**MATH 246: Chapter 1 Section 7: Approximation Methods**
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Main Topics:

- Euler's Method (The Left-Sum Method).
- The Runge-Trapezoid Method.
- The Runge-Midpoint Method.

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1. Euler's Method

   (a) Introduction Suppose we're dealing with the IVP given by:

   $$\frac{dy}{dt} = t + y \text{ with } y(1) = 2$$

   Suppose we'd really like to know $y(2)$.

   The DE tells us that at the point $(1, 2)$ the slope of the solution is $\frac{dy}{dt}(1, 2) = 3$. Of course the solution is not a straight line, meaning if we move right 1 we won't go up exactly 3, but if things aren't too bad then we would go up approximately 3. Thus we can conclude that $y(1 + 1) \approx 2 + 3$ or $y(2) \approx 5$.

   This approximately probably stinks, so what we can do instead is go to the right just 0.5 and up 0.5(3), then do the process again, now anchored at the new point. That is:

   At $(1, 2)$ the slope is $\frac{dy}{dt}(1, 2) = 3$ so we go over 0.5 and up 0.5(3) and now we're at $(1 + 0.5, 2 + 0.5(3)) = (1.5, 3.5)$.

   At $(1.5, 3.5)$ the slope is $\frac{dy}{dt}(1.5, 3.5) = 5$ so we go over 0.5 and up 0.5(5) and now we're at $(1.5 + 0.5, 3.5 + 0.5(5)) = (2, 6)$

   Then we conclude $y(2) \approx 6$. This approximation is probably better.

(b) Euler's Method.

This process is known as Euler's Method. We start with an IVP given by $\frac{dy}{dt} = f(t,y)$ with $y(t_0) = y_0$. and we choose a small $h$. We did $h = 1$ and then $h = 0.5$. We then proceed as follows:

$$(t_1, y_1) = (t_0 + h, y_0 + hf(t_0, y_0))$$
$$(t_2, y_2) = (t_1 + h, y_1 + hf(t_1, y_1))$$

Or, more generally:

Euler's Method

$$t_i = t_{i-1} + h$$
$$y_i \approx y_{i-1} + hf(t_{i-1}, y_{i-1})$$

**Example:** Again with $\frac{dy}{dt} = t + y$ with $y(1) = 2$. Let's approximate $y(2)$ using $n = 10$ steps of size $h = 0.1$.

This can all be put more nicely into a table as follows:

| 0 | 1 | 2 | y(1)=2 |
|---|---|---|---|
| $i$ | $t_i$ | $y_i \approx y_{i-1} + hf(t_{i-1}, y_{i-1})$ | So |
| 1 | $1 + 0.1 = 1.1$ | $2 + 0.3 = 2.3$ | $y(1.1) \approx 2.3$ |
| 2 | $1.1 + 0.1 = 1.2$ | $2.3 + 0.34 = 2.64$ | $y(1.2) \approx 2.64$ |
| 3 | $1.2 + 0.1 = 1.3$ | $2.64 + 0.384 = 3.024$ | $y(1.3) \approx 3.024$ |
| 4 | $1.3 + 0.1 = 1.4$ | $3.024 + 0.4324 = 3.4564$ | $y(1.4) \approx 3.4564$ |
| 5 | $1.4 + 0.1 = 1.5$ | $3.4564 + 0.48564 = 3.94204$ | $y(1.5) \approx 3.94204$ |
| 6 | $1.5 + 0.1 = 1.6$ | $3.94204 + 0.544204 = 4.48624$ | $y(1.6) \approx 4.48624$ |
| 7 | $1.6 + 0.1 = 1.7$ | $4.48624 + 0.608624 = 5.09487$ | $y(1.7) \approx 5.09487$ |
| 8 | $1.7 + 0.1 = 1.8$ | $5.09487 + 0.679487 = 5.77436$ | $y(1.8) \approx 5.77436$ |
| 9 | $1.8 + 0.1 = 1.9$ | $5.77436 + 0.757436 = 6.53179$ | $y(1.9) \approx 6.53179$ |
| 10 | $1.9 + 0.1 = 2$ | $6.53179 + 0.843179 = 7.37497$ | $y(2) \approx 7.37497$ |

Of course the further we go the less accurate we get but if the DE is not so bad then maybe we're good. The solution to the above DE (first-order linear) is $y(t) = 4e^{t-1} - t - 1$ and so $y(2) = 4e - 2 - 1 \approx 7.87312731383618094144114988541 06$ so our approximation is not terrible.

**Example:** Same IVP but we could to better by reducing $h$ and increasing the number of steps. Just for fun, compare to 1000 steps of size $h = 0.001$ each and see how close the approximation is at the end!

Note: This was generated in Python and some approximation and truncation is taking place.

| 0 | 1 | 2 | y(1)=2 |
|---|---|---|---|
| $i$ | $t_i$ | $y_i \approx y_{i-1} + hf(t_{i-1}, y_{i-1})$ | So |
| 1 | $1 + 0.001 = 1.001$ | $2 + 0.003 = 2.003$ | $y(1.001) \approx 2.003$ |
| 2 | $1.001 + 0.001 = 1.002$ | $2.003 + 0.003004 = 2.006$ | $y(1.002) \approx 2.006$ |
| 3 | $1.002 + 0.001 = 1.003$ | $2.006 + 0.003008 = 2.00901$ | $y(1.003) \approx 2.00901$ |
| ... | ... | ... | ... |
| 998 | $1.997 + 0.001 = 1.998$ | $7.83816 + 0.00983516 = 7.84799$ | $y(1.998) \approx 7.84799$ |
| 999 | $1.998 + 0.001 = 1.999$ | $7.84799 + 0.00984599 = 7.85784$ | $y(1.999) \approx 7.85784$ |
| 1000 | $1.999 + 0.001 = 2$ | $7.85784 + 0.00985684 = 7.8677$ | $y(2) \approx 7.8677$ |

2. Improving:

   First off recall that for a continuous function $y(t)$ the Fundamental Theorem of Calculus tells us that:

   $$\int_a^b \left[\frac{dy}{dt}\right] dt = y(b) - y(a)$$

   With our differential equation given that we're looking for some $y(t)$ satisfying $\frac{dy}{dt} = f(t, y(t))$ this translates to:

   $$\int_a^b f(t, y(t))\, dt = y(b) - y(a)$$

   Given that we started this whole process knowing $y_0$ and wanting $y_1$ we can write:

   $$y(t_1) - y(t_0) = \int_{t_0}^{t_1} f(t, y(t))\, dt$$

   which can then be rewritten as our *Basic Formula*:

   $$y_1 = y_0 + \int_{t_0}^{t_1} f(t, y(t))\, dt$$

   So the real question is how to tackle the integral.

   Let's revisit integrals. Suppose you wanted to know $\int_a^b g(x)\, dx$ but couldn't do it. One really bad approximation is just a left rectangle. That is

   $$\int_a^b g(x)\, dx \approx (b - a)g(a)$$

   Using this in the *Basic Formula* yields:

   $$\begin{aligned} y_1 &= y_0 + \int_{t_0}^{t_1} f(t, y(t))\, dt \\ y_1 &\approx y_0 + (t_1 - t_0) f(t_0, y(t_0)) \\ y_1 &\approx y_0 + (t_1 - t_0) f(t_0, y_0) \\ y_1 &\approx y_0 + h f(t_0, y_0) \end{aligned}$$

   Well then, we've just got Euler's Method!

   What this suggests is that better methods of approximating the integral yield better approximations for our IVP.

3. The Runge-Trapezoid Method:

A second way to approximate the integal would be to construct a trapezoid using the endpoints:

$$\int_a^b g(x)\,dx \approx \frac{1}{2}(b-a)(g(a)+g(b))$$

Using this in the *Basic Formula* yields:

$$\begin{aligned} y_1 &= y_0 + \int_{t_0}^{t_1} f(t,y(t))\,dt \\ y_1 &\approx y_0 + \frac{1}{2}(t_1-t_0)(f(t_0,y(t_0))+f(t_1,y(t_1))) \\ y_1 &\approx y_0 + \frac{1}{2}h(f(t_0,y_0)+f(t_0+h,y(t_1))) \end{aligned}$$

Which is all fun and games until we notice the right side has an $y(t_1)$ in it, and this is what we want. How can we resolve this? We do something slick and we plug in the result of Euler's Method into this:

$$y_1 \approx y_0 + \frac{1}{2}h(f(t_0,y_0)+f(t_0+h,\underbrace{y_0+hf(t_0,y_0)}_{\text{Euler:}}))$$

Haha what fun. What we're really doing is using one approximation of $y(t_1)$ to get what we think will be a better one.

Runge-Trapezoidal Method

$$t_i = t_{i-1}+h$$

$$y_i \approx y_{i-1} + \frac{1}{2}h\Big(f(t_{i-1},y_{i-1})+f(t_{i-1}+h,y_{i-1}+hf(t_{i-1},y_{i-1}))\Big)$$

Back to our first IVP $\frac{dy}{dt}=t+y$ with $y(1)=2$. If $h=0.1$ then proceeding one step gives us $t_1=0.1$ and:

$$\begin{aligned} y_1 &\approx y_0 + \frac{1}{2}h\left(f(t_0,y_0)+f(t_0+h,y_0+hf(t_0,y_0))\right) \\ &\approx 2+\frac{1}{2}(0.1)\left(f(1,2)+f(1+0.1,2+0.1f(1,2))\right) \\ &\approx 2+\frac{1}{2}(0.1)\left(1+2+f(1.1,2+0.1(1+2))\right) \\ &\approx 2+\frac{1}{2}(0.1)\left(1+2+1.1+2+0.1(1+2)\right)) = 2.32 \end{aligned}$$

Here's the Runge-Trapezoidal Method applied to our first IVP with 10 steps of size 0.1:

| 0 | 1 | 2 | y(1)=2 |
|---|---|---|---|
| $i$ | $t_i$ | $y_i$ | So |
| 1 | $1+0.1=1.1$ | 2.32 | $y(1.1)\approx 2.32$ |
| 2 | $1.1+0.1=1.2$ | 2.6841 | $y(1.2)\approx 2.6841$ |
| 3 | $1.2+0.1=1.3$ | 3.09693 | $y(1.3)\approx 3.09693$ |
| 4 | $1.3+0.1=1.4$ | 3.56361 | $y(1.4)\approx 3.56361$ |
| 5 | $1.4+0.1=1.5$ | 4.08979 | $y(1.5)\approx 4.08979$ |
| 6 | $1.5+0.1=1.6$ | 4.68171 | $y(1.6)\approx 4.68171$ |
| 7 | $1.6+0.1=1.7$ | 5.34629 | $y(1.7)\approx 5.34629$ |
| 8 | $1.7+0.1=1.8$ | 6.09116 | $y(1.8)\approx 6.09116$ |
| 9 | $1.8+0.1=1.9$ | 6.92473 | $y(1.9)\approx 6.92473$ |
| 10 | $1.9+0.1=2$ | 7.85632 | $y(2)\approx 7.85632$ |

Remember the exact value of $y(2)=4e-2-1\approx 7.8731273138361809414411498854106$.

4. The Runge-Midpoint Method:

A third way to approximate the integral is a midpoint rectangle:

$$\int_a^b g(x)\,dx \approx (b-a)g\left(\frac{a+b}{2}\right)$$

Using this in the *Basic Formula* and using the fact that our midpoint is $t_0 + \frac{1}{2}h$ yields:

$$\begin{aligned}
y_1 &= y_0 + \int_{t_0}^{t_1} f(t, y(t))\,dt \\
y_1 &\approx y_0 + (t_1 - t_0) f\left(t_0 + \frac{1}{2}h, y\left(t_0 + \frac{1}{2}h\right)\right) \\
y_1 &\approx y_0 + hf\left(t_0 + \frac{1}{2}h, y\left(t_0 + \frac{1}{2}h\right)\right)
\end{aligned}$$

Which again is all fun and games until we realize we don't know $y\left(t_0 + \frac{1}{2}h\right)$ so we swap in Euler's Method again using a half-step, that is $y_0 + \frac{1}{2}hf(t_0, y_0)$ and so

$$y_1 \approx y_0 + hf\Big(t_0 + \frac{1}{2}h, \underbrace{y_0 + \frac{1}{2}hf(t_0, y_0)}_{Euler}\Big)$$

Runge-Midpoint Method

$$\begin{aligned}
t_i &= t_{i-1} + h \\
y_i &\approx y_{i-1} + hf\left(t_{i-1} + \frac{1}{2}h, y_{i-1} + \frac{1}{2}hf(t_{i-1}, y_{i-1})\right)
\end{aligned}$$

Back to our first IVP $\frac{dy}{dt} = t + y$ with $y(1) = 2$. If $h = 0.1$ then proceeding one step gives us $t_1 = 0.1$ and:

$$\begin{aligned}
y_i &\approx y_0 + hf\left(t_0 + \frac{1}{2}h, y_0 + \frac{1}{2}hf(t_0, y_0)\right) \\
&\approx 2 + 0.1f\left(1 + \frac{1}{2}(0.1), 2 + \frac{1}{2}(0.1)f(1, 2)\right) \\
&\approx 2 + 0.1f\left(1 + \frac{1}{2}(0.1), 2 + \frac{1}{2}(0.1)(1 + 2)\right) \\
&\approx 2 + 0.1\left(1 + \frac{1}{2}(0.1) + 2 + \frac{1}{2}(0.1)(1 + 2)\right) = 2.32
\end{aligned}$$

This is actually the same as the Runge-Trapezoidal Method and in fact for this particular IVP the Runge-Midpoint Method applied to our first IVP actually gives the same result as the Runge-Trapezoidal Method, so we omit the full table.

5. Everything together:

Let $y(t)$ be the solution to $\frac{dy}{dt} = ty + t$ with $y(0) = 1$. Approximate $y(1)$ using $n = 10$ steps of size $h = 0.1$:

Euler

| 0 | 0 | 1 | y(0)=1 |
|---|---|---|---|
| $i$ | $t_i$ | $y_i \approx y_{i-1} + hf(t_{i-1}, y_{i-1})$ | So |
| 1 | $0 + 0.1 = 0.1$ | $1 + 0 = 1$ | $y(0.1) \approx 1$ |
| 2 | $0.1 + 0.1 = 0.2$ | $1 + 0.02 = 1.02$ | $y(0.2) \approx 1.02$ |
| 3 | $0.2 + 0.1 = 0.3$ | $1.02 + 0.0404 = 1.0604$ | $y(0.3) \approx 1.0604$ |
| 4 | $0.3 + 0.1 = 0.4$ | $1.0604 + 0.061812 = 1.12221$ | $y(0.4) \approx 1.12221$ |
| 5 | $0.4 + 0.1 = 0.5$ | $1.12221 + 0.0848885 = 1.2071$ | $y(0.5) \approx 1.2071$ |
| 6 | $0.5 + 0.1 = 0.6$ | $1.2071 + 0.110355 = 1.31746$ | $y(0.6) \approx 1.31746$ |
| 7 | $0.6 + 0.1 = 0.7$ | $1.31746 + 0.139047 = 1.4565$ | $y(0.7) \approx 1.4565$ |
| 8 | $0.7 + 0.1 = 0.8$ | $1.4565 + 0.171955 = 1.62846$ | $y(0.8) \approx 1.62846$ |
| 9 | $0.8 + 0.1 = 0.9$ | $1.62846 + 0.210277 = 1.83873$ | $y(0.9) \approx 1.83873$ |
| 10 | $0.9 + 0.1 = 1$ | $1.83873 + 0.255486 = 2.09422$ | $y(1) \approx 2.09422$ |

Runge-Trapezoidal

| 0 | 0 | 1 | y(0)=1 |
|---|---|---|---|
| $i$ | $t_i$ | $y_i$ | So |
| 1 | $0 + 0.1 = 0.1$ | 1.01 | $y(0.1) \approx 1.01$ |
| 2 | $0.1 + 0.1 = 0.2$ | 1.04035 | $y(0.2) \approx 1.04035$ |
| 3 | $0.2 + 0.1 = 0.3$ | 1.09197 | $y(0.3) \approx 1.09197$ |
| 4 | $0.3 + 0.1 = 0.4$ | 1.16645 | $y(0.4) \approx 1.16645$ |
| 5 | $0.4 + 0.1 = 0.5$ | 1.2661 | $y(0.5) \approx 1.2661$ |
| 6 | $0.5 + 0.1 = 0.6$ | 1.39414 | $y(0.6) \approx 1.39414$ |
| 7 | $0.6 + 0.1 = 0.7$ | 1.55478 | $y(0.7) \approx 1.55478$ |
| 8 | $0.7 + 0.1 = 0.8$ | 1.75355 | $y(0.8) \approx 1.75355$ |
| 9 | $0.8 + 0.1 = 0.9$ | 1.99751 | $y(0.9) \approx 1.99751$ |
| 10 | $0.9 + 0.1 = 1$ | 2.29576 | $y(1) \approx 2.29576$ |

Runge-Midpoint

| 0 | 0 | 1 | y(0)=1 |
|---|---|---|---|
| $i$ | $t_i$ | $y_i$ | So |
| 1 | $0 + 0.1 = 0.1$ | 1.01 | $y(0.1) \approx 1.01$ |
| 2 | $0.1 + 0.1 = 0.2$ | 1.0403 | $y(0.2) \approx 1.0403$ |
| 3 | $0.2 + 0.1 = 0.3$ | 1.09182 | $y(0.3) \approx 1.09182$ |
| 4 | $0.3 + 0.1 = 0.4$ | 1.16613 | $y(0.4) \approx 1.16613$ |
| 5 | $0.4 + 0.1 = 0.5$ | 1.26556 | $y(0.5) \approx 1.26556$ |
| 6 | $0.5 + 0.1 = 0.6$ | 1.39328 | $y(0.6) \approx 1.39328$ |
| 7 | $0.6 + 0.1 = 0.7$ | 1.55351 | $y(0.7) \approx 1.55351$ |
| 8 | $0.7 + 0.1 = 0.8$ | 1.75172 | $y(0.8) \approx 1.75172$ |
| 9 | $0.8 + 0.1 = 0.9$ | 1.99497 | $y(0.9) \approx 1.99497$ |
| 10 | $0.9 + 0.1 = 1$ | 2.2923 | $y(1) \approx 2.2923$ |

For reference the actual answer is $2e^{0.5} - 1 \approx 2.29744254140025629369730157562 83$.