MATH 246: Chapter 2 Section 6: Undetermined Coefficients
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Main Topics:

- General Idea
- Solution Building

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1. **Introduction**

   Remember where we are: We have a non-homogenous linear differential equation with constant coefficients. We know how to deal with the homogeneous version and we know that all we need to do is get ahold of a single solution to the non-homogeneous version denoted $Y_p(t)$ and then we can construct all solutions to the non-homogeneous version.

2. **General Idea**

   The Method of Undetermined Coefficients will only work if the right side of the diffential equation (the forcing function $f(t)$) is one of our nice forms like e^{2t} or t^2 or $e^{2t}\cos(5t)$. It is based on the premise that we know what the answer looks like and we only need to work out some coefficients.

   Just to warm up before we get all formal:

If $f(t) = e^{5t}$	then $Y_p(t)$ probably looks like Ce^{5t}
If $f(t) = \cos(2t)$	then $Y_p(t)$ probably looks like $C_1\cos(2t) + C_2\sin(2t)$
If $f(t) = te^{5t}$	then $Y_p(t)$ probably looks like $C_1e^{5t} + C_2te^{5t}$ or $(C_1 + C_2t)e^{5t}$

   The Method of Undetermined Coefficients will work as follows - we'll suggest what the solution looks like, with unknown constants, then we'll plug it into the DE and find the constants that make it work.

   Example: If we have $y'' - y = 2e^{5t}$ and we suggest that $Y_p(t) = Ce^{5t}$, then $Y_p'(t) = 5Ce^{5t}$ and $Y_p''(t) = 25Ce^{5t}$. If we plug these into the DE we get $25Ce^{5t} - Ce^{5t} = 2e^{5t}$ and so $24Ce^{5t} = 2e^{5t}$ and so $C = \frac{1}{12}$ and the solution is $Y_p(t) = \frac{1}{12}e^{5t}$. Wow, that was easy!

3. **Building A Soluion**

   This is very procedural and works as follows. Here $Q_n(t)$ means a known polynomial of degree n and GP_n means and unknown generic polynomial of degree n with undetermined coefficients for which we fill in A, B, C, etc. For example $GP_2 = At^2 + Bt + C$. This is actually far easier in practice than it looks, the most common mistake is forgetting to check the multiplicity.

   - If $f(t)$ has the form $Q_n(t)e^{rt}$, first find m the multiplicity of z as a root of the characteristic polynomial. Often $m = 0$.
     $\Longrightarrow Y_p(t) = t^m\,[GP_n]\,e^{rt}$
   - If $f(t)$ has the form $Q_n(t)e^{rt}\cos(st)$ or $Q_n(t)e^{rt}\cos(st)$, first find m the multiplicity of $r + si$ as a root of the characteristic polynomial. Often $m = 0$.
     $\Longrightarrow Y_p(t) = t^m\,[GP_n]\,e^{rt}\cos(st) + t^m\,[GP_n]\,e^{rt}\sin(st)$
     Here the GP_n are different with different coefficients!
   - If $f(t)$ is the sum of such forms, we add the resulting forms together. Make sure to never, ever, repeat undetermined coefficients.

4. **What to do with the Solution**

   Once we have our guess we plug it into the DE and simplify like mad. The result will have similar functions on the left and right but the coefficients on the left will be the unknowns. We then equate the coefficients on each side and solve for those unknowns.

   Theoretical Note: What allows us to do this last step is that the functions in the solution are linearly independent according to our construction of our solution.

5. **Examples:**

Example: Consider $y'' + 3y' + 2y = 2e^{3t}$.

Forcing: $f(t) = 2e^{3t}$.
CP: $p(z) = z^2 + 3z + 2 = (z+2)(z+1)$.

We have $r = 3$ which is not a root of $p(z)$ so $m = 0$. Because the coefficient polynomial 2 is degree 0, we have:

$$Y_p(t) = t^m \left[GP_0\right] e^{3t} = t^0(A)e^{3t} = Ae^{3t}$$

Then $Y_p' = 3Ae^{3t}$ and $Y_p'' = 9Ae^{3t}$. We plug these into the DE to get

$$\begin{aligned} 9Ae^{3t} + 3(3Ae^{3t}) + 2Ae^{3t} &= 2e^{3t} \\ 20Ae^{3t} &= 2e^{3t} \\ A &= \frac{1}{10} \end{aligned}$$

Then $Y_p(t) = \frac{1}{10}e^{3t}$.

Note: Because the fundamental set for the homogeneous version is $\left\{e^{-2t}, e^{-t}\right\}$ the general solution is $Y(t) = \frac{1}{10}e^{3t} + C_1e^{-2t} + C_2e^{-t}$.

Example: Consider $y'' - 3y' + 2y = 7e^{2t}$.

Forcing: $f(t) = 7e^{2t}$.
CP: $p(z) = z^2 - 3z + 2 = (z-2)(z-2)$.

We have $r = 2$ which is a root of $p(z)$ of multiplicity $m = 1$. Because the coefficient polynomial 7 is degree 0, we have:

$$Y_p(t) = t^m \left[GP_0\right] e^{2t} = t^1(A)e^{2t} = Ate^{2t}$$

Then $Y_p' = Ae^{2t} + 2Ate^{2t}$ and $Y_p'' = 4Ae^{2t} + 4Ate^{2t}$. We plug these into the DE to get

$$\begin{aligned} 4Ae^{2t} + 4Ate^{2t} - 3(Ae^{2t} + 2Ate^{2t}) + 2(Ate^{2t}) &= 7e^{2t} \\ Ae^{2t} &= 7e^{2t} \\ A &= 7 \end{aligned}$$

Then $Y_p(t) = 7te^{2t}$.

Note: Because the fundamental set for the homogeneous version is $\left\{e^{2t}, e^{t}\right\}$ the general solution is $Y(t) = 7te^{2t} + C_1e^{2t} + C_2e^{t}$.

Example: Consider $y''' - y'' = t^2$.

Forcing: $f(t) = t^2 = t^2 e^{0t}$.
CP: $p(z) = z^3 - z^2 = z^2(z-1)$.

We have $r = 0$ which is a root of $p(z)$ of multiplicity $m = 2$. Because the coefficient polynomial t^2 is degree 2, we have:

$$Y_p(t) = t^m \left[GP_2\right] e^{0t} = t^2(At^2 + Bt + C)e^{0t} = At^4 + Bt^3 + Ct^2$$

Then $Y_p' = 4At^3 + 3Bt^2 + 2Ct$, $Y_p'' = 12At^2 + 6Bt + 2C$ and $Y_p''' = 24At + 6B$. We plug these into the DE to get

$$\begin{aligned} 24At + 6B - (12At^2 + 6Bt + 2C) &= t^2 \\ -12At^2 + (24A - 6B)t + (6B - 2C) &= t^2 \end{aligned}$$

Therefore $-12A = 1$, $24A - 6B = 0$ and $6B - 2C = 0$, giving $A = -\frac{1}{12}$, $B = -\frac{1}{4}$ and $C = -\frac{3}{4}$.

Then $Y_p(t) = -\frac{1}{12}t^4 - \frac{1}{4}t^3 - \frac{3}{4}t^2$.

Note: Because the fundamental set for the homogeneous version is $\{1, t, e^t\}$ the general solution is $Y(t) = -\frac{1}{12}t^4 - \frac{1}{4}t^3 - \frac{3}{4}t^2 + C_1 + C_2 t + C_3 e^t$.

Example: Consider $y'' - 3y' + 2y = 5te^{4t}$.

Forcing: $f(t) = 5te^{4t}$.
CP: $p(z) = z^2 - 3z + 2 = (z-1)(z-1)$

We have $r = 4$ which is not a root of $p(z)$ so $m = 0$. Because the coefficient polynomial $5t$ is degree 1, we have:

$$Y_p(t) = t^m \left[GP_1\right] e^{4t} = t^0(At + B)e^{4t} = (At + B)e^{4t}$$

Then $Y_p' = Ae^{4t} + (4At + 4B)e^{4t}$ and $Y_p'' = 8Ae^{4t} + (16At + 16B)e^{4t}$. We plug these into the DE to get

$$\begin{aligned} 8Ae^{4t} + (16At + 16B)e^{4t} - 3(Ae^{4t} + (4At + 4B)e^{4t}) + 2((At + B)e^{4t}) &= 5te^{4t} \\ (5A + 6B)e^{4t} + (6A)te^{4t} &= 5te^{4t} \\ (5A + 6B) + (6A)t &= 5t \end{aligned}$$

Therefore $5A + 6B = 0$ and $6A = 5$, giving $A = \frac{5}{6}$ and $B = -\frac{25}{36}$.

Then $Y_p(t) = \left(\frac{5}{6}t - \frac{25}{36}\right) e^{4t}$.

Note: Because the fundamental set for the homogeneous version is $\left\{e^{2t}, e^t\right\}$ the general solution is $Y(t) = \left(\frac{5}{6}t - \frac{25}{36}\right) e^{4t} + C_1 e^{2t} + C_2 e^t$.

Example: Consider $y'' + y' = t + 3e^t$.

Forcing: $f(t) = t + 3e^t = te^{0t} + 3e^{1t}$ which has two parts.
CP: $p(z) = z^2 + z = z(z+1)$.

For the te^{0t} part we have $r = 0$ which is a root of $p(z) = z^2 + z = z(z+1)$ of multiplicity $m = 1$. Because the coefficient polynomial t is degree 1, we have:

$$\text{First Part: } Y_p(t) = t^1 \left[GP_1\right] e^{0t} = t(At + B)e^{0t} = At^2 + Bt$$

For the $3e^{1t}$ part we have $r = 1$ which is not a root of $p(z)$ so $m = 0$. Because the coefficient polynomial 3 is degree 0, we have:

$$\text{Second Part: } Y_p(t) = t^m \left[GP_0\right] e^t = t^0(C)e^t = Ce^t$$

Combining these we have:

$$Y_p(t) = At^2 + Bt + Ce^t$$

Then $Y_p' = 2At + B + Ce^t$ and $Y_p'' = 2A + Ce^t$. We plug these into the DE to get

$$\begin{aligned} 2A + Ce^t + 2At + B + Ce^t &= t + 3e^t \\ 2At + (2A + B) + 2Ce^t &= t + 3e^t \end{aligned}$$

Therefore $A = \frac{1}{2}$, $B = -1$ and $C = \frac{3}{2}$.

Then $Y_p(t) = \frac{1}{2}t^2 - t + \frac{3}{2}e^t$.

Note: Because the fundamental set for the homogeneous version is $\{1, e^{-t}\}$ the general solution is $Y(t) = \frac{1}{2}t^2 - t + \frac{3}{2}e^t + C_1 + C_2e^{-t}$.

Example: Consider $y'' + 2y' + 2y = 17\cos(3t)$.

Forcing: $f(t) = 17\cos(3t) = 17e^{0t}\cos(3t)$.
CP: $p(z) = z^2 + 2z + 2$ with roots $z = 1 \pm 1i$.

We have $r + si = 0 + 3i$ which is not a root of $p(z)$ so $m = 0$. Because the coefficient polynomial 17 is degree 0, we have:

$$\begin{aligned} Y_p(t) &= t^m \left[GP_0\right]\cos(3t) + t^m \left[GP_0\right]\sin(3t) \\ &= t^0(A)\cos(3t) + t^0(B)\sin(3t) = A\cos(3t) + B\sin(3t) \end{aligned}$$

Then $Y_p' = -3A\sin(3t) + 3B\cos(3t)$ and $Y_p'' = -9A\cos(3t) - 9B\sin(3t)$. We plug these into the DE to get

$$\begin{aligned} -9A\cos(3t) - 9B\sin(3t) + 2(-3A\sin(3t) + 3B\cos(3t)) + 2(A\cos(3t) + B\sin(3t)) &= 17\cos(3t) \\ (-7A + 6B)\cos(3t) + (-6A - 7B)\sin(3t) &= 17\cos(3t) \end{aligned}$$

Therefore $-7A + 6B = 17$ and $-6A - 7B = 0$, giving $A = -\frac{7}{5}$ and $B = \frac{6}{5}$.

Then $Y_p(t) = -\frac{7}{5}\cos(3t) + \frac{6}{5}\sin(3t)$.

Note: Because the fundamental set for the homogeneous version is $\{e^{-t}\cos(t), e^{-t}\sin(t)\}$ the general solution is $Y(t) = -\frac{7}{5}\cos(3t) + \frac{6}{5}\sin(3t) + C_1e^{-t}\cos(t) + C_2e^{-t}\sin(t)$

6. **Unfinished Examples:**

Example: Consider $y'' + 2y' + y = t^2 e^{-t} + 17te^t \cos(3t)$.

Forcing: $f(t) = t^2 e^{-t} + 17te^t \cos(3t)$ which has two parts.
CP: $p(z) = z^2 + 2z + 1 = (z+1)^2$.

For the $t^2 e^{-t}$ part we have $r = -1$ which is a root of $p(z)$ of multiplicity 2. Because the coefficient polynomial t^2 is degree 2 we have:

$$\text{First Part: } Y_p(t) = t^m \left[GP_2\right] e^{-t} = t^2(At^2 + Bt + C)e^{-t} = (At^4 + Bt^3 + Ct^2)e^{-t}$$

For the $17te^t \cos(3t)$ part we have $r + si = 1 + 3i$ which is not a root of $p(z)$ so $m = 0$. Because the coefficient polynomial $17t$ is degree 1 we have:

$$\begin{aligned}\text{Second Part: } Y_p(t) &= t^m \left[GP_1\right] e^t \cos(3t) + t^m \left[GP_1\right] e^t \sin(3t) \\ &= t^0(Dt + E)e^t \cos(3t) + t^0(Ft + G)e^t \sin(3t) \\ &= (Dt + E)e^t \cos(3t) + (Ft + G)e^t \sin(3t)\end{aligned}$$

Combining these we have:

$$Y_p(t) = (At^4 + Bt^3 + Ct^2)e^{-t} + (Dt + E)e^t \cos(3t) + (Ft + G)e^t \sin(3t)$$