MATH 241 Sections 03** Final Exam

Exam Submission:

1. Submit this exam to Gradescope.
2. Tag your problems!
3. You may print the exam, write on it, scan and upload.
4. Or you may just write on it on a tablet and upload.
5. Or you are welcome to write the answers on a separate piece of paper if other options don't appeal to you, then scan and upload.

Exam Rules:

1. You may ask me for clarification on questions but you may not ask me for help on questions!
2. You are permitted to use any non-interactive resources. This includes books, static pages on the internet, your notes, and YouTube videos.
3. You are not permitted to use any interactive resources. This includes your friends, your friends' friends, your calculator, Matlab, Wolfram Alpha, and online chat groups. Exception: Calculators are fine for basic arithmetic.
4. If you are unsure about whether a resource is considered "interactive" simply ask me and I'll let you (and everyone) know.
5. Petting small animals for stress relief is acceptable and is not considered an "interactive resource".

Work Shown:

1. Show all work as appropriate for and using techniques learned in this course.
2. Any pictures, work and scribbles which are legible and relevant will be considered for partial credit.

1. Consider the following three items:

$\mathcal{L}$: The line with symmetric equation $\frac{x-1}{2} = 3 - y = \frac{z}{8}$
$\mathcal{P}$: The plane with equation $x - 2y + 5z = 61$
$\mathcal{S}$: The sphere with equation $x^2 + (y-2)^2 + (z+1)^2 = 1$

(a) Determine where $\mathcal{L}$ meets $\mathcal{P}$. [10pts]

Solution:

If we rewrite the line as:

$$\begin{aligned} x &= 2t + 1 \\ y &= 3 - t \\ z &= 8t \end{aligned}$$

Then the line meets the plane when:

$$\begin{aligned} x - 2y + 5z &= 61 \\ 2t + 1 - 2(3 - t) + 5(8t) &= 61 \\ 2t + 1 - 6 + 2t + 40t &= 61 \\ 44t &= 66 \\ t &= \frac{3}{2} \end{aligned}$$

This then yields the point:

$$(2t + 1, 3 - t, 8t) = \left(4, \frac{3}{2}, 12\right)$$

(b) It turns out that $\mathcal{L}$ does not meet $\mathcal{S}$. You do not need to prove this. How close is $\mathcal{L}$ to the surface of $\mathcal{S}$? [15pts]

Solution:

If we assign, from the line, $P = (1, 3, 0)$ and $\bar{L} = 2\,\hat{\imath} - 1\,\hat{\jmath} + 8\,\hat{k}$, and we assign $Q = (0, 2, -1)$. Then the distance from the line to the center of the sphere is:

$$\begin{aligned} \frac{||\overline{PQ} \times \bar{L}||}{||\bar{L}||} &= \frac{||(-1\,\hat{\imath} - 1\,\hat{\jmath} - 1\,\hat{k}) \times (2\,\hat{\imath} - 1\,\hat{\jmath} + 8\,\hat{k})||}{||2\,\hat{\imath} - 1\,\hat{\jmath} + 8\,\hat{k}||} \\ &= \frac{||-9\,\hat{\imath} + 6\,\hat{\jmath} + 3\,\hat{k}||}{||2\,\hat{\imath} - 1\,\hat{\jmath} + 8\,\hat{k}||} \\ &= \frac{\sqrt{126}}{\sqrt{69}} \end{aligned}$$

Thus the distance to the surface of the sphere is:

$$\frac{\sqrt{126}}{\sqrt{69}} - 1 = \sqrt{\frac{42}{23}} - 1$$

Note: There are several other correct and acceptable ways to approach this problem.

2. Consider the two curves C_1 and C_2 parameterized by:

$$\begin{aligned} C_1: &\quad \bar{r}_1(t) = t^2\,\hat{\imath} + (t-2)\,\hat{\jmath} + 5\,\hat{k} \\ C_2: &\quad \bar{r}_2(t) = e^{t-1}\,\hat{\imath} + \cos(\pi t)\,\hat{\jmath} + 5t\,\hat{k} \end{aligned}$$

(a) Show that the curves meet at $t = 1$ and do not meet at any other t-values. [5pts]

Solution: To meet, they must be equal. Specifically the $\hat{k}$-components must be the same so $5 = 5t$ and so $t = 1$.

They do in fact meet at $t = 1$ since $\bar{r}_1(t) = 1\,\hat{\imath} - 1\,\hat{\jmath} + 5\,\hat{k}$ and $\bar{r}_2(t) = 1\,\hat{\imath} - 1\,\hat{\jmath} + 5\,\hat{k}$.

(b) Find the cosine of the angle between their velocity vectors at $t = 1$. [10pts]

Solution: We have:

$$\begin{aligned} \bar{r}_1'(t) &= 2t\,\hat{\imath} + 1\,\hat{\jmath} + 0\,\hat{k} \\ \bar{r}_1'(1) &= 2\,\hat{\imath} + 1\,\hat{\jmath} + 0\,\hat{k} \end{aligned}$$

and

$$\begin{aligned} \bar{r}_2'(t) &= e^{t-1}\,\hat{\imath} - \pi\sin(\pi t)\,\hat{\jmath} + 5\,\hat{k} \\ \bar{r}_2'(1) &= 1\,\hat{\imath} + 0\,\hat{\jmath} + 5\,\hat{k} \end{aligned}$$

Therefore we have:

$$\begin{aligned} \cos\theta &= \frac{\bar{r}_1'(1)\cdot\bar{r}_2'(1)}{||\bar{r}_1'(1)||\,||\bar{r}_2'(1)||} \\ &= \frac{2}{\sqrt{5}\sqrt{26}} = \frac{2}{\sqrt{130}} \end{aligned}$$

(c) Which of them curves is longer between $t = 1$ and $t = 2$? [10pts]

Note: There are two integrals involved and you need to explain why one is larger than the other. You don't need to actually integrate either, however.

Solution: The curve C_1 travels a distance of:

$$\int_1^2 \sqrt{4t^2+1}\,dt \le \int_1^2 \sqrt{4(2)^2+1}\,dt = \int_1^2 \sqrt{17}\,dt = \sqrt{17}$$

The curve C_2 travels a distance of:

$$\int_1^2 \sqrt{e^{2t-2} + \pi^2\sin^2(\pi t) + 25}\,dt \ge \int_1^2 \sqrt{0+0+25}\,dt = \sqrt{25}$$

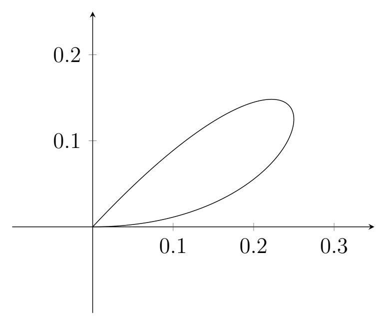
Thus the second is longer.

Note: It is insufficient to simply say the magnitude was bigger. It is insufficient to look at the distance between endpoints since that just gives a lower bound on the length of either curve. It is insufficient to use any form of approximation.

3. Suppose C is the closed curve parameterized by: [25pts]

$$\bar{r}(t) = (t - t^2)\,\hat{\imath} + (t^2 - t^3)\,\hat{\jmath} \text{ for } 0 \le t \le 1$$

This curve looks like this and is drawn counterclockwise:



Green's Theorem tells us that for a region R with counterclockwise edge C we have $\int_C (M\,\hat{\imath} + N\,\hat{\jmath}) \cdot d\bar{r} = \iint_R N_x - M_y\,dA$. Thus specifically we have:

$$\int_C (0\,\hat{\imath} + x\,\hat{\jmath}) \cdot d\bar{r} = \iint_R 1\,dA = \text{Area of } R$$

Use this line integral to calculate the area of the region R inside the curve.

Solution: We have:

$$\bar{r}'(t) = (1 - 2t)\,\hat{\imath} + (2t - 3t^2)\,\hat{\jmath}$$

Then we have:

$$\begin{aligned}
\text{Area} &= \iint_R 1\,dA \\
&= \int_C (0\,\hat{\imath} + x\,\hat{\jmath}) \cdot d\bar{r} \\
&= \int_0^1 (0\,\hat{\imath} + (t - t^2)\,\hat{\jmath}) \cdot ((1 - 2t)\,\hat{\imath} + (2t - 3t^2)\,\hat{\jmath})\,dt \\
&= \int_0^1 (t - t^2)(2t - 3t^2)\,dt \\
&= \int_0^1 2t^2 - 3t^3 - 2t^3 + 3t^4\,dt \\
&= \int_0^1 2t^2 - 5t^3 + 3t^4\,dt \\
&= \frac{2}{3}t^3 - \frac{5}{4}t^4 + \frac{3}{5}t^5 \Big|_0^1 \\
&= \frac{2}{3} - \frac{5}{4} + \frac{3}{5} \\
&= \frac{1}{60}
\end{aligned}$$

4. Suppose C is parameterized by: [25pts]

$$\bar{r}(t) = 4\cos t\,\hat{\imath} + (6 - 4\sin t)\,\hat{\jmath} + 4\sin t\,\hat{k} \text{ for } 0 \le t \le 2\pi$$

Using an appropriate surface apply Stokes' theorem to the following integral. Proceed until you have an iterated double integral and then stop. **Do Not Evaluate Your Final Iterated Integral!**

$$\int_C (xy\,\hat{\imath} - z^2\,\hat{\jmath} + xz\,\hat{k}\,) \cdot d\bar{r}$$

Solution:

The only reasonable surface is the part of the plane $y = 6 - z$ inside the cylinder $x^2 + z^2 \le 4$ so let Σ be that surface. This surface has orientation having negative $\hat{\jmath}$- and $\hat{k}$-components induced from C.

By Stokes' Theorem we then have:

$$\int_C (xy\,\hat{\imath} - z^2\,\hat{\jmath} + xz\,\hat{k}\,) \cdot d\bar{r} = \iint_\Sigma [2z\,\hat{\imath} - z\,\hat{\jmath} - x\,\hat{\jmath}] \cdot \bar{n}\,dS$$

We parametrize the surface by:

$$\bar{r}(r,\theta) = r\cos\theta\,\hat{\imath} + (6 - r\sin\theta)\,\hat{\jmath} + r\sin\theta\,\hat{k} \text{ for } 0 \le \theta \le 2\pi \text{ and } 0 \le r \le 4.$$

We then have:

$$\begin{aligned}
\bar{r}_r &= \cos\theta\,\hat{\imath} - \sin\theta\,\hat{\jmath} + \sin\theta\,\hat{k} \\
\bar{r}_\theta &= -r\sin\theta\,\hat{\imath} - r\cos\theta\,\hat{\jmath} + r\cos\theta\,\hat{k} \\
\bar{r}_r \times \bar{r}_\theta &= 0\,\hat{\imath} - r\,\hat{\jmath} - r\,\hat{k}
\end{aligned}$$

Note that this agrees with Σ's orientation. Thus we then have:

$$\begin{aligned}
\iint_\Sigma (2z\,\hat{\imath} - z\,\hat{\jmath} - x\,\hat{\jmath}) \cdot \bar{n}\,dS &= \iint_R \left(2r\sin\theta\,\hat{\imath} - r\sin\theta\,\hat{\jmath} - r\cos\theta\,\hat{k}\right) \cdot (0\,\hat{\imath} - r\,\hat{\jmath} - r\,\hat{k}\,)\,dA \\
&= \iint_r r^2\sin\theta + r^2\cos\theta\,dA \\
&= \int_0^{2\pi} \int_0^4 r^2(\sin\theta + \cos\theta)\,dr\,d\theta
\end{aligned}$$

5. Let C be the curve with parameterization: [25pts]

$$\bar{r}(t) = 12t\,\hat{\imath} + \cos(2\pi t)\,\hat{\jmath} + \sin(2\pi t)\,\hat{k} \text{ for } 0 \leq t \leq \tfrac{1}{12}$$

The following line integral can be done two ways that we know. Show that both ways yield the same result.

$$\int_C y\,dx + x\,dy + 3\,dz$$

Solution:

FTOLI: The vector field is conservative with potential function $f(x,y,z) = xy + 3z$. Since the curve starts at $\bar{r}(0) = 0\,\hat{\imath} + 1\,\hat{\jmath} + 0\,\hat{k}$ or $(0,1,0)$
and ends at $\bar{r}\left(\frac{1}{12}\right) = 1\,\hat{\imath} + \frac{\sqrt{3}}{2}\,\hat{\jmath} + \frac{1}{2}\,\hat{k}$ or $\left(1, \frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. We have:

$$\int_C y\,dx + x\,dy + 3\,dz = f\left(1, \frac{\sqrt{3}}{2}, \frac{1}{2}\right) - f(0,1,0) = \left(\frac{\sqrt{3}}{2} + \frac{3}{2}\right) - (0) = \frac{\sqrt{3}}{2} + \frac{3}{2}$$

Directly: On the other hand we can integrate directly:

$$\begin{aligned}\int_C y\,dx + x\,dy + 3\,dz &= \int_0^{1/12} \cos(2\pi t)(12) + 12t\left(-\sin(2\pi t)\right)(2\pi) + 3\cos(2\pi t)(2\pi)\,dt \\ &= \int_0^{1/12} 12\cos(2\pi t) - 24\pi t\sin(2\pi t) + 6\pi\cos(2\pi t)\,dt\end{aligned}$$

The awkward part of this is the middle term which requires IBP:

$$\begin{aligned}\int 2\pi t\sin(2\pi t)\,dt &= 24\pi t\left(-\frac{1}{2\pi}\cos(2\pi t)\right) - \int -\frac{1}{2\pi}\cos(2\pi t)(24\pi)\,dt \\ &= -12t\cos(2\pi t) + \int 12\cos(2\pi t)\,dt \\ &= -12t\cos(2\pi t) + \frac{6}{\pi}\sin(2\pi t) + C\end{aligned}$$

Thus our entire integral evaluates to:

$$\frac{6}{\pi}\sin(2\pi t) - \left[-12t\cos(2\pi t) + \frac{6}{\pi}\sin(2\pi t)\right] + 3\sin(2\pi t)\Big|_0^{1/12}$$

$$\begin{aligned}&= 12t\cos(2\pi t) + 3\sin(2\pi t)\Big|_0^{1/12} \\ &= \left[\frac{\sqrt{3}}{2} + \frac{3}{2}\right] - [0+0]\end{aligned}$$

6. **Instruction:**

Let A be the sum of the digits of your UID.
Let B be the largest single digit of your UID.

Write down your UID and the value(s) and mark them clearly. In the problem below, replace them by the appropriate value(s) before proceeding.

Define the function:

$$f(x,y) = Ax^2y + 2ABxy + By^2$$

(a) Let C be the level curve for $f(x,y) = B$. Determine where C meets the line $y = 1$. [10pts]

Solution:

The level curve is the curve:

$$Ax^2y + 2ABxy + By^2 = B$$

This meets the line $y = 1$ when we have:

$$\begin{aligned} Ax^2 + 2ABx + B &= B \\ Ax(x + 2B) &= 0 \end{aligned}$$

Therefore at $(0,1)$ and $(-2B,1)$.

(b) Find and categorize each of the critical point of f as relative maximum, relative minimum, or saddle point. You should have three such points. [15pts]

Solution:

We solve the system:

$$\begin{aligned} f_x(x,y) &= 2Axy + 2ABy = 0 \\ f_y(x,y) &= Ax^2 + 2ABx + 2By = 0 \end{aligned}$$

The first factors to $2Ay(x+B)$ which yields $y = 0$ or $x = -B$.
If $y = 0$ then the second becomes $Ax^2 + 2ABx = 0$ or $Ax(x+2B) = 0$ which yields $x = 0$ and $x = -2B$ for points $(0,0)$ and $(-2B,0)$.
If $x = -B$ then the second becomes $AB^2 - 2AB^2 + 2By = 0$ or $-AB + 2y = 0$ which yields $y = AB/2$ for the point $(-B, AB/2)$.
To check these points we define:

$$D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = (2Ay)(2B) - (2Ax + 2AB)^2$$

Then we proceed:

- $D(0,0) = -4A^2B^2 < 0$ which is a saddle point.
- $D(-2B,0) = -4A^2B^2 < 0$ which is a saddle point.
- $D(-B, AB/2) = 2A^2B^2 > 0$ and then $f_{xx}(-B, AB/2) = 2A(AB/2) = A^2B > 0$ which is a relative minimum.

7. **Instruction:**

Let E be the largest digit of your UID.
Let F be the smallest nonzero digit of your UID.

Write down your UID and the value(s) and mark them clearly. In the problem below, replace them by the appropriate value(s) before proceeding.

Let D be the solid object in the first octant and bounded by the surfaces: $Fx + y = F$ and $z = \sqrt{y}$. Let Σ be the surface of D. [25pts]

Suppose Σ is immersed in a fluid with flow $\bar{F}(x, y, z) = x\,\hat{\imath} + \frac{2E}{3}y^{3/2}\,\hat{\jmath} - z\,\hat{k}$. Find the rate at which $\bar{F}$ is flowing inwards through Σ.

Solution:

The rate of flow is:

$$\iint_\Sigma (x\,\hat{\imath} + \frac{2E}{3}y^{3/2}\,\hat{\jmath} - z\,\hat{k}\,) \cdot \bar{n}\,dS$$

where Σ has inwards orientation.

By the Divergence Theorem we have:

$$\iint_\Sigma \left(x\,\hat{\imath} + \frac{2E}{3}y^{3/2}\,\hat{\jmath} - z\,\hat{k}\right) \cdot \bar{n}\,dS = -\iiint_D E\sqrt{y}\,dV$$

We then parameterize using rectangular coordinates and evaluate:

$$\begin{aligned}
-\iiint_D E\,dV &= -\int_0^1 \int_0^{F-Fx} \int_0^{\sqrt{y}} E\sqrt{y}\,dz\,dy\,dx \\
&= -\int_0^1 \int_0^{F-Fx} Ez\sqrt{y}\Big|_0^{\sqrt{y}}\,dy\,dx \\
&= -\int_0^1 \int_0^{F-Fx} Ey\,dy\,dx \\
&= -\int_0^1 \frac{E}{2}y^2\Big|_0^{F-Fx}\,dx \\
&= -\int_0^1 \frac{E}{2}(F - Fx)^2\,dx \\
&= \frac{E}{6F}(F - Fx)^3\Big|_0^1 \\
&= \frac{E}{6F}\left[(F - F)^3 - (F - 0)^3\right] \\
&= -\frac{EF^2}{6}
\end{aligned}$$

Note: This can't be done easily as a surface integral because Σ, being the surface of D, has many components which would need to be parameterized separately.

8. **Instruction:**

Let G be the sum of the two smallest distinct digits of your UID.
Let H be the sum of the two largest distinct digits of your UID.

Write down your UID and the value(s) and mark them clearly. In the problem below, replace them by the appropriate value(s) before proceeding.

Let R be the region in the first quadrant bounded by the four curves: [25pts]

$$y = Gx^2 \qquad y = \tfrac{G}{x}$$
$$y = Hx^2 \qquad y = \tfrac{H}{x}$$

Use the change of variables:

$$u = \tfrac{y}{x^2} \text{ and } v = xy$$

to evaluate the integral:

$$\iint_R xy\, dA$$

Solution:

The new region S in the uv-plane is bounded by $u = G$, $u = H$, $v = G$, and $v = H$.

The new integrand is v.

The Jacobian is then:

$$1 \div \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = 1 \div \begin{vmatrix} -2x^{-3}y & x^{-2} \\ y & x \end{vmatrix} = 1 \div (-2x^{-2}y - x^{-2}y) = -\frac{x^2}{3y} = -\frac{1}{3u}$$

The resulting integral is then:

$$\begin{aligned}
\int_G^H \int_G^H v\frac{1}{3u}\, dv\, du &= \int_G^H \int_G^H \frac{v}{3u}\, dv\, du \\
&= \int_G^H \frac{v^2}{6u}\bigg|_G^H du \\
&= \int_G^H \frac{H^2 - G^2}{6u}\, du \\
&= \frac{H^2 - G^2}{6} \ln|u|\bigg|_G^H \\
&= \frac{H^2 - G^2}{6}(\ln H - \ln G)
\end{aligned}$$