MATH 246: Chapter 3 Section 1: Intro to First Order Systems
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Main Topics:

- Introduction
- Rewriting Single Higher Order as Systems
- Multiple Tank Problems

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1. **Introduction**

All that we've studied so far are DEs involving a single function y depending on a single variable t. In the real world things can get far more complicated. As a classic example consider a predator-prey situation. The rate of growth of the prey depends on both the number of prey and the number of predators, possibly as well as time, and similarly for the rate of growth of the predators.

What we'll study now are systems of first-order DEs. In the most basic case we'll have two functions $x_1(t)$ and $x_2(t)$ in which their derivatives x_1' and x_2' depend on both x_1 and x_2 themselves and maybe on other functions of t.

The goal will be to find two functions $x_1(t)$ and $x_2(t)$ which simultaneously satisfy the system.

Example:

$$x_1' = x_1 - x_2$$
$$x_2' = -3x_1 - x_2$$

In this example the pair $x_1(t) = 2e^{-2t}$ and $x_2(t) = 6e^{-2t}$ form a solution.

Example:

$$x_1' = 2t^2x_1 - 3x_2 + \cos t$$
$$x_2' = x_1 + 4tx_2 - 2t$$

In this example a solution pair is not easy at all.

We can also have more functions and equations.

Example:

$$x_1' = 2x_1 - 3x_2 - x_3 + \cos t$$
$$x_2' = x_1 + 4x_2 - 2t$$
$$x_3' = -x_1 + 20x_2 - 7x_3 + e^t$$

In addition we could have an initial value, which would mean an initial value for each of the functions.

Example: Here is an IVP.

$$x_1' = tx_1 - 3x_2$$
$$x_2' = x_1 + 4t^2x_2$$

With $x_1(0) = 2$ and $x_2(0) = -3$.

2. **Rewriting Single Higher-Order as Systems**

Single higher-order DEs can be rewritten as systems of first-order DEs. This may be useful as we go on to develop methods of solving systems. The general idea for an n^{th} order DE will be to rewrite it as a system of n first-order DEs.

(a) When dealing with just the DE part it's simple. For an n^{th} order system we assign:

$$\begin{aligned} x_1 &= y \\ x_2 &= Dy \\ x_3 &= D^2 y \\ \vdots &= \vdots \\ x_{n-1} &= D^{n-2} y \\ x_n &= D^{n-1} y \end{aligned}$$

The first $n-1$ then give us:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= x_4 \\ \vdots &= \vdots \\ x_{n-1}' &= x_n \end{aligned}$$

We get one more $x_n' = ...$ from the DE because $x_n' = y^{(n)}$, which we can find, and replacing all the other derivatives by their respective x_i.

Example: Consider $D^2y + tDy - 3y = t$. We assign:

$$\begin{aligned} x_1 &= y \\ x_2 &= Dy \end{aligned}$$

The first then gives us:

$$x_1' = y' = Dy = x_2$$

The differential equation gives us:

$$x_2' = D^2 y = t - tDy - 3y = t - tx_2 - 3x_1$$

Thus our final system is:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -3x_1 - tx_2 + t \end{aligned}$$

Example: Consider $D^3y - 2D^2y + tDy - e^t y = \sin t$. We assign and get:

$$x_1 = y$$
$$x_2 = Dy$$
$$x_3 = D^2y$$

The first two of these give us:

$$x_1' = x_2$$
$$x_2' = x_3$$

The third $x_3' = ...$ comes from the DE and is:

$$x_3' = D^3y = 2D^2y - tDy + e^t y + \sin t = 2x_3 - tx_2 + e^t x_1 + \sin t$$

All together we have the system:

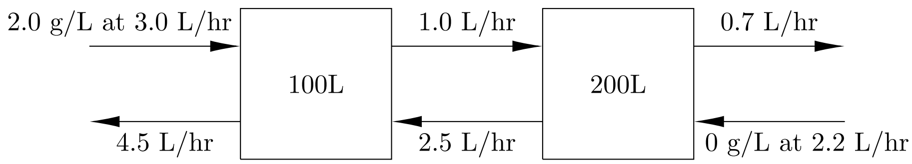
$$x_1' = x_2$$
$$x_2' = x_3$$
$$x_3' = 2x_3 - tx_2 + e^t x_1 + \sin t$$

(b) With Initial Values:
Initial values are easy to rewrite. Since we know $y(t_I), Dy(t_I), ..., D^{n-1}y(t_I)$ these just become $x_1(t_I), x_2(t_I), ..., x_n(t_I)$.

Example: Consider $D^2y + tDy - 3y = t$ again. Suppose we know that $y(1) = 2$ and $y'(1) = -3$. We set $x_1 = y$ and $x_2 = Dy$. Then we know $x_1(1) = 2$ and $x_2(1) = y'(1) = -3$.

3. **Tank Problems**

A classic example of these are tank problems. Imagine two tanks containing salt water. Water is being pumped into and out of these tanks in a variety of ways. For example in the following scenario there are two tanks. The one on the left contains 100L and the one on the right 200L. These quantities do not change in this example because the liters into each equals the gallons out.



The quantity 2.0 g/L at 3.0 L/hr indicates that salt water with that density is flowing into the left tank at that rate, and the quantity 0 g/L at 2.2 L/hr indicates the same for the right tank.

The other quantities do not have densities becase they are assumed to be mixtures from the tank.

Imagine now that x_1 is the amount of salt in the left tank and x_2 is the amount of salt in the right tank, each at time t. Then the density of salt in the left tank is $\frac{x_1}{100}$ and in the right tank is $\frac{x_2}{200}$. The entire scenario is then modeled by the system:

$$x_1' = [\text{Rate of Salt In}] - [\text{Rate of Salt Out}]$$
$$x_2' = [\text{Rate of Salt In}] - [\text{Rate of Salt Out}]$$

which is:

$$x_1' = +(2)(3) + (x_2/200)(2.5) - (x_1/100)(4.5) - (x_1/100)(1)$$
$$x_2' = +(x_1/100)(1) + (0)(2.2) - (x_2/200)(0.7) - (x_2/200)(2.5)$$

This simplifies to:

$$x_1' = -0.055x_1 + 0.0125x_2 + 6$$
$$x_2' = 0.01x_1 - 0.0475x_2$$

Suppose in addition we know that at time $t = 0$ there is 10g of salt in the left tank and 20g of salt in the right tank. Then we can add in the initial value $x_1(0) = 10$ and $x_2(0) = 20$.