MATH 246: Chapter 2 Section 9: Laplace Transforms
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Main Topics:

- Formal Definition
- Rules
- Reversing the Rules
- Derivative Rules
- Solving IVPs
- Step Functions with Laplace Transforms

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1. **Introduction:** Laplace transforms are a way of changing one function into another function. Basically we start with a function of t and change it to a function of s. We can also do the reverse. The Laplace transform has some really useful properties which will help us solve initial value problems.

2. **Formal Definition:** If $f(t)$ is a function then the Laplace transform of this function is formally defined by:

$$\mathcal{L}\left[y(t)\right](s) = \int_0^\infty y(t)e^{-st}\,dt = \lim_{b\to\infty}\int_0^b y(t)e^{-st}\,dt$$

For an unknown $y(t)$ we will often write $\mathcal{L}\left[y(t)\right]$ or just $\mathcal{L}\left[y\right]$ for readability. This formal definition is used to build a set of rules and the rules are what we'll use.

Example: If $y(t) = 1$ then we get:

$$\begin{aligned}
\mathcal{L}\left[1\right] &= \int_0^\infty 1e^{-st}\,dt \\
&= \lim_{b\to\infty}\int_0^b e^{-st}\,dt \\
&= \lim_{b\to\infty} -\frac{1}{s}e^{-st}\Big|_0^b \\
&= \lim_{b\to\infty} -\frac{1}{s}e^{-sb} + \frac{1}{s}e^{-s(0)} \\
&= \frac{1}{s}
\end{aligned}$$

Thus $\mathcal{L}\left[1\right] = \frac{1}{s}$.

Example: If $y(t) = t$ then we get:

$$\begin{aligned}
\mathcal{L}[t] &= \int_0^\infty te^{-st}\,dt \\
&= \lim_{b\to\infty} \int_0^b te^{-st}\,dt \\
&= \lim_{b\to\infty} t\left(-\frac{1}{s}\right)e^{-st}\Big|_0^b - \int_0^b \left(-\frac{1}{s}\right)e^{-st}\,dt \\
&= \lim_{b\to\infty} t\left(-\frac{1}{s}\right)e^{-st}\Big|_0^b - \frac{1}{s^2}e^{-st}\Big|_0^b \\
&= \lim_{b\to\infty} \left[-\frac{b}{s}e^{-sb} - 0\right] - \left[\frac{1}{s^2}e^{-sb} - \frac{1}{s^2}\right] \\
&= \frac{1}{s^2}
\end{aligned}$$

Thus $\mathcal{L}[1] = \frac{1}{s^2}$.

3. **Function Rules**

Using this same approach we can prove the following rules for common functions:

Function	Example
$\mathcal{L}[0] = 0$	n/a
$\mathcal{L}[c] = \frac{c}{s}$	$\mathcal{L}[42] = \frac{42}{s}$
$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$	$\mathcal{L}\left[t^3\right] = \frac{3!}{s^4}$
$\mathcal{L}[e^{at}] = \frac{1}{s-a}$	$\mathcal{L}\left[e^{5t}\right] = \frac{1}{s-5}$
$\mathcal{L}[\cos(bt)] = \frac{s}{s^2+b^2}$	$\mathcal{L}[\cos(7t)] = \frac{s}{s^2+49}$
$\mathcal{L}[\sin(bt)] = \frac{b}{s^2+b^2}$	$\mathcal{L}[\sin(7t)] = \frac{7}{s^2+49}$
$\mathcal{L}[e^{at}t^n] = \frac{n!}{(s-a)^{n+1}}$	$\mathcal{L}\left[e^{2t}t^4\right] = \frac{4!}{(s-2)^5}$
$\mathcal{L}[e^{at}\cos(bt)] = \frac{s-a}{(s-a)^2+b^2}$	$\mathcal{L}\left[e^{5t}\cos(3t)\right] = \frac{s-5}{(s-5)^2+9}$
$\mathcal{L}[e^{at}\sin(bt)] = \frac{b}{(s-a)^2+b^2}$	$\mathcal{L}\left[e^{5t}\sin(3t)\right] = \frac{3}{(s-5)^2+9}$
$\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)]$	$\mathcal{L}[2+5t] = \mathcal{L}[2] + 5\mathcal{L}[t] = \frac{2}{s} + 5\left(\frac{1}{s^2}\right)$

Notation: Sometimes if a function is denoted $y(t)$ then some sources use $Y(s)$ instead of $\mathcal{L}[y(t)]$ for convenience. I will tend to not do this much.

4. **Reversing** If we start with $\mathcal{L}[y]$ we can also work in reverse. Here are some examples with comments because sometimes we need to manipulate the function first. This can also be written with inverse notation.

Example: If $\mathcal{L}[y] = \frac{2}{s}$ then $y(t) = 2$. aka $\mathcal{L}^{-1}\left[\frac{2}{s}\right] = s$.

Example: If $\mathcal{L}[y] = \frac{1}{s^5}$ then first we rewrite $\mathcal{L}[y] = \frac{1}{4!}\left(\frac{4!}{s^5}\right)$ and then we see $y(t) = \frac{1}{24}t^4$. aka $\mathcal{L}^{-1}\left[\frac{1}{s^5}\right] = \frac{1}{24}t^4$.

Example: If $\mathcal{L}[y] = \frac{1}{s+4}$ we think of it as $\mathcal{L}[y] = \frac{1}{s-(-4)}$ then $y(t) = e^{-4t}$.

Example: If $\mathcal{L}[y] = \frac{-18}{s^2+9}$ then first rewrite to get $\mathcal{L}[y] = -6\left(\frac{3}{s^2+9}\right)$ and then we see $y(t) = -6\sin(3t)$.

Example: If $\mathcal{L}[y] = \frac{2}{s^2+s}$ then first we see $\mathcal{L}[y] = \frac{2}{s(s+1)}$ and then we need to rewrite with partial fractions first and then following this we need a bit more rewriting to fit the formulas so $\mathcal{L}[y] = \frac{2}{s} - \frac{2}{s+1} = \frac{2}{s} - 2\left(\frac{1}{s-(-1)}\right)$, and then we see $y(t) = 2 - 2e^{-t}$.

Example: If $\mathcal{L}[y] = \frac{4s+3}{s^2+25}$ then we need to break it up and rewrite a little to fit the formulas:
$\mathcal{L}[y] = \frac{4s+3}{s^2+25} = \frac{4s}{s^2+25} + \frac{3}{s^2+25} = 4\left(\frac{s}{s^2+25}\right) + \frac{3}{5}\left(\frac{5}{s^2+25}\right)$ and then we see that
$y(t) = 4\cos(5t) + \frac{3}{5}\sin(5t)$.

Example: If $\mathcal{L}[y] = \frac{s+1}{s^2-4s+5}$ then the denominator doesn't factor so instead we complete the square and then do a bit more rewriting to get $\mathcal{L}[y] = \frac{s+1}{(s-2)^2+1} = \frac{s-2}{(s-2)^2+1} + \frac{3}{(s-2)^2+1}$ and then we see that $y(t) = e^{2t}\cos(t) + 3e^{2t}\sin(t)$.

5. **Derivative Rules**

It turns out that the Laplace transfer is nice with derivatives of functions too.

Example: Observe that:

$$\begin{aligned}
\mathcal{L}\left[y'(t)\right] &= \lim_{b\to\infty} \int_0^b y'(t)e^{-st}\,dt \\
&= \lim_{b\to\infty} \left. y(t)e^{-st}\right|_0^b + s\int_0^b y(t)e^{-st}\,dt \\
&= \lim_{b\to\infty} \left[y(b)e^{-sb} - y(0)\right] + s\mathcal{L}\left[y(t)\right] \\
&= -y(0) + s\mathcal{L}\left[y(t)\right] \\
&= s\mathcal{L}\left[y(t)\right] - y(0)
\end{aligned}$$

In general we have the following pattern for an unknown $y(t)$:

$$\begin{aligned}
\mathcal{L}\left[y'\right] &= s\mathcal{L}\left[y(t)\right] - y(0) & &= s\mathcal{L}\left[y\right] - y(0) \\
\mathcal{L}\left[y''\right] &= s^2\mathcal{L}\left[y(t)\right] - sy(0) - y'(0) & &= s^2\mathcal{L}\left[y\right] - sy(0) - y'(0) \\
\mathcal{L}\left[y'''\right] &= s^3\mathcal{L}\left[y(t)\right] - s^2y(0) - sy'(0) - y''(0) & &= s^3\mathcal{L}\left[y\right] - s^2y(0) - sy'(0) - y''(0) \\
&\vdots & &\vdots
\end{aligned}$$

For example if $y(t)$ is unknown but we know $y(0) = 7$ and $y'(0) = -3$ then the second rule tells us that

$$\begin{aligned}
\mathcal{L}\left[y''\right] &= s^2\mathcal{L}\left[y\right] - sy(0) - y'(0) \\
&= s^2\mathcal{L}\left[y\right] - s(7) - (-3) \\
&= s^2\mathcal{L}\left[y\right] - 7s + 3
\end{aligned}$$

We'll see very soon why this is significant.

6. **Solving Initial Value Problems**

Laplace Transforms are incredibly useful for dealing with IVPs when $t_I = 0$. Other values of t_I can be dealt with using a function shift but we won't deal with those here.

When dealing with such an initial value problem our approach will be the following:

(a) Take the Laplace transform of each side.

(b) Apply the rules for functions and for derivatives to eliminate all the t, all the derivatives and substitute all the initial values.

(c) Solve the result for $\mathcal{L}[y]$.

(d) Reverse the Laplace transform to get the solution $y(t)$.

Example: Suppose we have $y' = 3$ with $y(0) = 1$. We do the following:

$$\begin{aligned} y' &= 3 \\ \mathcal{L}[y'] &= \mathcal{L}[3] \\ s\mathcal{L}[y] - y(0) &= \frac{3}{s} \\ s\mathcal{L}[y] - 1 &= \frac{3}{s} \\ s\mathcal{L}[y] &= \frac{3}{s} + 1 \\ \mathcal{L}[y] &= \frac{3}{s^2} + \frac{1}{s} \\ y(t) &= 3t + 1 \end{aligned}$$

And we've solved it! Notice that you really need to understand how the various tables are being used here. The Laplace tranform table is used at the beginning and end and the derivative rules are also used early on.

Example: Suppose we have $y'' - 2y' - 3y = 0$ with $y(0) = 1$ and $y'(0) = 4$. We do the following:

$$\begin{aligned}
y'' - 2y' - 3y &= 0 \\
\mathcal{L}[y''] - 2\mathcal{L}[y'] - 3\mathcal{L}[y] &= \mathcal{L}[0] \\
\left(s^2\mathcal{L}[y] - sy(0) - y'(0)\right) - 2\left(s\mathcal{L}[y] - y(0)\right) - 3\mathcal{L}[y] &= 0 \\
s^2\mathcal{L}[y] - s - 4 - 2s\mathcal{L}[y] + 2 - 3\mathcal{L}[y] &= 0 \\
\mathcal{L}[y]\left(s^2 - 2s - 3\right) - s - 2 &= 0 \\
\mathcal{L}[y]\left(s^2 - 2s - 3\right) &= s + 2 \\
\mathcal{L}[y] &= \frac{s+2}{s^2 - 2s - 3} \\
\mathcal{L}[y] &= \frac{s+2}{(s-3)(s+1)}
\end{aligned}$$

Now we need to do some manipulation with partial fractions:

$$\begin{aligned}
\frac{s+2}{(s-3)(s+1)} &= \frac{A}{s-3} + \frac{B}{s+1} \\
s + 2 &= A(s+1) + B(s-3)
\end{aligned}$$

At this point $s = -1$ gives us $B = -1/4$ and $s = 3$ gives us $A = 5/4$. Back to our problem with the most recent line rewritten:

$$\begin{aligned}
\mathcal{L}[y] &= \frac{s+2}{(s-3)(s+1)} \\
\mathcal{L}[y] &= \frac{5/4}{s-3} + \frac{-1/4}{s+1} \\
\mathcal{L}[y] &= \frac{5}{4}\left(\frac{1}{s-3}\right) - \frac{1}{4}\left(\frac{1}{s-(-1)}\right) \\
y(t) &= \frac{5}{4}e^{3t} - \frac{1}{4}e^{-t}
\end{aligned}$$

Example: Suppose we have $y'' + 4y = 2t$ with $y(0) = 1$ and $y'(0) = 0$. We do the following:

$$\begin{aligned} y'' + 4y &= 2t \\ \mathcal{L}\left[y''\right] + 4\mathcal{L}\left[y\right] &= \mathcal{L}\left[2t\right] \\ s^2\mathcal{L}\left[y\right] - sy(0) - y'(0) + 4\mathcal{L}\left[y\right] &= \frac{2}{s^2} \\ s^2\mathcal{L}\left[y\right] - s - 0 + 4\mathcal{L}\left[y\right] &= \frac{2}{s^2} \\ \mathcal{L}\left[y\right](s^2+4) &= \frac{2}{s^2} + s \\ \mathcal{L}\left[y\right] &= \frac{2}{s^2(s^2+4)} + \frac{s}{s^2+4} \end{aligned}$$

This doesn't look so nice. The second part is okay (it's from cos) but the first part is not in our table. Instead we need to break it up with partial fractions:

$$\begin{aligned} \frac{2}{s^2(s^2+4)} &= \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4} \\ 2 &= As(s^2+4) + B(s^2+4) + (Cs+D)s^2 \\ 2 &= (A+C)s^3 + (B+D)s^2 + 4As + 4B \end{aligned}$$

Comparing coefficients gives us $A + C = 0$, $B + D = 0$, $4A = 0$ and $4B = 2$ so that $B = 1/2$, $A = 0$, $D = -1/2$ and $C = 0$ and so back our process with the most recent line rewritten:

$$\begin{aligned} \mathcal{L}\left[y\right] &= \frac{2}{s^2(s^2+4)} + \frac{s}{s^2+4} \\ \mathcal{L}\left[y\right] &= \frac{1/2}{s^2} + \frac{-1/2}{s^2+4} + \frac{s}{s^2+4} \\ \mathcal{L}\left[y\right] &= \frac{1}{2}\left(\frac{1}{s^2}\right) - \frac{1}{4}\left(\frac{2}{s^2+4}\right) + \frac{s}{s^2+4} \\ y(t) &= \frac{1}{2}t - \frac{1}{4}\sin(2t) + \cos(2t) \end{aligned}$$

Compare this to before where we'd need to find the general solution to the homogeneous version of the differential equation, also find a specific solution to the nonhomogeneous version, add them, then use the initial values to find the constants. This way is significantly faster.

Sometimes it's good practice just to do the first part of a problem:

Example: Find the Laplace Transform of the solution to the initial value problem $y'' + y' - 3y = t + e^{2t}\cos(3t)$ with $y(0) = -1$ and $y'(0) = 2$.
Here all we need to get to is $\mathcal{L}[y]$. We do the following:

$$\begin{aligned}
y'' + y' - 3y &= t + e^{2t}\cos(3t) \\
\mathcal{L}[y''] + \mathcal{L}[y'] - 3\mathcal{L}[y] &= \mathcal{L}[t] + \mathcal{L}\left[e^{2t}\cos(3t)\right] \\
s^2\mathcal{L}[y] - sy(0) - y'(0) + s\mathcal{L}[y] - y(0) - 3\mathcal{L}[y] &= \frac{s-2}{(s-2)^2+9} \\
s^2\mathcal{L}[y] + s - 2 + s\mathcal{L}[y] + 1 - 3\mathcal{L}[y] &= \frac{2-s}{(s-2)^2+9} \\
\mathcal{L}[y](s^2 + s - 3) + s - 1 &= \frac{2-s}{(s-2)^2+9} \\
\mathcal{L}[y](s^2 + s - 3) &= \frac{2-s}{(s-2)^2+9} + 1 - s \\
\mathcal{L}[y] &= \frac{2-s}{((s-2)^2+9)(s^2+s-3)} + \frac{1-s}{s^2+s-3}
\end{aligned}$$

To finish, this would need to undergo a partial fractions decomposition and then the rules would need to be applied.

7. **Step Functions** The most basic step function is the function which returns 0 up until (but not including) $t = 0$ and then 1 after that. More specifically we have

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

There are other options. If we want to use a value other than 0 we denote it $u_c(t)$:

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$

Step functions are useful because they turn other functions on and off. For example the product function $u_\pi(t)\sin(t-\pi)$ is 0 for $t < \pi$ and $\sin(t-\pi)$ for $t \geq \pi$.

It may seem odd that we have $\sin(t-\pi)$ here rather than just $\sin(t)$ but there's a reason why this will usually happen. When a function "kicks in" at a certain t-value this usually means that before that t-value the function is 0 and then at that t-value the function begins as though 0 were plugged into it. So for example $u_\pi(t)\sin(t-\pi)$ equals 0 until $t = \pi$ at which point the $\sin(t-\pi)$ part starts behaving as if 0 were plugged in (because of the $t-\pi$ in there).

Example: Suppose a function equals 0 until $t = \pi/4$ and then starts behaving like the sine function, meaning like the sine function does at $t = 0$. This new function would be $u_{\pi/4}(t)\sin(t-\pi/4)$.

Example: Suppose a function equals 0 until $t = 3$ and then starts behaving like the exponential function e^t, meaning like ths exponential function does at $t = 0$. This new function would be $u_3(t)e^{t-3}$.

8. **Laplace Transforms and Step Functions**

Step functions have the following Laplace transform related behavior:

$$\mathcal{L}\left[u_c(t)f(t-c)\right] = e^{-cs}\mathcal{L}\left[f(t)\right]$$

This is a bit confusing so for the forward and backwards directions think:

Forward: Pull out the $u_c(t)$ which changes to e^{-cs} and change all $t-c$ to t then continue.

Example: $\mathcal{L}\left[u_3(t)(t-3)^5\right] = e^{-3s}\mathcal{L}\left[t^5\right] = e^{-3s}\left(\frac{5!}{s^6}\right)$

Example: $\mathcal{L}\left[u_\pi(t)\sin(t-\pi)\right] = e^{-\pi s}\mathcal{L}\left[\sin(t)\right] = e^{-\pi s}\left(\frac{1}{s^2+1}\right)$

Example: $\mathcal{L}\left[u_2(t)e^{4(t-2)}(t-2)^5\right] = e^{-2s}\mathcal{L}\left[e^{4t}t^5\right] = e^{-2s}\left(\frac{5!}{(s-4)^6}\right)$

Backward: For $\mathcal{L}\left[y\right] = e^{-cs}J(s)$ first find $j(t)$ with $\mathcal{L}\left[j(t)\right] = J(s)$, replace the t by $t-c$ and put a $u_c(t)$ in front.

Example: If $\mathcal{L}\left[y\right] = e^{-5s}\left(\frac{s}{s^2+49}\right)$ then we note $\mathcal{L}\left[\cos(7t)\right] = \frac{s}{s^2+49}$, replace the t by $t-5$ and put $u_5(t)$ in front, yielding $y(t) = u_5(t)\cos(7(t-5))$.

Example: If $\mathcal{L}\left[y\right] = e^{3s}\left(\frac{5!}{s^6}\right)$ then we note $\mathcal{L}\left[t^5\right] = \frac{5!}{s^6}$, replace the t by $t-(-3)$ and put $u_{(-3)}(t)$ in front, yielding $y(t) = u_{(-3)}(t)(t+3)^5$.

Example: If $\mathcal{L}\left[y\right] = e^{-5s}\left(\frac{6!}{(s-3)^7}\right)$ then we note $\mathcal{L}\left[e^{3t}t^6\right] = \frac{6!}{(s-3)^7}$, replace the t by $t-5$ and put $u_5(t)$ in front, yielding $y(t) = u_5(t)e^{3(t-5)}(t-5)^7$.

This can then be tied into initial value problems.

Example: Suppose $y' - 2y = f(t)$ where

$$f(t) = \begin{cases} 0 & t < 3 \\ t-3 & t \geq 3 \end{cases}$$

and where $y(0) = 0$.
We first note that $f(t) = u_3(t)(t-3)$ and then proceed:

$$\begin{aligned}
y' - 2y &= u_3(t)(t-3) \\
\mathcal{L}[y'] - 2\mathcal{L}[y] &= \mathcal{L}[u_3(t)(t-3)] \\
s\mathcal{L}[y] - y(0) - 2\mathcal{L}[y] &= e^{-3s}\mathcal{L}[t] \\
s\mathcal{L}[y] - 2\mathcal{L}[y] &= e^{-3s}\left(\frac{1}{s^2}\right) \\
\mathcal{L}[y](s-2) &= e^{-3s}\left(\frac{1}{s^2}\right) \\
\mathcal{L}[y] &= e^{-3s}\left(\frac{1}{s^2(s-2)}\right) \\
&\text{...Partial Fractions Not Shown...} \\
\mathcal{L}[y] &= e^{-3s}\underbrace{\left(\frac{-1/4}{s} - \frac{1/2}{s^2} + \frac{1/4}{s-2}\right)}_{\text{Yields: } -\frac{1}{4}-\frac{1}{2}t+\frac{1}{4}e^{2t}} \\
y(t) &= u_3(t)\left(-\frac{1}{4} - \frac{1}{2}(t-3) + \frac{1}{4}e^{2(t-3)}\right)
\end{aligned}$$

Example: Suppose $y'' - y' - 2y = f(t)$ where

$$f(t) = \begin{cases} 0 & t < 3 \\ 7 & t \geq 3 \end{cases}$$

and where $y(0) = 0$ and $y'(0) = -2$.
We first note that $f(t) = 7u_3(t)$ and then proceed:

$$\begin{aligned}
y'' - y' - 2y &= 7u_3(t) \\
\mathcal{L}\left[y''\right] - \mathcal{L}\left[y'\right] - 2\mathcal{L}\left[y\right] &= 7\mathcal{L}\left[u_3(t)\right] \\
s^2\mathcal{L}\left[y\right] - sy(0) - y'(0) - (s\mathcal{L}\left[y\right] - y(0)) - 2\mathcal{L}\left[y\right] &= 7e^{-3t} \\
s^2\mathcal{L}\left[y\right] - s + 2 - s\mathcal{L}\left[y\right] - 2\mathcal{L}\left[y\right] &= 7e^{-3t} \\
\mathcal{L}\left[y\right](s^2 - s - 2) - s + 2 &= 7e^{-3t} \\
\mathcal{L}\left[y\right](s^2 - s - 2) &= 7e^{-3t} + s - 2 \\
\mathcal{L}\left[y\right] &= 7e^{-3t}\left(\frac{1}{s^2 - s - 2}\right) + \frac{s-2}{s^2 - s - 2} \\
\mathcal{L}\left[y\right] &= 7e^{-3t}\left(\frac{1}{(s-2)(s+1)}\right) + \frac{1}{s+1} \\
&\text{...Partial Fractions Not Shown...} \\
\mathcal{L}\left[y\right] &= 7e^{-3t}\underbrace{\left(\frac{1/3}{s-2} - \frac{1/3}{s+1}\right)}_{\text{Yields: } \frac{1}{3}e^{2t} - \frac{1}{3}e^{-t}} + \frac{1}{s+1} \\
y(t) &= 7u_3(t)\left(\frac{1}{3}e^{2(t-3)} - \frac{1}{3}e^{-1(t-3)}\right) + e^{-t}
\end{aligned}$$

Example: Suppose $y'' - y' = f(t)$ where

$$f(t) = \begin{cases} 0 & t < \pi/4 \\ \cos(t - \pi/4) & t \geq \pi/4 \end{cases}$$

and wher $y(0) = 0$ and $y'(0) = 0$. We first note that $f(t) = u_{\pi/4}(t)\cos(t - \pi/4)$ and then proceed:

$$\begin{aligned}
y'' - y' &= u_{\pi/4}\cos(t - \pi/4) \\
\mathcal{L}[y''] - \mathcal{L}[y'] &= \mathcal{L}\left[u_{\pi/4}\cos(t - \pi/4)\right] \\
\left(s^2\mathcal{L}[y] - sy(0) - y'(0)\right) - (s\mathcal{L}[y] - y(0)) &= e^{-(\pi/4)s}\mathcal{L}[\cos(t)] \\
s^2\mathcal{L}[y] - s\mathcal{L}[y] &= e^{-(\pi/4)s}\left(\frac{s}{s^2+1}\right) \\
\mathcal{L}[y](s^2 - s) &= e^{-(\pi/4)s}\left(\frac{s}{s^2+1}\right) \\
\mathcal{L}[y] &= e^{-(\pi/4)s}\left(\frac{s}{(s^2+1)(s^2-s)}\right) \\
\mathcal{L}[y] &= e^{-(\pi/4)s}\left(\frac{s}{(s^2+1)s(s-1)}\right) \\
&\text{...Partial Fractions Not Shown...} \\
\mathcal{L}[y] &= e^{-(\pi/4)s}\left(\frac{-\frac{1}{2}s - \frac{1}{2}}{s^2+1} + \frac{\frac{1}{2}}{s-1}\right) \\
\mathcal{L}[y] &= -\frac{1}{2}e^{-(\pi/4)s}\left(\frac{s+1}{s^2+1} - \frac{1}{s-1}\right) \\
\mathcal{L}[y] &= -\frac{1}{2}e^{-(\pi/4)s}\underbrace{\left(\frac{s}{s^2+1} + \frac{1}{s^2+1} - \frac{1}{s-1}\right)}_{\text{Yields: } \cos(t)+\sin(t)-e^t} \\
y(t) &= -\frac{1}{2}u_{\pi/4}(t)\left(\cos(t - \pi/4) + \sin(t - \pi/4) - e^{(t-\pi/4)}\right)
\end{aligned}$$