MATH 246: Exam 3 Sample 1

1. Rewrite $y''+5y'+6y=0$ with $y(0)=2$, $y'(0)=-4$ as a system of first-order differential equations. Do not solve.

2. Find a fundamental pair of solutions to the system:
$$\begin{aligned} x' &= 2x + y \\ y' &= -3x + 4y \end{aligned}$$

3. Solve the initial value problem $\bar{x}' = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \bar{x}$ with $\bar{x}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

4. Tank 1 has volume of 100 Liters and Tank 2 has volume 500 Liters. Initially both are full with Tank 1 containing salt at 5 g/L and Tank 2 containing salt at 8 g/L. The Tank 1 mixture is flowing from Tank 1 to Tank 2 at 4 L/min while the Tank 2 mixture is flowing from Tank 2 to Tank 1 at 3 L/min. Fresh water is flowing into Tank 1 at 5 L/min while the Tank 1 mixture flows out to a drain at 4 L/min. Water at 4 g/L is flowing into Tank 2 at 6 L/min while the Tank 2 mixture flows out to a drain at 10 L/min. Let x_1 represent the amount of salt in Tank 1 at time t and x_2 represent the amount of salt in Tank 2 at time t. Draw a tank picture for this situation and write down the corresponding system with initial values.
Warning: Check if the volumes are constant.

5. Consider the system
$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned}$$
   (a) Find a criteria on a, b, c and/or d under which $(0,0)$ is neither a circle nor spiral.
   (b) Find a criteria on a, b, c and/or d under which there is an entire line of stationary solutions.

6. Use Hamiltonian Methods to sketch a family of solutions to the system:
$$\begin{aligned} x' &= x - 3xy^2 + 2y \\ y' &= y^3 - y \end{aligned}$$

7. Consider the predator-prey model given here:
$$\begin{aligned} x' &= (3 - 2x - y)x \\ y' &= (-1 + 2x)y \end{aligned}$$
This has stationary solutions $(0,0)$, $(1.5,0)$ and $(0.5,2)$. The first two of these have:
$$\partial^2 \bar{F}(0,0) = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \text{ with eigenpairs } \left(3, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \text{ and } \left(-1, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$
$$\partial^2 \bar{F}(1.5,0) = \begin{bmatrix} -3 & -1.5 \\ 0 & 2 \end{bmatrix} \text{ with eigenpairs } \left(-3, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \text{ and } \left(2, \begin{bmatrix} -3 \\ 10 \end{bmatrix}\right).$$
   (a) Find $\partial^2 \bar{F}(0.5,2)$ and its eigenvalues (not eigenvectors).
   (b) Sketch a reasonable family of solutions.

8. Why does it seem real-world unreasonable that a predator-prey model (with x the prey) could have an initial value problem with $\bar{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and with solution $\bar{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ satisfying $\lim_{t\to\infty} y(t) = 0$?

9. Suppose x and y are cooperating species where x is restricted by natural resources but y is not. Draw a reasonable family of solutions for such a situation. Pick one solution NOT on an axis and explain in real-world terms what is happening and why it is reasonable.