MATH 246: Chapter 3 Section 9: Linearization of Nonlinear Systems
Justin Wyss-Gallifent

Main Topics:

- Linearization
- Stationary Point Analysis

---

1. **Introduction**

   The goal of this section is to do a bit of analysis of nonlinear systems which are not necessarily Hamiltonian. The approach is similar though - find stationary solutions, find what they look like, fill them in, figure out what the remaining solutions look like.

2. **Linearization at the Stationary Solutions**

   This sounds far more complicated than it sounds. In Calculus suppose you know that $f(x) = x^2 - 9$ and you're investigating this function. You might notice that the x-intercepts are $x = \pm 3$ and you might want to know what happens at those points. You might notice that $f'(x) = 2x$ so $f'(-3) = -6$ and $f'(3) = 6$ and so the function is decreasing at $x = -3$ and increasing at $x = 3$. What you just did was that you *linearized* the function at $x = \pm 3$, meaning you sort of made it a line with slope ± 6 at those points.

   What we'll do is precisely the same thing but with a system of nonlinear differential equations.

   Given a system

   $$\frac{dx}{dt} = f(x, y)$$
   $$\frac{dy}{dt} = g(x, y)$$

   First a notation point - sometimes we write

   $$\bar{F}(x, y) = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix}$$

   Using this notation the linearization matrix for this system is the matrix:

   $$\partial \bar{F} = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}$$

   Example: Consider the system:

   $$x' = y$$
   $$y' = 4x - x^3$$

   Since $f(x, y) = y$ and $g(x, y) = 4x - x^3$ the linearization matrix is

   $$\partial \bar{F} = \begin{bmatrix} 0 & 1 \\ 4 - 3x^2 & 0 \end{bmatrix}$$

3. **Stationary Point Analysis**

   What's really cool about this linearization is this. Just like in calculus if you plug in a point the linearization matrix will tell you what's happening at that point. In our case we'll plug in the stationary points. The resulting matrix can be analyzed, more or less, just like the matrices in Chapter 3 Section 6. This means finding the eigenvalues, eigenvectors if necessary, and so on.

   Now then, it's not perfect, but basically we can know if the stationary points are nodal sources, nodal sinks, saddles, radial sources or sinks, spiral sources or sinks, or circles. Basically every case that doesn't have an eigenvalue of zero is still valid.

   If you're curious, this is just like if you discovered that $f'(3) = 6$ you know the function is increasing at that point but if you discovered that $f'(3) = 0$ then the function could be increasing or decreasing or neither at that point.

4. **Stationary Point Analysis**

What's really cool about this linearization is this. Just like in calculus if you plug in a point the linearization matrix will tell you what's happening at that point. In our case we'll plug in the stationary points. The resulting matrix can be analyzed, more or less, just like the matrices in Chapter 3 Section 6. This means finding the eigenvalues, eigenvectors if necessary, and so on.

Now then, it's not perfect, but basically we can know if the stationary points are nodal sources, nodal sinks, saddles, radial sources or sinks, spiral sources or sinks, or circles. Basically every case that doesn't have an eigenvalue of zero is still valid.

If you're curious, this is just like if you discovered that $f'(3) = 6$ you know the function is increasing at that point but if you discovered that $f'(3) = 0$ then the function could be increasing or decreasing or neither at that point.

Example:
Consider the system:

$$\begin{aligned} x' &= 1 - y \\ y' &= x^2 - y^2 \end{aligned}$$

Since $f(x, y) = 1 - y$ and $g(x, y) = x^2 - y^2$ the linearization matrix is

$$\partial \bar{F} = \begin{bmatrix} 0 & -1 \\ 2x & -2y \end{bmatrix}$$

The stationary solutions are $(1, 1)$ and $(-1, 1)$. We check the linearization matrix at those points:

- At $(1, 1)$: $\partial \bar{F}(1, 1) = \begin{bmatrix} 0 & -1 \\ 2 & -2 \end{bmatrix}$.
  The eigenvalues are $\lambda = -1 \pm i$ so the point is a counterclockwise spiral sink.
- At $(-1, 1)$: $\partial \bar{F}(-1, 1) = \begin{bmatrix} 0 & -1 \\ -2 & -2 \end{bmatrix}$. The eigenpairs are
  $\left(-1 + \sqrt{3}, \begin{bmatrix} 1 + \sqrt{3} \\ -2 \end{bmatrix} \approx \begin{bmatrix} 2.7 \\ -2 \end{bmatrix}\right)$ and $\left(-1 - \sqrt{3}, \begin{bmatrix} 1 - \sqrt{3} \\ -2 \end{bmatrix} \approx \begin{bmatrix} -1.7 \\ -2 \end{bmatrix}\right)$.
  This is a saddle.

Example: Consider the system:

$$x' = y$$
$$y' = 4x - x^3$$

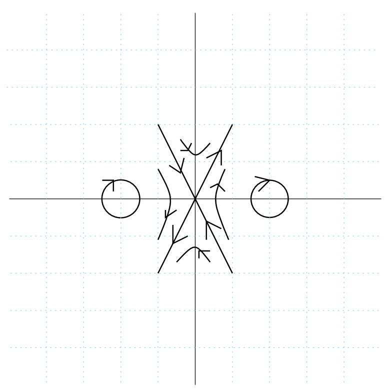
Since $f(x,y) = y$ and $g(x,y) = 4x - x^3$ the linearization matrix is

$$\partial \bar{F} = \begin{bmatrix} 0 & 1 \\ 4 - 3x^2 & 0 \end{bmatrix}$$

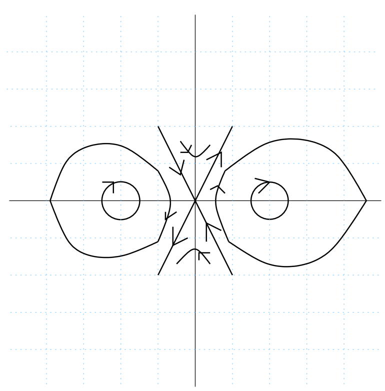
The stationary solutions are $(0,0)$, $(2,0)$ and $(-2,0)$. We check the linearization matrix at those points:

- At $(0,0)$: $\partial \bar{F}(0,0) = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}$. The eigenpairs are $\left(-2, \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right)$ and $\left(2, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$. This is a saddle.
- At $(2,0)$: $\partial \bar{F}(2,0) = \begin{bmatrix} 0 & 1 \\ -8 & 0 \end{bmatrix}$. The eigenvalues are $0 \pm i\sqrt{8}$. This is a clockwise circle since $a_{12} > 0$.
- At $(-2,0)$: $\partial \bar{F}(-2,0) = \begin{bmatrix} 0 & 1 \\ -8 & 0 \end{bmatrix}$. The eigenvalues are $0 \pm i\sqrt{8}$. This is a clockwise circle since $a_{12} > 0$.

Together we get the picture:



From here we can fill in a nice family of solutions:



Example: Consider the system:

$$\begin{aligned} x' &= (y-x)(x-1) \\ y' &= (3+2x-x^2)y \end{aligned}$$

Since $f(x,y) = xy - x^2 - y + x$ and $g(x,y) = 3y + 2xy - x^2y$ the linearization matrix is

$$\partial \bar{F} = \begin{bmatrix} y-2x+1 & x-1 \\ 2y-2xy & 3+2x-x^2 \end{bmatrix}$$

The stationary solutions are $(0,0)$, $(-1,-1)$, $(1,0)$ and $(3,3)$. We check the linearization at those poinst:

- At $(0,0)$: $\partial\bar{F}(0,0) = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix}$. The eigenpairs are $\left(1, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ and $\left(3, \begin{bmatrix} 1 \\ -2 \end{bmatrix}\right)$. This is a source. Solutions close to $(0,0)$ are tangent to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
- At $(-1,-1)$: $\partial\bar{F}(-1,-1) = \begin{bmatrix} 2 & -2 \\ -4 & 0 \end{bmatrix}$. The eigenpairs are $\left(-2, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$ and $\left(4, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$. This is a saddle.
- At $(1,0)$: $\partial\bar{F}(1,0) = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$. The eigenpairs are $\left(-1, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ and $\left(4, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$. This is a saddle.
- At $(3,3)$: $\partial\bar{F}(3,3) = \begin{bmatrix} -2 & 2 \\ 12 & 0 \end{bmatrix}$. The eigenvalues are $-1 \pm i\sqrt{23}$. Since $a_{12} > 0$ this is a clockwise spiral sink.

Together we get the picture: