1. Let R be the region bounded by the lines $y = x$, $y = 12 - 2x$ and $x = 0$. Evaluate and simplify the integral $\iint_R x\,dA$. [20 pts]

   This is the only integral you need to evaluate/simplify!

   Solution:

   The functions meet at $x = 4$ and vertically simple is easiest. We have:

$$
\begin{aligned}
\int_R x\,dA &= \int_0^4 \int_x^{12-2x} x\,dy\,dx \\
&= \int_0^4 xy\Big|_x^{12-2x}\,dx \\
&= \int_0^4 x(12-2x) - x(x)\,dx \\
&= \int_0^4 12x - 2x^2 - x^2\,dx \\
&= \int_0^4 12x - 3x^2\,dx \\
&= 6x^2 - x^3\Big|_0^4 \\
&= 6(4)^2 - (4)^3 \\
&= 16(6-4) \\
&= 32
\end{aligned}
$$

2. (a) Let R be the region inside $r = 2\cos\theta$ and outside $r = 1$. Write down an iterated double integral in polar coordinates for $\iint_R \frac{y}{x}\, dA$. **Do not evaluate.** [10 pts]

Solution:

The solution is:

$$\int_{-\pi/3}^{\pi/3} \int_{1}^{2\cos\theta} \frac{r\sin\theta}{r\cos\theta} r\, dr\, d\theta$$

(b) Let D be the solid inside the cylinder $r = \sin\theta$, above the xy-plane, and below the paraboloid $z = 100 - x^2 - y^2$. Set up an iterated triple integral in cylindrical coordinates for $\iiint_D z\, dV$. **Do not evaluate.** [10 pts]

Solution:

The solution is:

$$\int_{0}^{\pi} \int_{0}^{\sin\theta} \int_{0}^{100-r^2} zr\, dz\, dr\, d\theta$$

3. (a) Reparametrize the following polar iterated integral as a vertically simple iterated integral. **Do not evaluate.** [10 pts]

$$\int_{\pi/4}^{\pi/2} \int_{0}^{2} r\cos\theta \, r \, dr \, d\theta$$

Solution:

The region R is inside the circle $r = 2$, or $x^2 + y^2 = 4$, above the line $y = x$ and to the right of the line $x = 0$. The new integral is:

$$\int_{0}^{\sqrt{2}} \int_{x}^{\sqrt{4-x^2}} x \, dy \, dx$$

(b) Let D be the solid below the cone $z = \sqrt{x^2 + y^2}$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 4$. Write down an iterated triple integral in spherical coordinates for $\iiint_D x \, dV$. **Do not evaluate.** [10 pts]

Solution:

The solution is:

$$\int_{0}^{2\pi} \int_{\pi/4}^{\pi/2} \int_{0}^{2\csc\phi} \rho\sin\phi\cos\theta\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

4. (a) Let Σ be the portion of the plane $y + z = 9$ inside the cylinder $x^2 + z^2 = 9$, Write down a parametrization of Σ. [5 pts]

Solution:

The cleanest solution is:

$$\mathbf{r}(t) = r\cos\theta\,\mathbf{i} + (9 - r\sin\theta)\,\mathbf{j} + r\sin\theta\,\mathbf{k}$$

$$0 \leq \theta \leq 2\pi$$

$$0 \leq r \leq 3$$

(b) Let D be the solid bounded by the planes $y = x$, $y = 2x$, $y = 6$, $z = 0$ and $z = 10$. Set up the iterated triple integral in rectangular coordinates for $\iiint_D z^2\,dV$. Horizontally simple is best. **Do not evaluate.** [15 pts]

Solution:

The solution is:

$$\int_0^6 \int_{y/2}^{y} \int_0^{10} z^2\,dz\,dx\,dy$$

5. Let R be the region in the first quadrant bounded by the functions $y = \frac{1}{x}$, $y = \frac{5}{x}$, $y = \frac{1}{4}x$, and $y = 3x$. Use a change of variables to convert the integral [20 pts]

$$\iint_R y^2 \, dA$$

into an iterated double integral over a rectangular region. **Do not evaluate.**

Solution:

We rewrite the functions as: $xy = 1$, $xy = 5$, $\frac{y}{x} = \frac{1}{4}$ and $\frac{y}{x} = 3$.

We assign $u = xy$ and $v = \frac{y}{x} = x^{-1}y$.

The new region S is then bounded by the lines $u = 1$, $u = 5$, $v = \frac{1}{4}$ and $v = 3$.

Noting that $uv = y^2$ we don't necessarily need to solve for x and y and so:

$$\begin{aligned} J &= 1 \div \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} \\ &= 1 \div \begin{vmatrix} y & x \\ -x^{-2}y & x^{-1} \end{vmatrix} \\ &= 1 \div (2x^{-1}y) \\ &= \frac{x}{2y} \\ &= \frac{1}{2v} \end{aligned}$$

Then we have:

$$\iint_R y^2 \, dA = \iint_S uv \left| \frac{1}{2v} \right| \, dA = \int_1^5 \int_{1/4}^3 \frac{u}{2} \, dv \, du$$