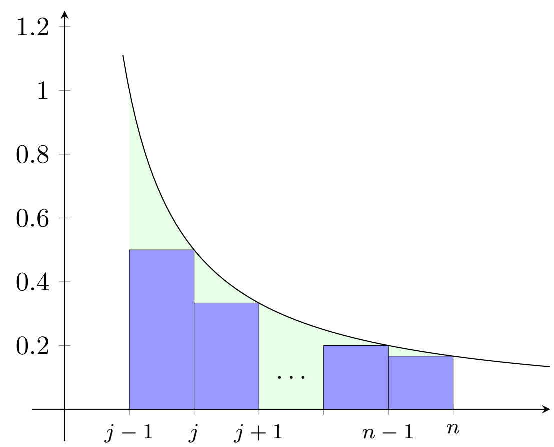
3.5 Integration

Some basic rules:

$$
\begin{aligned}
\int x^r &= \frac{1}{r+1}x^{r+1} + C \quad \text{for } r \neq -1 \\
\int x^{-1} &= \ln x + C \\
\int u\,dv &= uv - \int v\,du \text{ (Integration by Parts)}
\end{aligned}
$$

3.6 Integral Bounds for Sums

Consider the following picture of a decreasing function $f(x)$ and some rectangles below it:



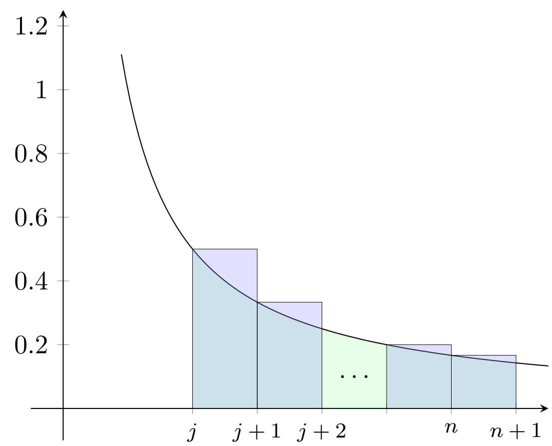
It's clear from this picture that the sum of the areas of the rectangles is smaller than the area under the curve from $j-1$ to n. If we define $a_i = f(i)$ then the sum of the areas of the rectangles (all have width 1) is:

$$(1)f(j) + (1)f(j+1) + ... + (1)f(n) = a_j + a_{j+1} + ... + a_n$$

which is clearly smaller than the area under the curve between $x = j-1$ and $x = n$.
Thus we have:

$$\sum_{i=j}^{n} a_i \leq \int_{j-1}^{n} f(x)\,dx$$

Likewise consider this picture with the rectangles all shifted to the right and the same function.



The sum of the areas of the rectangles (all have width 1) doesn't change, it still is:

$$(1)f(j) + (1)f(j+1) + ... + (1)f(n) = a_j + a_{j+1} + ... + a_n$$

which is clearly greater than the area under the curve between $x = j$ and $x = n + 1$.
Thus we have:

$$\int_{i=j}^{n+1} f(x)\,dx \leq \sum_{i=j}^{n} a_i$$

Together we have:

Theorem 3.6.1. As a general rule if a_i (and its corresponding $f(x)$ having $f(i) = a_i$) are decreasing then:

$$\int_{j}^{n+1} f(x)\,dx \;\leq\; \sum_{i=j}^{n} a_i \;\leq\; \int_{j-1}^{n} f(x)\,dx$$

Example 3.4. Suppose we want to get integral-related bounds for:

$$\sum_{i=3}^{100} \frac{1}{i}$$

Observe that we have:

$$\int_{3}^{101} \frac{1}{x}\,dx \leq \sum_{i=3}^{100} \frac{1}{i} \leq \int_{2}^{100} \frac{1}{x}\,dx$$

Calculating the left and right sides yields:

$$3.5166 \approx \ln(101) - \ln(3) \leq \sum_{i=2}^{100} \frac{1}{i} \leq \ln(100) - \ln(2) \approx 3.9120$$

Just for reference note that:

$$\sum_{i=3}^{100} \frac{1}{i} = 3.68737751...$$

Warning: Note that the sum is from $i = j$ to $i = n$ but one of the integrals goes from $j - 1$ and the other goes to $n + 1$. It's entirely possible that the function $f(x)$ is undefined at either $j-1$ or $n+1$ or both in which case we need to tweak a bit.

Example 3.5. Suppose we wanted an upper bound for:

$$\sum_{i=1}^{20} \frac{1}{i^2}$$

It might be tempting to simply use the right-hand inequality:

$$\sum_{i=1}^{20} \frac{1}{i^2} \leq \int_0^{20} \frac{1}{x^2}\, dx$$

but the function is undefined at $x = 0$. While the integral may stil be (via improper integrals) it may not be and in any case is more work than we need. The approach is to simply separate out the first term of the sequence first:

$$\sum_{i=1}^{20} \frac{1}{i^2} = \frac{1}{1^2} + \sum_{i=2}^{20} \frac{1}{i^2} \leq 1 + \int_1^{20} \frac{1}{x^2}\, dx$$

Theorem 3.6.2. If a_n (and its corresponding $f(x)$) are increasing then what would the inequalities look like?

4 Thoughts, Problems, Ideas

1. Prove using weak induction that:

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

2. Verify and find the corresponding constants using structural induction for:

$$\sum_{i=0}^{n} 3^i = \alpha 3^n + \beta$$

3. Define a set of graphs S by: An single node is in S and if $G \in S$ then the result of adding an edge and a new node to a node already in S is in S. Prove that $V = E + 1$.

4. How many possible comparisons of the form $x < y$ are there if $x, y \in \{a, b, c, d\}$?

5. How many possible comparisons of the form $x - y < z$ are there if $x, y, z \in \{a, b, c, d, e, f, g\}$?

6. How many possible comparisons of the form $x + y < z$ are there if $x, y, z \in \{a, b, c, d, e, f, g\}$ if we assume $x + y$ and $y + x$ are equivalent?

7. Suppose an algorithm adds any set of numbers given to it. If the input to the algorithm could be one of three sets, either $\{1, 2, 3\}$, $\{4, 5, 1\}$, and $\{0, 2, 3, 10\}$, with probabilitites 0.5, 0.3 and 0.2 respectively, what is the expected value of the output of the algorithm?

8. Suppose algorithm A can accept any nonnegative integer and finds the square root of its input. If the probabilities of the input being $0, 1, 2, 3, \ldots$ are $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$ respectively what can you say about the expected value of the output of the algorithm?

9. Calculate the sum:

$$\sum_{i=2}^{2n} (3i - 1)^2$$

10. Calculate the sum:

$$\sum_{i=3}^{100} (0.2)^{5i}$$

11. Suppose b is an unknown base and you know $\log_b 3 = \alpha$ and $\log_b 4 = \beta$. Calculate each of:

$$\log_b 36, \, \lg 3, \, \log_8 \tfrac{81}{4}$$

12. Calculate the derivative:

$$\frac{d}{dx} x^2 + 3xe^{2x}$$

13. Calculate the integral:

$$\int_0^1 xe^x \, dx$$

14. Calculate the integral:

$$\int_2^4 \ln x \, dx$$

Hint: Use IBP with $u = \ln x$ and $dv = 1\, dx$.

15. Calculate the integral:

$$\int_2^4 x \ln x \, dx$$

16. Find integral-related bounds for:

$$\sum_{i=1}^{20} \frac{1}{i}$$

17. Find integral-related bounds for:

$$\sum_{i=1}^{20} \frac{1}{i+3}$$

18. Find integral-related bounds for:

$$\sum_{i=1}^{100} \frac{1}{i^3}$$

19. Find integral-related bounds for:

$$\sum_{i=1}^{10} \frac{1}{\sqrt{i}}$$

20. Calculate and write down the inequality pair that corresponds to the theorem for increasing functions. Show work, including pictures as necessary.

21. Find integral-related bounds for:

$$\sum_{i=1}^{5} i + \sqrt{i}$$

22. Find integral-related bounds for:

$$\sum_{i=1}^{5} \ln(i)$$