Math 241 Fall 2021 Final Exam

- Be sure your name, section number and problem number are on each answer sheet and that you have copied and signed the honor pledge on the first answer sheet.
- Follow the instructions as to which problem goes on which answer sheet. You may use the back of the answer sheets but if you do so, please write "See Back" or something similar on the bottom of the front so we know!
- No calculators or formula sheets are permitted.
- For problems with multiple parts, whether the parts are related or not, be sure to go on to subsequent parts even if there is some part you cannot do.
- Simplification of answers is not necessary. Please leave answers such as $5\sqrt{2}$ or 3π in terms of radicals and π and *do not convert to decimals.*

Please put problem 1 on answer sheet 1

1. (a) Find the projection of the vector $\mathbf{b} = 1\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}} + 0\hat{\boldsymbol{k}}$ onto the vector $\mathbf{a} = 2\hat{\boldsymbol{\imath}} - 1\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}}$. [10 pts]

   Solution:

   We have:

$$\mathrm{Pr}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\mathbf{a} = \frac{1}{9}(2\hat{\boldsymbol{\imath}} - 1\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}})$$

   (b) Find symmetric equations for the line that contains the point $(-3, -3, 1)$ and is perpendicular to the plane $2x - 3y + 4z = 7$. [10 pts]

   Solution:

   Since the line is perpendicular it has direction vector:

$$\mathbf{d} = 2\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}} + 4\hat{\boldsymbol{k}}$$

   Hence the parametric form would be:

$$\begin{aligned} x &= -3 + 2t \\ y &= -3 - 3t \\ z &= 1 + 4t \end{aligned}$$

   Hence the symmetric equations are:

$$\frac{x+3}{2} = \frac{y+3}{-3} = \frac{z-1}{4}$$

Please put problem 2 on answer sheet 2

2. (a) Find an equation for the plane that contains both the lines $x = 1 + 2t, y = t, z = 4 + 4t$, and $x = 1 - t, y = 2t, z = 4 + 3t$. Write this in the form $ax + by + cz = d$. [10 pts]

Solution:

There are a variety of ways to go about it but in all cases we need a point on the line. We can use $(1, 0, 4)$ for example.

Since the plane contains both the lines, and since the lines are not parallel, its normal vector will be the cross product of the two direction vectors:

$$\begin{aligned}\mathbf{d}_1 &= 2\hat{\boldsymbol{\imath}} + 1\hat{\boldsymbol{\jmath}} + 4\hat{\boldsymbol{k}} \\ \mathbf{d}_2 &= -1\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} + 3\hat{\boldsymbol{k}} \\ \mathbf{n} &= -5\hat{\boldsymbol{\imath}} - 10\hat{\boldsymbol{\jmath}} + 5\hat{\boldsymbol{k}}\end{aligned}$$

Thus the equation is:

$$\begin{aligned}-5(x-1) - 10(y-0) + 5(z-4) &= 0 \\ -5x - 10y + 5z &= 25 \\ x + 2y - z &= -5\end{aligned}$$

2 points for identifying a point on the plane
3 points for finding a normal vector
3 points for the plane equation
2 points for simplification to desired form

(b) Suppose the points $(1, 2, 3)$ and $(11, 10, -5)$ are on opposite sides of a sphere (the center of the sphere is the midpoint of the two points). Find the equation of the sphere. [10 pts]

Solution:

The center will be the midpoint:

$$\left(\frac{1+11}{2}, \frac{2+10}{2}, \frac{3-5}{2}\right) = (6, 6, -1)$$

The radius will be the distance from the center to either point:

$$\sqrt{(6-1)^2 + (6-2)^2 + (-1-3)^2} = \sqrt{25 + 16 + 16} = \sqrt{57}$$

Thus the equation is:

$$(x-6)^2 + (y-6)^2 + (z+1)^2 = 57$$

Please put problem 3 on answer sheet 3

3. A particle with initial position $\mathbf{r}(0) = 0\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}} + 1\hat{\boldsymbol{k}}$ has velocity $\mathbf{v}(t) = t\hat{\boldsymbol{\imath}} + 2t^2\hat{\boldsymbol{\jmath}} + \sqrt{t}\hat{\boldsymbol{k}}$ for any time $t \geq 0$.

   (a) Find the particle's position vector $\mathbf{r}(t)$ at any time $t \geq 0$. [10 pts]
   Solution:
   We have:

$$\begin{aligned} \mathbf{r}(t) &= \int \mathbf{v}(t) \\ &= \int t\hat{\boldsymbol{\imath}} + 2t^2\hat{\boldsymbol{\jmath}} + t^{1/2}\hat{\boldsymbol{k}} \\ &= \frac{1}{2}t^2\hat{\boldsymbol{\imath}} + \frac{2}{3}t^3\hat{\boldsymbol{\jmath}} + \frac{2}{3}t^{3/2}\hat{\boldsymbol{k}} + \mathbf{C} \end{aligned}$$

   Then:

$$\mathbf{r}(0) = \mathbf{C} = 0\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}} + 1\hat{\boldsymbol{k}}$$

   And so:

$$\mathbf{r}(t) = \frac{1}{2}t^2\hat{\boldsymbol{\imath}} + \frac{2}{3}t^3\hat{\boldsymbol{\jmath}} + \left(\frac{2}{3}t^{3/2} + 1\right)\hat{\boldsymbol{k}}$$

   (b) Find the tangential component of acceleration at $t = 1$. [10 pts]
   Solution:
   We have:

$$\mathbf{v}(1) = 1\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}} + 1\hat{\boldsymbol{k}}$$

   And we have:

$$\begin{aligned} \mathbf{a}(t) &= 1\hat{\boldsymbol{\imath}} + 4t\hat{\boldsymbol{\jmath}} + \frac{1}{2}t^{-1/2}\hat{\boldsymbol{k}} \\ \mathbf{a}(1) &= 1\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}} + \frac{1}{2}\hat{\boldsymbol{k}} \end{aligned}$$

   The tangential component of acceleration is then:

$$\frac{\mathbf{v}(1) \cdot \mathbf{a}(1)}{||\mathbf{v}(1)||} = \frac{1 + 8 + \frac{1}{2}}{\sqrt{6}}$$

Please put problem 4 on answer sheet 4

4. Let $\mathbf{r}(t) = \sin(t)\hat{\boldsymbol{\imath}} + \cos(t)\hat{\boldsymbol{\jmath}} + t^{3/2}\hat{\boldsymbol{k}}$ for $\frac{\pi}{6} \le t \le \frac{\pi}{2}$ be the parameterization of a curve.

   (a) Determine whether the parameterization is smooth, piecewise smooth or neither. [5 pts]

   Solution:

   We have:

   $$\mathbf{r}'(t) = \cos(t)\hat{\boldsymbol{\imath}} - \sin(t)\hat{\boldsymbol{\jmath}} + \frac{3}{2}t^{1/2}\hat{\boldsymbol{k}}$$

   Since this is defined and continuous whereever $\mathbf{r}(t)$ is defined and since it is never $\mathbf{0}$, the parameterization is smooth.

   (b) Find the length of the curve. [15 pts]

   Solution:

   We have:

   $$\begin{aligned}
   \text{Length} &= \int_{\pi/6}^{\pi/2} ||\mathbf{r}'(t)||\, dt \\
   &= \int_{\pi/6}^{\pi/2} \sqrt{\cos^2 t + \sin^2 t + \frac{9}{4}t}\, dt \\
   &= \int_{\pi/6}^{\pi/2} \sqrt{1 + \frac{9}{4}t}\, dt \\
   &= \frac{2}{3}\cdot\frac{4}{9}\left(1 + \frac{9}{4}t\right)^{3/2}\Big|_{\pi/6}^{\pi/2} \\
   &= \frac{8}{27}\left(1 + \frac{9}{4}\left(\frac{\pi}{2}\right)\right)^{3/2} - \frac{8}{27}\left(1 + \frac{9}{4}\left(\frac{\pi}{6}\right)^{3/2}\right)
   \end{aligned}$$

Please put problem 5 on answer sheet 5

5. Use the method of Lagrange multipliers to find the point on the ellipse $2x^2 + xy + y^2 = 7$ with the largest possible x-coordinate. [20 pts]

   Solution:

   We set $f(x, y) = x$ and $g(x, y) = 2x^2 + xy + y^2$. The system is then:

$$\begin{aligned} 1 &= \lambda(4x + y) \\ 0 &= \lambda(x + 2y) \\ 2x^2 + xy + y^2 &= 7 \end{aligned}$$

   The second tells us that either $\lambda = 0$, which contradicts the first, or $x + 2y = 0$, which then must be true. Then $x = 2y$ which we can plug into the third:

$$\begin{aligned} 2(2y)^2 + (2y)y + y^2 &= 7 \\ 8y^2 + 2y^2 + y^2 &= 7 \\ y^2 &= \frac{7}{11} \\ y &= \pm\sqrt{\frac{7}{11}} \end{aligned}$$

   Thus we have two possible points:

$$\left(2\sqrt{\frac{7}{11}}, \sqrt{\frac{7}{11}}\right) \text{ and } \left(-2\sqrt{\frac{7}{11}}, -\sqrt{\frac{7}{11}}\right)$$

   The first is clearly the one with the largest x-coordinate.

Please put problem 6 on answer sheet 6

6. Let $f(x, y, z) = x^3 + 3xy + xe^z$, and suppose $x(t)$, $y(t)$, and $z(t)$ are functions satisfying $x(0) = 1$, $y(0) = 0$, $z(0) = 0$, $x'(0) = 1$, $y'(0) = 2$, and $z'(0) = 3$. Calculate the following: [20 pts]

$$\frac{df}{dt} \text{ at } t = 0$$

Solution:

By the chain rule we have:

$$\begin{aligned}\frac{df}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \\ &= (3x^2 + 3y + e^z)\frac{dx}{dt} + (3x)\frac{dy}{dt} + (xe^z)\frac{dz}{dt}\end{aligned}$$

When $t = 0$ we then have:

$$\left.\frac{df}{dt}\right|_{t=0} = (3(1)^2 + 3(0) + e^0)(1) + (3(1))(2) + (1e^0)(3)$$

Please put problem 7 on answer sheet 7 '

7. Reverse the order of integration and evaluate the following integral: [20 pts]

$$\int_0^{\sqrt{\pi}/2} \int_y^{\sqrt{\pi}/2} \cos(x^2)\, dx\, dy$$

Solution:

The region R is iterated as horizontally simple and between $y = 0$ and $y = \sqrt{\pi}/2$, to the right of $x = y$ and to the left of $x = \sqrt{\pi}/2$. We re-iterate it as vertically simple:

$$\begin{aligned}&\int_0^{\sqrt{\pi}/2} \int_0^{x} \cos(x^2)\, dy\, dx \\ &= \int_0^{\sqrt{\pi}/2} y\cos(x^2)\Big|_0^x \, dx \\ &= \int_0^{\sqrt{\pi}/2} x\cos(x^2)\, dx \\ &= \frac{1}{2}\sin(x^2)\Big|_0^{\sqrt{\pi}/2} \\ &= \frac{1}{2}\sin\left(\frac{\pi}{4}\right) + \frac{1}{2}\sin(0) \\ &= \frac{1}{2}\cdot\frac{\sqrt{2}}{2}\end{aligned}$$

Please put problem 8 on answer sheet 8

8. Let D be the solid above the xy-plane, below the cone $z = \sqrt{3x^2 + 3y^2}$, and between the spheres $x^2 + y^2 + z^2 = 16$ and $x^2 + y^2 + z^2 = 25$. Evaluate the following integral: [20 pts]

$$\iiint_D \frac{36\sqrt{x^2+y^2+z^2}}{\sqrt{x^2+y^2}}\, dV$$

Solution:

If we parameterize the solid using spherical coordinates we have:

$$\begin{gathered} 0 \le \theta \le 2\pi \\ \tfrac{\pi}{6} \le \phi \le \tfrac{\pi}{2} \\ 4 \le \rho \le 5 \end{gathered}$$

Thus we have:

$$\begin{aligned}
&\iiint_D \frac{36\sqrt{x^2+y^2+z^2}}{\sqrt{x^2+y^2}}\, dV \\
&\qquad = \int_0^{2\pi} \int_{\pi/6}^{\pi/4} \int_4^5 \frac{36\rho}{\rho\sin\phi}\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta \\
&\qquad = \int_0^{2\pi} \int_{\pi/6}^{\pi/4} \int_4^5 36\rho^2 \, d\rho\, d\phi\, d\theta \\
&\qquad = \int_0^{2\pi} \int_{\pi/6}^{\pi/4} 12\rho^3\Big|_4^5 \, d\phi\, d\theta \\
&\qquad = \int_0^{2\pi} \int_{\pi/6}^{\pi/4} 12(125) - 12(64)\, d\phi\, d\theta \\
&\qquad = \int_0^{2\pi} \int_{\pi/6}^{\pi/4} 12(61)\, d\phi\, d\theta \\
&\qquad = \ldots \\
&\qquad = 12(61)\left(\frac{\pi}{6} - \frac{\pi}{4}\right)(2\pi)
\end{aligned}$$

Please put problem 9 on answer sheet 9

9. (a) Let C be any curve from $(1,-2,3)$ to $(0,5,2)$ not passing through $z=0$. Evaluate the following integral: [10 pts]

$$\int_C 2x\,dx + \left(1+\frac{1}{z}\right)dy - \frac{y}{z^2}\,dz$$

Solution:

The vector field is conservative with potential function:

$$f(x,y,z) = x^2 + y + \frac{y}{z}$$

Thus by the Fundamental Theorem of Line Integrals we get the result:

$$f(0,5,2) - f(1,-2,3) = \left(0^2+5+\frac{5}{2}\right) - \left(1^2-2-\frac{2}{3}\right)$$

(b) Let C be the counterclockwise curve consisting of the semicircle $x^2+y^2=9$ with $x \geq 0$ along with the line segment joining the endpoints of that semicircle. Use Green's Theorem to evaluate the following integral: [10 pts]

$$\int_C 6y\,dx + 15x\,dy$$

Solution:

By Green's Theorem:

$$\int_C 6y\,dx + 15x\,dy = \iint_R 15 - 6\,dA$$

Since the integrand is a constant and R is a semicircle we can use the area:

$$\iint_R 9\,dA = 9\iint_R 1\,dA = 9\cdot\frac{1}{2}\pi(3)^2$$

Please put problem 10 on answer sheet 10

10. Let C be the intersection of the plane $z = 9 - y$ with the cylinder $x^2 + y^2 = 4$ with counter-clockwise orientation when viewed from above. Consider the integral: [20 pts]

$$\int_C z^2\,dx + x\,dy + z\,dz$$

Use Stokes' Theorem to rewrite the line integral as an iterated double integral. Do not evaluate this integral.

Solution:

We see that C is the edge of the surface Σ where Σ is the part of the plane inside the cylinder. By Stokes' Theorem we then have:

$$\begin{aligned}\int_C z^2\,dx + x\,dy + z\,dz &= \iint_\Sigma ((0-0)\hat{\boldsymbol{\imath}} - (0-2z)\hat{\boldsymbol{\jmath}} + (1-0)\hat{\boldsymbol{k}}) \cdot \mathbf{n}\,dS \\ &= \iint_\Sigma (0\hat{\boldsymbol{\imath}} + 2z\hat{\boldsymbol{\jmath}} + 1\hat{\boldsymbol{k}}) \cdot \mathbf{n}\,dS\end{aligned}$$

Here Σ has orientation up and to the right, induced by C.

We then parameterize Σ by:

$$\begin{gathered}\mathbf{r}(r,\theta) = r\cos\theta\hat{\boldsymbol{\imath}} + r\sin\theta\hat{\boldsymbol{\jmath}} + (9 - r\sin\theta)\hat{\boldsymbol{k}} \\ 0 \le \theta \le 2\pi \\ 0 \le r \le 2\end{gathered}$$

We have:

$$\begin{aligned}\mathbf{r}_r &= \cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}} - \sin\theta\hat{\boldsymbol{k}} \\ \mathbf{r}_\theta &= -r\sin\theta\hat{\boldsymbol{\imath}} + r\cos\theta\hat{\boldsymbol{\jmath}} - r\cos\theta\hat{\boldsymbol{k}} \\ \mathbf{r}_r \times \mathbf{r}_\theta &= 0\hat{\boldsymbol{\imath}} + r\hat{\boldsymbol{\jmath}} + r\hat{\boldsymbol{k}}\end{aligned}$$

These match the orientation of Σ and hence, letting R represent the set of inequalities:

$$\begin{aligned}\iint_\Sigma (0\hat{\boldsymbol{\imath}} + 2z\hat{\boldsymbol{\jmath}} + 1\hat{\boldsymbol{k}}) \cdot \mathbf{n}\,dS &= +\iint_R (0\hat{\boldsymbol{\imath}} + 2(9 - r\sin\theta)j + 1\hat{\boldsymbol{k}}) \cdot (0\hat{\boldsymbol{\imath}} + r\hat{\boldsymbol{\jmath}} + r\hat{\boldsymbol{k}})\,dA \\ &= \int_0^{2\pi}\int_0^2 2(9 - r\sin\theta)r + r\,dr\,d\theta\end{aligned}$$