MATH 246: Chapter 3 Section 5: Using Eigenpairs to Construct Solutions
Justin Wyss-Gallifent

Main Topics:

- Introduction
- Real Eigenvalues with Multiplicty 1
- Real Eigenvalues with Multiplicity 2
- Pairs of Complex Eigenvalues

---

1. **Using Eigenpairs to Construct Solutions:**

   If we go back to $\bar{x}' = A\bar{x}$ observe that if $(\lambda, \bar{v})$ is an eigenpair then it turns out that $\bar{x} = e^{\lambda t}\bar{v}$ is a solution:

   $$\bar{x}' = \frac{d}{dt}\left(e^{\lambda t}\bar{v}\right) = e^{\lambda t}\lambda\bar{v} = e^{\lambda t}A\bar{v} = Ae^{\lambda t}\bar{v} = A\bar{x}$$

   This tells that an eigenpair yields a solution.

   However there are some nuances. The process gets signficantly harder at the 3×3 case and above as we have to start to consider things like eigenspace dimensions and obtaining linearly independent sets of eigenvalues. Consequently we will stay with the 2×2 case.

2. Two Real Eigenvalues

   If we have two real eigenpairs with distinct eigenvectors:

   $$(\lambda_1, \bar{v}_1) \text{ and } (\lambda_2, \bar{v}_2) \text{ with } \lambda_1 \neq \lambda_2$$

   So our two solutions are:

   $$\{\bar{x}_1, \bar{x}_2\} = \left\{e^{\lambda_1 t}\bar{v}_1, e^{\lambda_2 t}\bar{v}_2\right\}$$

   and they will form a fundamental set.

   Example: The system

   $$\bar{x}' = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \bar{x}$$

   has a matrix with eigenpairs

   $$\left(5, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \text{ and } \left(1, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$$

   Therefore we have two solutions

   $$\{\bar{x}_1, \bar{x}_2\} = \left\{e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, e^{t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$$

   So the general solution is:

   $$\bar{x} = C_1 e^{5t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

3. One Real Eigenvalue

   If we have just one eigenpair $(\lambda, \bar{v})$ with multiplicity 2 then the situation is trickier. This eigenpair will give us one solution

$$\bar{x}_1 = \bar{v}e^{\lambda t}$$

   It turns out that a second solution can be obtained by choosing $\bar{w}$ to be a nonzero multiple of $\bar{v}$ and then assigning:

$$\bar{x}_2 = e^{\lambda t}\left(\bar{w} + t\left(A - \lambda I\right)\bar{w}\right)$$

   The proof of why this works is not at all obvious.

   So our two solutions are:

$$\{\bar{x}_1, \bar{x}_2\} = \left\{e^{\lambda t}\bar{v}, e^{\lambda t}\left(\bar{w} + t\left(A - \lambda I\right)\bar{w}\right)\right\}$$

Example:

$$\bar{x}' = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}\bar{x}$$

has matrix with eigenpair:

$$\left(3, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

This gives us one solution

$$\bar{x}_1 = e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

To find another choose $\bar{w}$ to be any non-multiple of $\bar{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, for example $\bar{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Then a second solution is:

$$\begin{aligned} x_2 &= e^{\lambda t}\left(\bar{w} + t\left(A - \lambda I\right)\bar{w}\right) \\ &= e^{3t}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \\ &= e^{3t}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \\ &= e^{3t}\begin{bmatrix} 1+t \\ t \end{bmatrix} \end{aligned}$$

So the general solution is

$$\bar{x} = C_1 e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{3t}\begin{bmatrix} 1+t \\ t \end{bmatrix}$$

4. Two Complex Conjugate Eigenvalues

If we have two complex conjugate eigenpairs then we do get two solutions but they are not real solutions. We've seen this issue before.

To understand the method we will work through an example the long way and then point out that there's a short method. Then we will work through another example with the short method.

Example:

$$\bar{x}' = \begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix} \bar{x}$$

has matrix with eigenpairs

$$\left(3+2i, \begin{bmatrix} 1 \\ i \end{bmatrix}\right) \text{ and } \left(3-2i, \begin{bmatrix} 1 \\ -i \end{bmatrix}\right)$$

Long Way: The first gives us the solution:

$$\begin{aligned}
\bar{x} &= e^{(3+2i)t} \begin{bmatrix} 1 \\ i \end{bmatrix} \\
&= e^{3t} \left(\cos(2t) + i\sin(2t)\right) \begin{bmatrix} 1 \\ i \end{bmatrix} \\
&= e^{3t} \begin{bmatrix} \cos(2t) + i\sin(2t) \\ i\cos(2t) - \sin(2t) \end{bmatrix} \\
&= e^{3t} \begin{bmatrix} \cos(2t) + i\sin(2t) \\ -\sin(2t) + i\cos(2t) \end{bmatrix} \quad ①
\end{aligned}$$

The second gives us the solution:

$$\begin{aligned}
\bar{x} &= e^{(3-2i)t} \begin{bmatrix} 1 \\ -i \end{bmatrix} \\
&= e^{3t} \left(\cos(-2t) + i\sin(-2t)\right) \begin{bmatrix} 1 \\ -i \end{bmatrix} \\
&= \left(e^{3t}\cos(2t) - ie^{3t}\sin(2t)\right) \begin{bmatrix} 1 \\ -i \end{bmatrix} \\
&= e^{3t} \begin{bmatrix} \cos(2t) - i\sin(2t) \\ -i\cos(2t) - \sin(2t) \end{bmatrix} \\
&= e^{3t} \begin{bmatrix} \cos(2t) - i\sin(2t) \\ -\sin(2t) - i\cos(2t) \end{bmatrix} \quad ②
\end{aligned}$$

Since linear combinations of solutions are solutions if we take half of the sum of these we get the solution:

$$\bar{x}_1 = \frac{1}{2}\left[① + ②\right] = e^{3t} \begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix}$$

and if we take $\frac{1}{2i}$ times the difference we get the solution:

$$\bar{x}_1 = \frac{1}{2i}\left[① - ②\right] = e^{3t} \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}$$

Short Way: If we look back at ① and break it into real and imaginary parts then the two solutions can be extracted from the real and imaginary parts:

$$\bar{x} = e^{3t}\begin{bmatrix} \cos(2t) + i\sin(2t) \\ -\sin(2t) + i\cos(2t) \end{bmatrix}$$
$$= \underbrace{e^{3t}\begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix}}_{\bar{x}_1} + i \; \underbrace{e^{3t}\begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}}_{\bar{x}_2}$$

So the general solution is:

$$\bar{x} = C_1 e^{3t}\begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix} + C_2 e^{3t}\begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}$$

We can see now that the short method would be to take one eigenpair:

$$(r + si, \bar{v})$$

Use this to calculate ① and extract the real and imaginary parts. But where did ① come from? It came from doing the calculation:

$$e^{rt}\left(\cos(st) + i\sin(st)\right)\bar{v}$$

So the key is to do this calculation and extract the real and imaginary parts and those will be $\bar{x}_1$ and $\bar{x}_2$.

Example:

$$\bar{x}' = \begin{bmatrix} 2 & 1 \\ -5 & 2 \end{bmatrix} \bar{x}$$

has matrix with eigenpairs

$$\left(2 + i\sqrt{5}, \begin{bmatrix} i\sqrt{5} \\ -5 \end{bmatrix}\right) \text{ and } \left(2 - i\sqrt{5}, \begin{bmatrix} -i\sqrt{5} \\ -5 \end{bmatrix}\right)$$

We then calculate:

$$e^{2t}\left(\cos\left(\sqrt{5}t\right) + i\sin\left(\sqrt{5}t\right)\right)\begin{bmatrix} i\sqrt{5} \\ -5 \end{bmatrix}$$

$$= e^{2t}\begin{bmatrix} -\sqrt{5}\sin\left(\sqrt{5}t\right) + i\sqrt{5}\cos\left(\sqrt{5}t\right) \\ -5\cos\left(\sqrt{5}t\right) - 5i\sin\left(\sqrt{5}t\right) \end{bmatrix}$$

From here we extract the real and imaginary parts:

$$\bar{x}_1 = e^{2t}\begin{bmatrix} -\sqrt{5}\sin\left(\sqrt{5}t\right) \\ -5\cos\left(\sqrt{5}t\right) \end{bmatrix}$$

$$\bar{x}_2 = e^{2t}\begin{bmatrix} -\sqrt{5}\cos\left(\sqrt{5}t\right) \\ -5\sin\left(\sqrt{5}t\right) \end{bmatrix}$$

So the general solution is

$$\bar{x} = C_1 e^{2t}\begin{bmatrix} -\sqrt{5}\sin\left(\sqrt{5}t\right) \\ -5\cos\left(\sqrt{5}t\right) \end{bmatrix} + C_2 e^{2t}\begin{bmatrix} -\sqrt{5}\cos\left(\sqrt{5}t\right) \\ -5\sin\left(\sqrt{5}t\right) \end{bmatrix}$$

Note: Other valid answers can look quite different from this since any multiple of an eigenvector is an eigevector and since complex multiples can look quite surprising.

5. **An Initial Value Problem:** Since we haven't done one from start to finish, here is an initial value problem:

Example: Solve

$$\bar{x}' = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \text{ with } \bar{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

(a) Find the eigenvalues:

$$p(z) = \det \begin{bmatrix} z-5 & -1 \\ 3 & z-1 \end{bmatrix} = (z-5)(z-1) - (-1)(3)$$
$$= z^2 - 6z + 8 = (z-2)(z-4)$$

So the eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 4$.

(b) Find the eigenvectors:

For $\lambda_1 = 2$ choose a nonzero column of $A - \lambda_2 I = \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix}$ so $\bar{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

For $\lambda_2 = 4$ choose a nonzero column of $A - \lambda_1 I = \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix}$ so $\bar{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(c) Write down the general solution:
We have

$$\bar{x} = C_1 e^{2t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + C_2 e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(d) Plug in the initial value and solve:

$$\bar{x}(0) = C_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} \bar{c} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

so that:

$$\bar{c} = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{1}{-2} \begin{bmatrix} 1 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} -1 \\ -3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 3/2 \end{bmatrix}$$

(e) Write down the answer:

$$\bar{x} = \frac{1}{2} e^{2t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \frac{3}{2} e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$