MATH 246: Chapter 3 Section 2: Basic Theory and Notation for Systems
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Main Topics:

- Linear Algebra Notation
- Theory for Homogeneous
- The Fundamental Matrix
- Comment on Nonhomogeneous

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1. **Matrix and Vector Notation for Systems of First Order Linear DEs**

   Consider the following:

   - A first order linear system of differential equations may be written:
   $$\bar{x}' = A(t)\bar{x} + \bar{f}(t)$$
   - This system is homogeneous when $\bar{f}(t) = \bar{0}$ and we say the system has constant coefficients when the matrix $A(t)$ is all constants.
   - An initial value can then be written as $\bar{x}(t_I) = \bar{x}_I$,
   - The solution can then be given as a single $\bar{x}$.

Example: Consider the following initial value problem:

$$x_1' = 3x_1 + 2tx_2 + t \qquad x_1(0) = 1$$
$$x_2' = t^2x_1 + 3x_2 \qquad x_2(0) = -2$$

This can be rewritten as:

$$\underbrace{\begin{bmatrix} x_1' \\ x_2' \end{bmatrix}}_{\bar{x}'} = \underbrace{\begin{bmatrix} 3 & 2t \\ t^2 & 3 \end{bmatrix}}_{A(t)} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\bar{x}} + \underbrace{\begin{bmatrix} t \\ 0 \end{bmatrix}}_{\bar{f}(t)} \text{ with } \underbrace{\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}}_{\bar{x}(0)} = \underbrace{\begin{bmatrix} 1 \\ -2 \end{bmatrix}}_{\bar{x}_I}$$

Thus more simply:

$$\bar{x}' = \begin{bmatrix} 3 & 2t \\ t^2 & 3 \end{bmatrix}\bar{x} + \begin{bmatrix} t \\ 0 \end{bmatrix} \text{ with } \bar{x}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Example: Consider the following solution:

$$\bar{x}' = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}\bar{x}$$

One solution to this is:

$$\bar{x} = \begin{bmatrix} e^{5t} \\ e^{5t} \end{bmatrix}$$

This can be checked with a matrix calculation:

$$\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}\bar{x} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} e^{5t} \\ e^{5t} \end{bmatrix} = \begin{bmatrix} 3e^{5t} + 2e^{5t} \\ 2e^{5t} + 3e^{5t} \end{bmatrix} = \begin{bmatrix} 5e^{5t} \\ 5e^{5t} \end{bmatrix} = \bar{x}'$$

2. **Theory for Homogeneous - Fundamental Sets and Fundamental Matrices**

   Note: In what follows I've written $n = 2$ to mean that I'm giving a specific example that generalizes. You could substitute $n = 3, 4, ...$ and the theory would still be good. In cases where it's not clear what happens for 3,4,... I've said more.

   Theory: A homogeneous system of $n = 2$ DEs has a fundamental pair/set consisting of $n = 2$ solutions $\bar{x}_1$ and $\bar{x}_2$ (more if $n \geq 3$)

   A fundamental set has nonzero Wronskian where

   $$W[\bar{x}_1, \bar{x}_2] = |\bar{x}_1 \ \ \bar{x}_2|$$

   That determinant is just found by dumping the vectors $\bar{x}_1$ and $\bar{x}_2$ together in in a matrix and going from there.

   When we have a fundamental pair/set the matrix used to determine the Wronskian is called the *fundamental matrix* and is usually denoted Ψ or $\Psi(t)$.

   The general solution to the system then consists of all linear combinations of those $n = 2$ solutions, this can be written several ways:

   $$\bar{x}(t) = c_1\bar{x}_1 + c_2\bar{x}_2 = [\bar{x}_1 \ \ \bar{x}_2] \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \Psi(t) \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \Psi(t)\bar{c}$$

   Example: Consider the system

   $$\bar{x}' = \begin{bmatrix} t^2 & 2t - t^4 \\ 1 & -t^2 \end{bmatrix} \bar{x}$$

   This has solutions (calculation omitted) $\{\bar{x}_1, \bar{x}_2\} = \left\{ \begin{bmatrix} 1 + t^3 \\ t \end{bmatrix}, \begin{bmatrix} t^2 \\ 1 \end{bmatrix} \right\}$.

   These form a fundamental pair because $W[\bar{x}_1, \bar{x}_2] = \begin{vmatrix} 1 + t^3 & t^2 \\ t & 1 \end{vmatrix} = 1 \not\equiv 0$.

   Consequently the general solution to the system is:

   $$\bar{x}(t) = c_1 \begin{bmatrix} 1 + t^3 \\ t^2 \end{bmatrix} + c_2 \begin{bmatrix} t^2 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1(1 + t^3) + c_2t^2 \\ c_1t^2 + c_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 + t^3 & t^2 \\ t^2 & 1 \end{bmatrix}}_{\Psi(t)} \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{\bar{c}}$$

   It's worth noting that if we go back and think of the original problem as:

   $$x_1' = t^2x_1 + (2t - t^4)x_2$$
   $$x_2' = x_2 - t^2x_2$$

   Then the general solution is:

   $$x_1 = c_1(1 + t^3) + c_2t^2$$
   $$x_2 = c_1t + c_2$$

Example 1: Consider the system

$$\bar{x}' = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \bar{x}$$

This has solutions (calculation omitted) $\{\bar{x}_1, \bar{x}_2\} = \left\{ \begin{bmatrix} e^{5t} \\ e^{5t} \end{bmatrix}, \begin{bmatrix} e^{t} \\ -e^{t} \end{bmatrix} \right\}$.

These form a fundamental pair because $W[\bar{x}_1, \bar{x}_2] = \begin{vmatrix} e^{5t} & e^{t} \\ e^{5t} & -e^{t} \end{vmatrix} = -2e^{6t} \not\equiv 0$.

Consequently the general solution to the system is:

$$\bar{x}(t) = c_1 \begin{bmatrix} e^{5t} \\ e^{5t} \end{bmatrix} + c_2 \begin{bmatrix} e^{t} \\ -e^{t} \end{bmatrix} = \begin{bmatrix} c_1 e^{5t} + c_2 e^{t} \\ c_1 e^{5t} - c_2 e^{t} \end{bmatrix} = \underbrace{\begin{bmatrix} e^{5t} & e^{t} \\ e^{5t} & -e^{t} \end{bmatrix}}_{\Psi(t)} \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{\bar{c}}$$

If we turn this into an initial value problem with the initial condition:

$$\bar{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Then we can find the specific solution using a simple matrix calculation:

$$\begin{aligned} \bar{x}(0) &= \Psi(0)\bar{c} \\ \begin{bmatrix} 1 \\ 2 \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \bar{c} \\ \bar{c} &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ \bar{c} &= \frac{1}{-2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ \bar{c} &= -\frac{1}{2} \begin{bmatrix} -3 \\ 1 \end{bmatrix} \\ \bar{c} &= \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix} \end{aligned}$$

Hence the specific solution can be written a few ways:

$$\bar{x}(t) = \Psi(t)\bar{c} = \begin{bmatrix} e^{5t} & e^{t} \\ e^{5t} & -e^{t} \end{bmatrix} \begin{bmatrix} 3/2 \\ -1/2 \end{bmatrix} = \frac{3}{2} \begin{bmatrix} e^{5t} \\ e^{5t} \end{bmatrix} - \frac{1}{2} \begin{bmatrix} e^{t} \\ -e^{t} \end{bmatrix} = \begin{bmatrix} \frac{3}{2}e^{5t} - \frac{1}{2}e^{t} \\ \frac{3}{2}e^{5t} + \frac{1}{2}e^{t} \end{bmatrix}$$

It's worth noting that if we go back and think of the original problem as the IVP:

$$\begin{aligned} x_1' &= 3x_1 + 2x_2 & \qquad x_1(0) &= 1 \\ x_2' &= 2x_1 + 3x_2 & \qquad x_2(0) &= 2 \end{aligned}$$

Then the specific solution is:

$$\begin{aligned} x_1 &= \frac{3}{2}e^{5t} - \frac{1}{2}e^{t} \\ x_2 &= \frac{3}{2}e^{5t} + \frac{1}{2}e^{t} \end{aligned}$$

3. **Natural Fundamental Sets and Matrices**

The natural fundamental matrix associated to a specific t_I is a specific fundamental matrix which is incredibly useful.

Suppose we solve the two initial value problems:

$$\bar{x}' = A\bar{x} \text{ with } \bar{x}(t_I) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \longrightarrow \text{call this solution } x_1$$

and

$$\bar{x}' = A\bar{x} \text{ with } \bar{x}(t_I) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \longrightarrow \text{call this solution } x_2$$

we get what are known together as the *natural fundamental set associated to* t_I and if we put these together in a matrix we get the *natural fundamental matrix associated to* t_I which is denoted $\Phi(t)$ or just Φ. There is only one of these for a given t_I.

Example: Consider the system:

$$\bar{x}' = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \bar{x}$$

This has natural fundamental matrix associated to $t_I = 0$ of:

$$\Phi(t) = \begin{bmatrix} \frac{1}{2}e^{5t} + \frac{1}{2}e^t & \frac{1}{2}e^{5t} - \frac{1}{2}e^t \\ \frac{1}{2}e^{5t} - \frac{1}{2}e^t & \frac{1}{2}e^{5t} + \frac{1}{2}e^t \end{bmatrix}$$

This matrix $\Phi(t)$ is incredibly useful. To see this, suppose we have a system given by

$$\bar{x}' = A(t)\bar{x}$$

If we have the natural fundamental matrix $\Phi(t)$ assoicated to some t_I and we wish to solve the initial value problem with:

$$\bar{x}(t_I) = \bar{x}_I = \begin{bmatrix} a \\ b \end{bmatrix}$$

Consider the vector:

$$\bar{x}(t) = \Phi(t)\bar{x}_I$$

Observe that by definition of matrix/vector multiplication we have:

$$\bar{x}(t) = \Phi(t)\bar{x}_I = [\bar{x}_1 \ \bar{x}_2] \begin{bmatrix} a \\ b \end{bmatrix} = a\bar{x}_1 + b\bar{x}_2$$

Since $\bar{x}$ is a linear combination of $\bar{x}_1$ and $\bar{x}_2$ it is a solution to the DE. Moreover observe that:

$$\bar{x}(t_I) = \Phi(t_I)\bar{x}_I = \Phi(t_I)\begin{bmatrix} a \\ b \end{bmatrix} = a\bar{x}_1(t_I) + b\bar{x}_2(t_I) = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} = \bar{x}_I$$

It follows that the solution to the IVP is simply given by:

$$\bar{x}(t) = \Phi(t)\bar{x}_I$$

This is handy when we need to solve repeated initial value problems with the same t_I.

Example:
Revisit the system:

$$\bar{x}' = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}\bar{x}$$

We saw this has natural fundamental matrix associated to $t_I = 0$ of:

$$\Phi(t) = \begin{bmatrix} \frac{1}{2}e^{5t} + \frac{1}{2}e^t & \frac{1}{2}e^{5t} - \frac{1}{2}e^t \\ \frac{1}{2}e^{5t} - \frac{1}{2}e^t & \frac{1}{2}e^{5t} + \frac{1}{2}e^t \end{bmatrix}$$

So now we can throw out solutions to IVPs easily:

- If we have: $\bar{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
  Then the solution is:
  $$\bar{x} = \Phi(t)\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}e^{5t} + \frac{1}{2}e^t & \frac{1}{2}e^{5t} - \frac{1}{2}e^t \\ \frac{1}{2}e^{5t} - \frac{1}{2}e^t & \frac{1}{2}e^{5t} + \frac{1}{2}e^t \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2}e^{5t} - \frac{1}{2}e^t \\ \frac{3}{2}e^{5t} + \frac{1}{2}e^t \end{bmatrix}$$

- If we have: $\bar{x}(0) = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$
  Then the solution is:
  $$\bar{x} = \Phi(t)\begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}e^{5t} + \frac{1}{2}e^t & \frac{1}{2}e^{5t} - \frac{1}{2}e^t \\ \frac{1}{2}e^{5t} - \frac{1}{2}e^t & \frac{1}{2}e^{5t} + \frac{1}{2}e^t \end{bmatrix}\begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} e^{5t} + 3e^t \\ e^{5t} - 3e^t \end{bmatrix}$$

What's even better is that if we have any fundamental matrix $\Psi(t)$ then we can find the natural fundamental matrix for any t_I by calculating:

$$\Phi(t) = \Psi(t)\Psi(t_I)^{-1}$$

Example:
The system:

$$\bar{x}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \bar{x}$$

has fundamental matrix:

$$\Psi(t) = \begin{bmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{bmatrix}$$

Suppose we wish to solve the IVP with $\bar{x}(0) = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$.
We first find Φ associated to $t_I = 0$ by doing the following:

$$\begin{aligned} \Phi &= \Psi(t)\Psi(0)^{-1} \\ &= \begin{bmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{bmatrix} \frac{1}{-1} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \end{aligned}$$

Then the solution to the IVP is given by:

$$\bar{x} = \Phi \bar{x}_I = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 2\cos x + 7\sin x \\ -2\sin x + 7\cos x \end{bmatrix}$$