Math 241 Fall 2018 Final Exam Solution

1. Parts (a) and (b) are independent.

   (a) Use the dot product to show that the three points $A = (-2, 3, -4), B = (0, 11, -1)$, and $C = (1, 10, -5)$ form a right triangle. Then find the length of the hypotenuse. [10 pts]
   Solution: We have:

$$\overline{AB} = 2\,\mathbf{i} + 8\,\mathbf{j} + 3\,\mathbf{k}$$
$$\overline{AC} = 3\,\mathbf{i} + 7\,\mathbf{j} - 1\,\mathbf{k}$$
$$\overline{BC} = 1\,\mathbf{i} - 1\,\mathbf{j} - 4\,\mathbf{k}$$

   And so:

$$\overline{AB} \cdot \overline{AC} \neq 0$$
$$\overline{AB} \cdot \overline{BC} \neq 0$$
$$\overline{AC} \cdot \overline{BC} = 0$$

   So we have a right triangle. The length of the hypotenuse is then

$$|AB| = \sqrt{2^2 + 8^2 + 3^2}$$

   (b) Find the distance between the point $Q = (-2, 1, 3)$ and the line $x = 2$, $\frac{y+3}{-2} = z - 4$. [10 pts]
   Solution: The point $P = (2, -3, 4)$ is on the line and $\mathbf{L} = 0\,\mathbf{i} - 2\,\mathbf{j} + 1\,\mathbf{k}$. We have $\overline{PQ} = -4\,\mathbf{i} + 4\,\mathbf{j} - 1\,\mathbf{k}$ so that

$$\begin{aligned} \text{dist} &= \frac{||\overline{PQ} \times \mathbf{L}||}{||\mathbf{L}||} \\ &= \frac{||2\,\mathbf{i} + 4\,\mathbf{j} + 8\,\mathbf{k}||}{||0\,\mathbf{i} - 2\,\mathbf{j} + 1\,\mathbf{k}||} \\ &= \frac{\sqrt{4 + 16 + 64}}{\sqrt{0 + 4 + 1}} \end{aligned}$$

2. Parts (a) and (b) are independent.

(a) An 80 pound force and a 50 pound force are applied to an object at the same point with an angle of $\frac{\pi}{6}$ between them. Find the magnitude of the resultant force on the object. [10 pts]
Solution: If we put the object at the origin and the 50lb force on the positive x axis and the 80lb force in the first quadrant then $\mathbf{F}_1 = 50\,\mathbf{i} + 0\,\mathbf{j}$ and $\mathbf{F}_2 = 80\cos(pi/6)\,\mathbf{i} + 80\sin(\pi/6)\,\mathbf{j} = 40\sqrt{3}\,\mathbf{i} + 40\,\mathbf{j}$ and so

$$||\mathbf{F}_1 + \mathbf{F}_2|| = \sqrt{(50 + 40\sqrt{3})^2 + (40)^2}$$

(b) Show that the two lines $\mathbf{r}_1(t) = (t+1)\,\mathbf{i} + 2t\,\mathbf{j} + 3\,\mathbf{k}$ and $\mathbf{r}_2(s) = s\,\mathbf{i} + (4-s)\,\mathbf{j} + (s+1)\,\mathbf{k}$ intersect and are not parallel, then find the equation of the plane containing them. [10 pts]
Solution: The lines meet when $t + 1 = s$, $2t = 4 - s$ and $3 = s + 1$. The last gives $s = 2$ and so $t = 1$.

Noting $\mathbf{r}_1(1) = 2\,\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$ they meet at $(2, 2, 3)$.

The vectors are $\mathbf{L}_1 = 1\,\mathbf{i} + 2\,\mathbf{j} + 0\,\mathbf{k}$ and $\mathbf{L}_2 = 1\,\mathbf{i} - 1\,\mathbf{j} + 1\,\mathbf{k}$ and these are not multiples so the lines are not parallel.

We use $\mathbf{n} = \mathbf{L}_1 \times \mathbf{L}_2 = 2\,\mathbf{i} - 1\,\mathbf{j} - 3\,\mathbf{k}$ and so the plane equation is

$$2(x-2) - 1(y-2) - 3(z-3) = 0$$

3. Consider the curve C parametrized by $\mathbf{r}(t) = e^t \cos t\,\mathbf{i} + e^t \sin t\,\mathbf{j}$.

(a) Find the tangent vector $\mathbf{T}(t)$. Simplify. [10 pts]
Solution: We have:

$$\begin{aligned}
\mathbf{r}'(t) &= (e^t \cos t - e^t \sin t)\,\mathbf{i} + (e^t \sin t + e^t \cos t)\,\mathbf{j} \\
||\mathbf{r}'(t)|| &= \sqrt{e^{2t}\cos^2 t - 2e^{2t}\sin t\cos t + e^{2t}\sin^2 t + e^{2t}\cos^2 t + 2e^{2t}\sin t\cos t + e^{2t}\sin^2 t} \\
&= \sqrt{2e^{2t}} \\
&= e^t\sqrt{2}
\end{aligned}$$

and so

$$\mathbf{T}(t) = \frac{1}{\sqrt{2}}(\cos t - \sin t)\,\mathbf{i} + \frac{1}{\sqrt{2}}(\sin t + \cos t)\,\mathbf{j}$$

(b) Find the normal vector $\mathbf{N}(t)$. [10 pts]
Solution: We have:

$$\begin{aligned}
\mathbf{T}'(t) &= -\frac{1}{\sqrt{2}}(-\sin t - \cos t)\,\mathbf{i} + \frac{1}{\sqrt{2}}(\cos t - \sin t)\,\mathbf{j} \\
||\mathbf{T}'(t)|| &= \sqrt{\frac{1}{2}(\sin^2 t + 2\sin t\cos t + \cos^2 t) + \frac{1}{2}(\cos^2 t - 2\sin t\cos t + \sin^2 t)} \\
&= 1
\end{aligned}$$

and so

$$\mathbf{N}(t) = -\frac{1}{\sqrt{2}}(-\sin t - \cos t)\,\mathbf{i} + \frac{1}{\sqrt{2}}(\cos t - \sin t)\,\mathbf{j}$$

4. Let C be the curve defined as the portion of the parabola $y = x^2$ in the plane $z = -2$ between the points $(2, 4, -2)$ and $(3, 9, -2)$.

   (a) Find a parametrization of C. [10 pts]
   Solution: We have $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} - 2\,\mathbf{k}$ for $2 \le t \le 3$.

   (b) Set up the iterated integral that computes the length of C. **Do not evaluate the integral!** [10 pts]
   Solution: We have

$$\begin{aligned} \mathbf{r}'(t) &= 1\,\mathbf{i} + 2t\,\mathbf{j} + 0\,\mathbf{k} \\ ||\mathbf{r}'(t)|| &= \sqrt{1 + 4t^2} \\ \text{Length} &= \int_2^3 \sqrt{1 + 4t^2}\,dt \end{aligned}$$

5. Use Lagrange Multipliers to find the extreme values of $f(x, y) = 3x - y$ subject to the constraint $x^2 + 2y^2 = 1$. You may assume these values exist. [20 pts]

Solution: We set $g(x, y) = x^2 + 2y^2$ and then solve the system

$$\begin{aligned} 3 &= \lambda 2x \\ -1 &= \lambda 4y \\ x^2 + 2y^2 &= 1 \end{aligned}$$

We can't have $x = 0$ or $y = 0$ since these would contradict the first and second, therefore the first tells us $\lambda = \frac{3}{2x}$ and the second tells us $\lambda = -\frac{1}{4y}$. Thus

$$\begin{aligned} \frac{3}{2x} &= -\frac{1}{4y} \\ 12y &= -2x \\ x &= -6y \end{aligned}$$

Plugging this into the third tells us

$$\begin{aligned} 36y^2 + 2y^2 &= 1 \\ y &= \pm 1/\sqrt{38} \end{aligned}$$

This yields the points $(-6/\sqrt{38}, 1/\sqrt{38})$ and $(6/\sqrt{38}, -1/\sqrt{38})$. Then:

$$\begin{aligned} f(-6/\sqrt{38}, 1/\sqrt{38}) &= -19/\sqrt{38} \text{ Min} \\ f(6/\sqrt{38}, -1/\sqrt{38}) &= 19/\sqrt{38} \text{ Max} \end{aligned}$$

6. If $f(x,y) = x^2y + xy + y^3$ use tangent plane approximation at $(1,2)$ to approximate $f(0.95, 2.1)$. [20 pts]
Solution: We have

$$\begin{aligned} f_x &= 2xy + y \\ f_y &= x^2 + x + 3y^2 \end{aligned}$$

and so

$$\begin{aligned} f(1,2) &= 12 \\ f_x(1,2) &= 6 \\ f_y(1,2) &= 14 \end{aligned}$$

Therefore

$$\begin{aligned} f(0.95, 2.1) &\approx f(1,2) + f_x(1,2)(0.95 - 1) + f_y(1,2)(2.1 - 2) \\ &\approx 12 + 6(0.95 - 1) + 14(2.1 - 2) \end{aligned}$$

7. Let D be the solid that lies inside the sphere $x^2+y^2+z^2=2$ and outside the cylinder $x^2+y^2=1$.

(a) Set up an iterated triple integral in spherical coordinates that evaluates the volume of D. [10 pts]
Do not evaluate the integral!
Solution: In spherical the equations are $\rho = \sqrt{2}$ and $\rho^2 \sin^2\phi = 1$ (or $\rho \sin\phi = 1$ or $\rho = \csc\phi$) respectively. These meet when $\sqrt{2}\sin\phi = 1$ or $\phi = \frac{\pi}{4}, \frac{3\pi}{4}$. Therefore the volume is given by

$$\int_0^{2\pi} \int_{\pi/4}^{3\pi/4} \int_{\csc\phi}^{\sqrt{2}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

(b) Set up an iterated triple integral in cylindrical coordinates that evaluates $\iiint_D x^2 \, dV$. [10 pts]
Do not evaluate the integral!
Solution: In cylindrical the equations of the sphere yields $z = \pm\sqrt{2-r^2}$ and so:

$$\iiint_D x^2 \, dV = \int_0^{2\pi} \int_1^{\sqrt{2}} \int_{-\sqrt{2-r^2}}^{+\sqrt{2-r^2}} (r\cos\theta)^2 r \, dz \, dr \, d\theta$$

8. Let R be the region enclosed by the ellipse given by $9x^2 + y^2 = 9$. Using an appropriate change of variables, evaluate $\iint\limits_R x^2\, dA$. Make sure you specify the change of variables, and draw the new region. **Evaluate the integral!** [20 pts]

Solution: Set $u = 3x$ and $v = y$ and then the ellipse becomes $u^2 + v^2 = 9$. We have $x = \frac{1}{3}u$ and $y = v$ and so the Jacobian of the change of variables is

$$\begin{vmatrix} 1/3 & 0 \\ 0 & 1 \end{vmatrix} = 1/3$$

and so:

$$\iint\limits_R x^2\, dA = \iint\limits_S \frac{1}{9}u^2|1/3|\, dA$$

We rewrite this in polar:

$$\begin{aligned}
\iint\limits_S \frac{1}{9}u^2|1/3|\, dA &= \frac{1}{27}\int_0^{2\pi}\int_0^3 r^2\cos^2\theta\, r\, dr\, d\theta \\
&= \frac{1}{27}\int_0^{2\pi} \frac{1}{4}r^4\cos^2\theta\Big|_0^3\, d\theta \\
&= \frac{3}{4}\int_0^{2\pi} \cos^2\theta\, d\theta \\
&= \frac{3}{4}\int_0^{2\pi} \frac{1}{2}(1+\cos(2\theta))\theta\, d\theta \\
&= \frac{3}{8}(\theta + \frac{1}{2}\sin(2\theta))\Big|_0^{2\pi} \\
&= \frac{3}{8}(2\pi)
\end{aligned}$$

9. Let Σ be the portion of the plane $z = 9 - x$ inside the cylinder $r = 2\cos\theta$. Let R be the edge of Σ with counterclockwise orientation when viewed from above. Use Stokes' Theorem to rewrite the integral $\int_C 2x\,dx + yz\,dy + x^2z^2\,dz$ as a surface integral, parametrize the surface and then proceed until you have an iterated double integral. **Do not evaluate the integral!** [20 pts]

Solution: Stokes' Theorem tells us that

$$\int_C 2x\,dx + yz\,dy + x^2z^2\,dz = \iint_\Sigma \left[(0-y)\,\mathbf{i} - (2xz^2 - 0)\,\mathbf{j} + (0-0)\,\mathbf{k}\right] \cdot \mathbf{n}\,dS$$

where Σ is the part of the plane inside the cylinder with upwards orientation. We parametrize Σ as:

$$\mathbf{r}(r,\theta) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} + (9 - r\cos\theta)\,\mathbf{k}$$
$$-\tfrac{\pi}{2} \le \theta \le \tfrac{\pi}{2}$$
$$0 \le r \le 2\cos\theta$$

Then

$$\begin{aligned}
\mathbf{r}_r &= \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} - \cos\theta\,\mathbf{k} \\
\mathbf{r}_\theta &= -r\sin\theta\,\mathbf{i} + r\cos\theta\,\mathbf{j} + r\sin\theta\,\mathbf{k} \\
\mathbf{r}_r \times \mathbf{r}_\theta &= r\,\mathbf{i} - 0\,\mathbf{j} + r\,\mathbf{k}
\end{aligned}$$

Since this matches Σ's orientation the integral becomes
$\iint_\Sigma \left[(0-y)\,\mathbf{i} - (2xz^2 - 0)\,\mathbf{j} + (0-0)\,\mathbf{k}\right] \cdot \mathbf{n}\,dS$

$$\begin{aligned}
&= +\iint_R \left[-r\sin\theta\,\mathbf{i} - 2(r\cos\theta)(9 - r\cos\theta)^2\,\mathbf{j} + 0\,\mathbf{k}\right] \cdot \left[r\,\mathbf{i} - 0\,\mathbf{j} + r\,\mathbf{k}\right]\,dA \\
&= \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} -r^2\sin\theta\,dr\,d\theta
\end{aligned}$$

10. Parts (a) and (b) are independent.

(a) Use Green's Theorem to evaluate $\int_C xy\,dx + x\,dy$ where C is the triangle with vertices $(0,0)$, $(3,0)$, and $(3,6)$, oriented clockwise. **Evaluate the integral!** [10 pts]

Solution: If R is the region inside the triangle then because of the orientation we have

$$\begin{aligned}
\int_c xy\,dx + x\,dy &= -\iint_R 1 - x\,dA \\
&= -\int_0^3 \int_0^{2x} 1 - x\,dy\,dx \\
&= -\int_0^3 y - xy\Big|_0^{2x} dx \\
&= -\int_0^3 2x - x(2x)\,dx \\
&= -\int_0^3 x^2 - \frac{2}{3}x^3\Big|_0^3 \\
&= -(3^2 - \frac{2}{3}(3)^3)
\end{aligned}$$

(b) Let Σ be the portion of the paraboloid $z = 16 - x^2 - y^2$ restricted by $0 \le x \le 2$ and $0 \le y \le 3$. Write down an iterated double integral for the surface area of Σ. **Do not evaluate the integral!** [10 pts]

Solution: The surface is parametrized by

$$\begin{gathered}
\mathbf{r}(x,y) = x\,\mathbf{i} + y\,\mathbf{j} + (16 - x^2 - y^2)\,\mathbf{k} \\
0 \le x \le 2 \\
0 \le y \le 3
\end{gathered}$$

So we have

$$\begin{aligned}
\mathbf{r}_x &= 1\,\mathbf{i} + 0\,\mathbf{j} - 2x\,\mathbf{k} \\
\mathbf{r}_y &= 0\,\mathbf{i} + 1\,\mathbf{j} - 2y\,\mathbf{k} \\
\mathbf{r}_x \times \mathbf{r}_y &= 2x\,\mathbf{i} + 2y\,\mathbf{j} + 1\,\mathbf{k} \\
||\mathbf{r}_x \times \mathbf{r}_y|| &= \sqrt{4x^2 + 4y^2 + 1}
\end{aligned}$$

and so the surface area is

$$\begin{aligned}
\iint_\Sigma 1\,dS &= \iint_R \sqrt{4x^2 + 4y^2 + 1}\,dA \\
&= \int_0^2 \int_0^3 \sqrt{4x^2 + 4y^2 + 1}\,dy\,dx
\end{aligned}$$