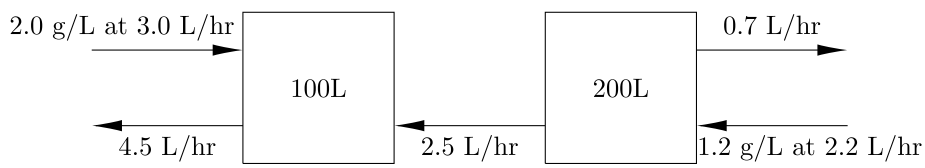
**MATH 246: Exam 3 Sample 2**

1. Rewrite the following second-order differential equation as a system of first-order:
$$2D^2y + tDy - 5y = \sin t$$

2. Find a fundamental pair of solutions to the system:
$$\bar{x}' = \begin{bmatrix} 5 & -2 \\ 2 & 1 \end{bmatrix} \bar{x}$$

3. Solve the initial value problem
$$\begin{aligned} x_1' &= 2x_1 + 2x_2 \qquad\qquad & x_1(0) &= 2 \\ x_2' &= 3x_1 + 7x_2 & x_2(0) &= -1 \end{aligned}$$

4. Write down the differential equation corresponding to the following tank diagram. Assume the tanks start (do they remain?) full.




5. Consider the system:
$$\bar{x}' = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} \bar{x}$$
A fundamental matrix for this system is:
$$\Psi(t) = \begin{bmatrix} 1 & 2t+1 \\ 1 & 2t \end{bmatrix}$$
   (a) Find the natural fundamental matrix $\Phi(t)$ associated to $t_I = 1$.
   (b) Use your answer to (a) to find the solutions to the IVP having $\bar{x}(1) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

6. Use Hamiltonian Methods to sketch a family of solutions to the system:
$$\begin{aligned} x' &= 2x^2 + 2y \\ y' &= -1 - 4xy \end{aligned}$$

7. Consider the competing species model given here:
$$\begin{aligned} x' &= (20 - 2x - y)x \\ y' &= (20 - 2y)y \end{aligned}$$
This has stationary solutions $(0,0)$, $(10,0)$, $(0,10)$ and $(5,10)$. The first three of these have:
$$\partial^2\bar{F}(0,0) = \begin{bmatrix} 20 & 0 \\ 0 & 20 \end{bmatrix} \text{ with eigenpairs } \left(20, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \text{ and } \left(20, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$
$$\partial^2\bar{F}(10,0) = \begin{bmatrix} -20 & -10 \\ 0 & 20 \end{bmatrix} \text{ with eigenpairs } \left(20, \begin{bmatrix} 1 \\ -4 \end{bmatrix}\right) \text{ and } \left(-20, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right).$$
$$\partial^2\bar{F}(0,10) = \begin{bmatrix} 10 & 0 \\ 0 & -20 \end{bmatrix} \text{ with eigenpairs } \left(-20, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \text{ and } \left(10, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right).$$
   (a) Find $\partial^2\bar{F}(5,10)$ and its eigenvalues (not eigenvectors).
   (b) Sketch a reasonable family of solutions.
   (c) Explain in real-world terms what would happen to an intial population of $(4, 0.1)$.
   (d) Explain in real-world terms what would happen to an intial population of $(100, 0.1)$.