MATH 246: Chapter 2 Section 7: Variation of Parameters
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Main Topics:

- Restrictions.
- General Idea and Method.

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1. **Introduction**

   For the last chapter we've been focusing on finding a single solution $Y_P(t)$ to a non-homogeneous linear differential equation with constant coefficients where $f(t)$ is a familiar form.

   What we're going to do now is remove both the restriction that the coefficients be constant and the restriction that $f(t)$ is of our familiar form. We will restrict to second-order though, and we'll make sure the coefficient of y'' is 1 (linear normal form), which can easily be attained through division. The goal will be the same, to find some $Y_P(t)$, because again the general solution will be $Y(t) = Y_p(t) + c_1Y_1(t) + c_2Y_2(t)$ where $\{Y_1(t), Y_2(t)\}$ is the fundamental set for the homogeneous version.

   It may seem like if we could do this then why would we need the previous chapter? The answer is that the method of this section gets extremely complicated for third and higher order and can be computationally intensive even for second order. However it has the main advantage of providing a formulaic solution.

2. **General Idea**

   The general idea is to start with the fundamental set $\{Y_1, Y_2\}$ for the homogeneous version and ask a simple question - is it possible to find two function $u_1(t)$ and $u_2(t)$ such that $y = u_1Y_1 + u_2Y_2$ is a solution to the nonhomogeneous version?

   It turns out that simply plugging this y into the DE leaves us with quite a mess:

$$\begin{aligned}
y'' + a(t)y' + b(t)y &= f(t) \\
(u_1Y_1 + u_2Y_2)'' + a(t)(u_1Y_1 + u_2Y_2)' + b(t)(u_1Y_1 + u_2Y_2) &= f(t) \\
\underbrace{(u_1'Y_1 + u_1Y_1' + u_2'Y_2 + u_2Y_2')' + a(t)(u_1'Y_1 + u_1Y_1' + u_2'Y_2 + u_2Y_2') + b(t)(u_1Y_1 + u_2Y_2)}_{\text{Quite a Mess!}} &= f(t)
\end{aligned}$$

   However if we look at this mess we notice that $u_1'Y_1 + u_2'Y_2$ shows up in two places. It turns out that if $u_1'Y_1 + u_2'Y_2 = 0$ then the above "Quite a Mess" tidies up:

$$\begin{aligned}
(u_1Y_1' + u_2Y_2')' + a(t)(u_1Y_1' + u_2Y_2') + b(t)(u_1Y_1 + u_2Y_2) &= f(t) \\
u_1'Y_1' + u_1Y_1'' + u_2'Y_2' + u_2Y_2'' + a(t)(u_1Y_1' + u_2Y_2') + b(t)(u_1Y_1 + u_2Y_2) &= f(t) \\
u_1\underbrace{(Y_1'' + a(t)Y_1' + b(t)Y_1)}_{=0\text{ bc homog soln}} + u_2\underbrace{(Y_2'' + a(t)Y_2' + b(t)Y_2)}_{=0\text{ bc homog soln}} + u_1'Y_1' + u_2'Y_2' &= f(t) \\
u_1'Y_1' + u_2'Y_2' &= f(t)
\end{aligned}$$

The practical upshot of all this is that if we can find u_1 and u_2 satisfying the system

$$\begin{aligned} u_1'Y_1 + u_2'Y_2 &= 0 \\ u_1'Y_1' + u_2'Y_2' &= f(t) \end{aligned}$$

then $Y_p = u_1Y_1 + u_2Y_2$ will be a solution to the nonhomogeneous DE.

Conveniently this is easy to solve because it's a system of two equations and two unknowns where the unknowns are u_1' and u_2'.

Linear algebra gives us a generic formula because this is a matrix equation:

$$\begin{aligned} \begin{bmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} &= \begin{bmatrix} 0 \\ f(t) \end{bmatrix} \\ \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} &= \begin{bmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ f(t) \end{bmatrix} \\ \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} &= \frac{1}{W[Y_1, Y_2]} \begin{bmatrix} Y_2' & -Y_2 \\ -Y_1' & Y_1 \end{bmatrix} \begin{bmatrix} 0 \\ f(t) \end{bmatrix} \\ \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} &= \frac{1}{W[Y_1, Y_2]} \begin{bmatrix} -Y_2 f(t) \\ Y_1 f(t) \end{bmatrix} \end{aligned}$$

Thus:

$$u_1' = -\tfrac{Y_2 f(t)}{W[Y_1,Y_2]} \text{ and } u_2' = \tfrac{Y_1 f(t)}{W[Y_1,Y_2]}$$

Then

$$u_1 = -\int \tfrac{Y_2 f(t)}{W[Y_1,Y_2]}\ dt \text{ and } u_2 = \int \tfrac{Y_1 f(t)}{W[Y_1,Y_2]}\ dt$$

and the final result is:

$$Y_p = u_1Y_1 + u_2Y_2$$

It's worth noting that it's sometimes messy but we can directly write down a formula for $Y_p(t)$:

$$Y_p(t) = -Y_1 \int \frac{Y_2 f(t)}{W[Y_1, Y_2]}\ dt + Y_2 \int \frac{Y_1 f(t)}{W[Y_1, Y_2]}\ dt$$

3. **Examples**

Example: Consider $y'' + y = \sec t$.

Since the characteristic polynomial is $z^2 + 1$ with roots $0 \pm 1i$ the fundamental set for the homogeneous version is $\{\cos t, \sin t\}$.

We find

$$W[Y_1, Y_2] = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1$$

and then we simply evaluate:

$$\begin{array}{ll} u_1 = -\int \frac{(\sin t)(\sec t)}{1}\ dt = -\int \tan t\ dt = \ln|\cos t| + C & \text{Choose } u_1 = \ln|\cos t| \\ u_2 = \int \frac{(\cos t)(\sec t)}{1}\ dt = \int 1\ dt = t + C & \text{Choose } u_2 = t \end{array}$$

Thus a particular solution to the nonhomogeneous version is

$$Y_p(t) = u_1 Y_1 + u_2 Y_2 = (\ln|\cos t|)\cos t + t \sin t$$

and the general solution to the nonhomogeneous version is

$$Y_p(t) = (\ln|\cos t|)\cos t + t\sin t + c_1 \cos t + c_2 \sin t$$

Example: Consider $y'' - 3y' + 2y = t$.

Since the characteristic polynomial is $z^2 - 3z + 2 = (z-1)(z-2)$ with roots $1, 2$ the fundamental set for the homogeneous version is $\{e^t, e^{2t}\}$.

We find

$$W[Y_1, Y_2] = \begin{vmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{vmatrix} = e^{3t}$$

and then we simply evaluate (some IBP here):

$$\begin{array}{ll} u_1 = -\int \frac{e^{2t} t}{e^{3t}}\ dt = -\int te^{-t}\ dt = te^{-t} - e^{-t} + C & \text{Choose } u_1 = te^{-t} - e^{-t} \\ u_2 = \int \frac{e^t t}{e^{3t}}\ dt = \int te^{-2t} = -\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t} + C & \text{Choose } u_2 = -\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t} \end{array}$$

Thus a particular solution to the nonhomogeneous version is

$$Y_p(t) = u_1 Y_1 + u_2 Y_2 = \left(te^{-t} - e^{-t}\right)e^t + \left(-\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t}\right)e^{2t} = \frac{1}{2}t - \frac{3}{4}$$

and the general solution to the nonhomogeneous version is

$$Y(t) = \frac{1}{2}t - \frac{3}{4} + c_1 e^t + c_2 e^{3t}$$

Side Note: The Method of Undetermined Coefficients is much nicer for this problem.

Example: Consider $(t^2+1)y'' - 2ty' + 2y = (t^2+1)^2$.

First we rewrite as $y'' - \frac{2t}{t^2+1}y' + \frac{2}{t^2+1}y = t^2+1$. It's worth noting that even though this looks uglier the only thing it affects that we need is the right side. We have no technique for finding the fundamental set for the homogeneous version so I'll just give it to you, it's $\{t, t^2-1\}$.

We find

$$W[Y_1, Y_2] = \begin{vmatrix} t & t^2-1 \\ 1 & 2t \end{vmatrix} = t^2+1$$

and then we simply evaluate:

$$\begin{array}{ll} u_1 = -\int \frac{(t^2-1)(t^2+1)}{t^2+1}\ dt = -\int t^2-1\ dt = -\frac{1}{3}t^3 + t + C & \text{Choose } u_1 = -\frac{1}{3}t^3 + t \\ u_2 = \int \frac{(t)(t^2+1)}{t^2+1}\ dt = \int t\ dt = \frac{1}{2}t^2 + C & \text{Choose } u_2 = \frac{1}{2}t^2 \end{array}$$

Thus a particular solution to the nonhomogeneous version is

$$Y_p(t) = u_1Y_1 + u_2Y_2 = \left(-\frac{1}{3}t^3 + t\right)t + \left(\frac{1}{2}t^2\right)(t^2-1) = \frac{1}{6}t^4 + \frac{1}{2}t^2$$

and the general solution to the nonhomogeneous version is

$$Y(t) = \frac{1}{6}t^4 + \frac{1}{2}t^2 + c_1t + c_2(t^2-1)$$

We can make this a pretty nice IVP by adding the condition $Y(1) = 0$ and $Y'(1) = 1$. Since $Y'(t) = \frac{2}{3}t^3 + t + c_1 + 2c_2t$ we then have

$$\begin{aligned} Y(1) &= \frac{1}{6} + \frac{1}{2} + c_1 = 0 \\ Y'(1) &= \frac{2}{3} + 1 + c_1 + 2c_2 = 1 \end{aligned}$$

so then $c_1 = -\frac{2}{3}$ and $c_2 = 0$ so the specific solution to the IVP is

$$Y(t) = \frac{1}{6}t^4 + \frac{1}{2}t^2 - \frac{2}{3}t$$