MATH 246: Chapter 1 Section 2: Linear First-Order DEs
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Main Topics:

- Linear First-Order DEs and Linear Normal Form.
- General approach.
- Initial Value Problems.
- Theory

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1. Linear first-order ODEs.

   Recall that these will all have the form $a_1(t)y' + a_0(t)y = c(t)$ where a_1, a_0, c can be any functions of t.

   Example: $4y' + 5y = 0$
   Example: $4ty' + e^t y = \sin t$

2. Linear Normal Form.

   (a) Introduction:
   We will usually divide through by $a_1(t)$ and re-label a bit to get what is known as the *linear normal form*:

   $$y' + a(t)y = f(t) \text{ for functions } a(t) \text{ and } f(t)$$

   (b) General Solution.
   These we can actually handle, and most of you did in Calculus II though it may be rusty.
   Method: If we let $A(t)$ be an antiderivative (any antiderivative, meaning use $+0$ for the constant) of $a(t)$ so that $A'(t) = a(t)$ then observe:

   $$\begin{aligned} y' + a(t)y &= f(t) \\ e^{A(t)}y' + e^{A(t)}a(t)y &= f(t)e^{A(t)} \\ \frac{d}{dt}\left(e^{A(t)}y\right) &= f(t)e^{A(t)} \\ e^{A(t)}y &= \int f(t)e^{A(t)}\,dt \\ y &= e^{-A(t)}\int f(t)e^{A(t)}\,dt \end{aligned}$$

   The only step that might concern you here is from line 2 to line 3. This is just the reverse of the product rule with a bit of chain rule thrown in. Reading it from line 3 to line 2 might be easier. We'll see this sort of thing happen again with exact DEs later.
   This process can either be repeated for each problem or treated simply as a recipe, meaning find $A(t)$ and then use the formula at the end of the calculation.
   Be careful though, the $e^{-A(t)}$ is multiplied by the entire integral, meaning the $+C$ too when you integrate.
   We'll call this final expression the *integral-form solution*:

   $$y = e^{-A(t)}\int f(t)e^{A(t)}\,dt$$

Example: Consider $y' + 5y = 2$. We see that $a(t) = 5$ so $A(t) = 5t$ and the solution is

$$\begin{aligned} y &= e^{-5t}\int 2e^{5t}\ dt \\ &= e^{-5t}\left(\frac{2}{5}e^{5t} + C\right) \quad \leftarrow \text{Note the parentheses!!!} \\ &= \frac{2}{5} + Ce^{-5t} \end{aligned}$$

If you'd have forgotten the parentheses you'd have got an incorrect answer which I won't even write here!

Example: Consider $ty' + 2y = t^4$ with $t > 0$. This is not in linear normal form so we divide by t to get $y' + \frac{2}{t}y = t^3$. Then $a(t) = \frac{2}{t}$ so $A(t) = 2\ln t$ and the solution is

$$\begin{aligned} y &= e^{-2\ln t}\int t^3 e^{2\ln t}\ dt \\ y &= e^{-2\ln t}\int t^3 e^{2\ln t}\ dt \\ &= t^{-2}\int t^5\ dt \\ &= t^{-2}\left(\frac{1}{6}t^6 + C\right) \\ &= \frac{1}{6}t^4 + \frac{C}{t^2} \end{aligned}$$

Here's one with an IVP:

Example: Consider $y' - 6y = e^t$ with $y(0) = 2$. We see that $a(t) = -6$ so $A(t) = -6t$ and the general solution is

$$\begin{aligned} y &= e^{-(-6t)}\int e^t e^{-6t}\ dt \\ &= e^{6t}\int e^{-5t}\ dt \\ &= e^{6t}\left(-\frac{1}{5}e^{-5t} + C\right) \\ &= -\frac{1}{5}e^t + Ce^{6t} \end{aligned}$$

At this point $y(0) = -\frac{1}{5}e^0 + Ce^0 = -\frac{1}{5} + C = 2$ so that $C = \frac{11}{5}$ so the specific solution is

$$y = -\frac{1}{5}e^t + \frac{11}{5}e^{6t}$$

At this point you can probably see that solving a first-order linear ODE is as easy (or as hard) as first finding $A(t)$ and then finding $\int f(t)e^{A(t)}\ dt$.

(c) Note about the choice of $A(t)$. You might wonder what happened if you didn't choose $+0$ as your constant when choosing $A(t)$. In fact it makes no difference. Suppose we took $A(t)$ and adjusted it by adding some number like $+7$. The solution would then be:

$$y = e^{-(A(t)+7)} \int f(t)e^{A(t)+7}\ dt = e^{-7}e^{-A(t)} \int f(t)e^{7}e^{A(t)}\ dt = e^{-A(t)} \int f(t)e^{A(t)}\, dt$$

which is exactly the same.

3. Theory!

   The Second Fundamental Theorem of Calculus states that if a function is continuous on an open interval then it has an antiderivative on that interval and that antiderivative will be continuous. What this means is that if $a(t)$ is continuous then $A(t)$ will exist and therefore so will $e^{-A(t)}$ and then provided that $f(t)$ is continuous then so will $\int f(t)e^{A(t)}\, dt$.

   Warning! This doesn't mean that these things are easy to calculate, just that they exist!

   What this means is that if we have an initial value $y(t_I) = y_I$ then the interval of existence of the solution will be the largest open interval containing t_I on which both $f(t)$ and $a(t)$ are continouous. As before this lets us find the IE even when we can't solve the IVP.

   Example: Consider $y' + \frac{1}{t^2}y = \frac{1}{t-5}$ with $y(2) = 17$. Here $a(t) = \frac{1}{t^2}$ and $f(t) = \frac{1}{t-5}$. The largest open interval containing $t_I = 2$ on which both are continuous is $(0, 5)$ so this is the IE of the solution. Finding the solution is a different matter entirely but it exists on $(0, 5)$!

4. Integration Comment.

   As a final note observe that there are two antiderivatives involved in the problem, finding $A(t)$ and finding $\int f(t)e^{A(t)}\, dt$. This latter one will often involve simplification involving e and $\ln$ as well as substitution and integration by parts.